Mathematical Logic

Part Two

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Take out a sheet of paper!

What's the truth table for the → connective?

What's the negation of $p \rightarrow q$?

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

These blue terms are called constant symbols. Unlike propositional variables, they refer to *objects*, not propositions.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

The red things that look like function calls are called *predicates*. Predicates take objects as arguments and evaluate to true or false.

$$Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \vee ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \wedge TookBackTo(Him, Me, EastAtlanta)$$

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

$$Cute(Quokka)$$

$$ArgueIncessantly(Democrats, Republicans)$$

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$Cute(a) \to Dikdik(a) \lor Kitty(a) \lor Puppy(a)$$

$$Succeeds(You) \leftrightarrow Practices(You)$$

$$x < 8 \to x < 137$$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this way.

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as $\to$ and $\neg$ are.
- Examples:

$$\mathit{TomMarvoloRiddle} = \mathit{LordVoldemort}$$

$$\mathit{MorningStar} = \mathit{EveningStar}$$

- Equality can only be applied to ***objects***; to state that two ***propositions*** are equal, use $\leftrightarrow$.

Let's see some more examples.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

These purple terms are *functions*. Functions take objects as input and produce objects as output.

Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

$$ColorOf(Money)$$

$$MedianOf(x, y, z)$$

$$x + y$$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

Objects and Predicates

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.
- You cannot apply connectives to objects:

⚠ $Venus \to TheSun$ ⚠

- You cannot apply functions to propositions:

⚠ $StarOf(IsRed(Sun) \land IsGreen(Mars))$ ⚠

- Ever get confused? *Just ask!*

The Type-Checking Table

	… operate on …	… and produce
Connectives (↔, ∧, etc.) …	propositions	a proposition
Predicates (=, etc.) …	objects	a proposition
Functions …	objects	an object

One last (and major) change

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

The Existential Quantifier

- A statement of the form

$$\exists x.\ \textbf{\textit{some-formula}}$$

  is true if, for *some* choice of x, the statement ***some-formula*** is true when that x is plugged into it.

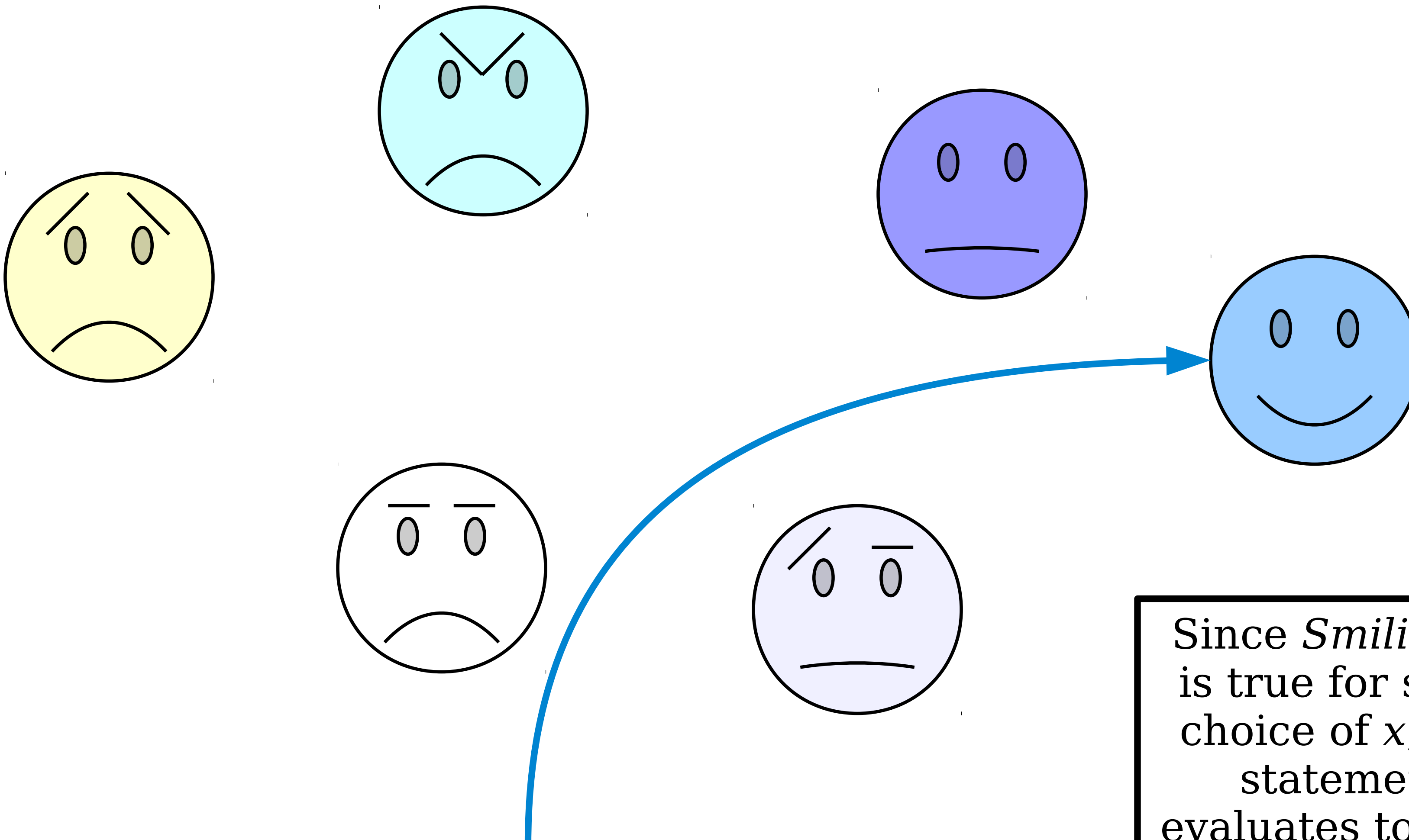
- Examples:

$$\exists x.\ (Even(x) \land Prime(x))$$

$$\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$$

$$(\exists w.\ Will(w)) \to (\exists x.\ Way(x))$$

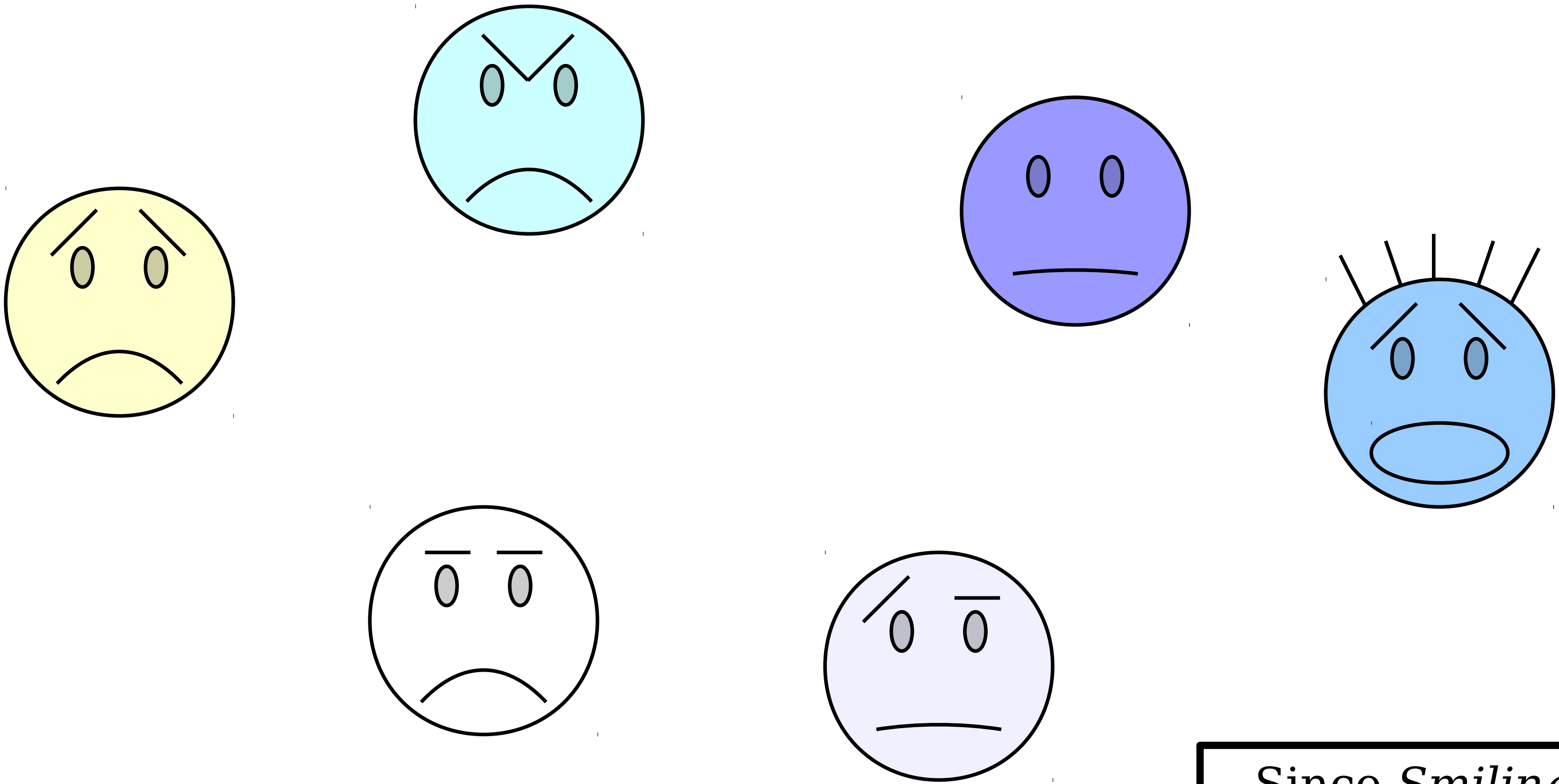
The Existential Quantifier



$\exists x.\ \mathit{Smiling}(x)$

Since $\mathit{Smiling}(x)$ is true for some choice of x, this statement evaluates to true.

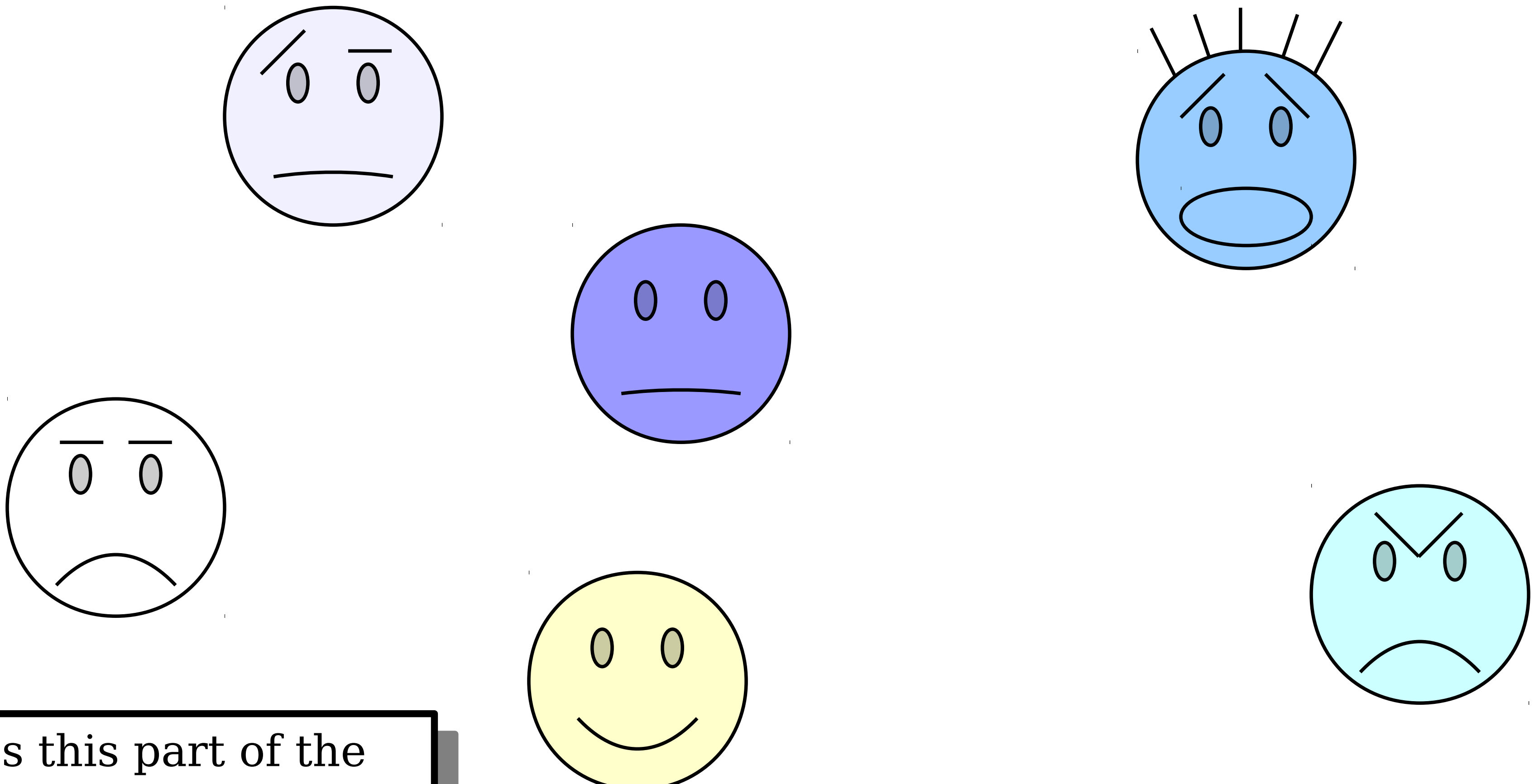
The Existential Quantifier



~~$\exists x.\ Smiling(x)$~~

Since $Smiling(x)$ is not true for any choice of x, this statement evaluates to false.

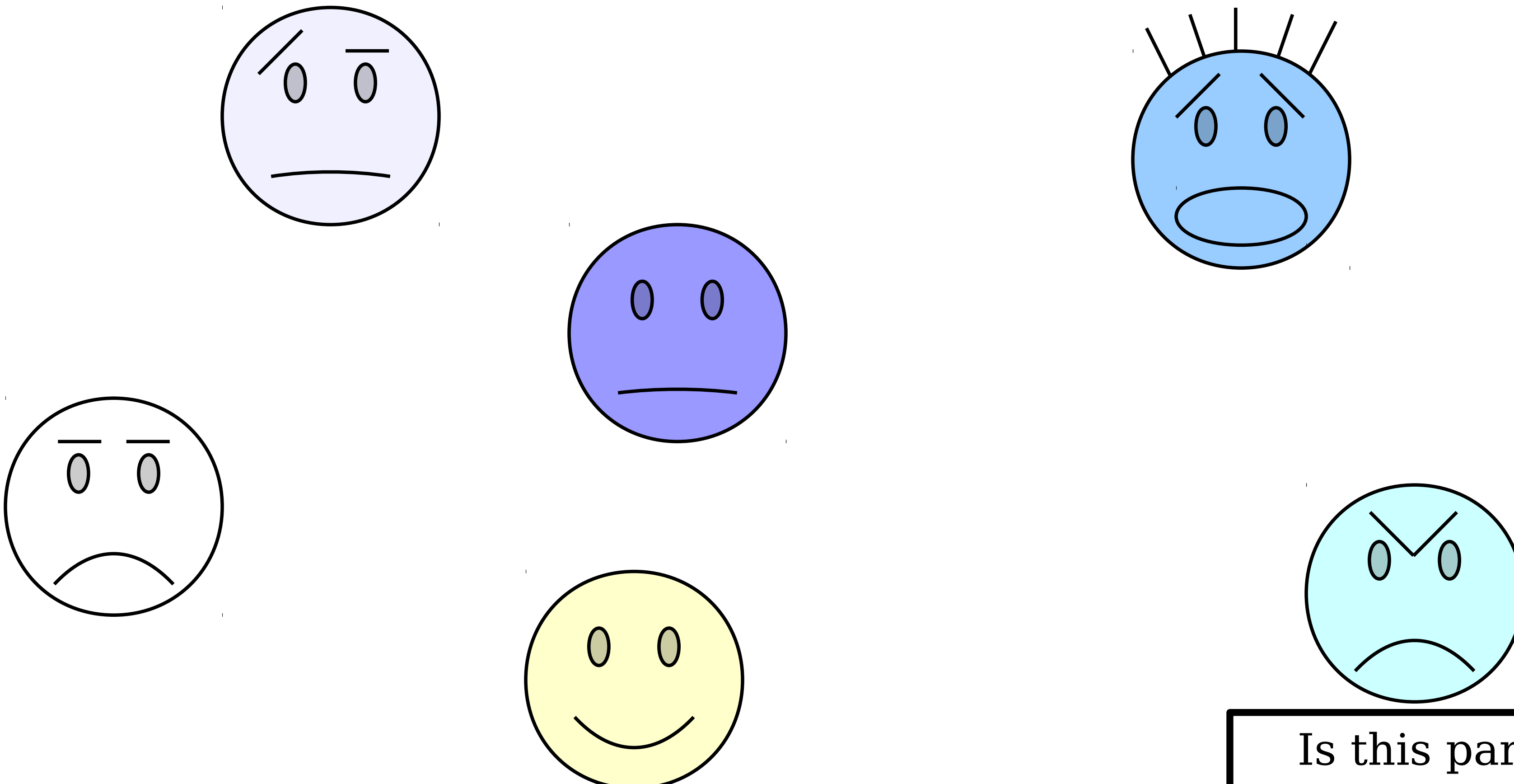
The Existential Quantifier



Is this part of the statement true or false?

$$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$$

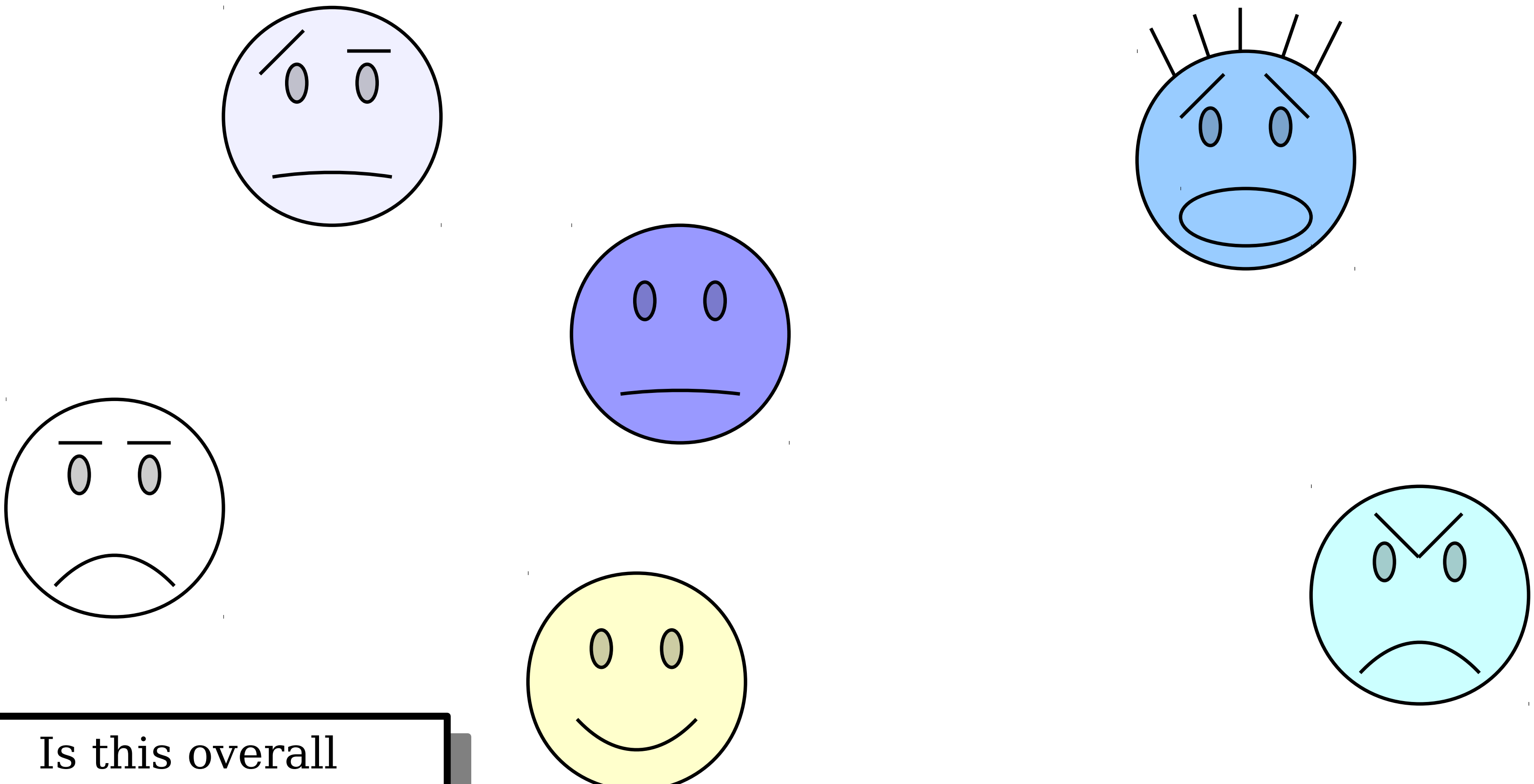
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ \mathit{Smiling}(x)) \to$ ~~$(\exists y.\ \mathit{WearingHat}(y))$~~

The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$~~

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since it's not possible to choose an object!

~~$\exists x.\ Smiling(x)$~~

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ \mathit{Loves}(\mathit{You}, x)) \land (\exists y.\ \mathit{Loves}(y, \mathit{You}))$$

The variable x just lives here.

The variable y just lives here.

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x.\ Loves(You, x)) \land (\exists x.\ Loves(x, You))$$

The variable x just lives here.

A different variable, also named x, just lives here.

Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below $\neg$.
- The statement

$$\exists x.\ P(x) \land R(x) \land Q(x)$$

is parsed like this:

$$(\exists x.\ P(x)) \ \land\ (R(x) \land Q(x))$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x.\ (P(x) \land R(x) \land Q(x))$$

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier***
and says "for any choice of n,
the following is true."

The Universal Quantifier

- A statement of the form

  $$\forall x.\ \textit{some-formula}$$

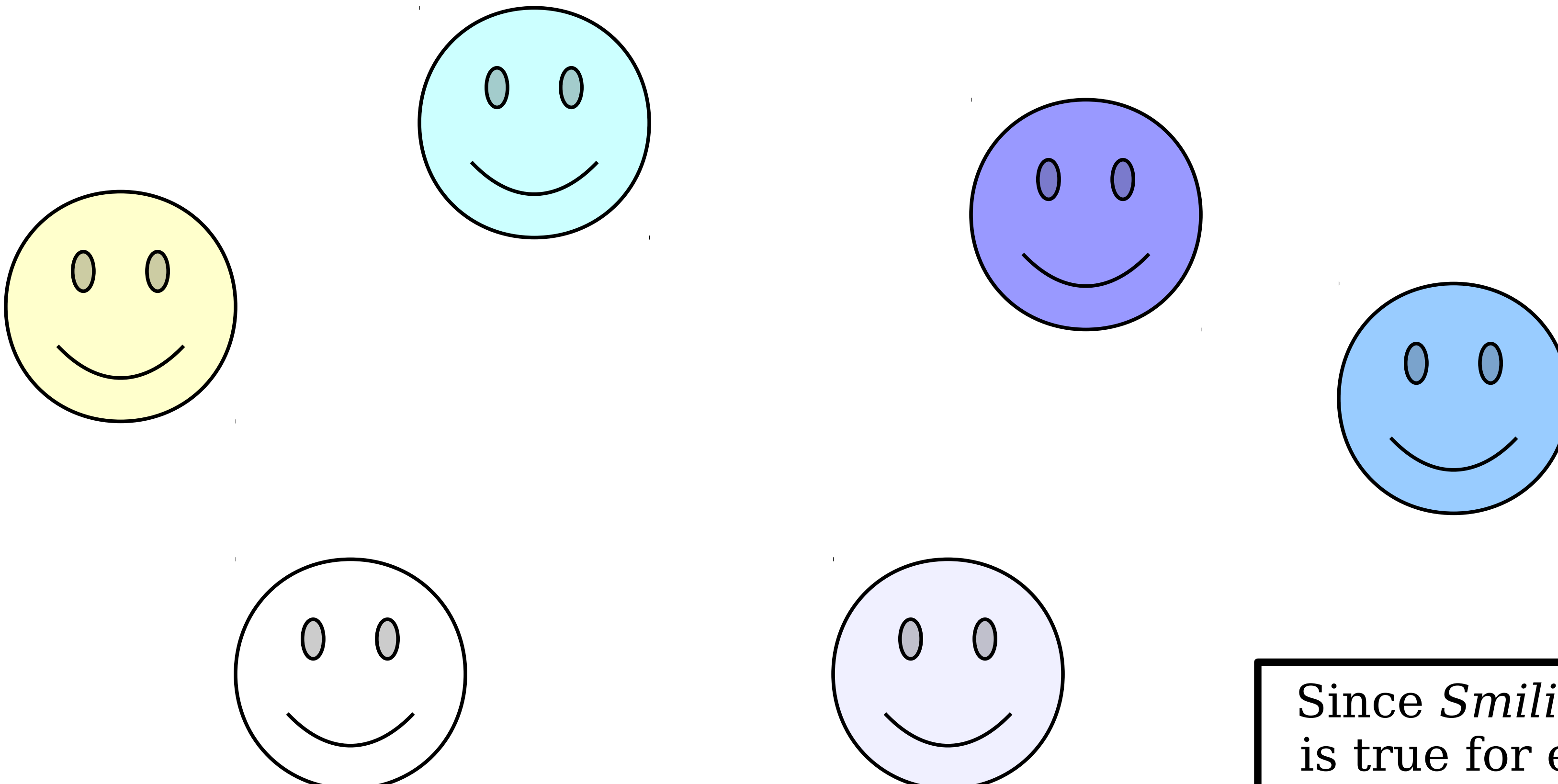
  is true if, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.
- Examples:

  $$\forall p.\ (\mathit{Puppy}(p) \to \mathit{Cute}(p))$$

  $$\forall a.\ (\mathit{EatsPlants}(a) \lor \mathit{EatsAnimals}(a))$$

  $$\mathit{Tallest}(\mathit{SultanK\ddot{o}sen}) \to \forall x.\ (\mathit{SultanK\ddot{o}sen} \neq x \to \mathit{ShorterThan}(x, \mathit{SultanK\ddot{o}sen}))$$

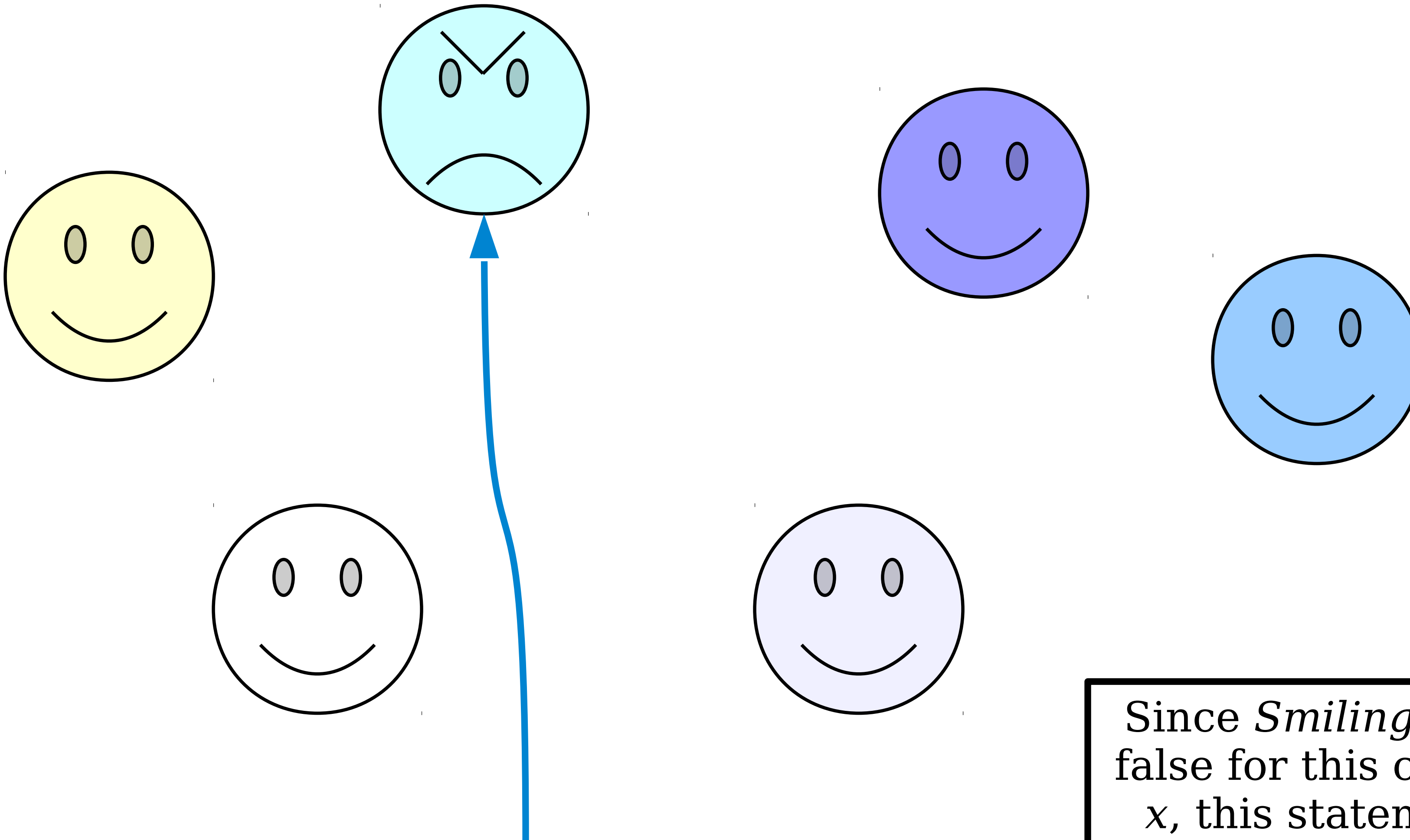
The Universal Quantifier



Since *Smiling*(x) is true for every choice of x, this statement evaluates to true.

$\forall x.\ Smiling(x)$

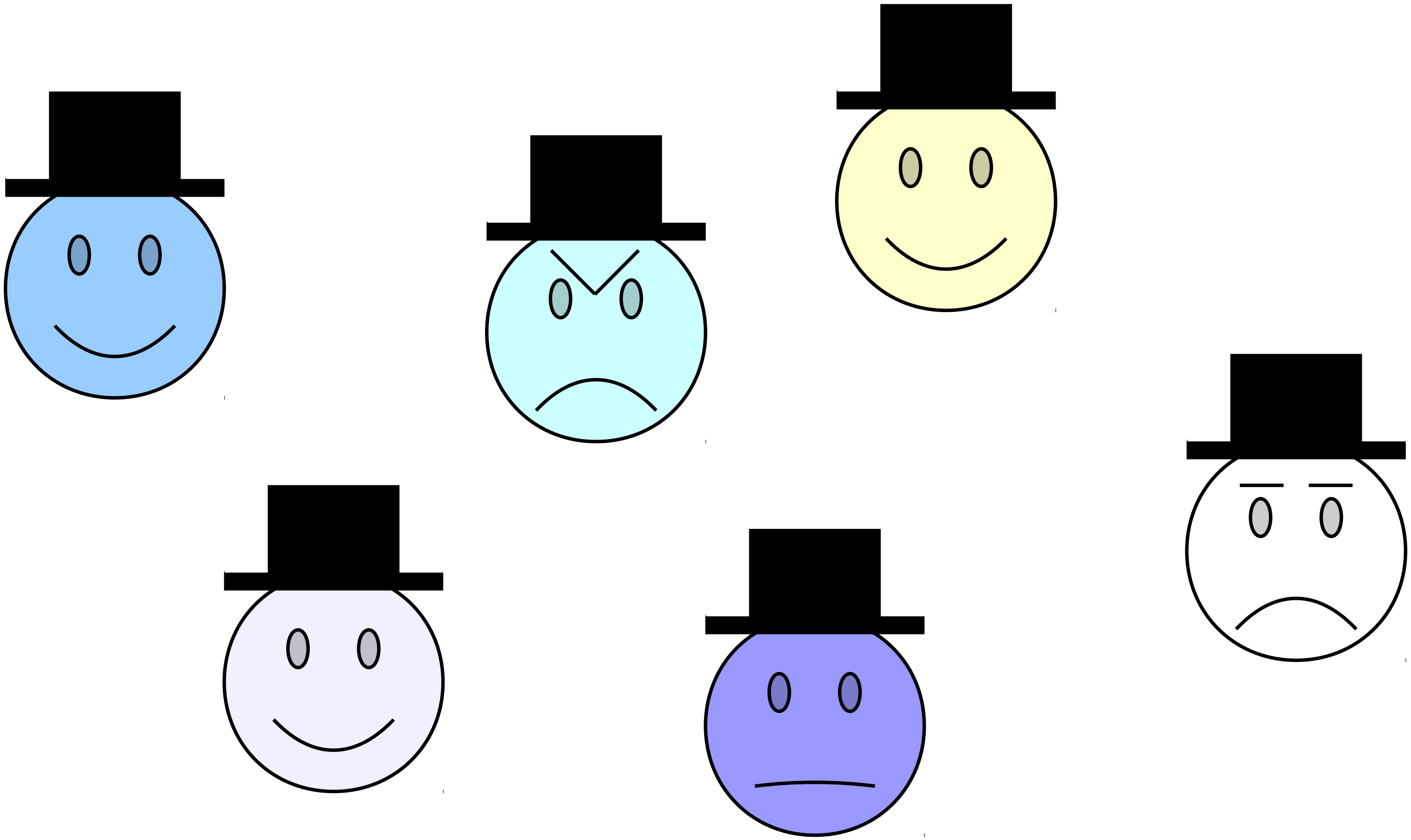
The Universal Quantifier



~~$\forall x.\ Smiling(x)$~~

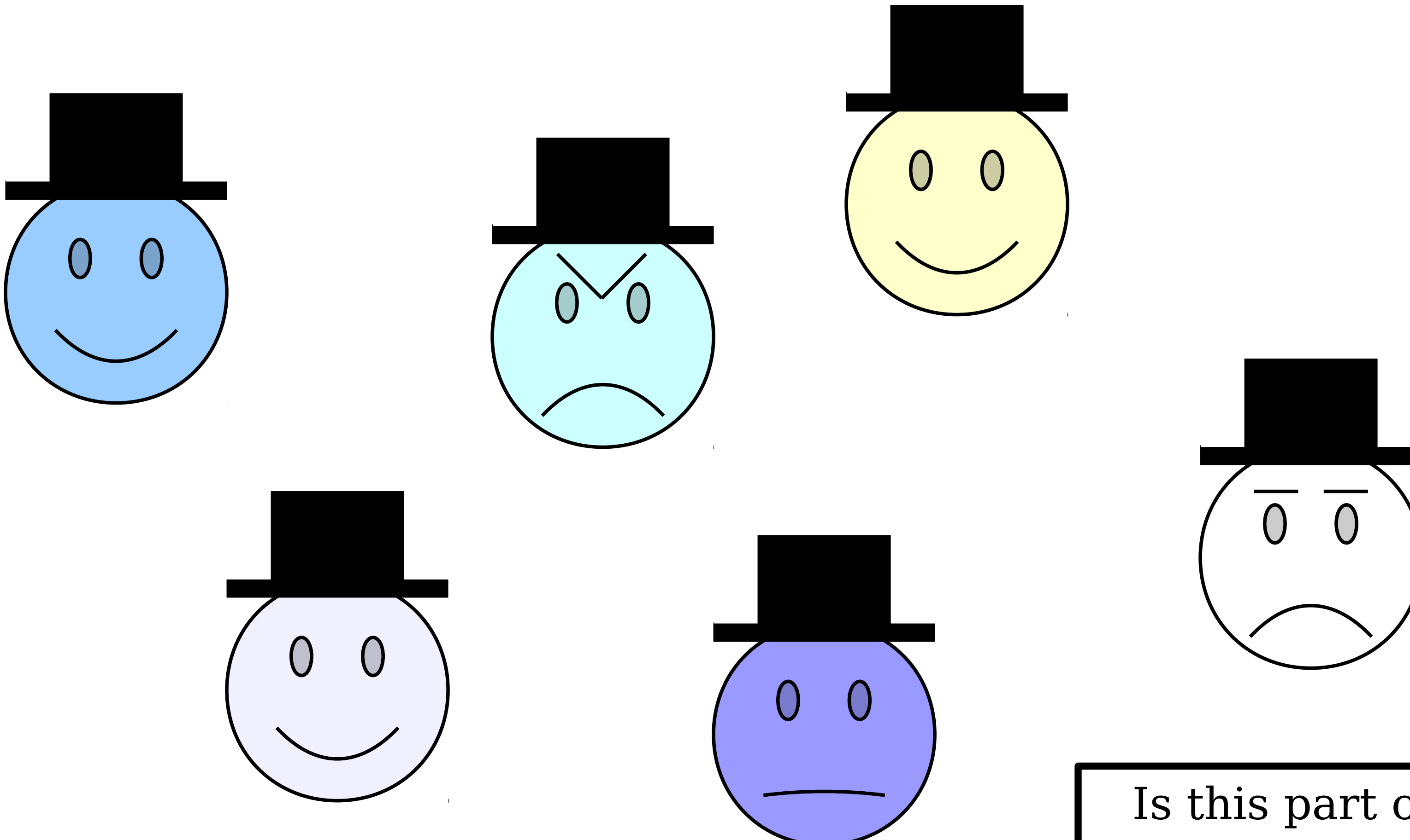
Since *Smiling*(*x*) is false for this choice *x*, this statement evaluates to false.

The Universal Quantifier



$$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$$

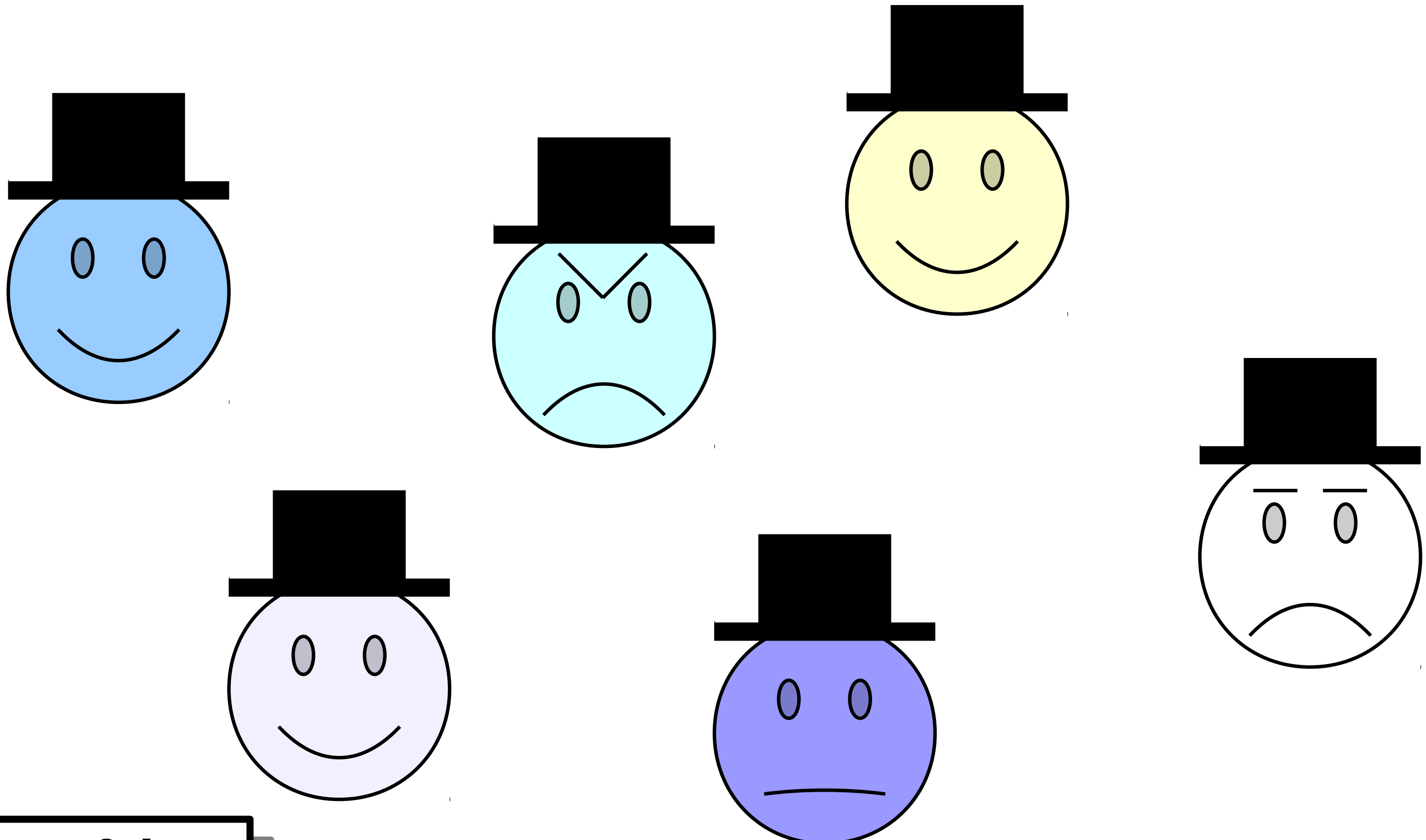
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$

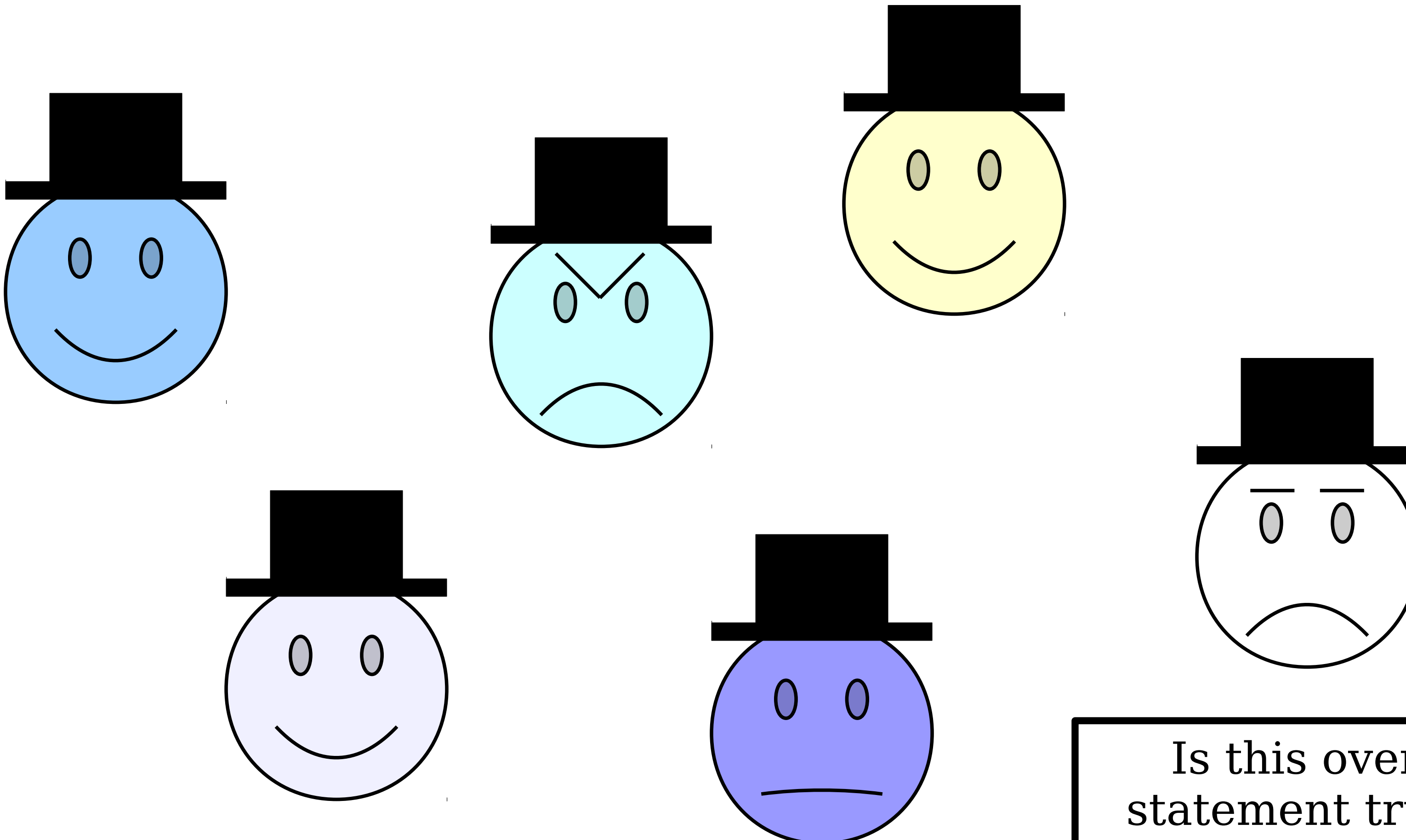
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x.\ Smiling(x))$~~ $\rightarrow (\forall y.\ WearingHat(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$$

Fun with Edge Cases

Universally-quantified statements are ***vacuously true*** in empty worlds.

$$\forall x.\ Smiling(x)$$

Time-Out for Announcements!

hopes

HUNTINGTON'S OUTREACH PROJECT
FOR EDUCATION, AT STANFORD

HOPES is hiring student researchers, graphic designers, and web developers.

The Huntington's Outreach Project for Education, at Stanford (HOPES) is an educational service project working to build a web resource on Huntington's disease (HD). Our mission is to make scientific information about HD more accessible to patients, their families, and the general public.

Apply by **Sunday, October 7th at 11:59 pm!** Please send a resume, letter of application, and unofficial transcript to HOPES project leader Cole Holderman (jcoleh@stanford.edu) with the subject line "YOUR LAST NAME - HOPES application." The letter should include a candid discussion of your qualifications, other time commitments, leadership skills, and reasons for interest in the position.

Student researchers: Please attach two writing samples that are science-related or research-based in nature.

Graphic designers: Please send in 3 recent designs with a description about each (tools used, time spent, purpose/client)

Web developers: Please send links to any of your web-design work.
For more information, please visit hopes.stanford.edu or email the project leader.

health++

Stanford's Health Hackathon

November 3-4, 2018

Stanford University | Huang Engineering Center

We're bringing together engineers, designers, healthcare professionals, and business experts to

Collaborate
with interdisciplinary teams

Design
innovative solutions to validated needs

Create
prototypes and business models

for healthcare affordability, domestically and worldwide

Interested in Participating?
Sign-up below!

Participate: bit.ly/2O4sJS3

healthplusplus.stanford.edu

- The Brown Institute for Media Innovation is holding a showcase this Friday at 5PM at the Gates building.
- Interested in seeing the intersection of technology, journalism, and media? Come check it out!
- RSVP is requested. Use **this link**.

Checkpoints Graded

- The Problem Set One checkpoint problem has been graded. Feedback is now available in GradeScope.
- ***You need to look over our feedback as soon as possible.***
  - The purpose of the checkpoint is to help you see where to focus and how to improve.
  - If you don't review the feedback you received, you risk making the same mistakes in the future.

Your Questions

“Suggestions for combating impostor syndrome? Especially in CS?”

Yes! I'll draw some pictures to illustrate these points:

1. Don't confuse unions and intersections.
2. Don't confuse talent for experience.
3. Don't confuse relative and absolute performance.

"How did you ask your first girlfriend out?"

Over AOL Instant Messenger. I then asked my parents if I could get a ride because I didn't have my license yet.

Ah, the joys of being 15.

Back to CS103!

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

Translating Into Logic

- ***Translating statements into first-order logic is a lot more difficult than it looks.***
- There are a lot of nuances that come up when translating into first-order logic.
- We'll cover examples of both good and bad translations into logic so that you can learn what to watch for.
- We'll also show lots of examples of translations so that you can see the process that goes into it.

Using the predicates

- $Puppy(p)$, which states that p is a puppy, and
- $Cute(x)$, which states that x is cute,

write a sentence in first-order logic that means “all puppies are cute.”

An Incorrect Translation

All puppies are cute!

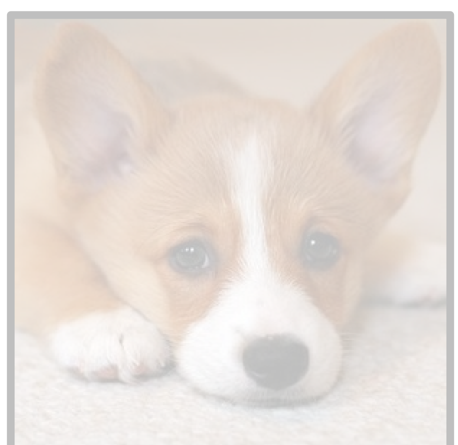
$$\forall x. (Puppy(x) \land Cute(x))$$

This should work for <u>any</u> choice of x, including things that aren't puppies.

An Incorrect Translation

All puppies are cute!

$\forall x.$ (~~$Puppy(x) \land Cute(x)$~~)

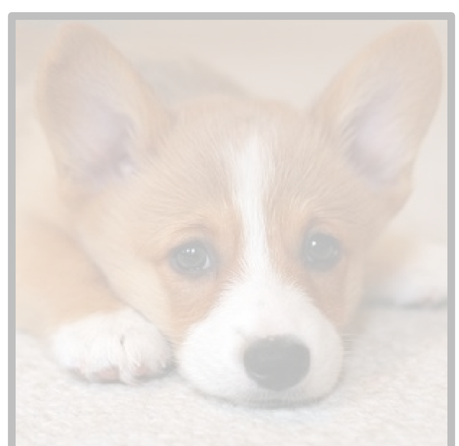


This should work for *any* choice of x, including things that aren't puppies.

An Incorrect Translation

All puppies are cute!

~~$\forall x. (Puppy(x) \land Cute(x))$~~



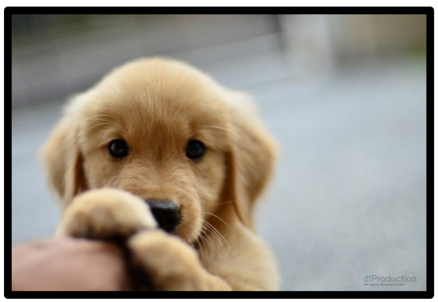
A statement of the form

$\forall x.$ ***something***

is true only when *something* is true for <u>*every*</u> choice of x.

An Incorrect Translation

All puppies are cute!

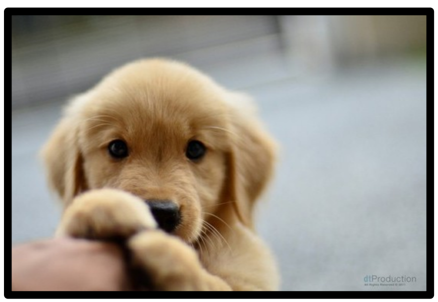


~~$\forall x. \ (Puppy(x) \land Cute(x))$~~

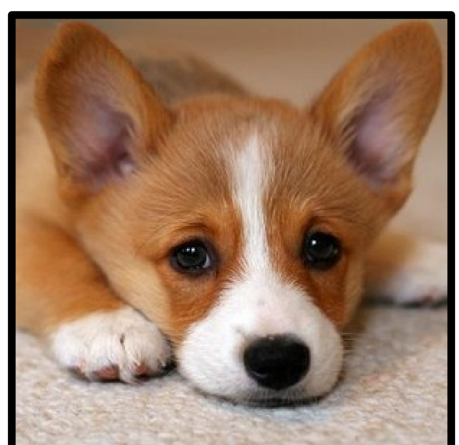
This first-order statement is false even though the English statement is true. Therefore, it can't be a correct translation.

An Incorrect Translation

All puppies are cute!

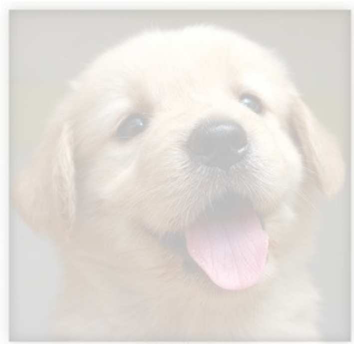


$$\forall x. \, (Puppy(x) \land Cute(x))$$



The issue here is that this statement asserts that everything is a puppy. That's too strong of a claim to make.

A Better Translation



All puppies are cute!

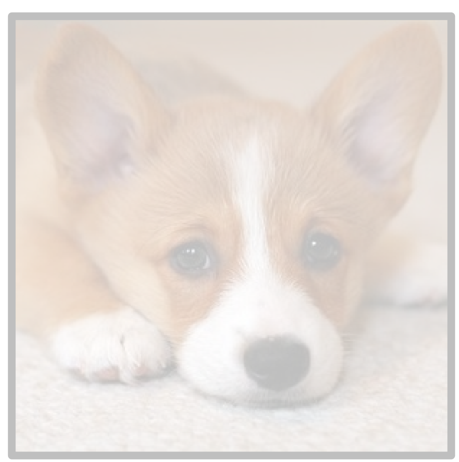
$$\forall x.\ (\cancel{Puppy(x)} \to Cute(x))$$

This should work for *any* choice of x, including things that aren't puppies.

A Better Translation

All puppies are cute!

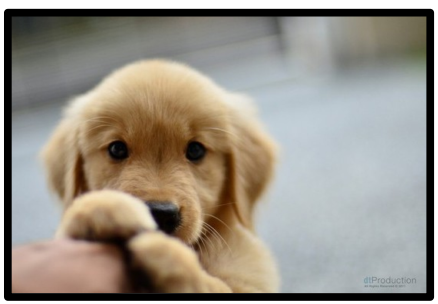
$\forall x.\ (Puppy(x) \rightarrow Cute(x))$



This should work for *any* choice of x, including things that aren't puppies.

A Better Translation

All puppies are cute!



$\forall x.\ (Puppy(x) \rightarrow Cute(x))$

A statement of the form

$\forall x.$ ***something***

is true only when *something* is true for <u>every</u> choice of x.

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \rightarrow Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \rightarrow Q(x))$$

If x is a counterexample, it *must* have property *P* but not have property *Q*.

Using the predicates

- $Blobfish(b)$, which states that b is a blobfish, and
- $Cute(x)$, which states that x is cute,

write a sentence in first-order logic that means “some blobfish is cute.”

An Incorrect Translation

Some blobfish is cute.

$\exists x. \, (Blobfish(x) \rightarrow Cute(x))$

An Incorrect Translation

Some blobfish is cute.

$$\exists x.\ (\cancel{Blobfish(x)} \to Cute(x))$$

An Incorrect Translation

Some blobfish is cute.

$$\exists x.\ (\mathit{Blobfish}(x) \to \mathit{Cute}(x))$$

An Incorrect Translation

Some blobfish is cute.

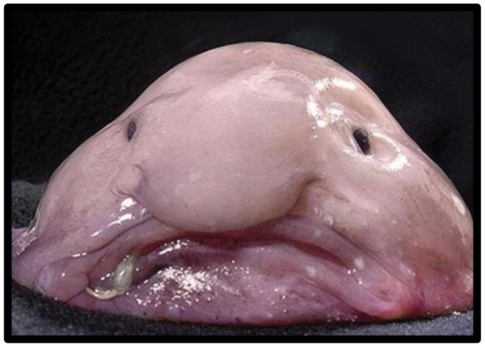
$\exists x.\ (Blobfish(x) \to Cute(x))$

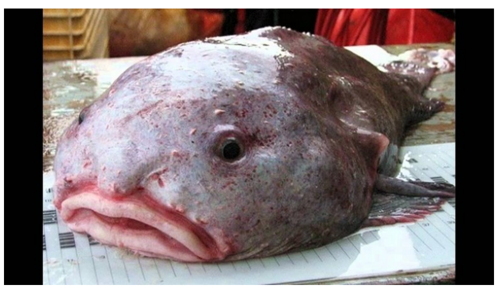
A statement of the form

$\exists x.$ ***something***

is true only when *something* is true for *at least one* choice of x.

An Incorrect Translation





Some blobfish is cute.

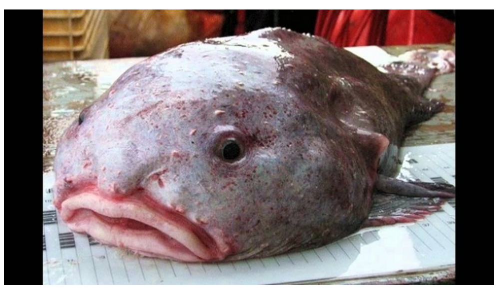
$\exists x.\ (Blobfish(x) \to Cute(x))$

This first-order statement is true even though the English statement is false. Therefore, it can't be a correct translation.

An Incorrect Translation

Some blobfish is cute.

$$\exists x.\ (Blobfish(x) \rightarrow Cute(x))$$



The issue here is that implications are true whenever the antecedent is false. This statement "accidentally" is true because of what happens when x isn't a blobfish.

A Correct Translation

Some blobfish is cute.

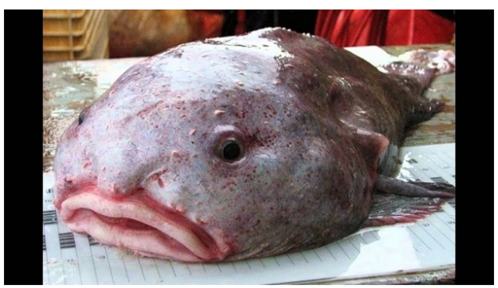
$\exists x.\ ($ ~~$Blobfish(x)$~~ $\land\ Cute(x))$

A Correct Translation

Some blobfish is cute.

$\exists x.$ (~~$Blobfish(x) \land Cute(x)$~~)

A Correct Translation



Some blobfish is cute.

~~$\exists x.\ (\mathit{Blobfish}(x) \land \mathit{Cute}(x))$~~

A statement of the form

$\exists x.$ ***something***

is true only when

something is true for

at least one choice of x.

“Some P is a Q”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If x is an example, it *must* have property P on top of property Q.

Good Pairings

- The ∀ quantifier *usually* is paired with →.

$$\forall x.\ (P(x) \to Q(x))$$

- The ∃ quantifier *usually* is paired with ∧.

$$\exists x.\ (P(x) \land Q(x))$$

- In the case of ∀, the → connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of ∃, the ∧ connective prevents the statement from being *true* when speaking about some object you don't care about.

Next Time

- ***First-Order Translations***
  - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
  - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
  - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
  - How do we say there’s just one object of a certain type?