Direct Proofs

Outline for Today

- ***Mathematical Proof***
  - What is a mathematical proof? What does a proof look like?
- ***Direct Proofs***
  - A versatile, powerful proof technique.
- ***Universal and Existential Statements***
  - What exactly are we trying to prove?
- ***Proofs on Set Theory***
  - Formalizing our reasoning.

What is a Proof?

A ***proof*** is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.

$*54 \cdot 43.$ $\vdash :. \alpha, \beta$ [illegible] $\beta = \Lambda \,.\, \equiv \,.\, \alpha \cup \beta \in 2$

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$\vdash$ [illegible] $y) \,.\, \alpha = \iota`x \,.\, \beta = \iota`y \,.$ [illegible] $2 \,.\, \equiv \,.\, \alpha \cap$ [illegible] (2)

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From this p[illegible]on it will follow, when a[illegible]tion has been defined, that 1 + [illegible]

Modern Proofs

Two Quick Definitions

- An integer n is ***even*** if there is an integer k such that $n = 2k$.
  - This means that 0 is even.
- An integer n is ***odd*** if there is an integer k such that $n = 2k + 1$.
  - This means that 0 is not odd.
- We'll assume the following for now:
  - Every integer is either even or odd.
  - No integer is both even and odd.

Our First Direct Proof

Theorem: If n is an even integer, then n^2 is even.

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This symbol means "end of proof"

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This is the definition of an even integer. When writing a mathematical proof, it's common to call back to the definitions.

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Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.

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Our First Direct Proof

Our ultimate goal is to prove that **n^2** is even. This means that we need to find some **m** such that **$n^2 = 2m$**. Here, we're explicitly showing how we can do that.

Theorem: If n … en.
Proof: Let n b…

Since n…
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This me… that n … $(2k)$ … $4k$ … $2(2k^2)$.

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Hey, that's what we were trying to show! We're done now.

Therefore, n^2 is even. ■

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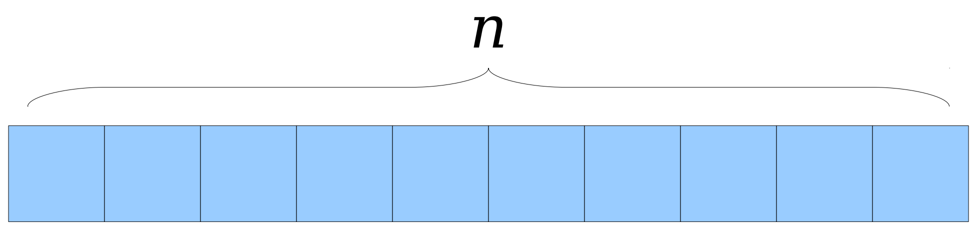
A Visual Intuition



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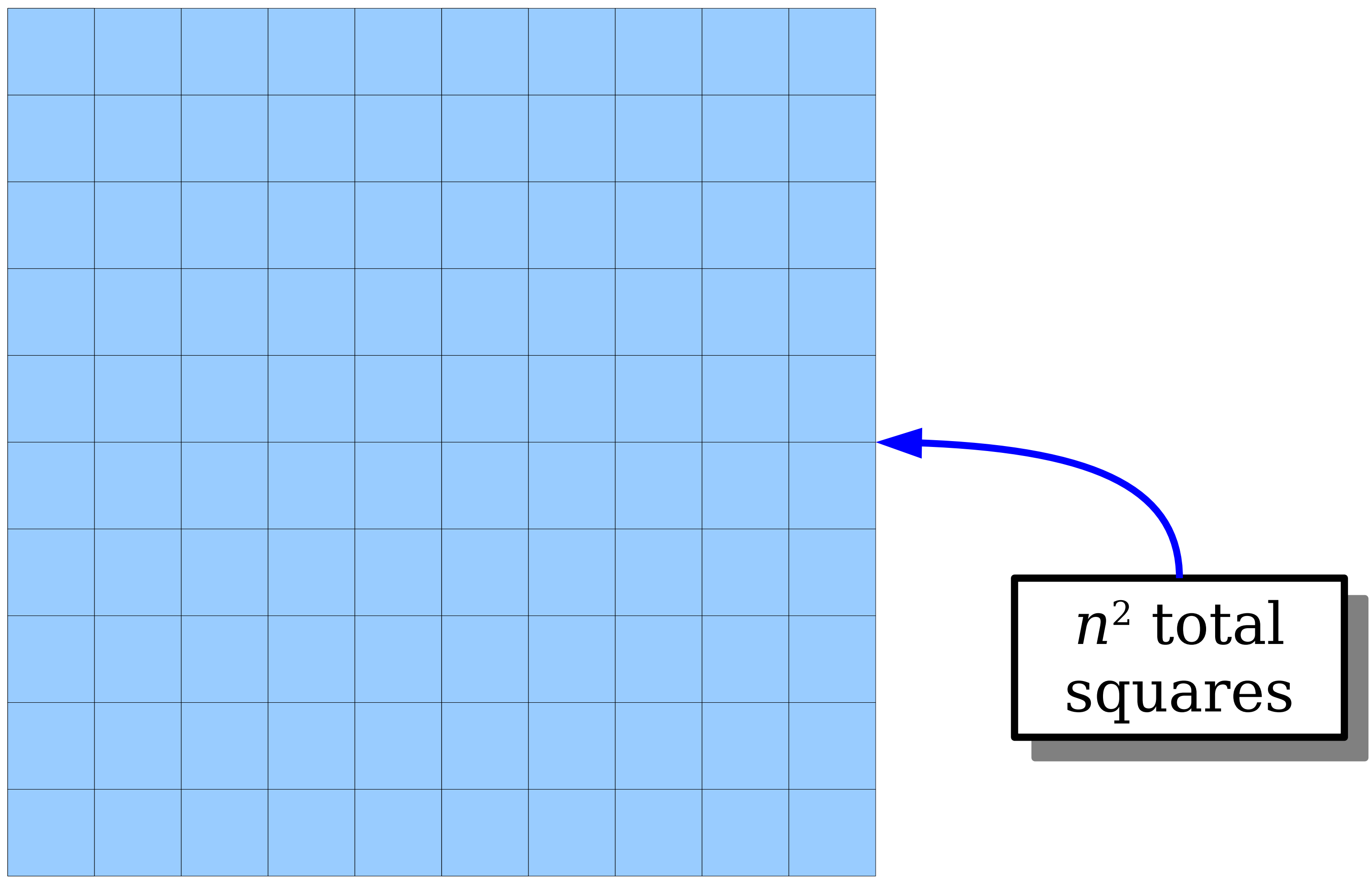

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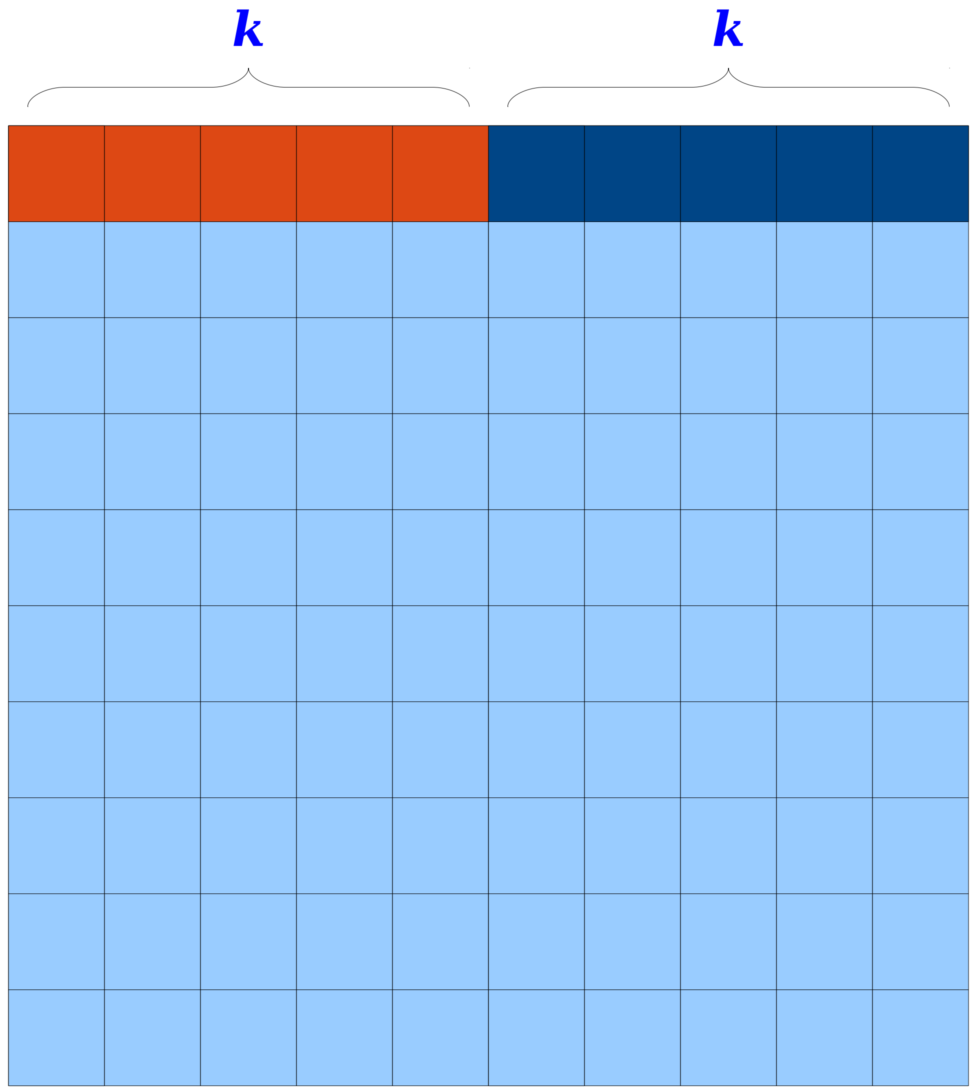

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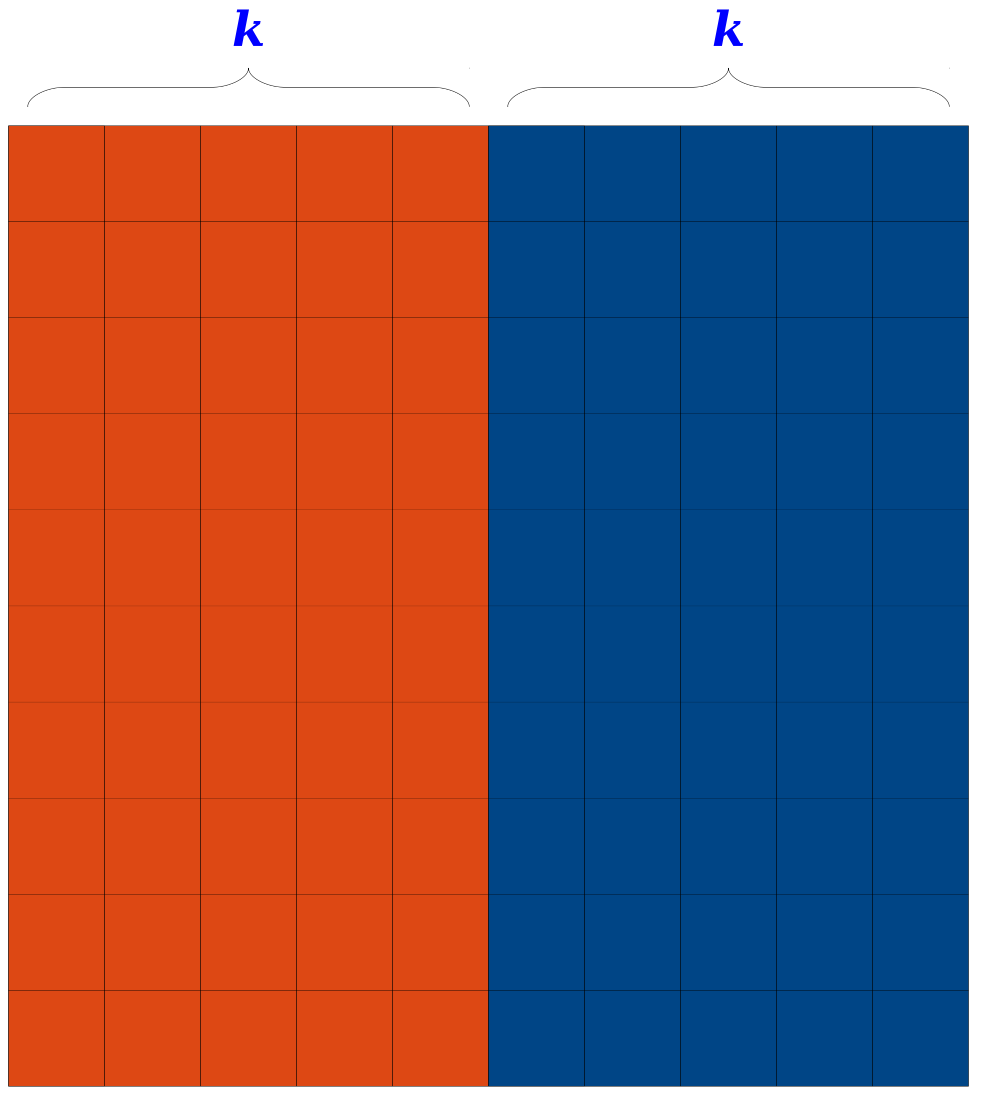

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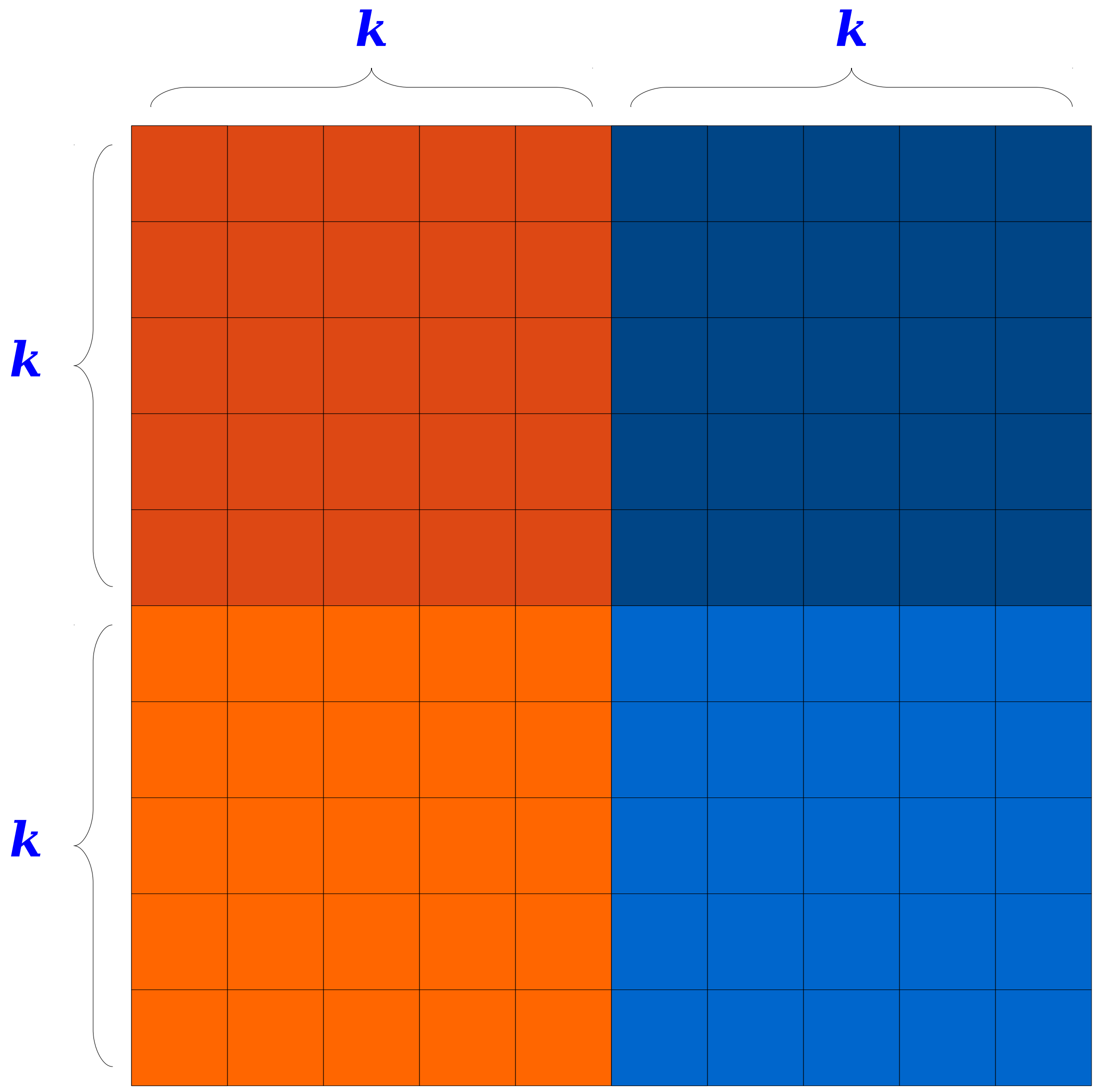

$$(2k)^2 = 2(2k^2)$$

That wasn't so bad! Let's do another one.

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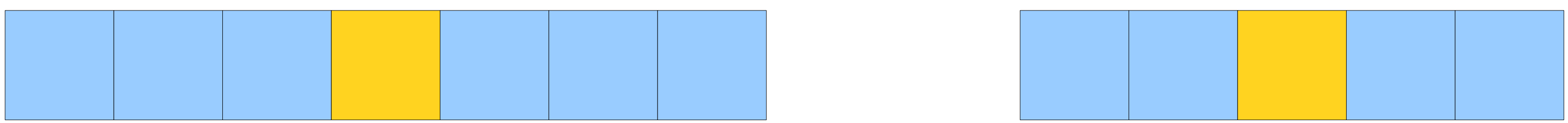

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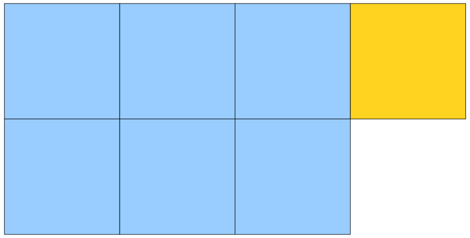

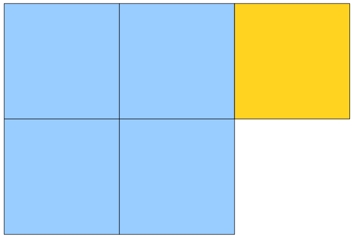

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How do we prove that this is true for *any* integers?

Proving Something Always Holds

- Many statements have the form

For any *x*, `[some-property]` holds of *x*.

- Examples:

For all integers n, if n is even, n^2 is even.

For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

For all sets S: $|S| < |\wp(S)|$.

Everything that drowns me makes me wanna fly.

- How do we prove these statements when there are (potentially) infinitely many cases to check?

Arbitrary Choices

- To prove that some property holds true for all possible x, show that no matter what choice of x you make, that property must be true.
- Start the proof by choosing x ***arbitrarily***:
  - "Let x be an arbitrary even integer."
  - "Let x be any set containing 137."
  - "Consider any x."
  - "Pick an odd integer x."
- Demonstrate that the property holds true for this choice of x.

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By picking ***m*** and ***n*** arbitrarily, anything we prove about ***m*** and ***n*** will generalize to all possible choices we could have made.

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To prove a statement of the form

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Assume that ***P*** is true, then show that ***Q*** must be true as well.

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Proof: Consider any arbitrary integers m and n where m and n are odd. Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

Similarly, because n is odd there must be some integer r such that

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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Notice that we use k in the first equality and r in the second equality. That's because we know that n is twice something plus one, but we can't say for sure that it's k specifically.

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This is a grammatically correct and complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

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Good exercise: look back at the visual intuition for this proof. What do k and r correspond to? Where does the +1 come from?

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Proof by Exhaustion

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$$\ldots \quad \boxed{-3 \quad -2} \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad \ldots$$

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Case 1: n is even.

Case 2: n is odd.

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Proof: Pick any two consecutive integers n and $n+1$. We'll prove that their product $n(n+1)$ is even. Let's consider two cases:

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This is called a proof by cases (alternatively, a proof by exhaustion) and works by showing that the theorem is true regardless of what specific outcome arises.

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Therefore, there is an integer m (namely, $k(n+1)$) such that $n(n+1) = 2m$, so $n(n+1)$ is even.

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Case 2: n is odd. Then there is an integer k where $n = 2k+1$.

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After splitting into cases, it's a good idea to summarize what you just did so that the reader knows what to take away from it.

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Some Little Exercises

- Here’s a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we’ll just take these results for granted. Feel free to use them in the problem sets.
- If you’d like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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Proof: Pick any odd integer n.

This is a very different sort of request than what we've seen in the past. How on earth do we go about proving something like this?

Universal vs. Existential Statements

- A ***universal statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existential statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

- How do you prove an existential statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existential statement of the form

  There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.

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Proof: Pick any odd integer n.

Goal: Discover some choice of r and s that makes this claim hold true.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Pick any odd integer n. Since n is odd, we know there is some integer k where $n = 2k + 1$.

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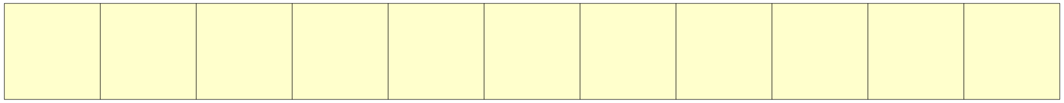

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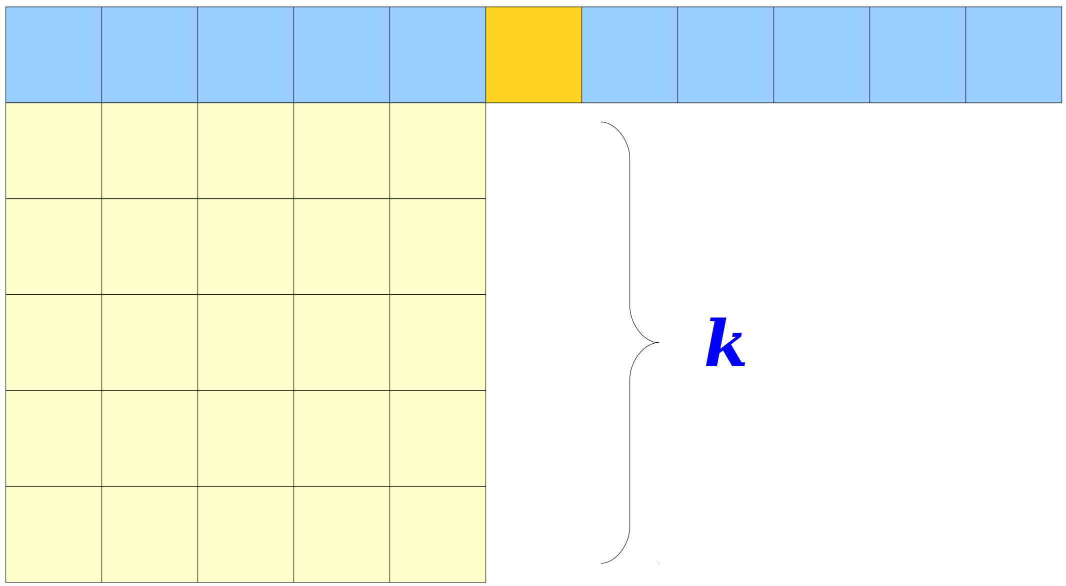

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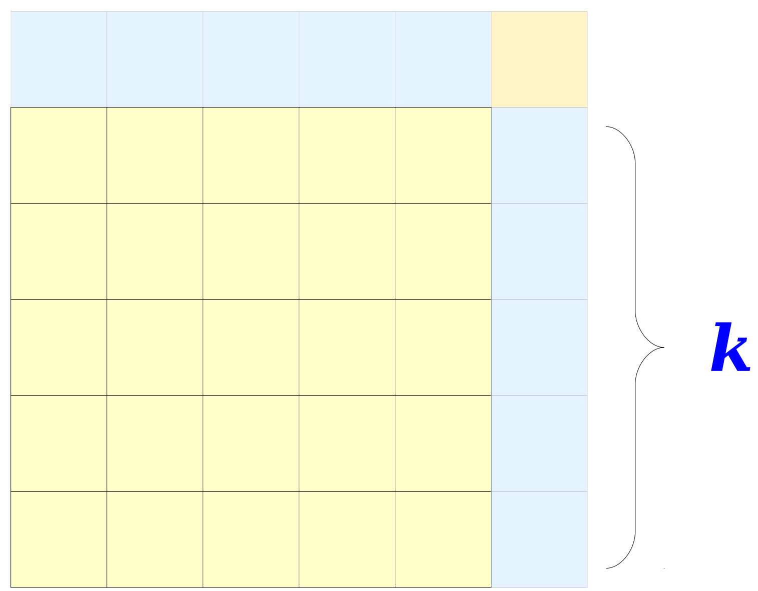

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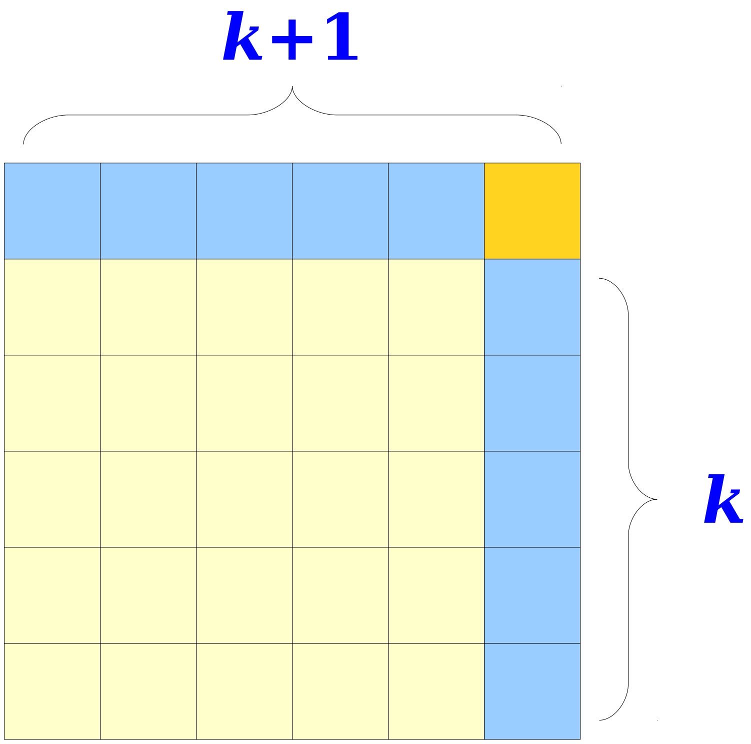
Our guess:

$(k+1)^2 - k^2 = n$



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Now, let $r = k+1$ and $s = k$.

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Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 \quad = \quad (k+1)^2 - k^2$$

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This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Follow-Up Question: There are some integers that can't be written as $r^2 - s^2$ for any integers r and s.

Can you prove that every integer can be formed by adding and subtracting some combination of at most *three* perfect squares?

Time-Out for Announcements!

Campus-Wide Announcements

WiCS
Frosh Intern Program

{ Curious about CS? Looking for a community on campus? Excited about the WiCS mission? }

Apply for the WiCS Frosh Intern Program at bit.ly/wics-frosh-intern-1819

{ Frosh interns rotate through different WiCS teams, work on meaningful projects, and join a community of lifelong friends and mentors }

Applications are due Friday, Oct. 5 at 11:59 PM

Stanford Women in Computer Science {W}

[tl;dr] CS+Social Good is looking for bright, driven, and excited individuals to join our leadership team! Apply!

Have you ever thought that critically thinking about the impact of technology of society and/or using technology to effect meaningful change are maybe kinda, like, lowkey good ideas? If so, join the club! CS+Social Good is looking for folks who are committed to social impact, excited about technology, and motivated to inspire change.

☞Apply here! ☜

You'll be leading some of the most exciting projects and events on campus, all centered around how technology can be a force for good. This year, we're teaching a class that builds projects in partnership with nonprofits & social ventures, funding summer fellowships for students to work on impactful projects all over the world, helping high school teachers teach tech for good in AP CS classes — and much more! Read more about what we did last year here or visit our website at cs4good.org.

If you have any questions, drop us a line at cs4good@cs.stanford.edu. Applications are open here until 11:59 PM on Friday, October 5.

Engineers for a Sustainable World

DESIGN • EDUCATE • BUILD

Interested in engineering and sustainable development?

Join Engineers for a Sustainable World!

ESW is a student group that runs projects addressing global and local challenges around engineering, sustainability, and social impact.

Past and future projects include:
-Building remote monitoring systems with Indonesian rural development NGO IBEKA
- Partnerships with Engineers Without Borders Malaysia
- Sunpower solar panel installations
- Patagonia Worn Wear Truck
- Repair Cafes

Find out more at our first meeting:
Tuesday October 2, 8:00 PM, Crothers Big Lounge

To learn more, fill out this form:
https://goo.gl/forms/Q6QqESh0CD6X02qx2 or reach out to riyav@stanford.edu

Interested in improving online education software for underprivileged students across the world? Enjoy growing a social forum for migrants to become more comfortable in their new cultures? Wish you could help UNICEF visualize mobility data?

Code the Change is a student-led group that works on year-long social good projects. We are looking for developers and designers to help social good organizations with their CS needs.

If interested, consider applying **here** and check out our **website**. Applications are due October 3rd. Be sure to also attend our info session this Saturday in the Haas Center DK Room from 4-5pm (there will be pizza!).

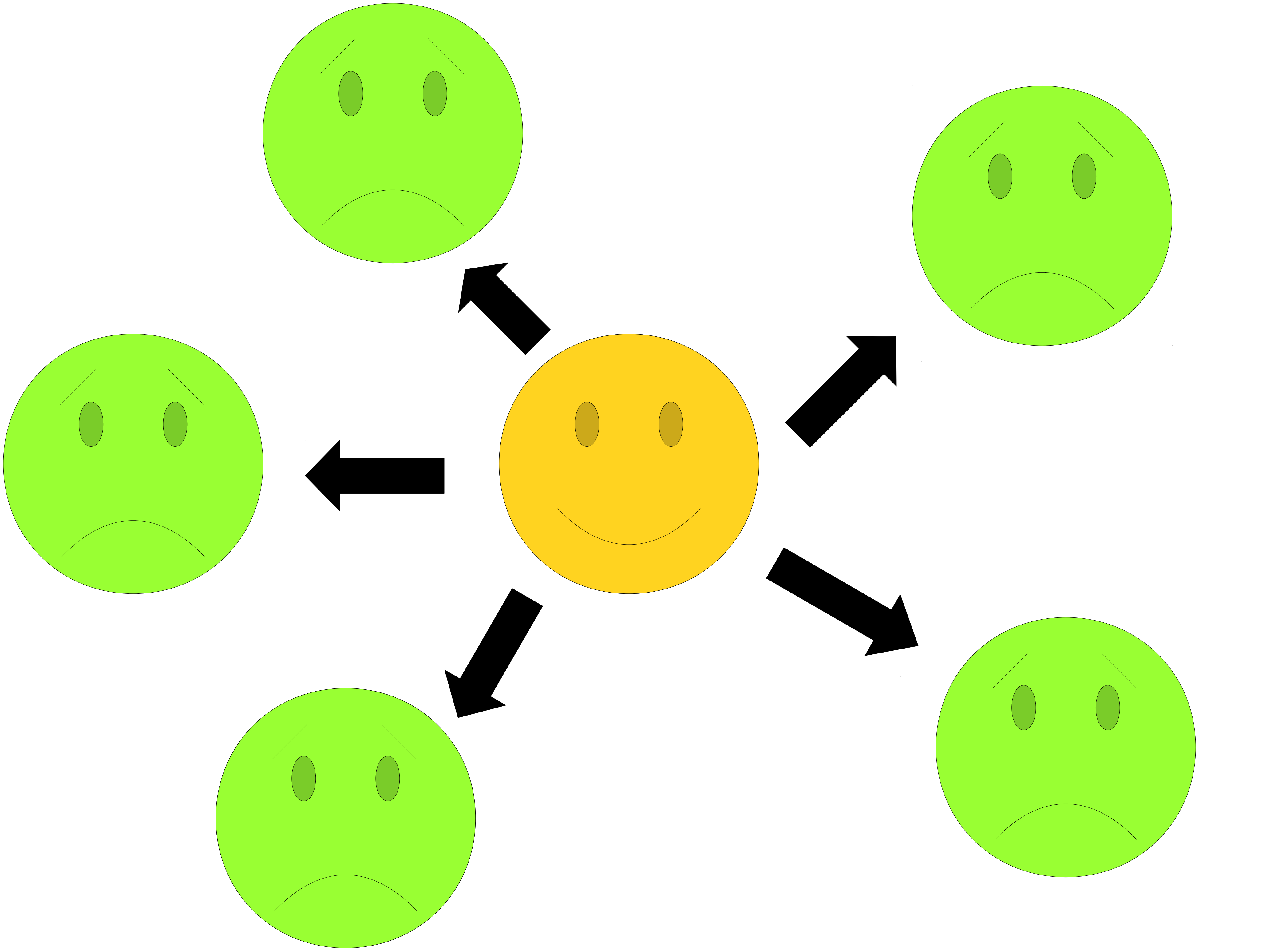
If you have any questions, feel free to email **drewgreg@stanford.edu**.

CURIS Poster Session

- CURIS, the CS department’s undergrad research program, is hosting a poster session showcasing last summer’s research projects.
- Held ***Friday, September 28*** from ***3:00PM – 5:00PM*** on the lawn of the Packard building.
- Interested in learning what research is like? Want to see what undergraduates have been up to? Stop on by!

Register to Vote

- Want to register to vote in Santa Clara county? Feel free to pick up a voter registration form up front.
- If you are eligible to vote in the US, please do so! It's really important.



Flu Shots!

- The *selfish* reason to get a flu shot: the flu is horrible. Don’t get it.
- The *altruistic* reason to get a flu shot: the flu is horrible. Don’t give it to your friends or family.
- Stanford offers free flu shots. Stop by Vaden between 3:00PM – 6:00PM on Monday, October 1st to get one.
- There are lots of other times; for more information, check **this link**.

Some CS103 Announcements!

Reading Recommendations

- We’ve released two handouts online that you should read over:
  - Handout 06: How to Succeed in CS103
  - Handout 07: Guide to Set Theory Proofs
- Additionally, if you haven’t yet read over the Guide to Elements and Subsets, we’d recommend doing so.
- Finally, we strongly recommend reading over Chapter 1 and Chapter 2 of the online course reader to get some more background with proofs and set theory.

Piazza

- We have a Piazza site for CS103.
- Sign in to www.piazza.com and search for the course CS103 to sign in.
- Feel free to ask us questions!
- ***Use the site to find a partner for the problem sets!***

Qt Creator Help Session

- The lovely CS106B/X folks have invited all y'all to join them for a Qt Creator Help Session this evening if you're having trouble getting Qt Creator up and running on your system.
- Runs ***7:00PM – 9:00PM*** in the Tresidder first floor lounge.
- SCPD students – please reach out to us if you need help setting things up. We'll do our best to help out.

Problem Set 0

- Problem Set 0 went out on Monday. It's due this Friday at 2:30PM.
  - Even though this just involves setting up your compiler and submitting things, please start this one early. If you start things on Friday morning, we can't help you troubleshoot Qt Creator issues!
  - There's a very detailed troubleshooting guide up on the CS103 website and a Piazza post detailing common fixes. If you're still having trouble, please feel free to ask on Piazza!

Back to CS103!

Proofs on Sets

Set Theory Review

- Recall from last time that we write $x \in S$ if x is an element of set S and $x \notin S$ if x is not an element of set S.
- If S and T are sets, we say that S is a subset of T (denoted $S \subseteq T$) if the following statement is true:

  For every object x, if $x \in S$, then $x \in T$.

- Let's explore some properties of the subset relation.

Theorem: For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

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To prove a statement of the form

"If *P*, then *Q*"

Assume that ***P*** is true, then show that ***Q*** must be true as well.

Theorem: For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let A, B, and C be arbitrary sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$.

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Notice that the original theorem says nothing about a variable x, but our proof needs one anyway. If you proceed slowly and "unpack" definitions like we're doing here, you'll often find yourself introducing extra variables.

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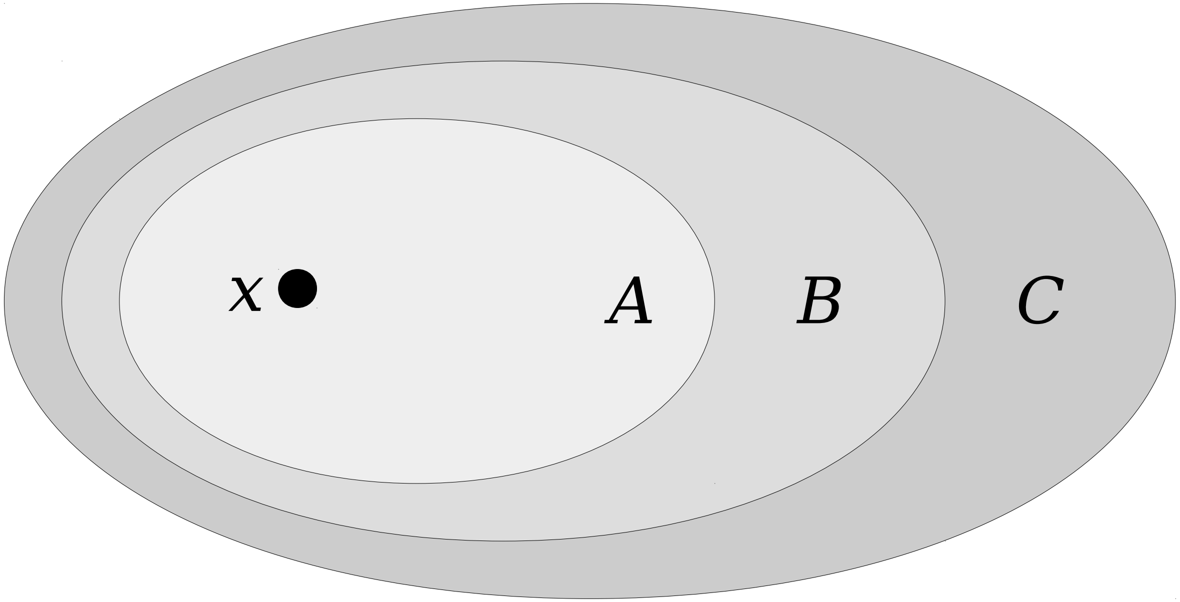

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Since $A \subseteq B$ and $x \in A$, we see that $x \in B$.

Theorem: For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let A, B, and C be arbitrary sets where $A \subseteq B$ and $B \subseteq C$. We need to prove that $A \subseteq C$. To do so, consider any $x \in A$. We will prove that $x \in C$.

Since $A \subseteq B$ and $x \in A$, we see that $x \in B$. Also, because $B \subseteq C$ and $x \in B$, we see that $x \in C$, which is what we needed to show.

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This property of the subset relation is called *transitivity*. We'll revisit transitivity in a couple of weeks.

In this first case, we are *proving* that $A \subseteq C$. That means we pick a new variable $x \in A$ and prove $x \in C$.

Theorem: For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Let A, B, and C be arbitrary sets where $A \subseteq B$ and $B \subseteq C$. **We need to prove that $A \subseteq C$. To do so, consider any $x \in A$.** We will prove that $x \in C$.

Since $A \subseteq B$ and $x \in A$, we see that $x \in B$. Also, because $B \subseteq C$ and $x \in B$, we see that $x \in C$, which is what we needed to show. ■

In this second case, we are *harnessing* the fact that $A \subseteq B$. That means we take an existing variable and learn something new about it.

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Theorem: For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Question to ponder: is this theorem still true if we replace $\subseteq$ with $\in$?

Proof: Let A, B, and C b… B and $B \subseteq C$. We need t… consider any $x \in A$. We will prove that $x \in C$.

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Set Equality and Lemmas

Set Equality

- As we mentioned on Monday, two sets A and B are equal when they have exactly the same elements.
- Here's a little theorem that's very useful for showing that two sets are equal:

  Theorem: If A and B are sets where $A \subseteq B$ and $B \subseteq A$, then $A = B$.

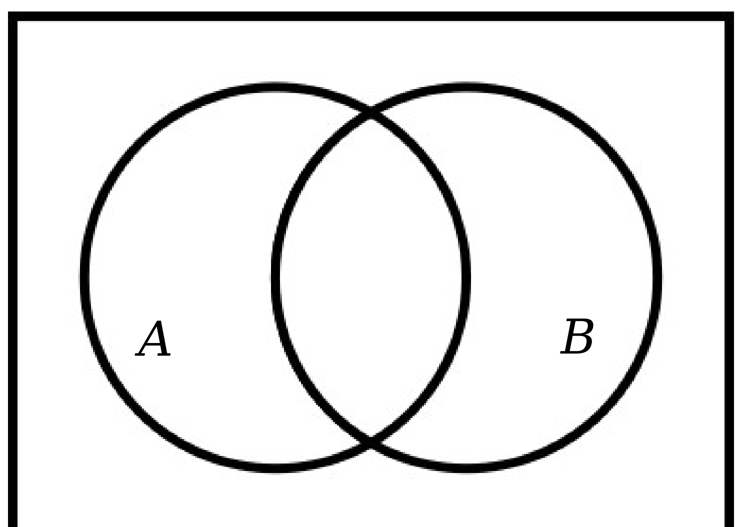
- We've included a proof of this result as an appendix to this slide deck. You should read over it on your own time.

A Trickier Theorem

- Our last theorem for today is this one, which comes to us from the annals of set theory:

Theorem: If A and B are sets and $A \cup B \subseteq A \cap B$, then $A = B$.

- Unlike our previous theorem, this one is a lot harder to see using Venn diagrams alone.



Tackling our Theorem

Theorem: If A and B are sets and $A \cup B \subseteq A \cap B$, then $A = B$.

- Before we Flail and Panic, let's see if we can tease out some info about what this proof might look like.

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Reasonable guess: let's try proving that $A \subseteq B$ and that $B \subseteq A$.

- We're going to pick arbitrary sets A and B.
- We're going to assume $A \cup B \subseteq A \cap B$.
- We're going to prove that $A = B$.

Lemma: If S and T are sets and $S \cup T \subseteq S \cap T$, then $S \subseteq T$.

A *lemma* is a smaller proof that's designed to build into a larger one. Think of it like program decomposition, except for proofs!

Lemma: If S and T are sets and $S \cup T \subseteq S \cap T$, then $S \subseteq T$.

Proof:

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Since $x \in S$, we know that $x \in S \cup T$ because x belongs to at least one of S and T.

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Lemma: If S and T are sets and $S \cup T \subseteq S \cap T$, then $S \subseteq T$.

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What We've Covered

- ***What is a mathematical proof?***
  - An argument – mostly written in English – outlining a mathematical argument.
- ***What is a direct proof?***
  - It's a proof where you begin from some initial assumptions and reason your way to the conclusion.
- ***What are universal and existential statements?***
  - Universal statements make a claim about all objects of one type. Existential statements make claims about at least one object of some type.
- ***How do we write proofs about set theory?***
  - By calling back to definitions! Definitions are key.

Your Action Items

- ***Read "How to Succeed in CS103."***
  - There's a lot of valuable advice in there – take it to heart!
- ***Read "Guide to Proofs on Set Theory."***
  - This picks up where we left off in today's lecture.
- ***Read "Guide to $\in$ and $\subseteq$."***
  - You'll want to have a handle on how these concepts are related, and on how they differ.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if *P*, then *Q*" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

Appendix: Set Equality

Set Equality

- If A and B are sets, we say that $A = B$ precisely when the following statement is true:

For any object x, $x \in A$ if and only if $x \in B$.

- (This is called the ***axiom of extensionality***.)
- In practice, this definition is tricky to work with.
- It's often easier to use the following result to show that two sets are equal:

For any sets A and B,
if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

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Theorem: For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Let A and B be arbitrary sets where $A \subseteq B$ and $B \subseteq A$. We need to prove $A = B$.

Theorem: For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Let A and B be arbitrary sets where $A \subseteq B$ and $B \subseteq A$. We need to prove $A = B$. To do so, we will prove for all x that $x \in A$ if and only if $x \in B$.

Theorem: For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Let A and B be arbitrary sets where $A \subseteq B$ and $B \subseteq A$. We need to prove $A = B$. To do so, we will prove for all x that $x \in A$ if and only if $x \in B$.

First, we'll prove that if $x \in A$, then $x \in B$.

Theorem: For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Let A and B be arbitrary sets where $A \subseteq B$ and $B \subseteq A$. We need to prove $A = B$. To do so, we will prove for all x that $x \in A$ if and only if $x \in B$.

First, we'll prove that if $x \in A$, then $x \in B$. To do so, take any $x \in A$.

Theorem: For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Let A and B be arbitrary sets where $A \subseteq B$ and $B \subseteq A$. We need to prove $A = B$. To do so, we will prove for all x that $x \in A$ if and only if $x \in B$.

First, we'll prove that if $x \in A$, then $x \in B$. To do so, take any $x \in A$. Since $A \subseteq B$ and $x \in A$, we see that $x \in B$, as required.

Theorem: For any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

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Since we've proven both directions of implication, we see that $A = B$.

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