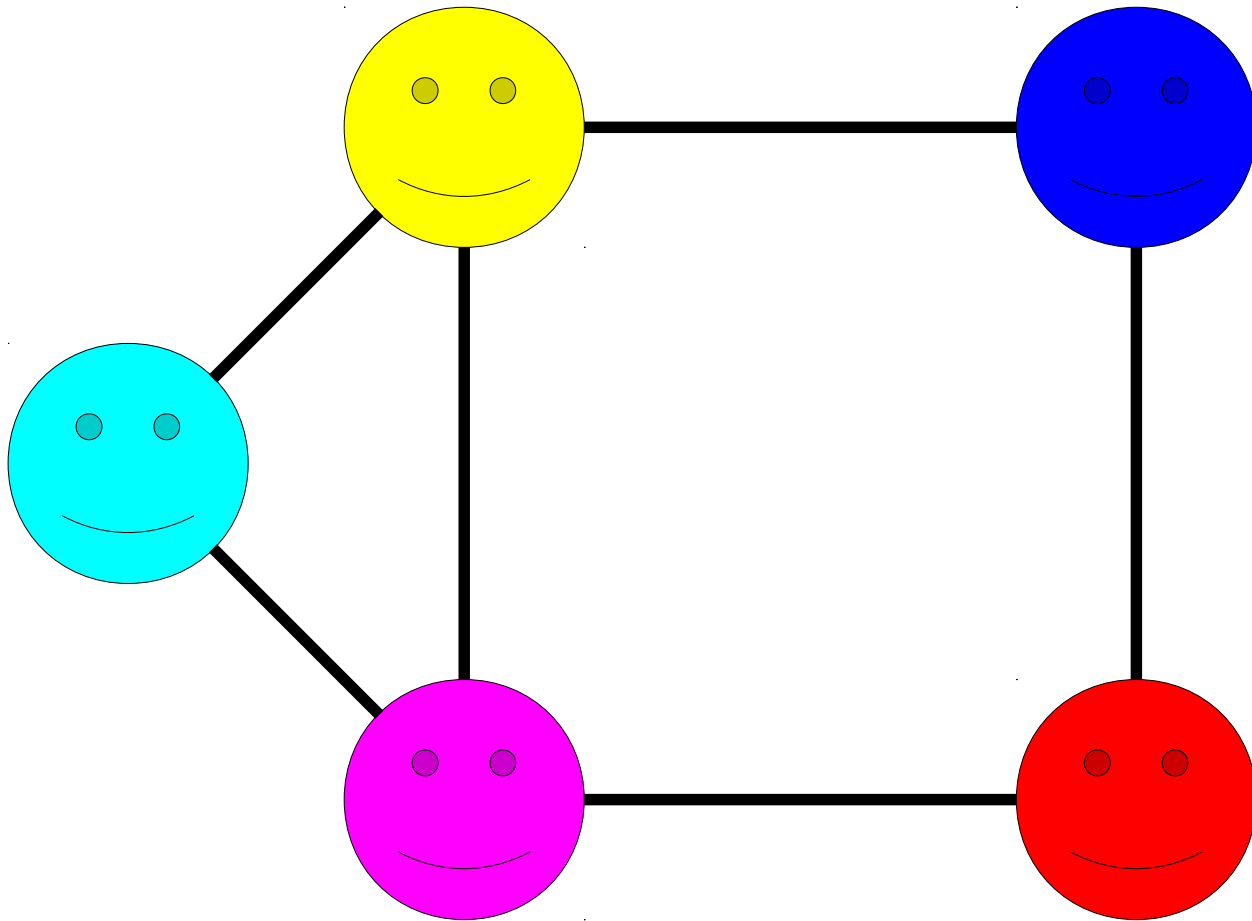


Graph Theory

Part Two

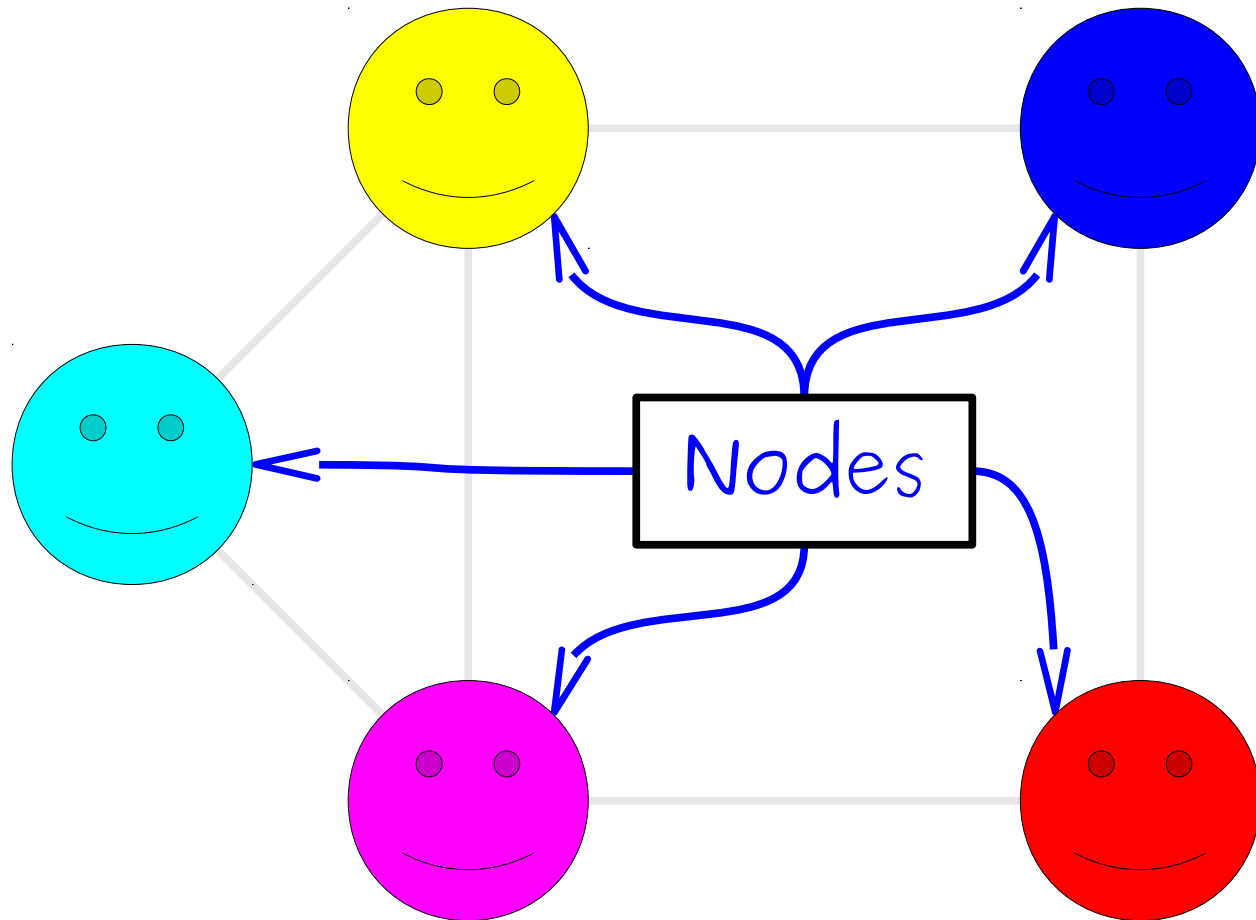
Recap from Last Time

A **graph** is a mathematical structure for representing relationships.



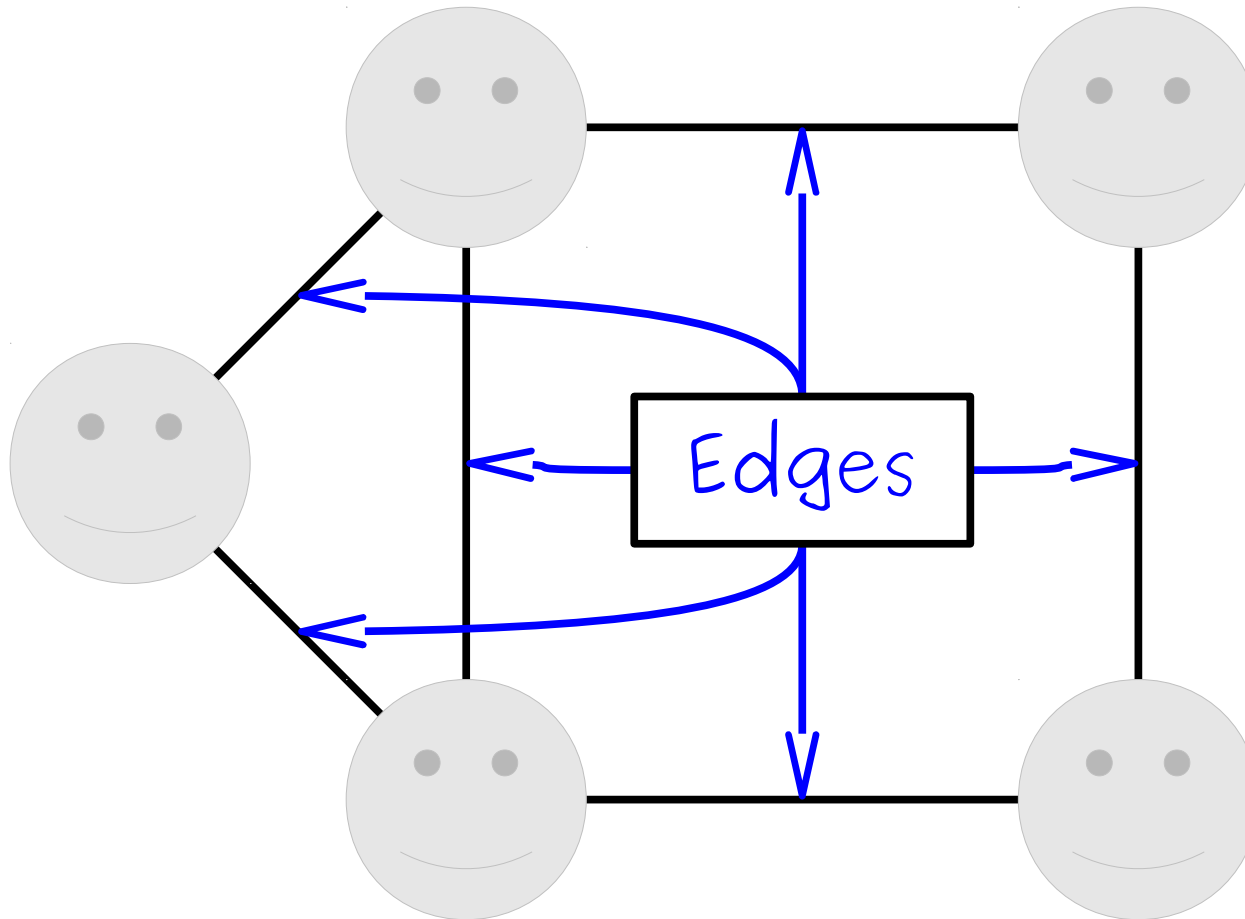
A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

A **graph** is a mathematical structure for representing relationships.



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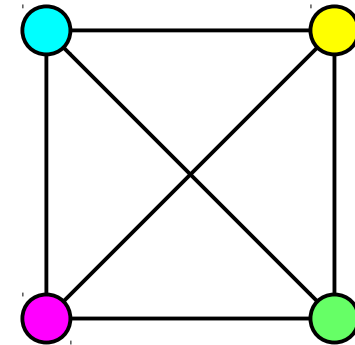
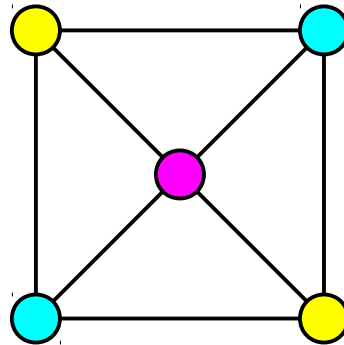
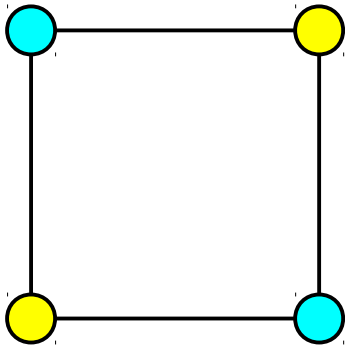
A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**)

Adjacency and Connectivity

- Two nodes in a graph are called ***adjacent*** if there's an edge between them.
- Two nodes in a graph are called ***connected*** if there's a path between them.
 - A path is a series of one or more nodes where consecutive nodes are adjacent.

k -Vertex-Colorings

- If $G = (V, E)$ is a graph, a **k -vertex-coloring** of G is a way of assigning colors to the nodes of G , using at most k colors, so that no two nodes of the same color are adjacent.



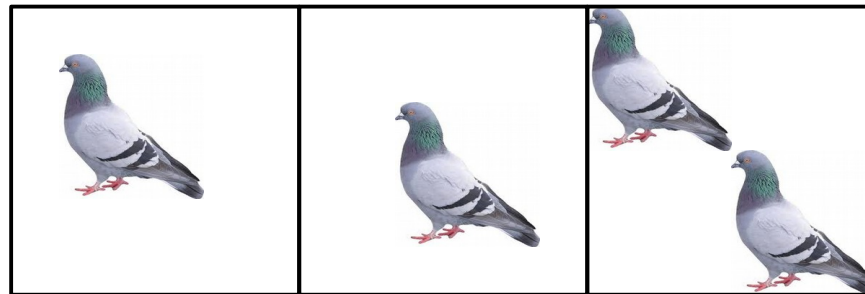
- The **chromatic number** of G , denoted $\chi(G)$, is the minimum number of colors needed in any k -coloring of G .
- Today, we're going to see several results involving coloring parts of graphs. They don't necessarily involve k -vertex-colorings of graphs, so feel free to ask for clarifications if you need them!

New Stuff!

The Pigeonhole Principle

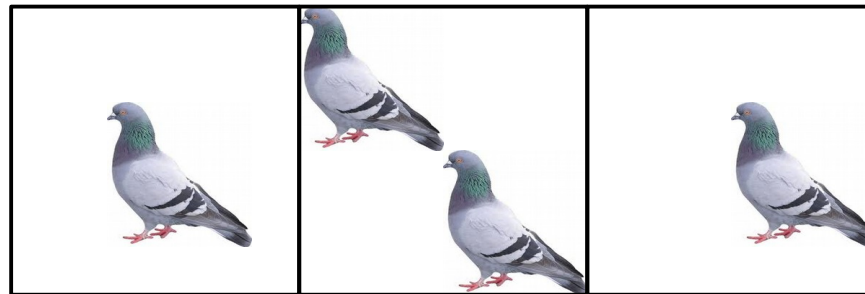
The Pigeonhole Principle

- ***Theorem (The Pigeonhole Principle):***
If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



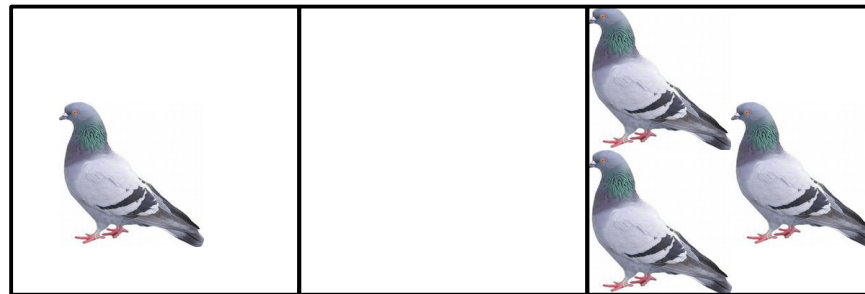
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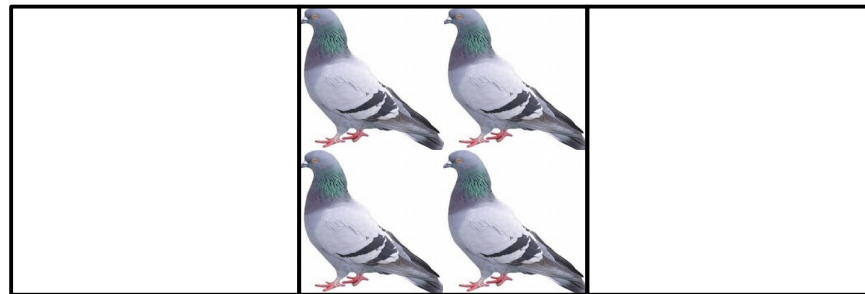
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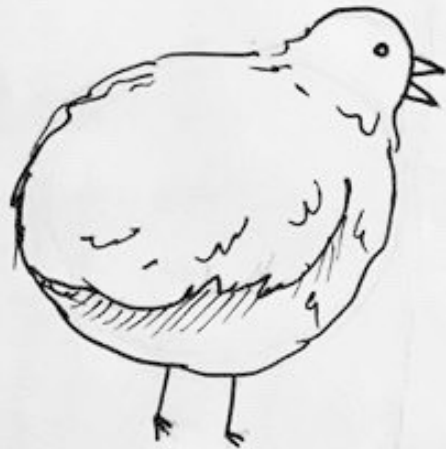


The Pigeonhole Principle

- ***Theorem (The Pigeonhole Principle):***
If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



NO MORE
- PIGEON HOLES?!



$$m = 4, n = 3$$

Thanks to Amy Liu for this awesome drawing!

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
 - 366 possible birthdays (pigeonholes)
 - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
 - Maximum number of hairs ever found on a human head is no greater than 500,000.
 - There are over 800,000 people in San Francisco.

Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.

Let A and B be finite sets (sets whose cardinalities are natural numbers) and assume $|A| > |B|$. How many of the following statements are true?

- (1) If $f : A \rightarrow B$, then f is injective.
- (2) If $f : A \rightarrow B$, then f is not injective.
- (3) If $f : A \rightarrow B$, then f is surjective.
- (4) If $f : A \rightarrow B$, then f is not surjective.

Proving the Pigeonhole Principle

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . There are m objects in total, so we know that

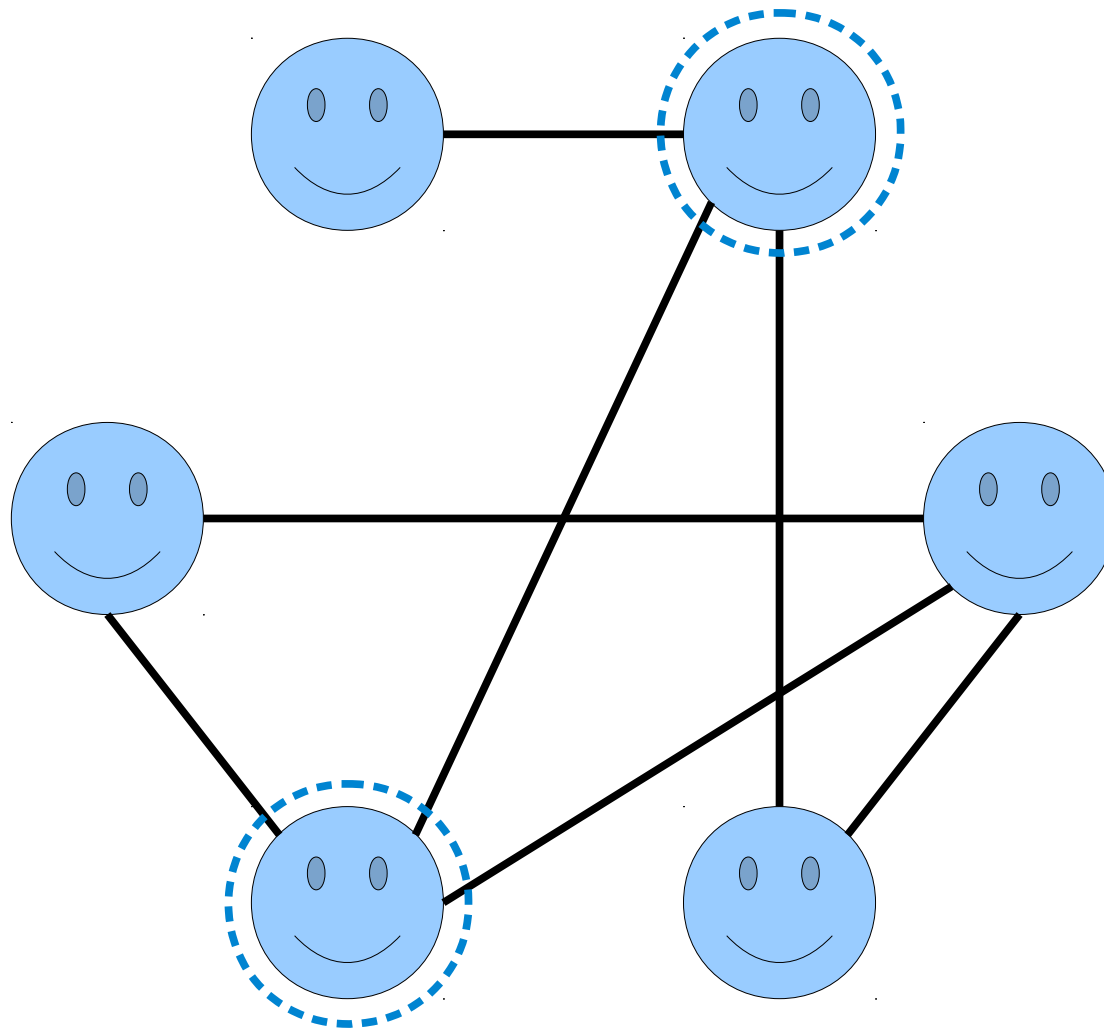
$$m = x_1 + x_2 + \dots + x_n.$$

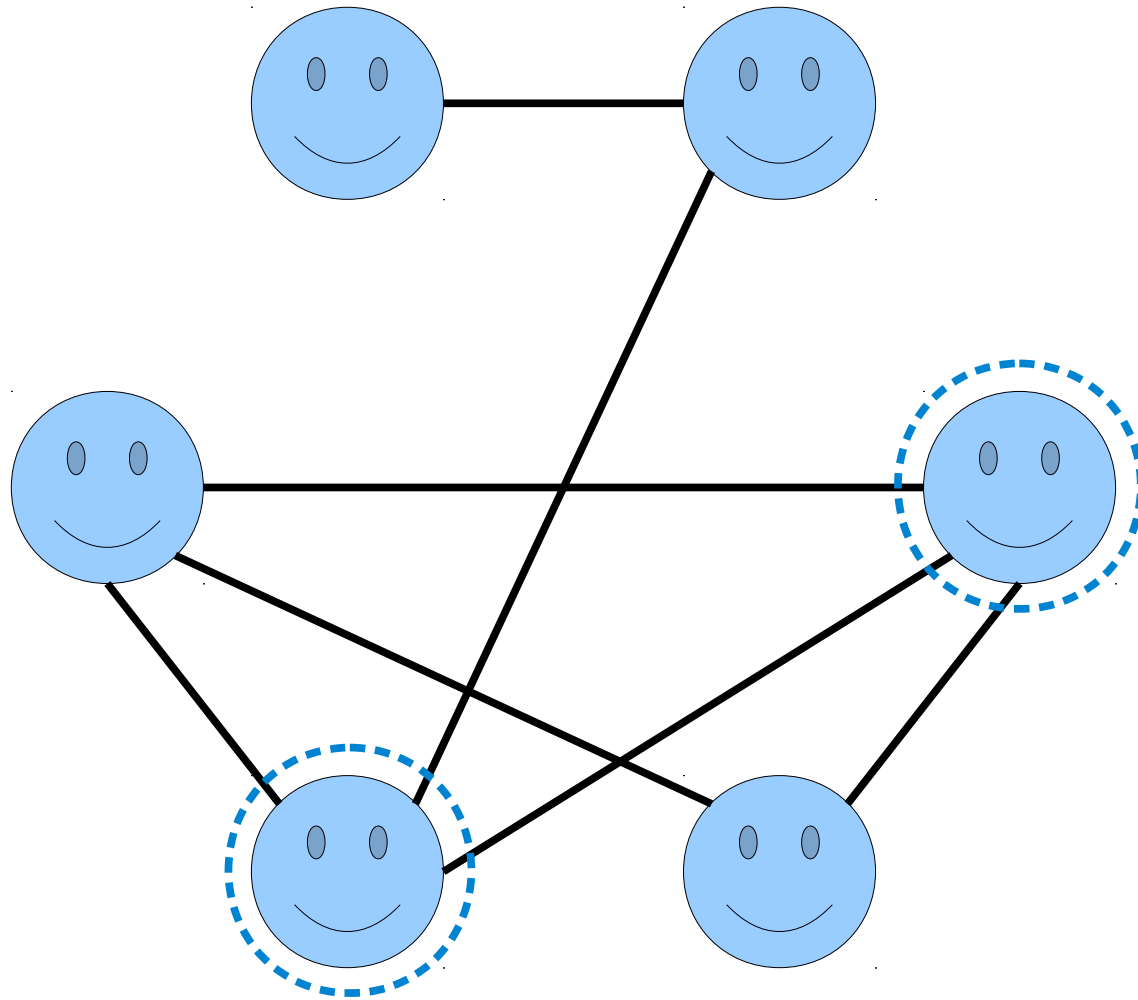
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i . This means that

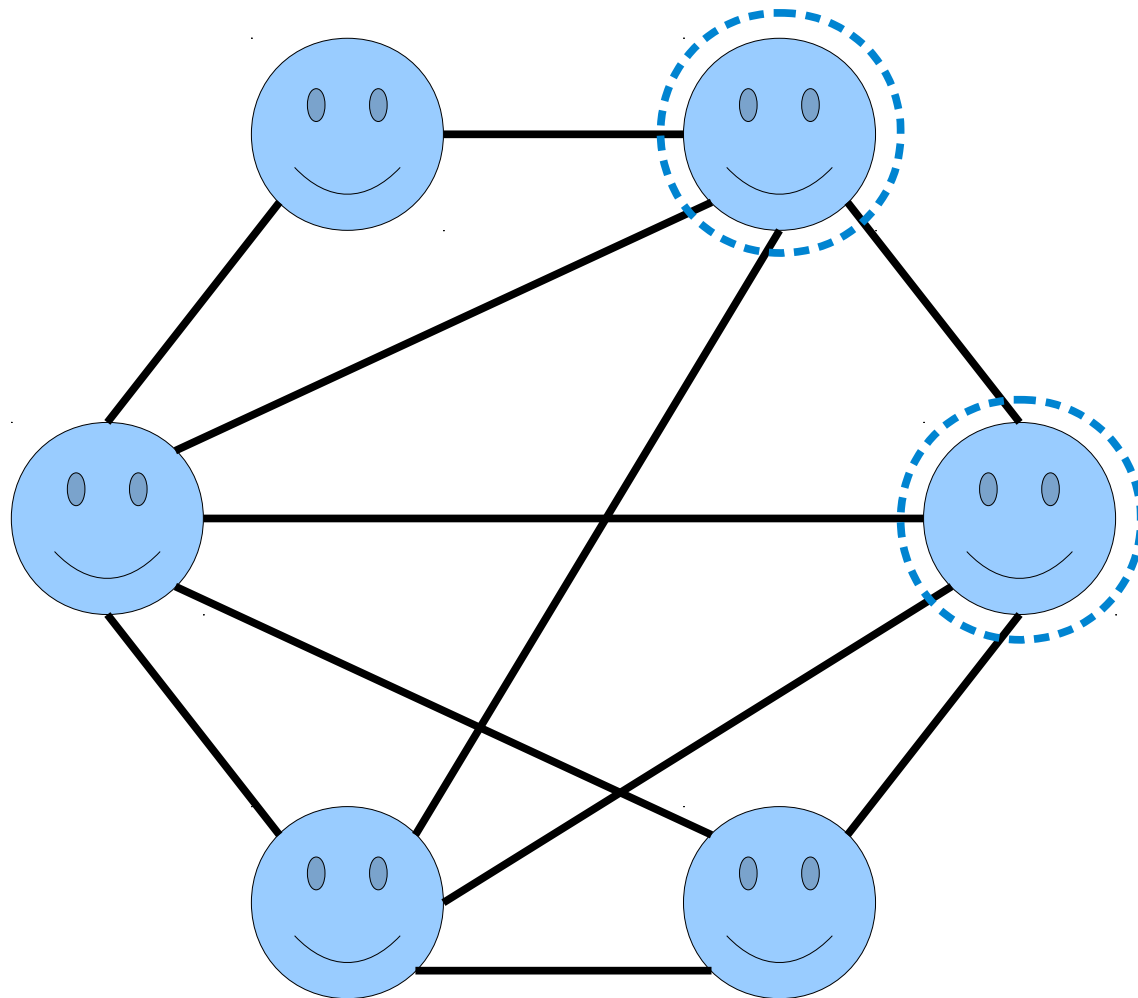
$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &\leq 1 + 1 + \dots + 1 \quad (n \text{ times}) \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks

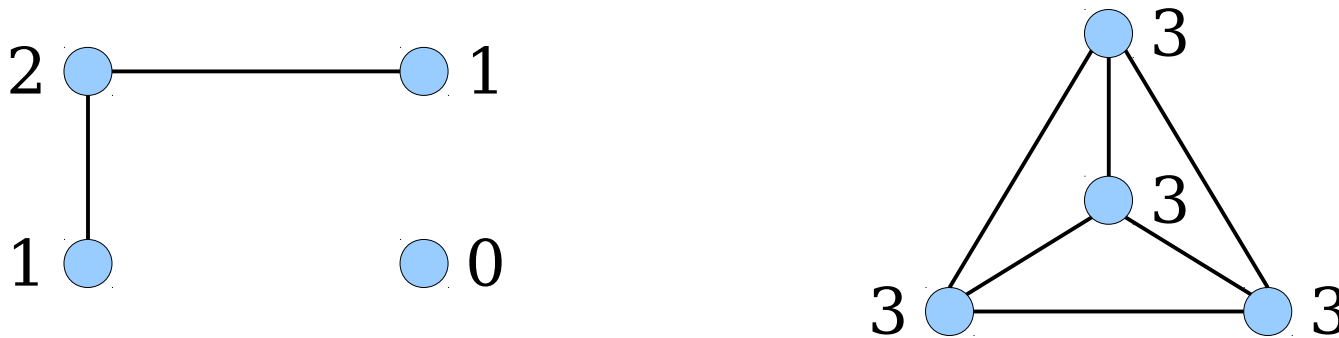




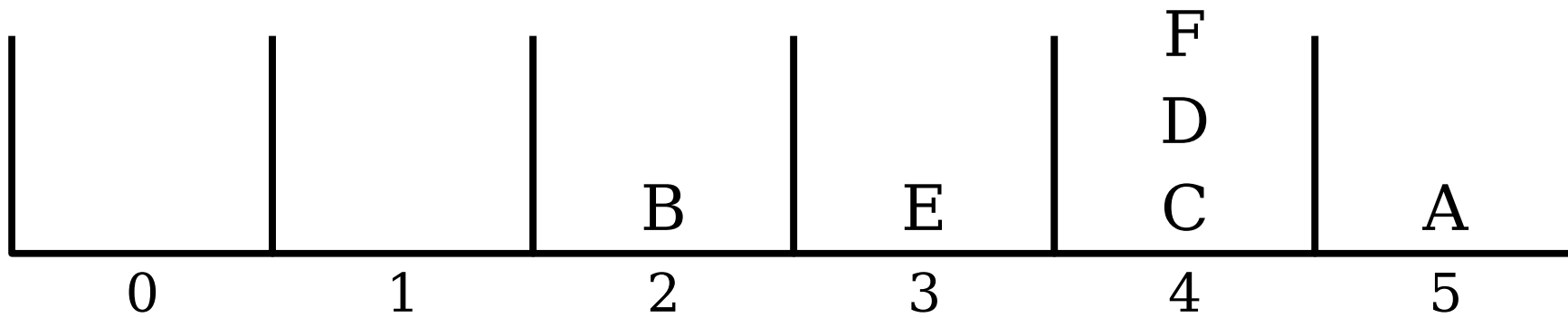
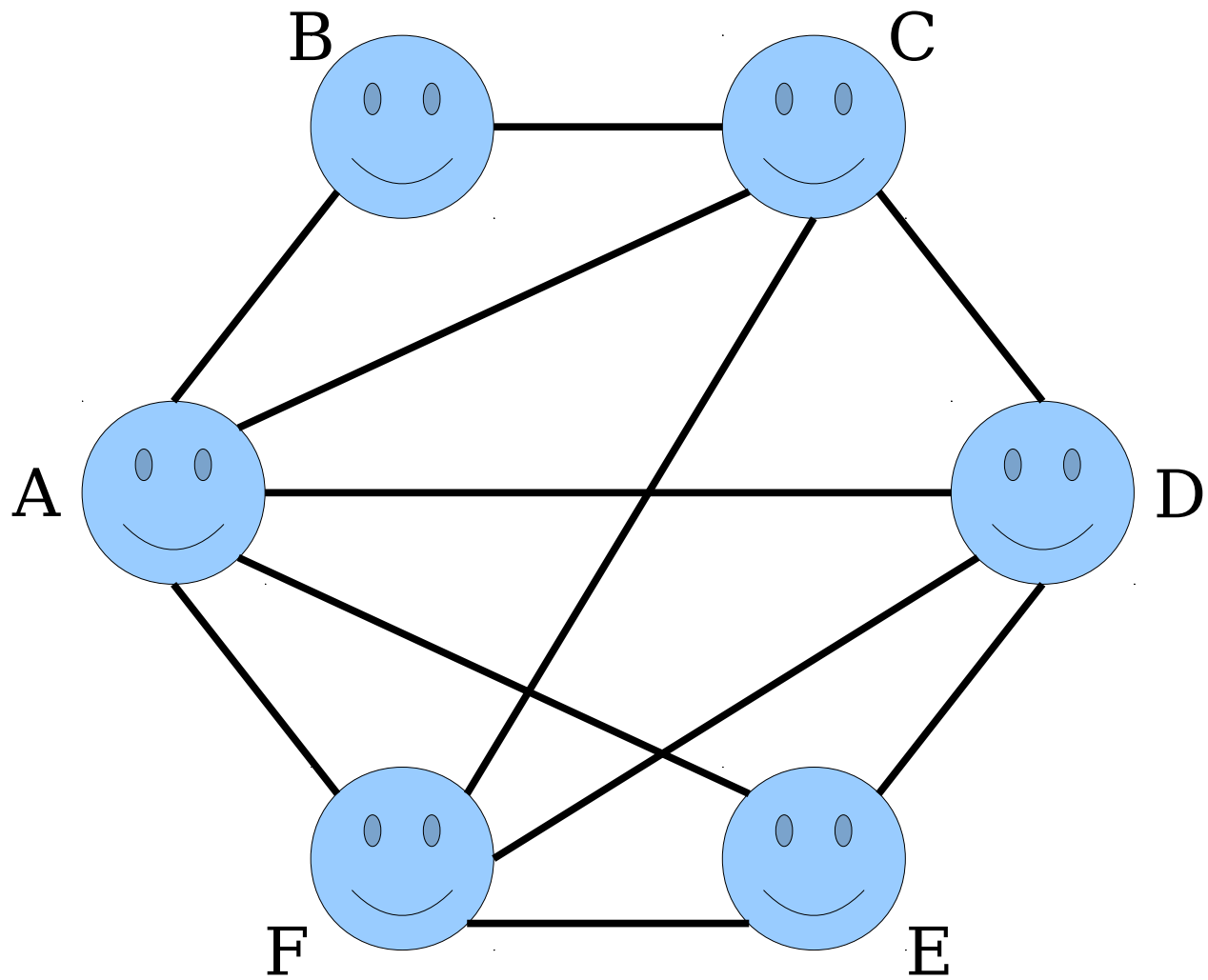


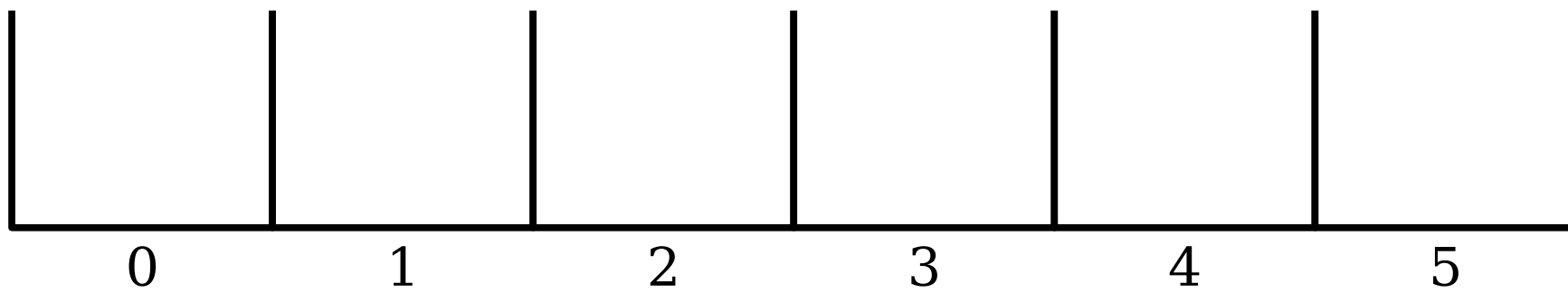
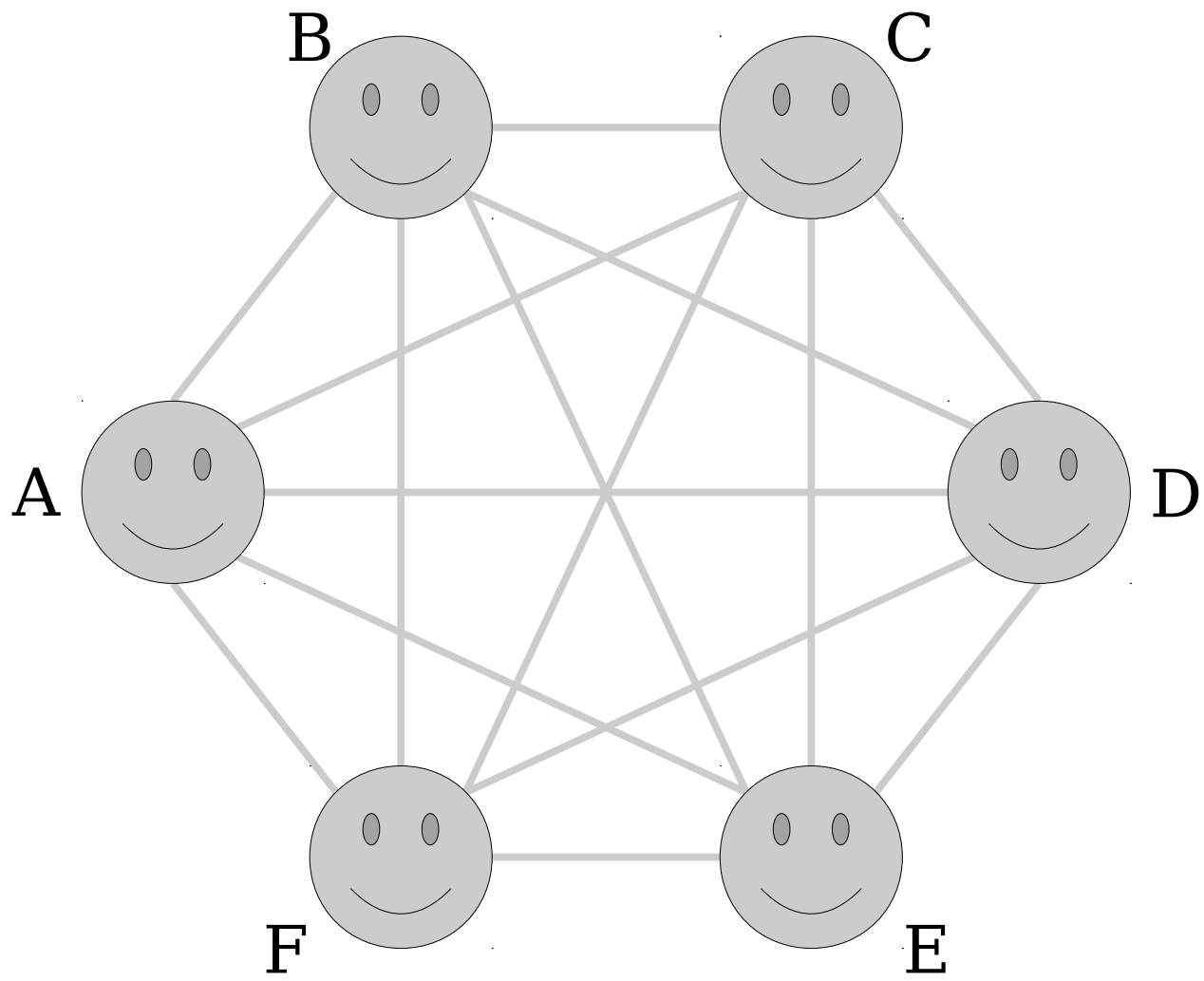
Degrees

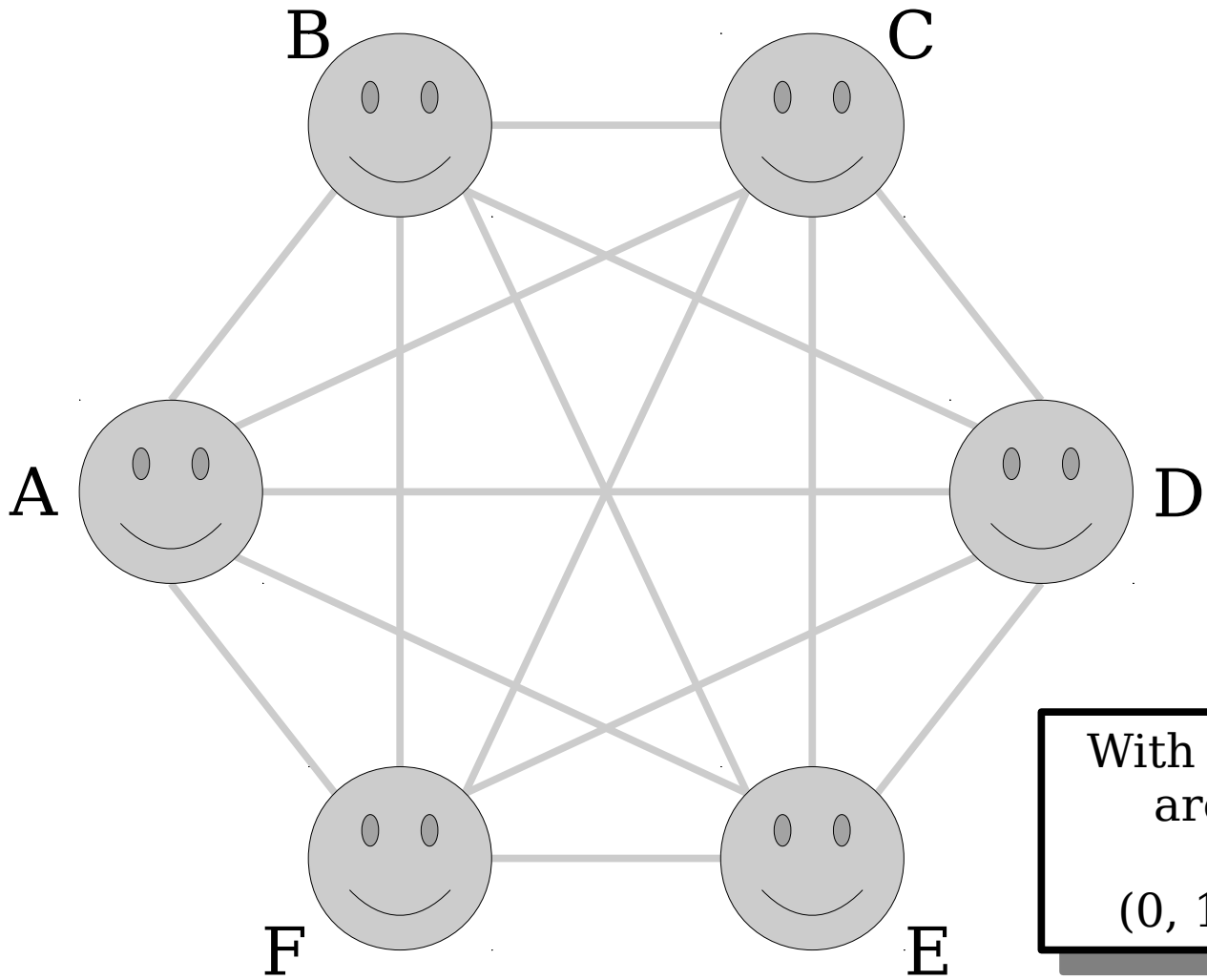
- The **degree** of a node v in a graph is the number of nodes that v is adjacent to.



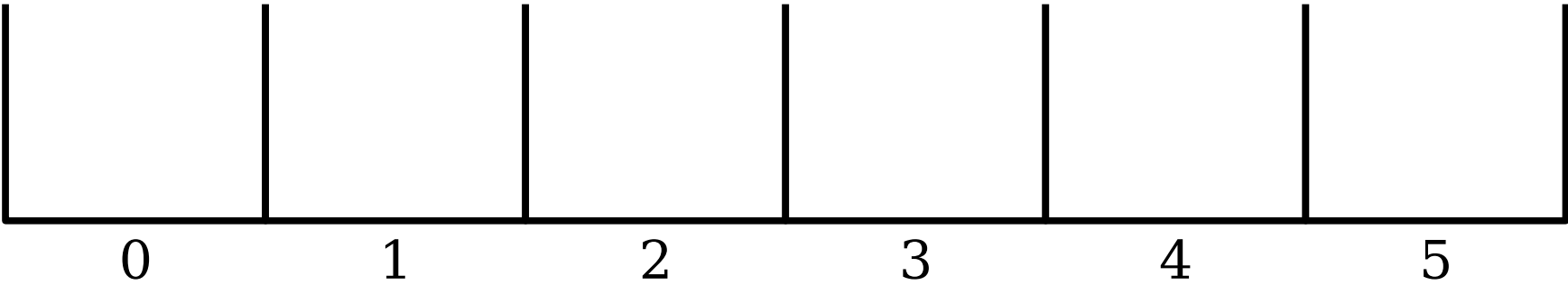
- **Theorem:** Every graph with at least two nodes has at least two nodes with the same degree.
 - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.

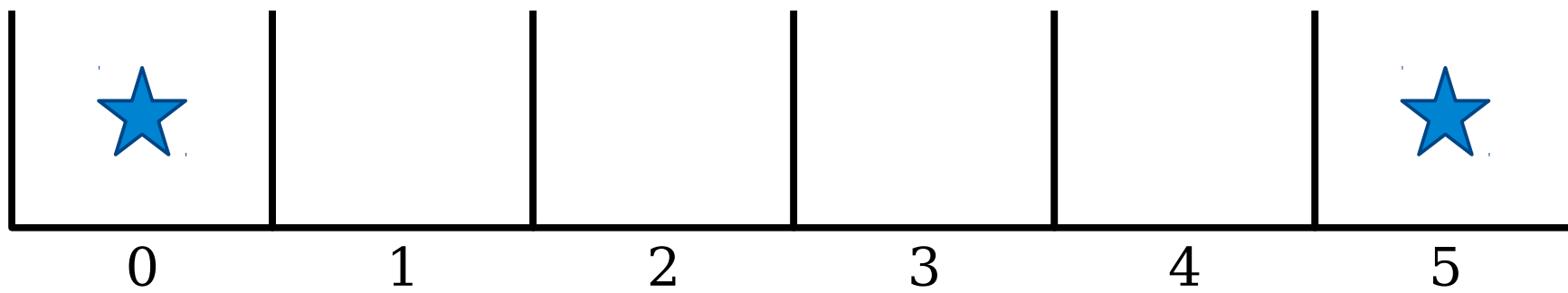
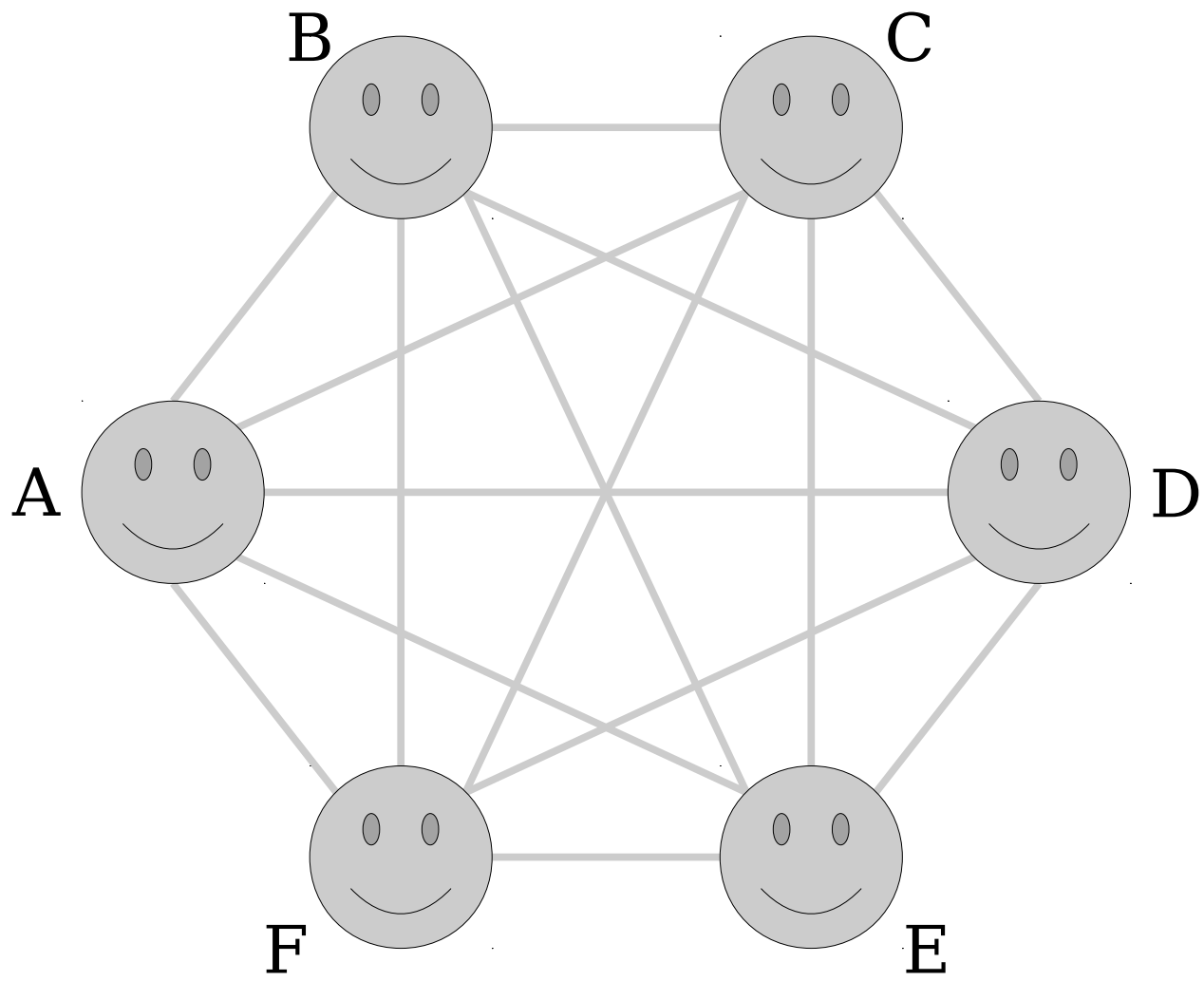


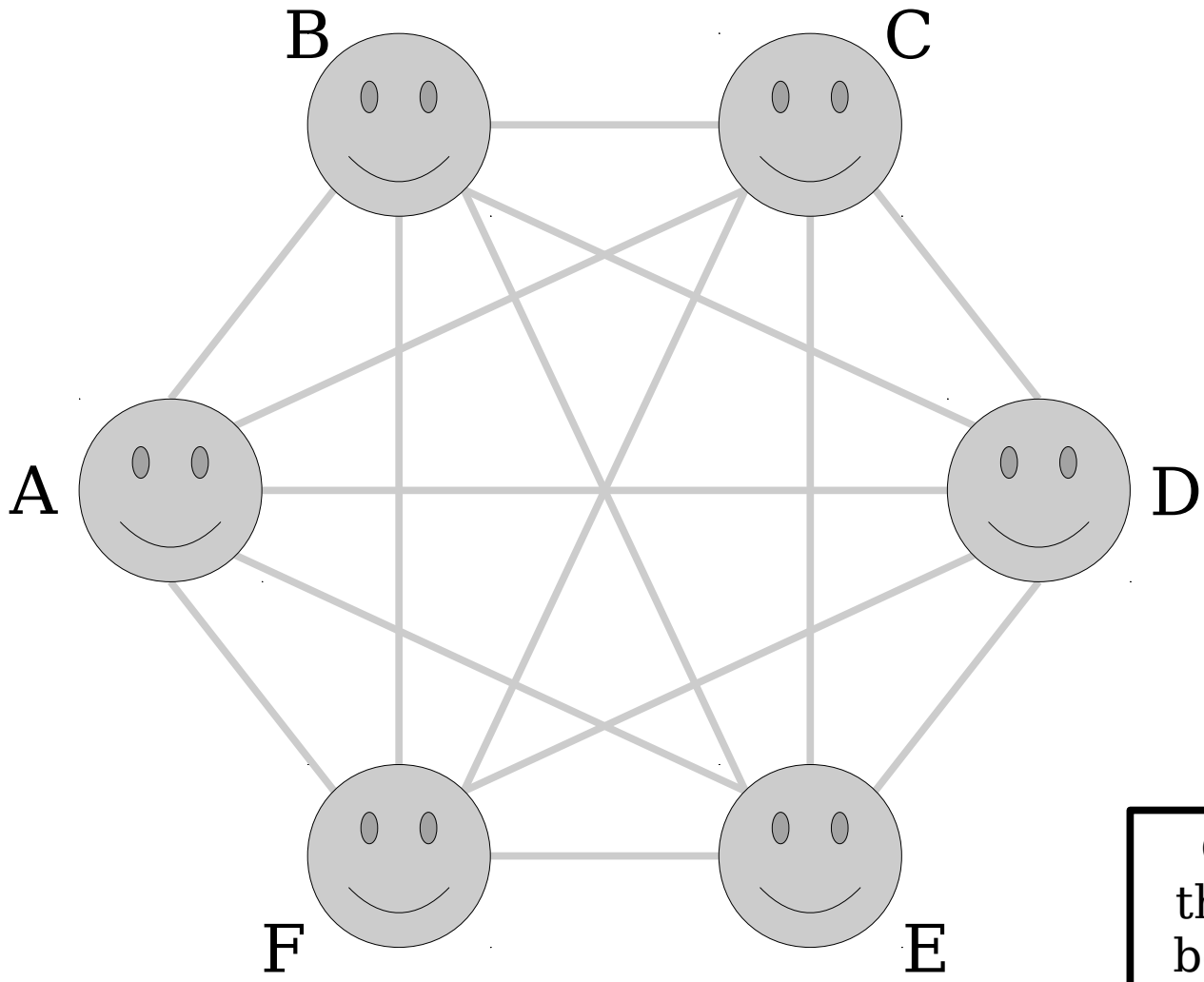




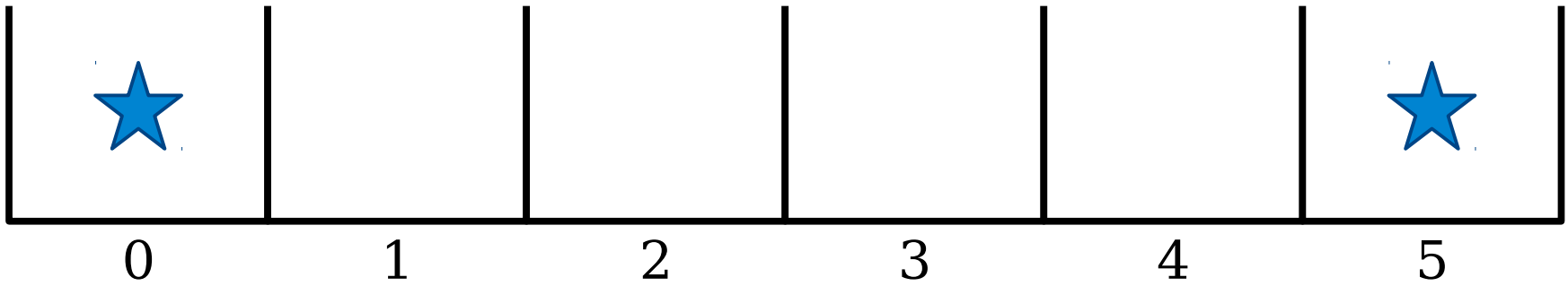
With n nodes, there are n possible degrees
(0, 1, 2, ..., $n - 1$)

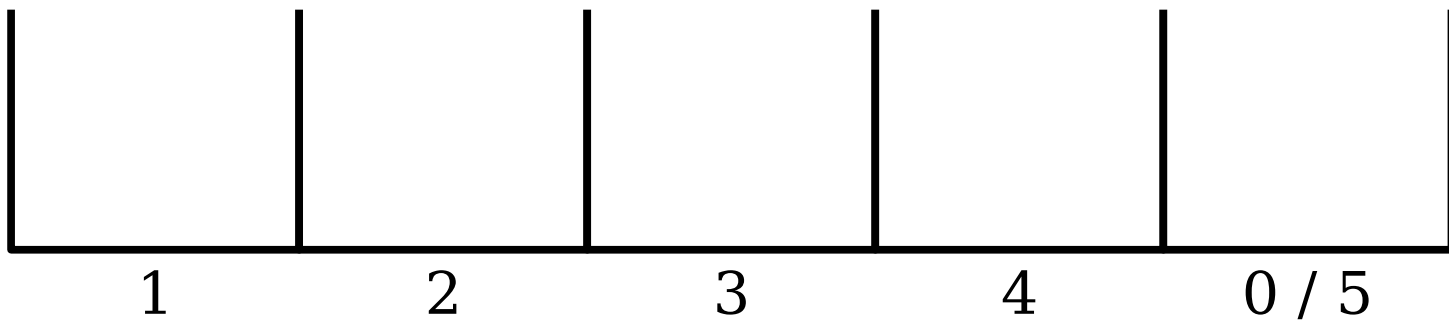
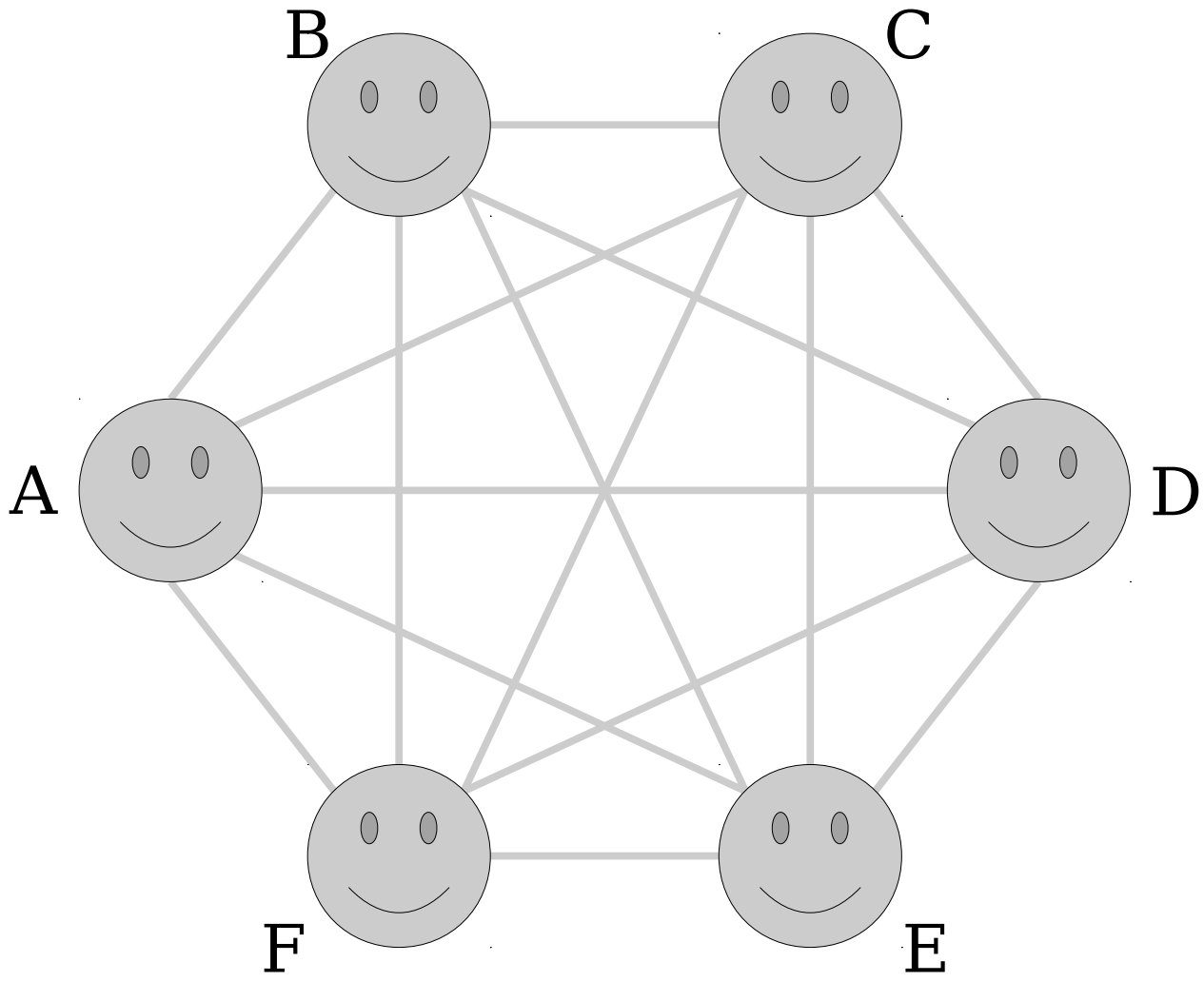






Can both of these buckets be nonempty?





Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

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Proof 1:

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We claim that G cannot simultaneously have a node u of degree 0 and a node v of degree $n - 1$:

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We claim that G cannot simultaneously have a node u of degree 0 and a node v of degree $n - 1$: if there were such nodes, then node u would be adjacent to no other nodes and node v would be adjacent to all other nodes, including u . (Note that u and v must be different nodes, since v has degree at least 1 and u has degree 0 .)

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We therefore see that the possible options for degrees of nodes in G are either drawn from $0, 1, \dots, n - 2$ or from $1, 2, \dots, n - 1$.

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We therefore see that the possible options for degrees of nodes in G are either drawn from $0, 1, \dots, n - 2$ or from $1, 2, \dots, n - 1$. In either case, there are n nodes and $n - 1$ possible degrees, so by the pigeonhole principle two nodes in G must have the same degree.

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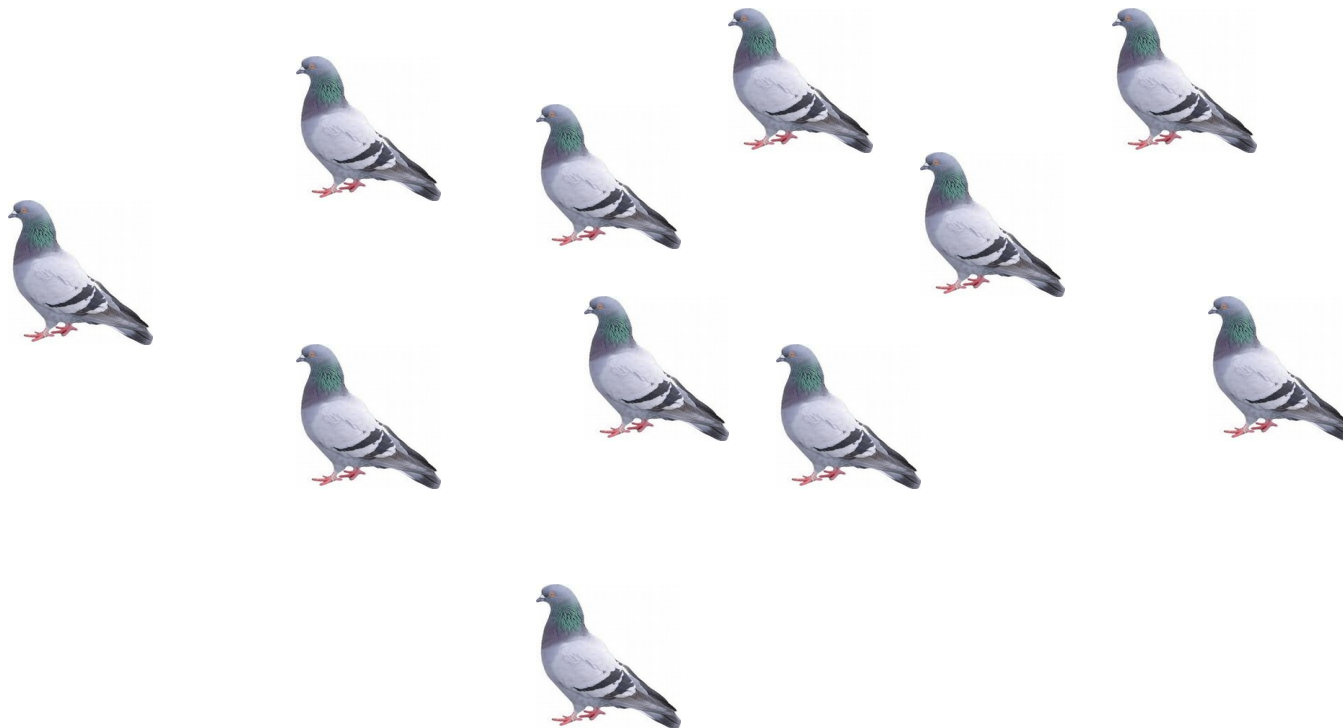
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G , namely $0, 1, 2, \dots, n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

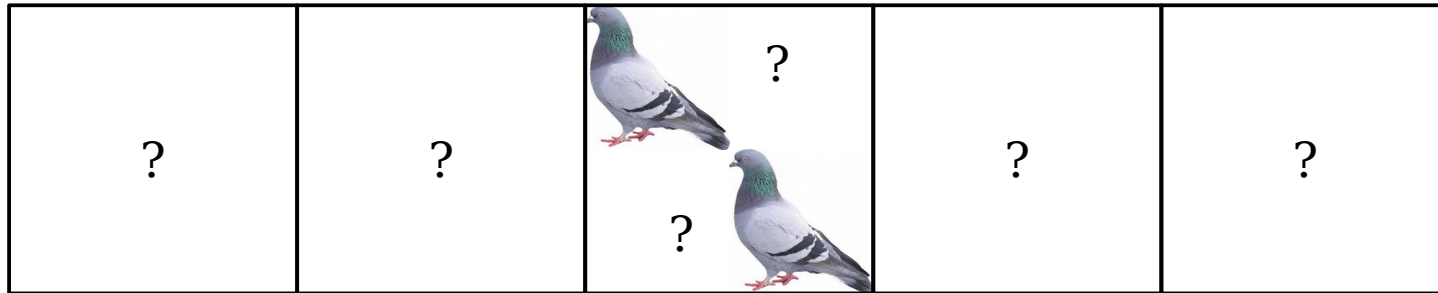
We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

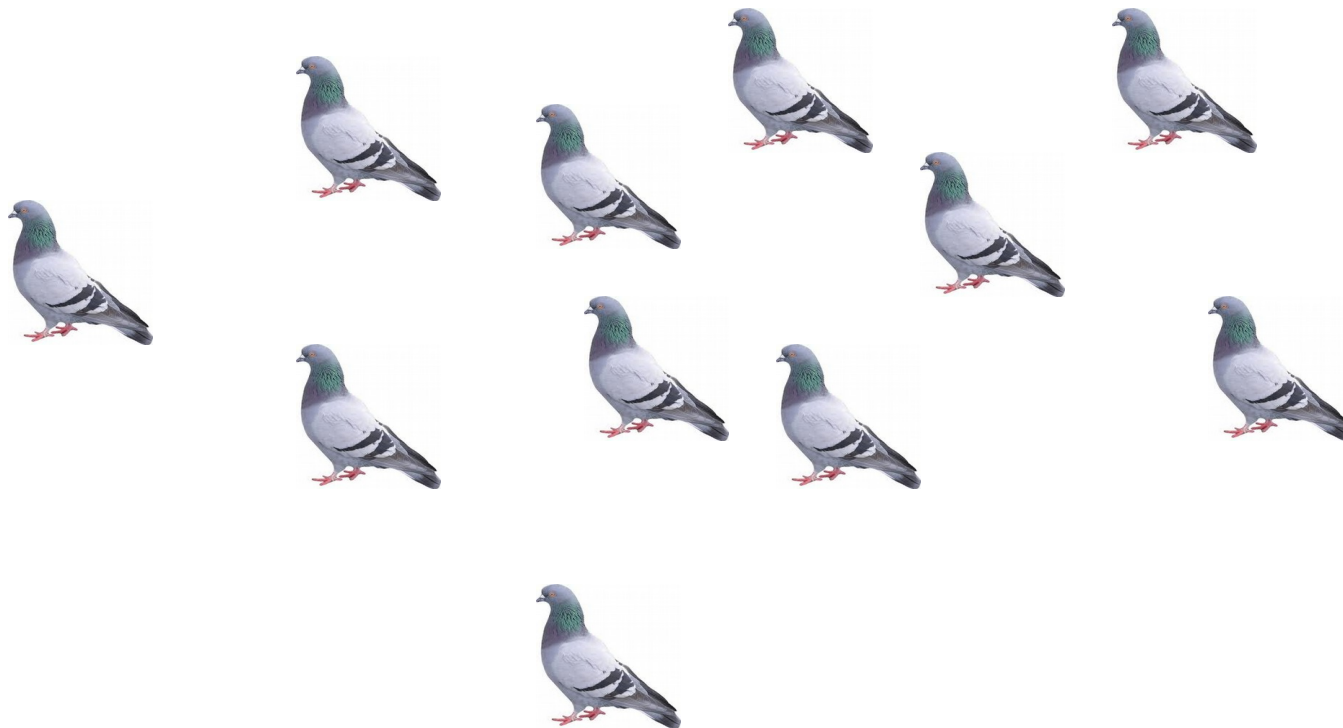
The Pigeonhole Principle



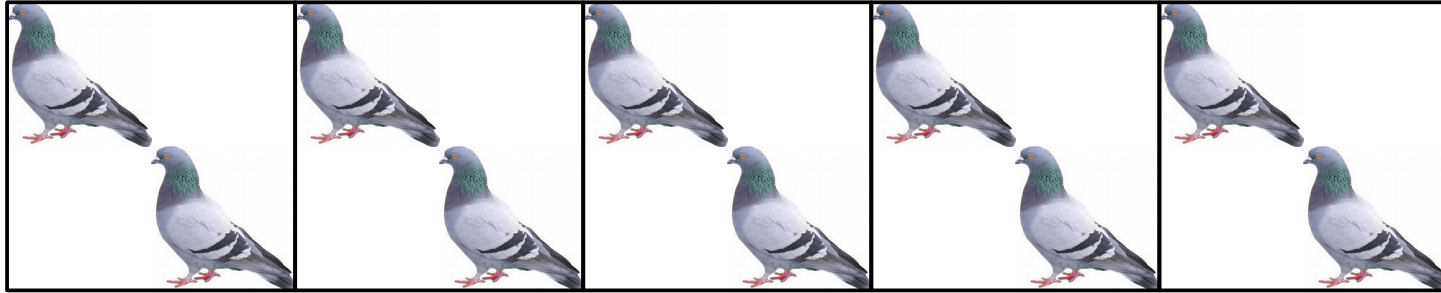
The Pigeonhole Principle



The Pigeonhole Principle



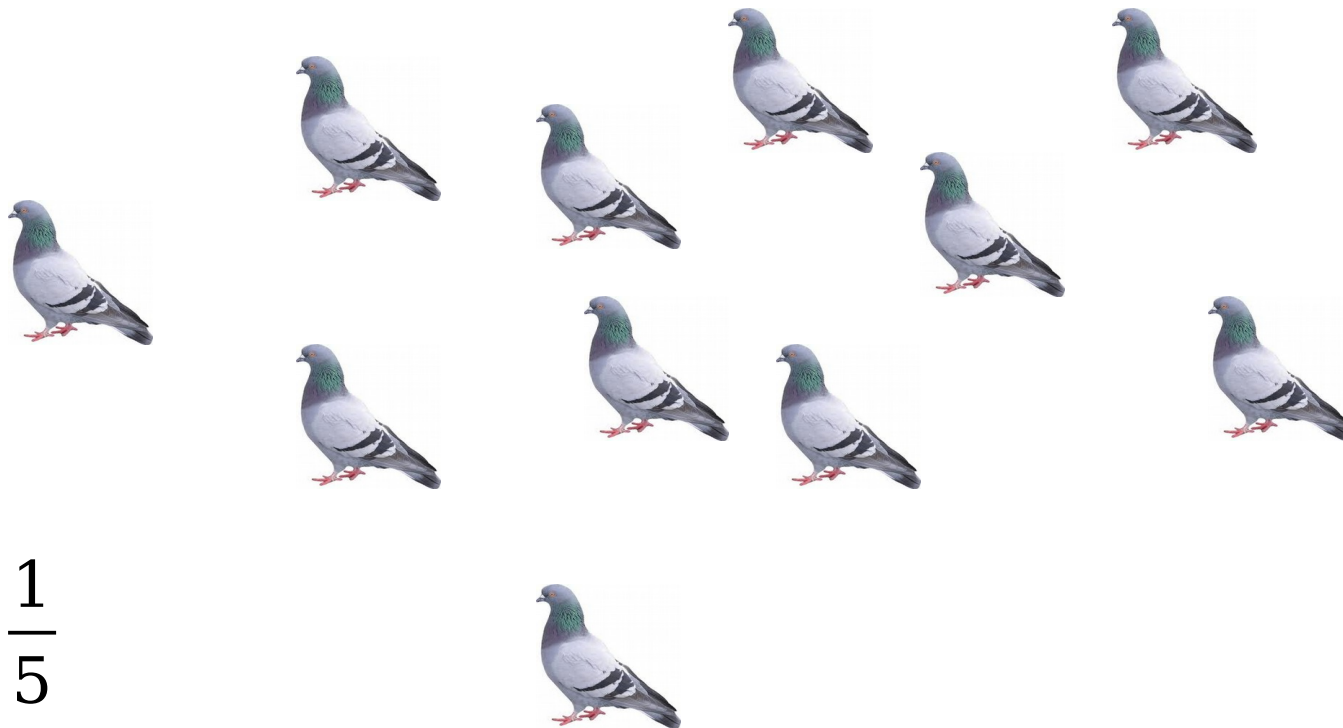
The Pigeonhole Principle



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The Pigeonhole Principle

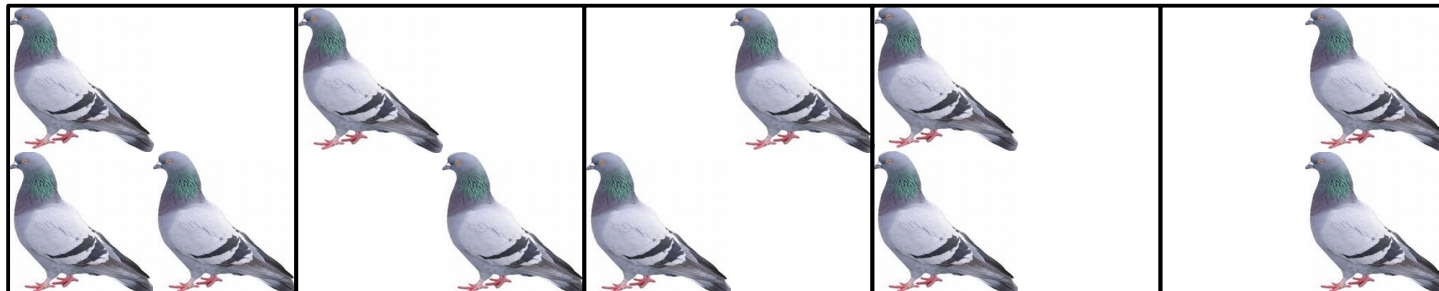


$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The **generalized pigeonhole principle** says that if you distribute m objects into n bins, then
 - some bin will have at least $\lceil m/n \rceil$ objects in it, and
 - some bin will have at most $\lfloor m/n \rfloor$ objects in it.

$\lceil m/n \rceil$ means “ m/n , rounded up.”
 $\lfloor m/n \rfloor$ means “ m/n , rounded down.”



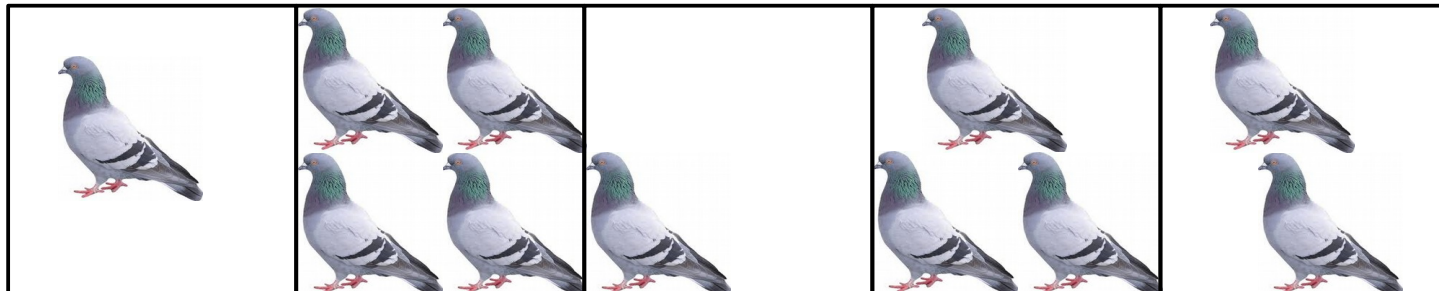
$$m = 11$$
$$n = 5$$

$$\lceil m / n \rceil = 3$$
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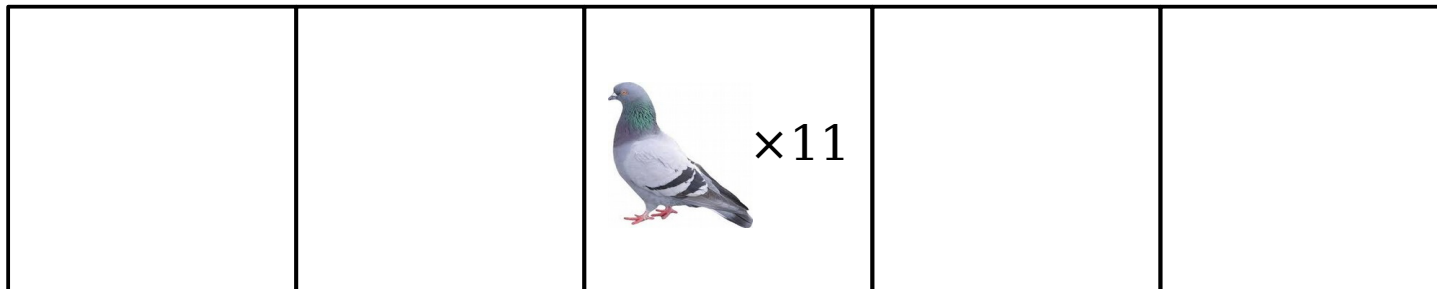
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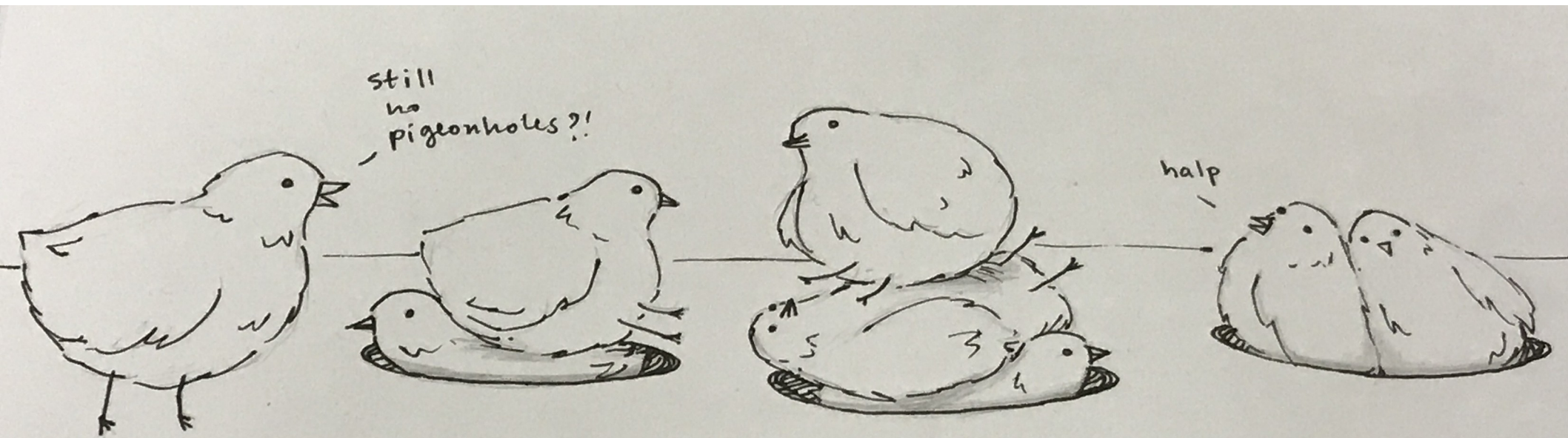
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$$m = 11$$
$$n = 5$$

$$\lceil m / n \rceil = 3$$
$$\lfloor m / n \rfloor = 2$$



$$m = 8, n = 3$$

Thanks to Amy Liu for this awesome drawing!

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least $\lceil m/n \rceil$ objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n , there is a way to distribute m objects into n bins such that each bin contains fewer than $\lceil m/n \rceil$ objects.

Number the bins $1, 2, 3, \dots, n$ and let x_i denote the number of objects in bin i . Since there are m objects in total, we know that

$$m = x_1 + x_2 + \dots + x_n.$$

Since each bin contains fewer than $\lceil m/n \rceil$ objects, we see that $x_i < \lceil m/n \rceil$ for each i . Therefore, we have that

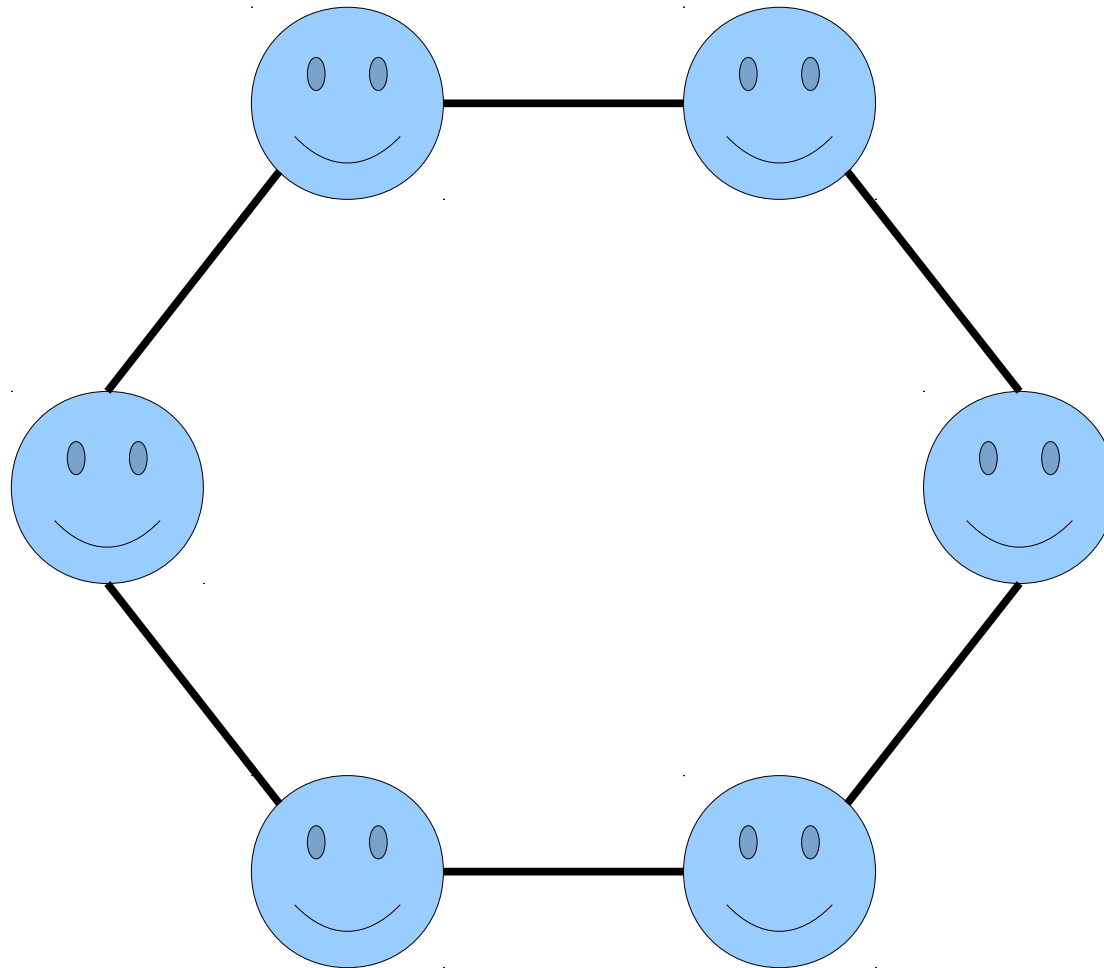
$$\begin{aligned} m &= x_1 + x_2 + \dots + x_n \\ &< \lceil m/n \rceil + \lceil m/n \rceil + \dots + \lceil m/n \rceil \quad (n \text{ times}) \\ &= m. \end{aligned}$$

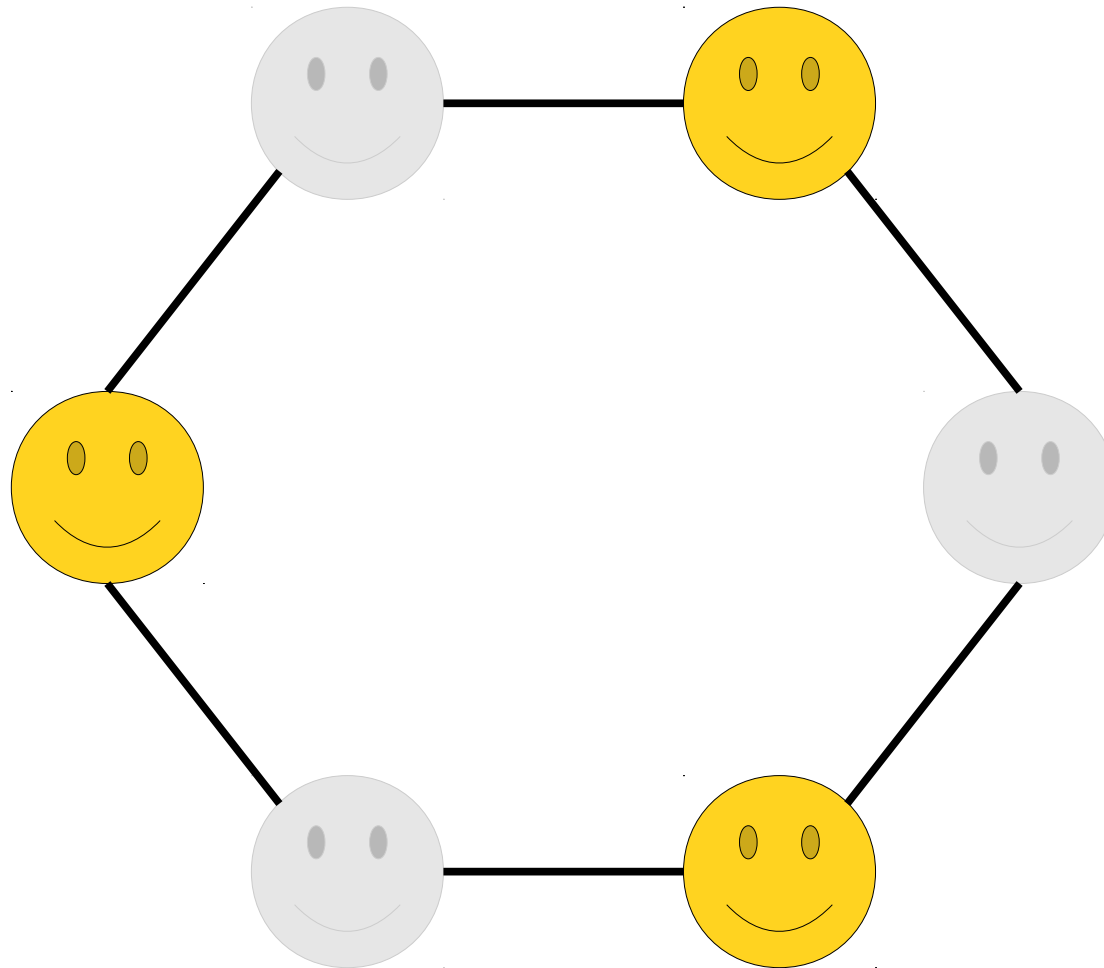
But this means that $m < m$, which is impossible. We have reached a contradiction, so our initial assumption must have been wrong. Therefore, if m objects are distributed into n bins, some bin must contain at least $\lceil m/n \rceil$ objects. ■

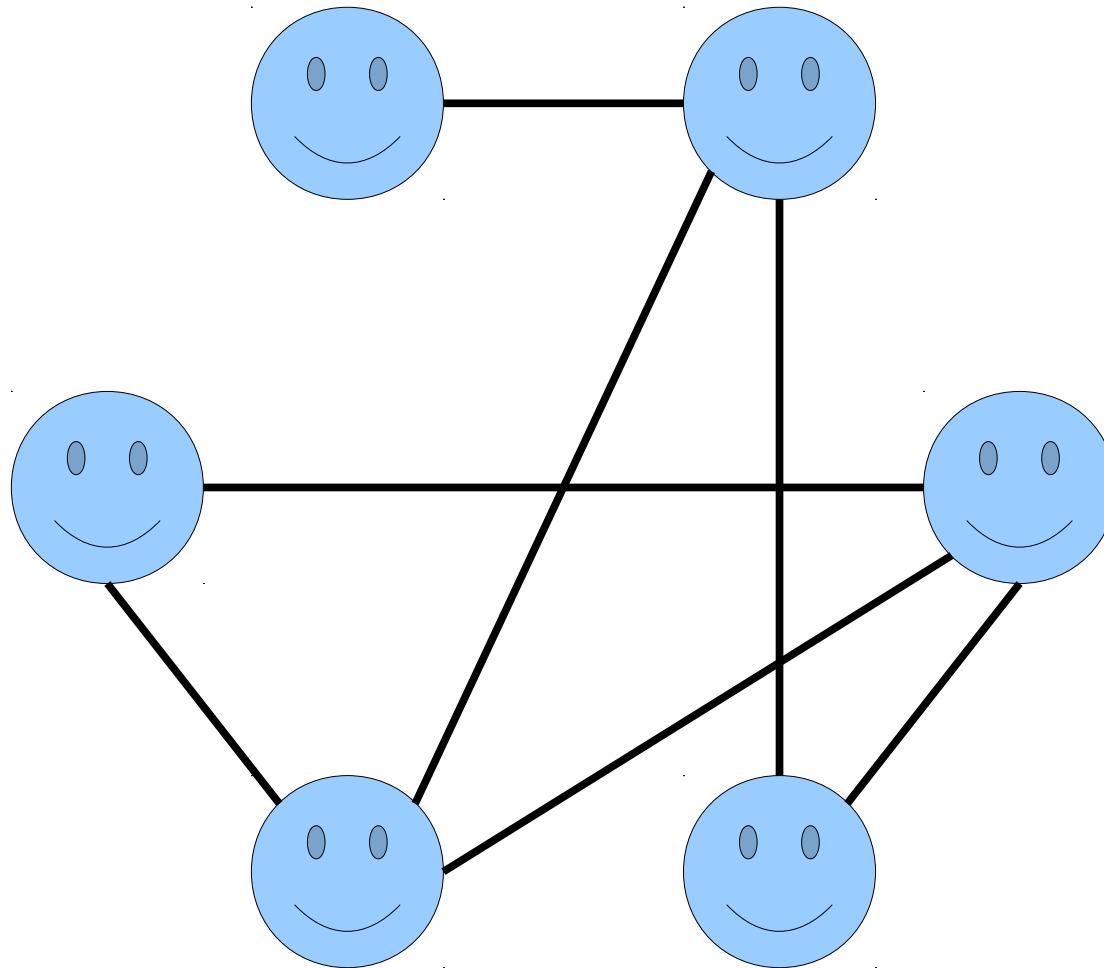
An Application: Friends and Strangers

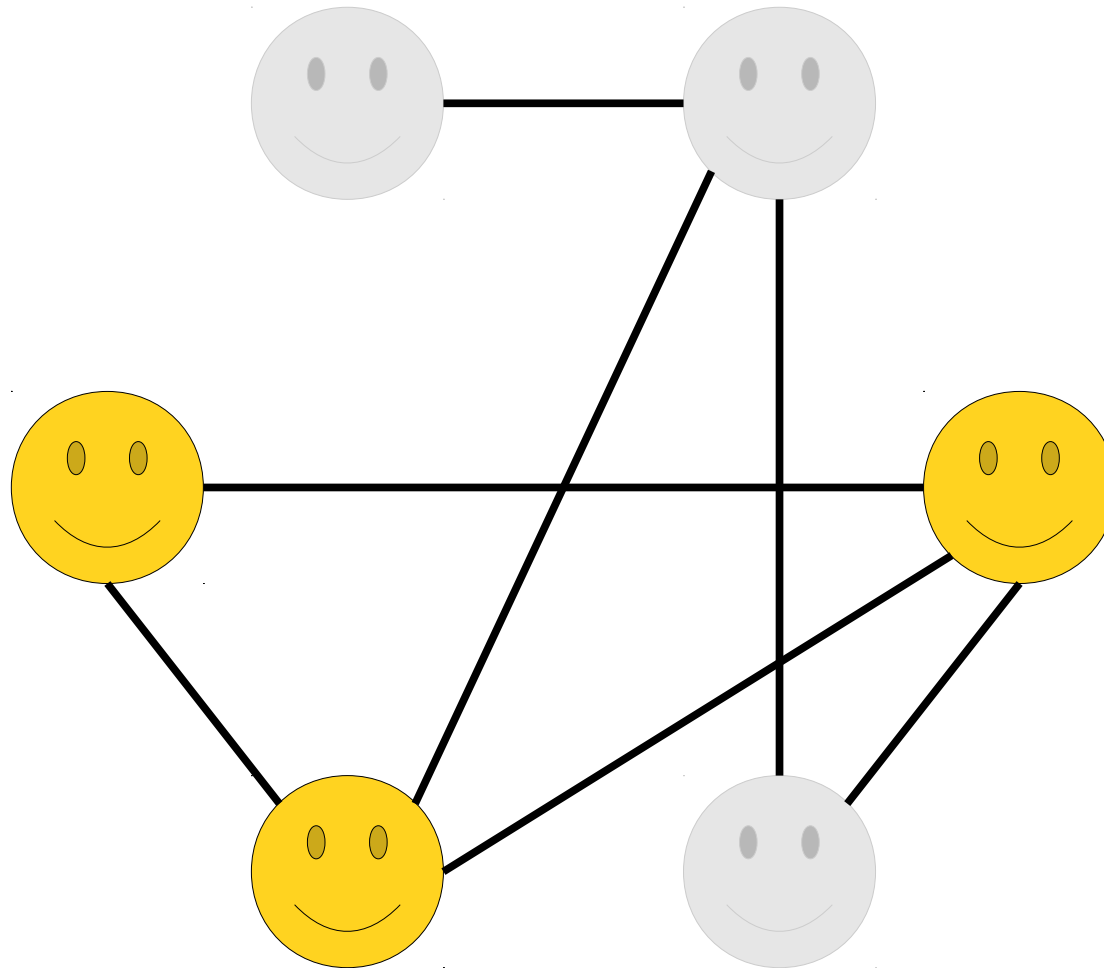
Friends and Strangers

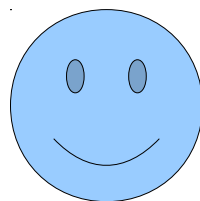
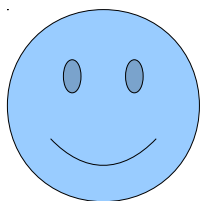
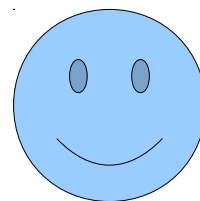
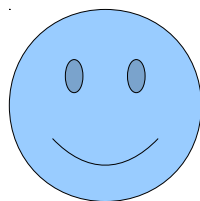
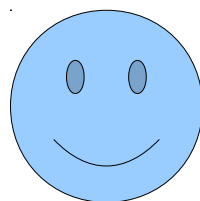
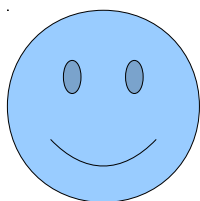
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

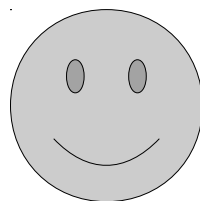
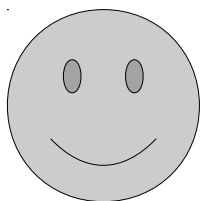
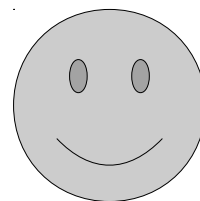
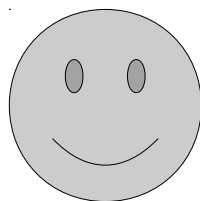
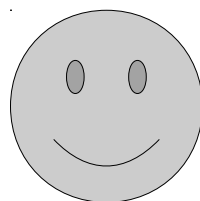
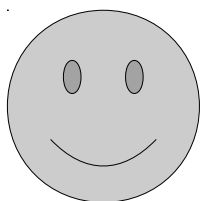


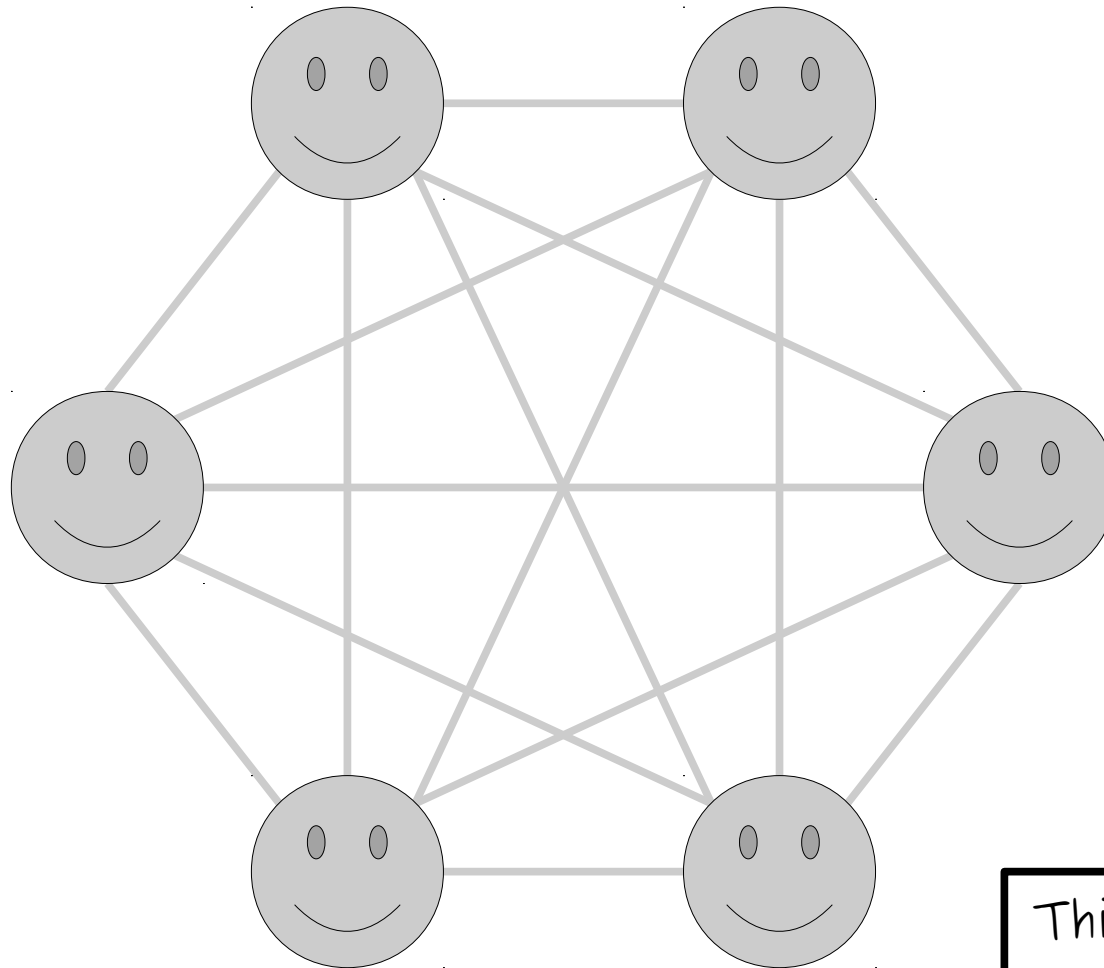




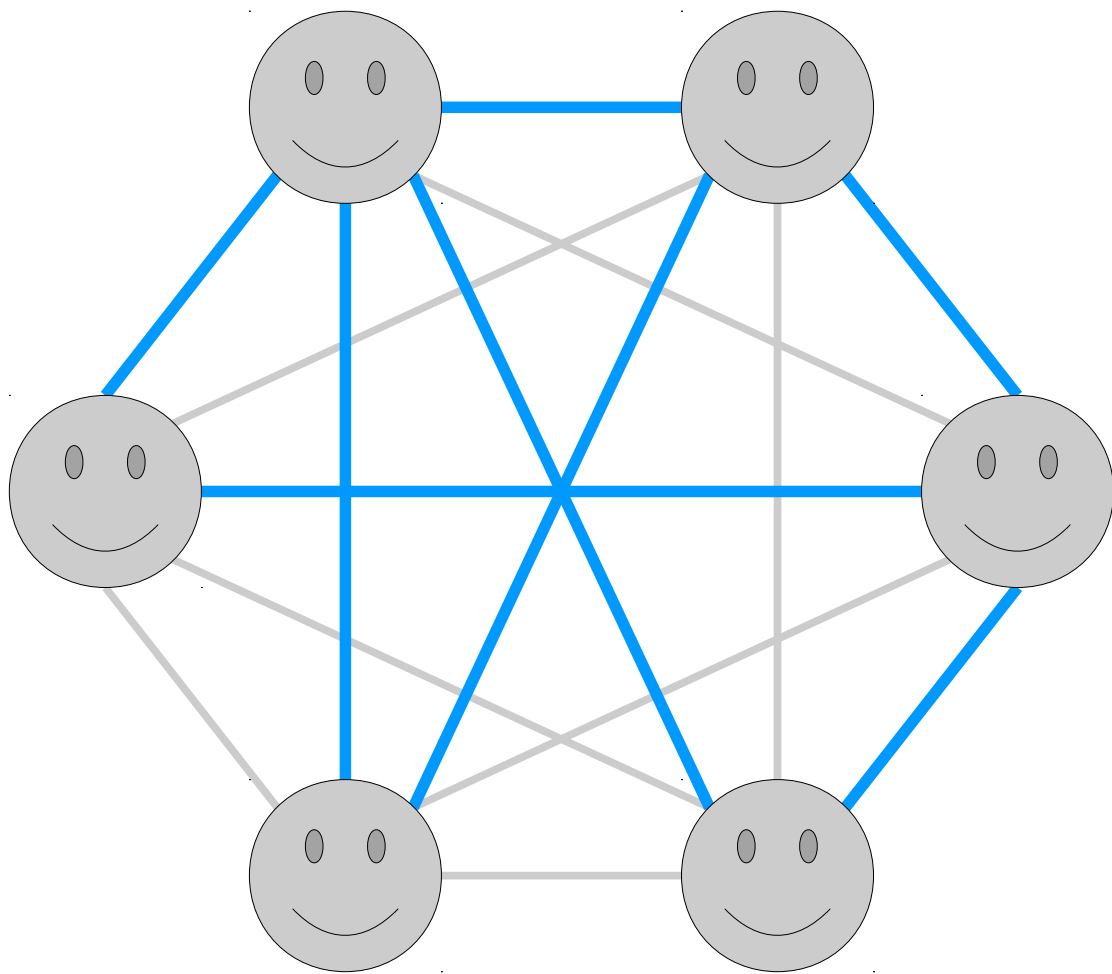


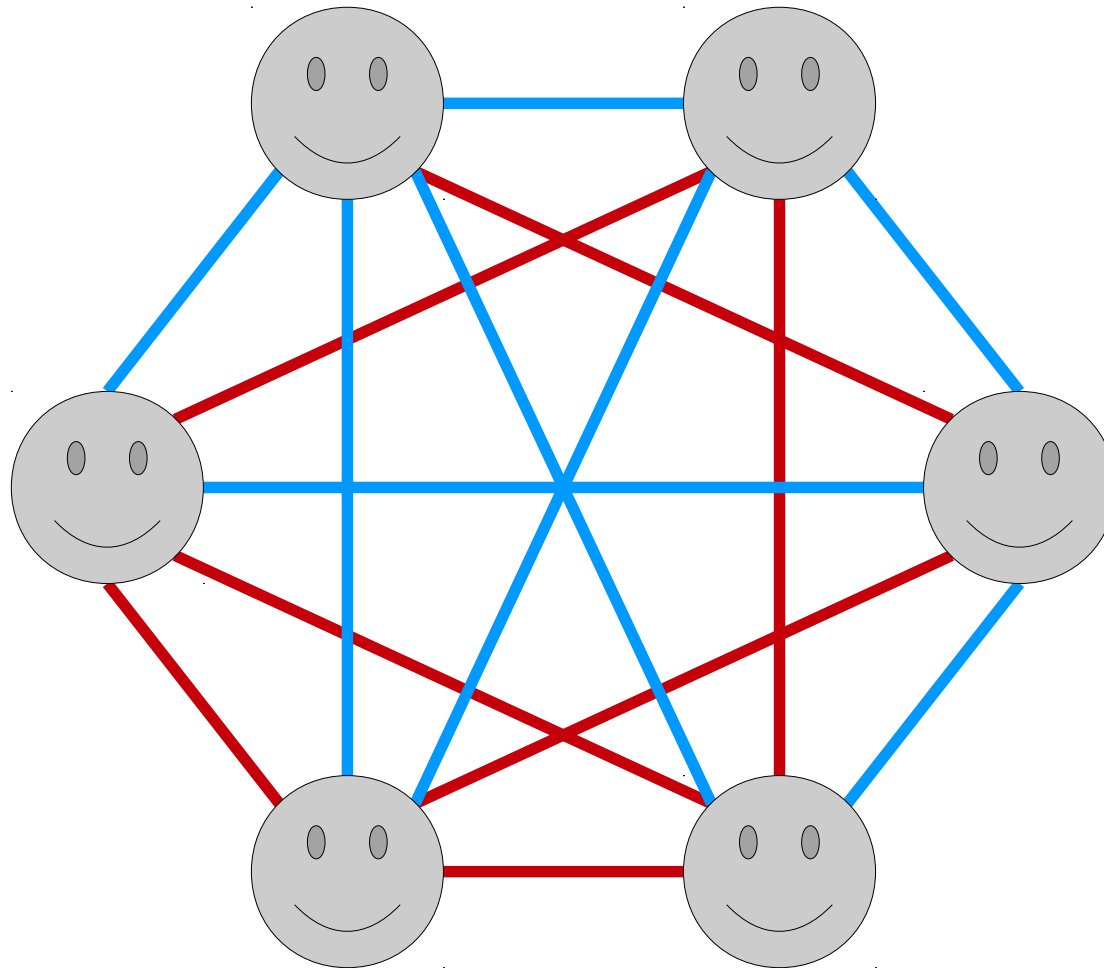


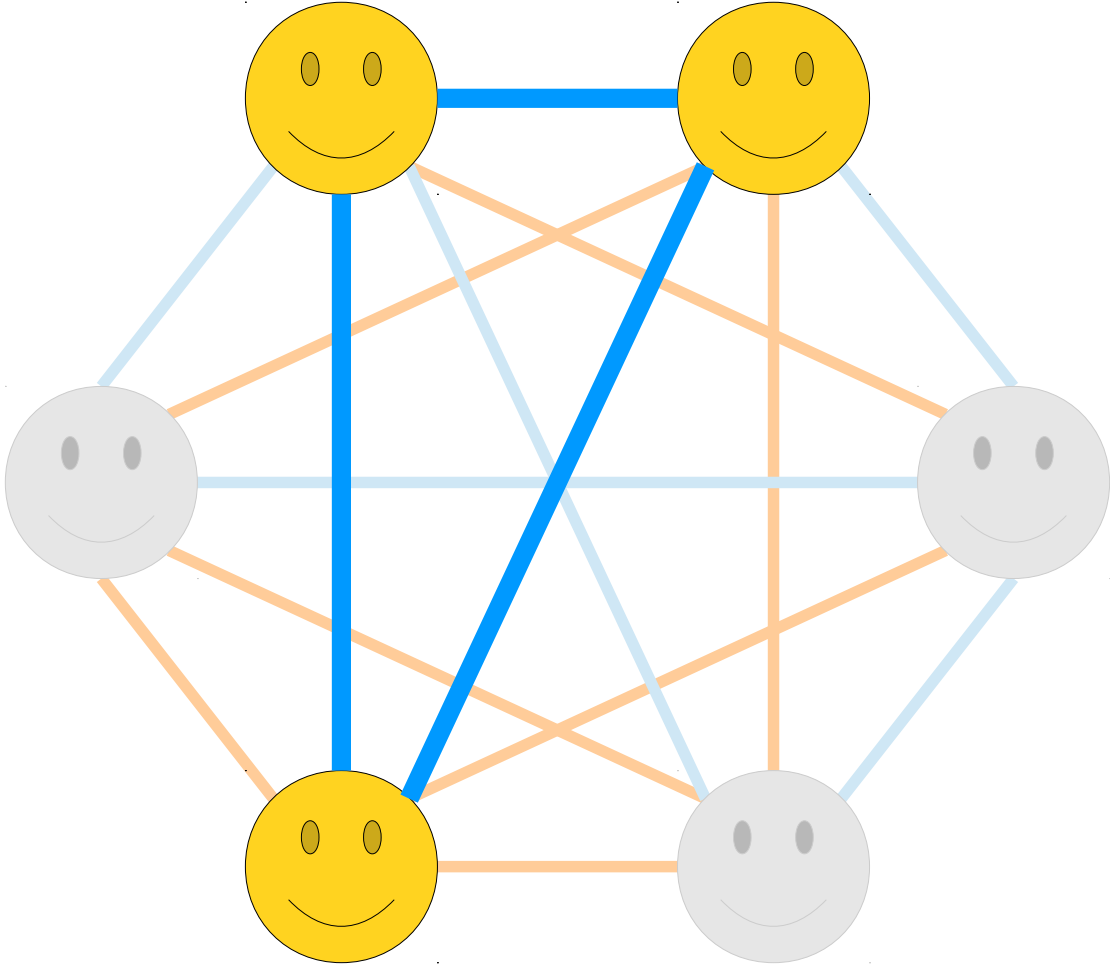


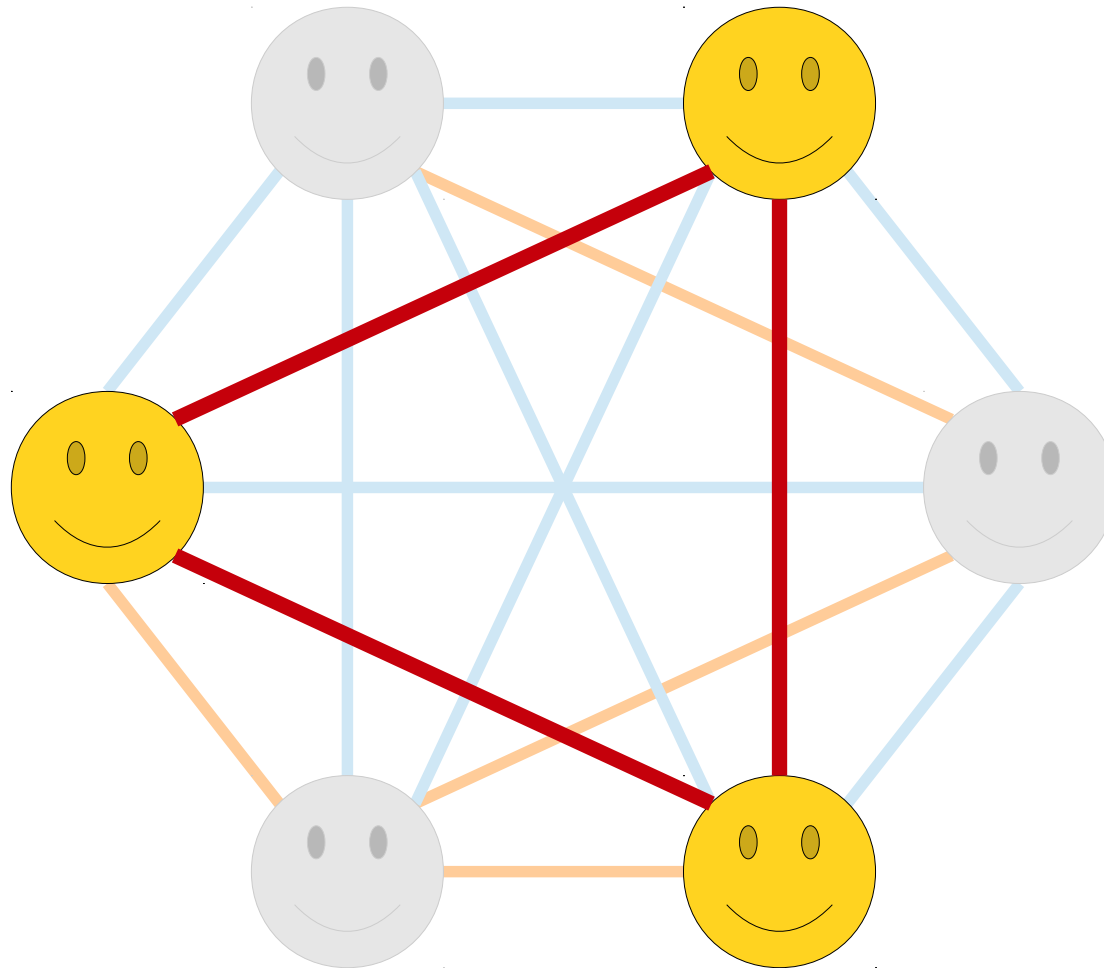


This graph is called
a *6-clique*, by the
way.







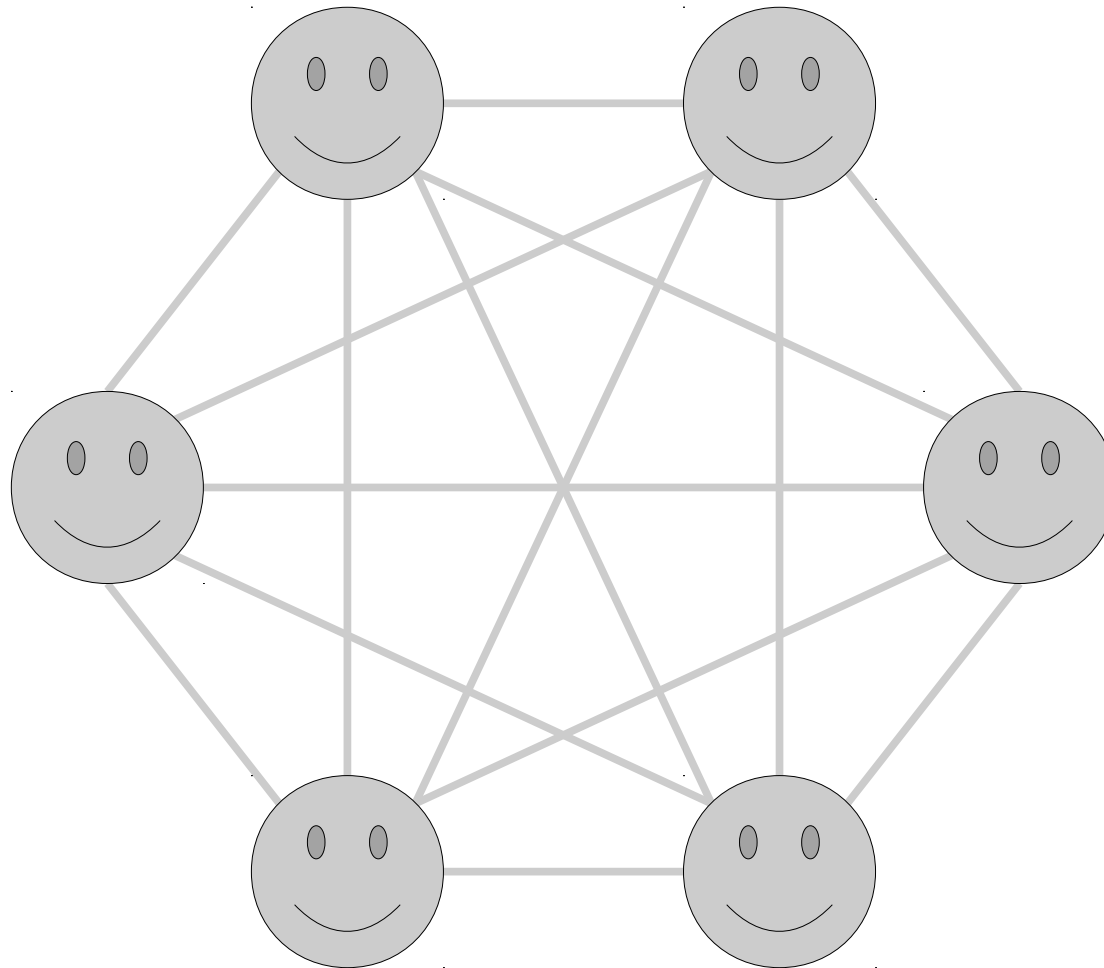


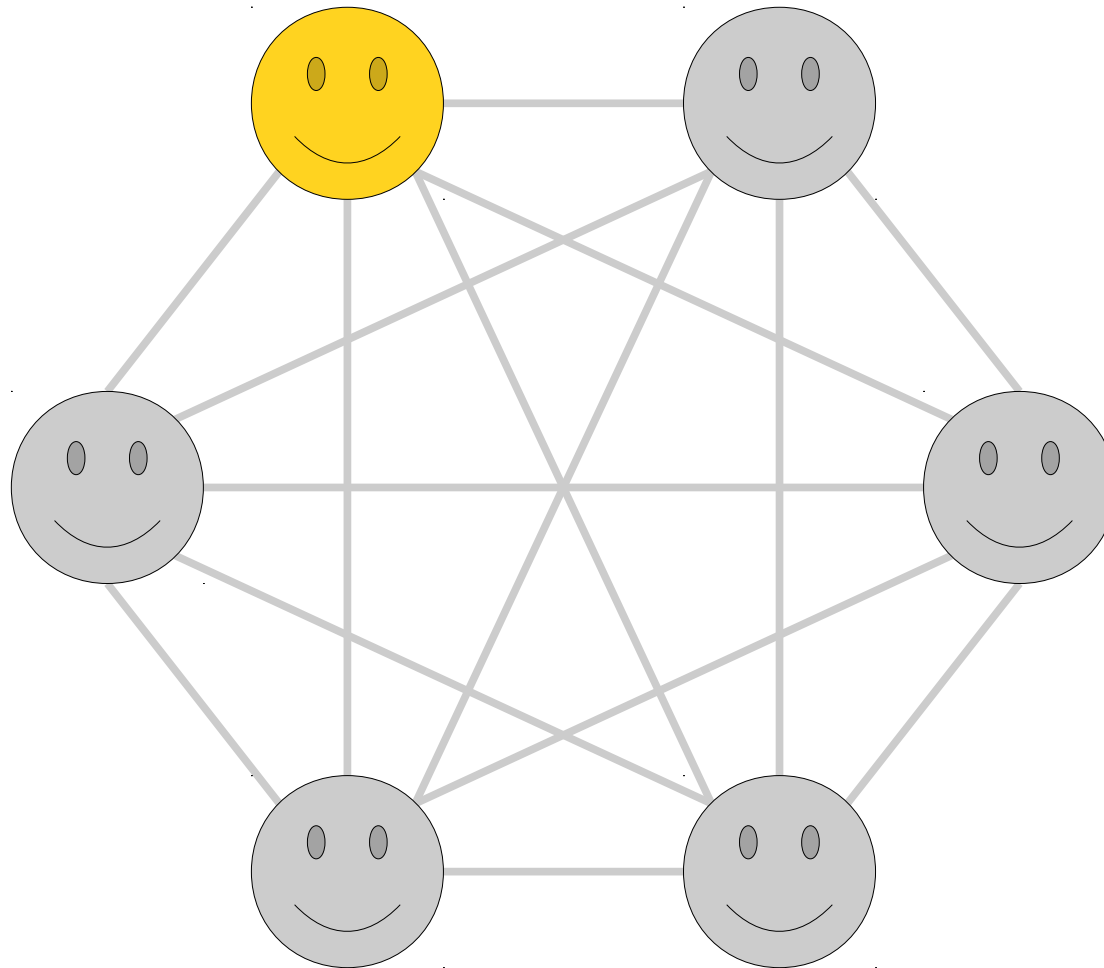
Friends and Strangers Restated

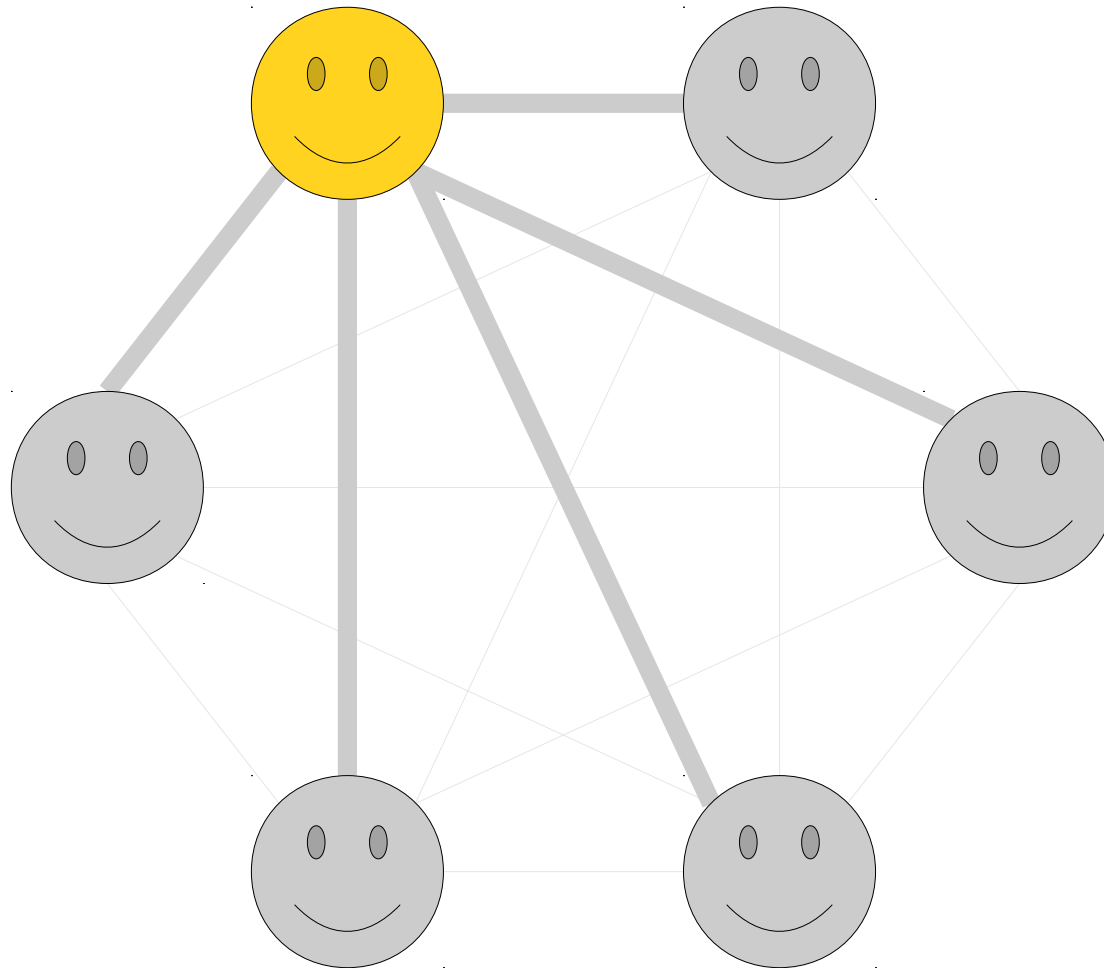
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

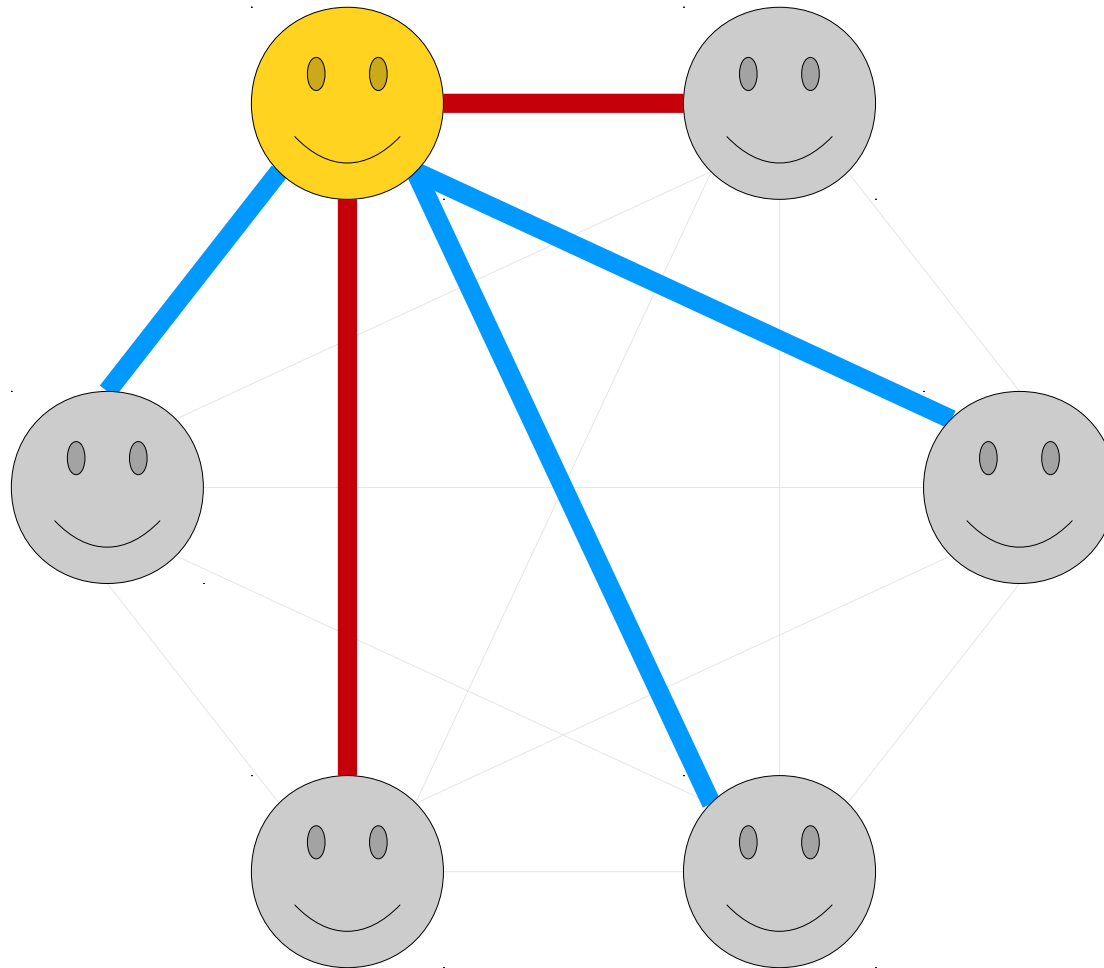
Theorem: Any 6-clique whose edges are colored red and blue contains a red triangle or a blue triangle (or both).

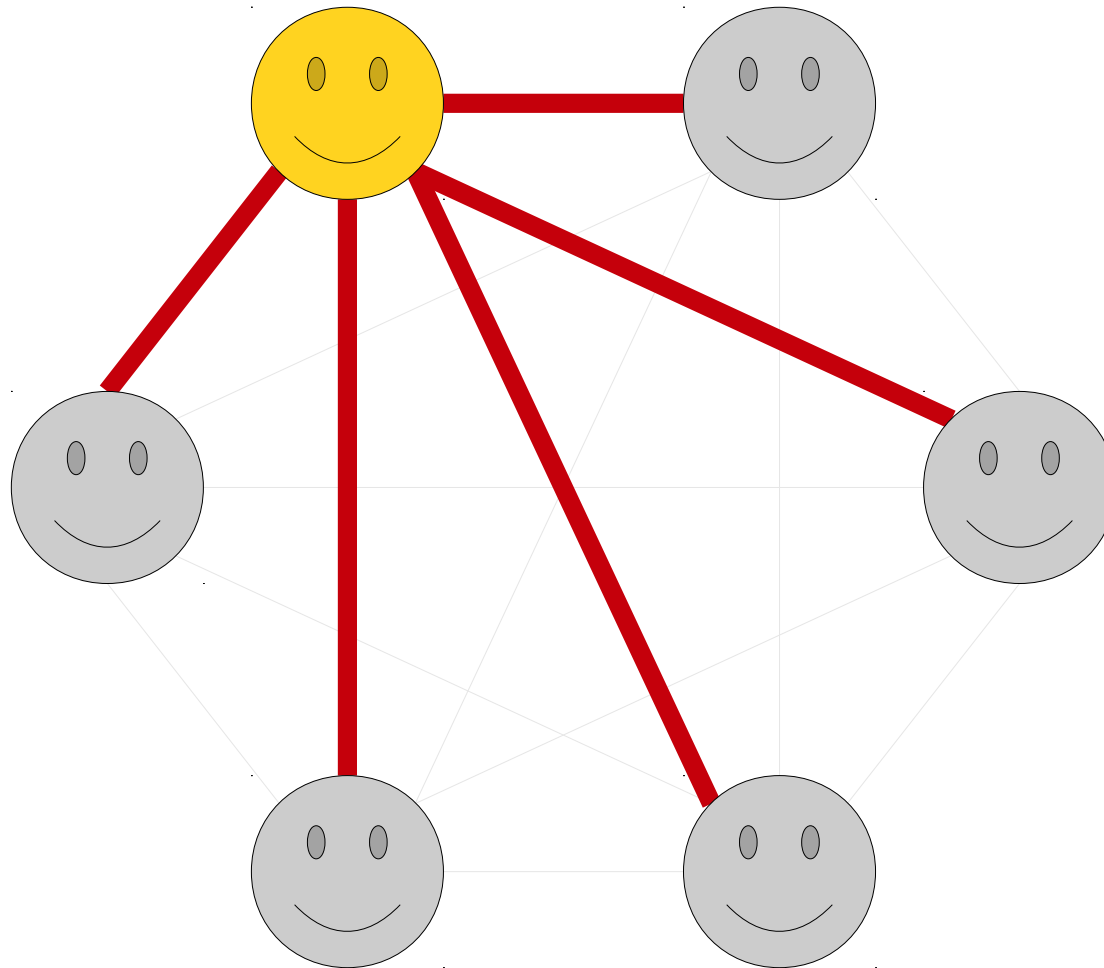
- How can we prove this?

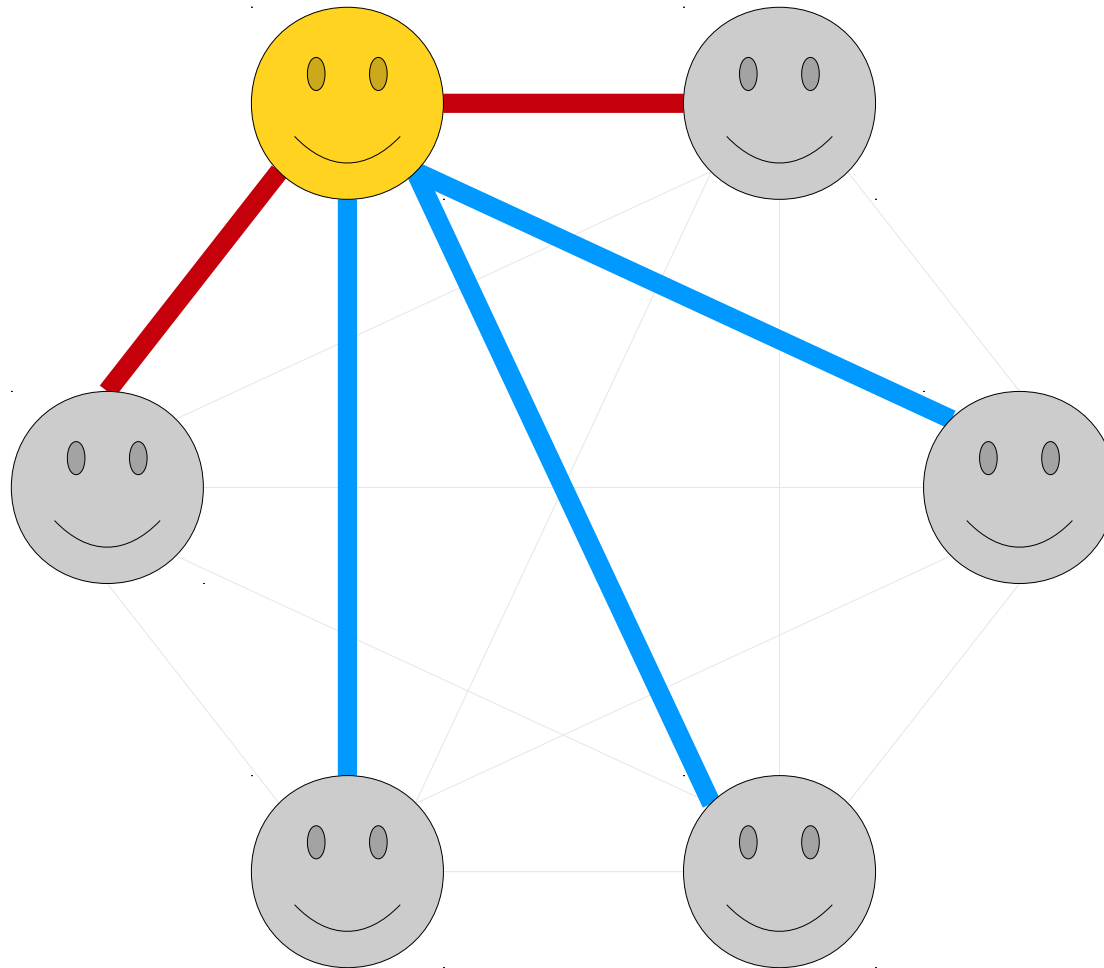


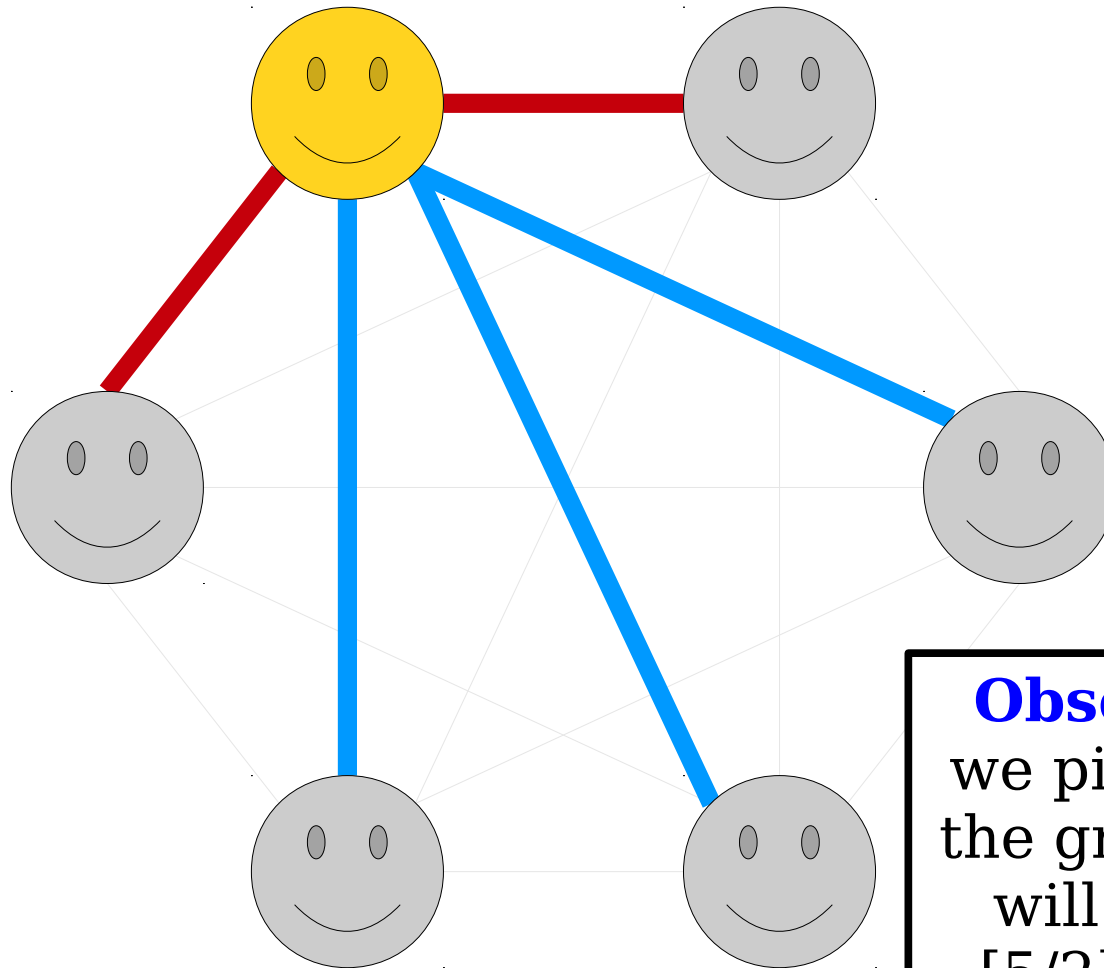




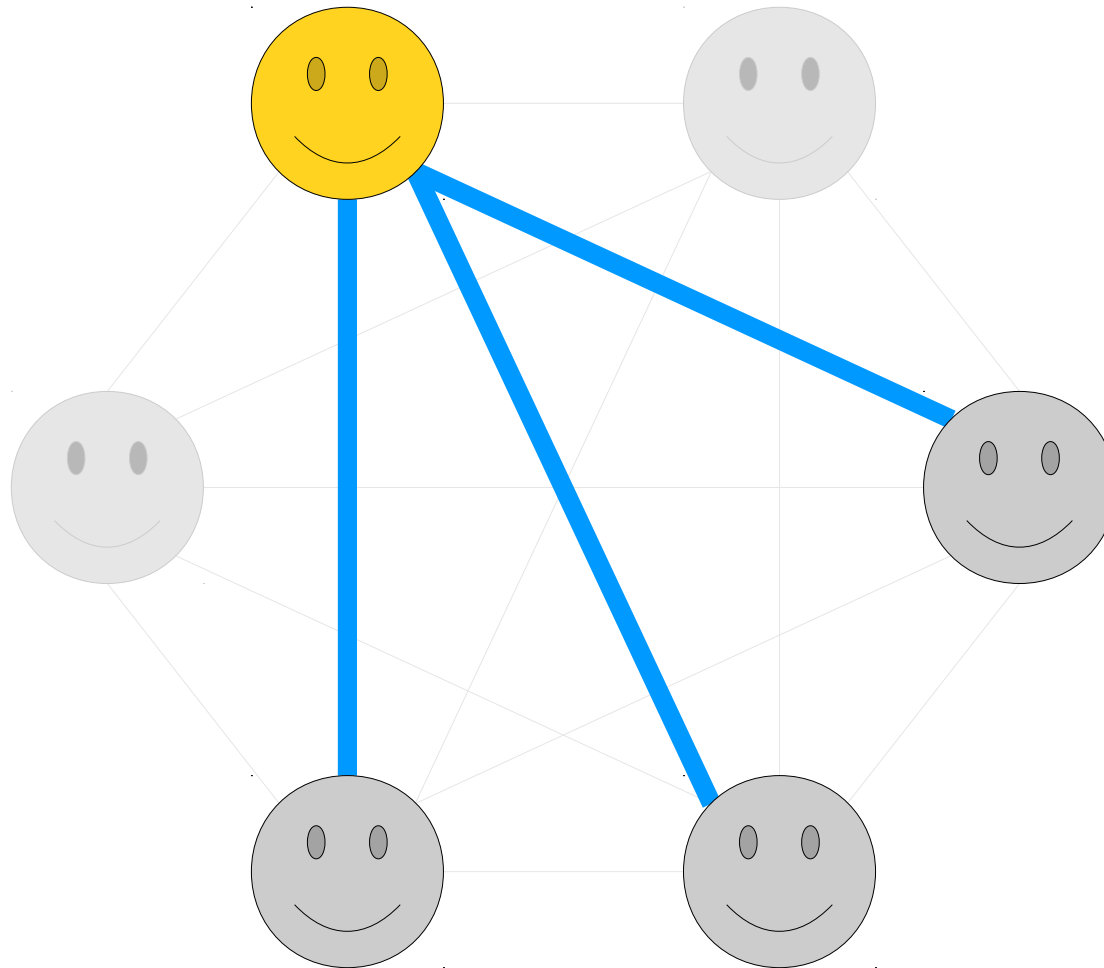


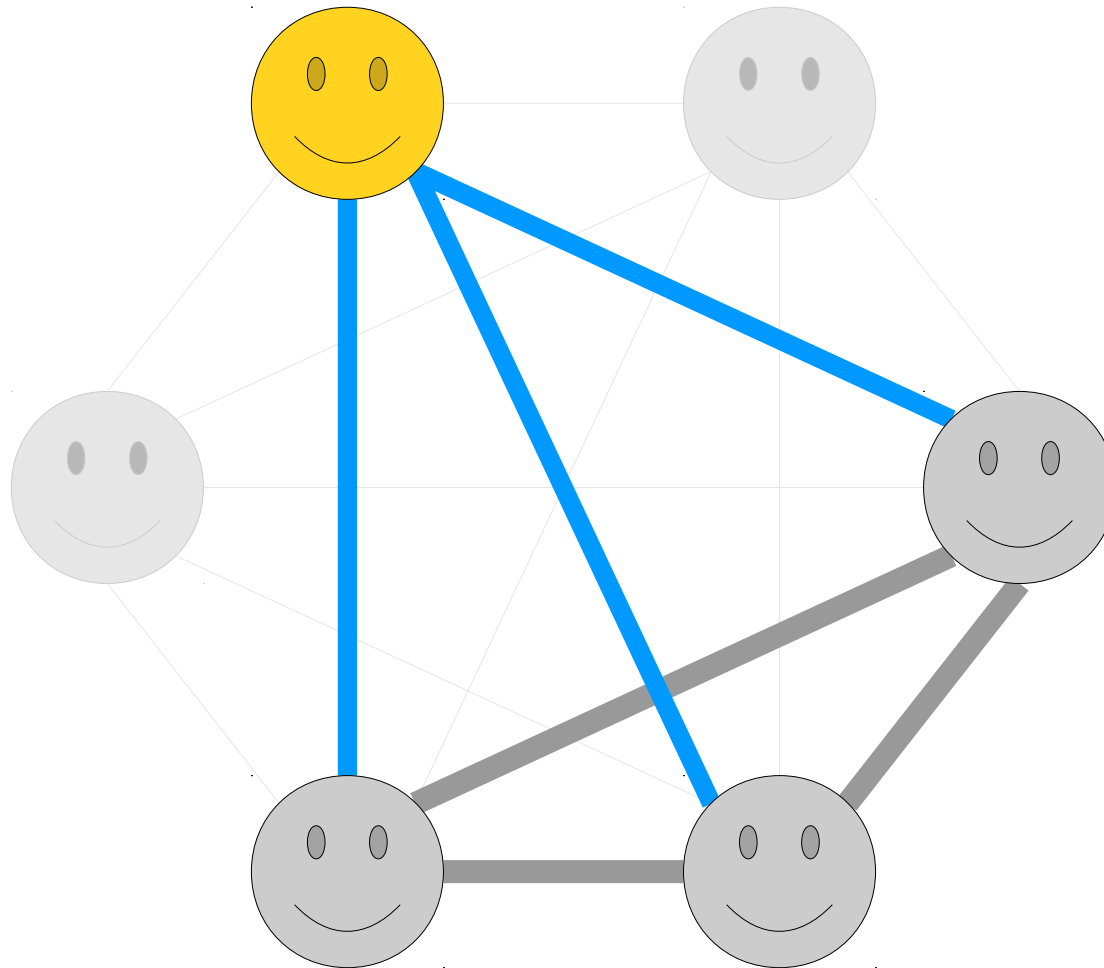


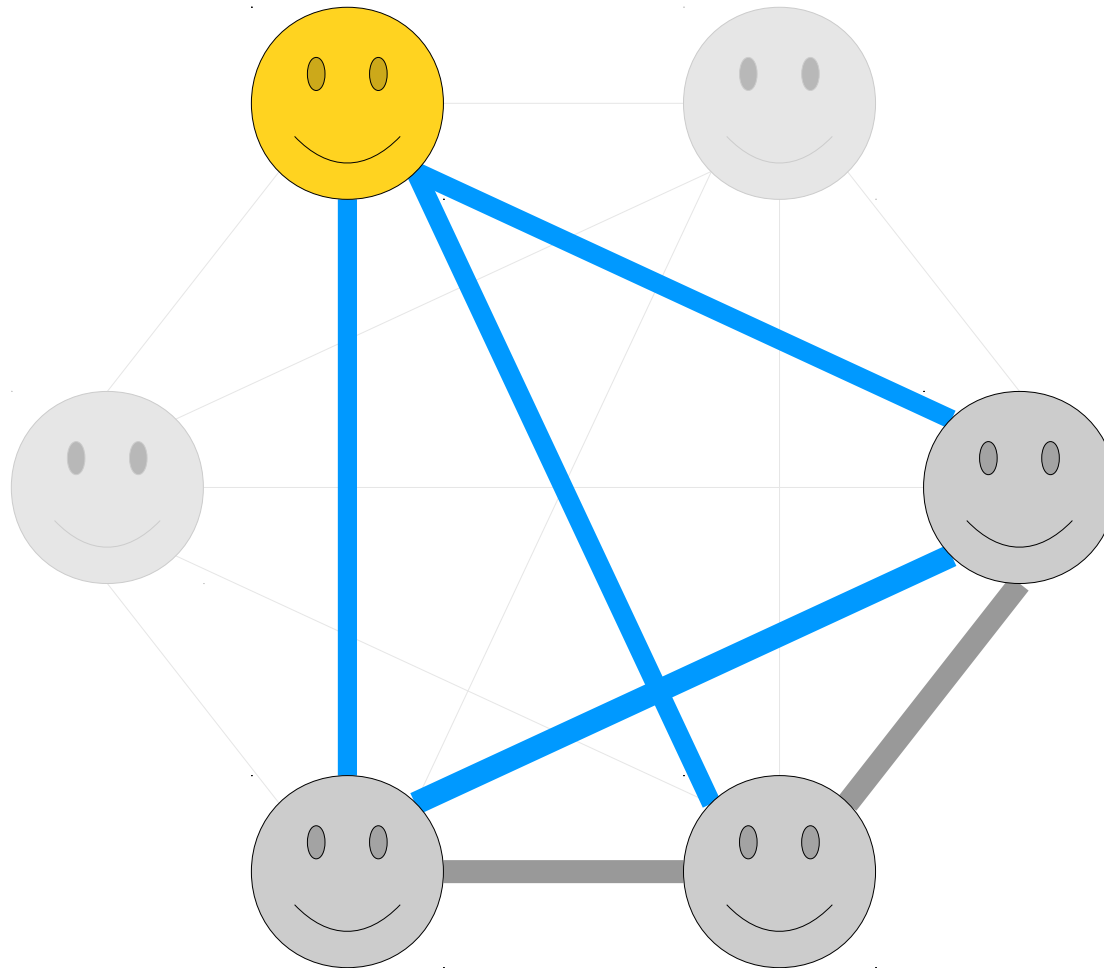


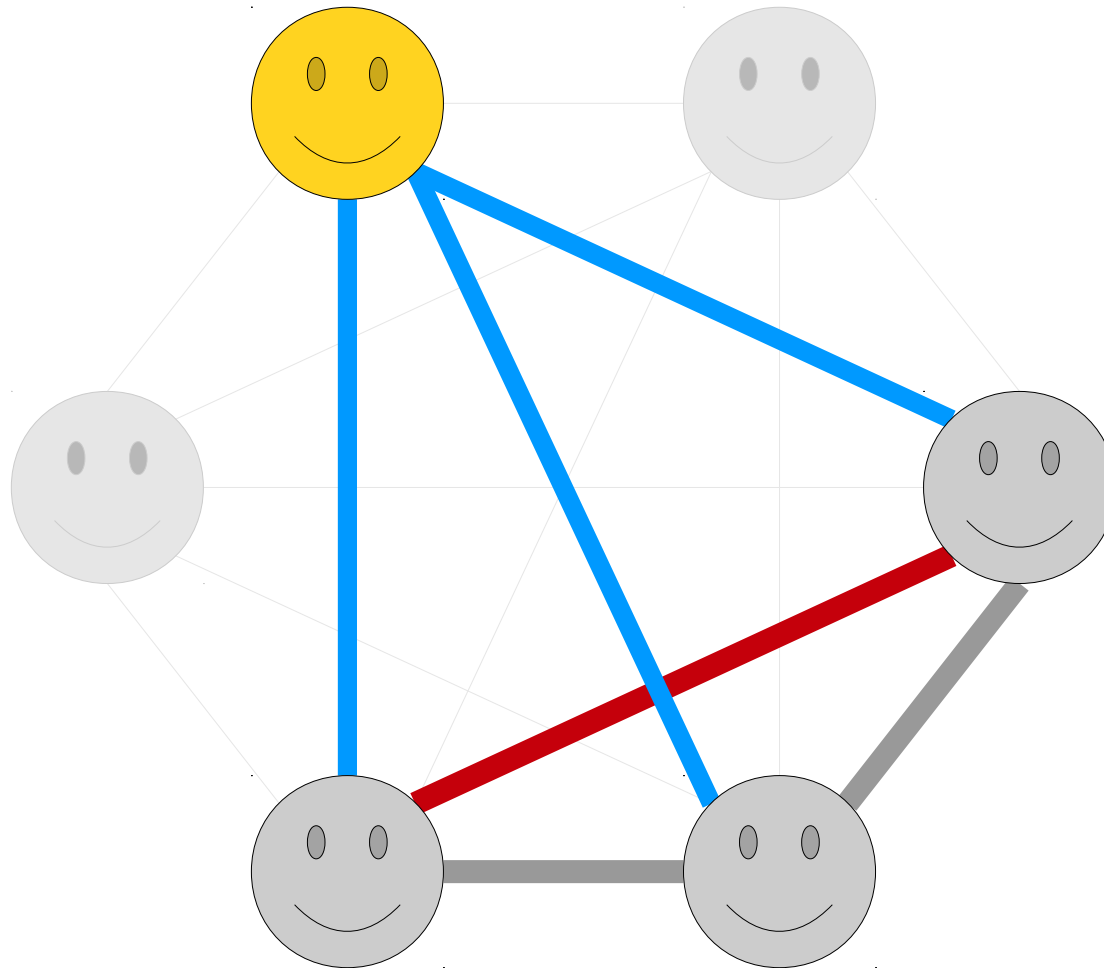


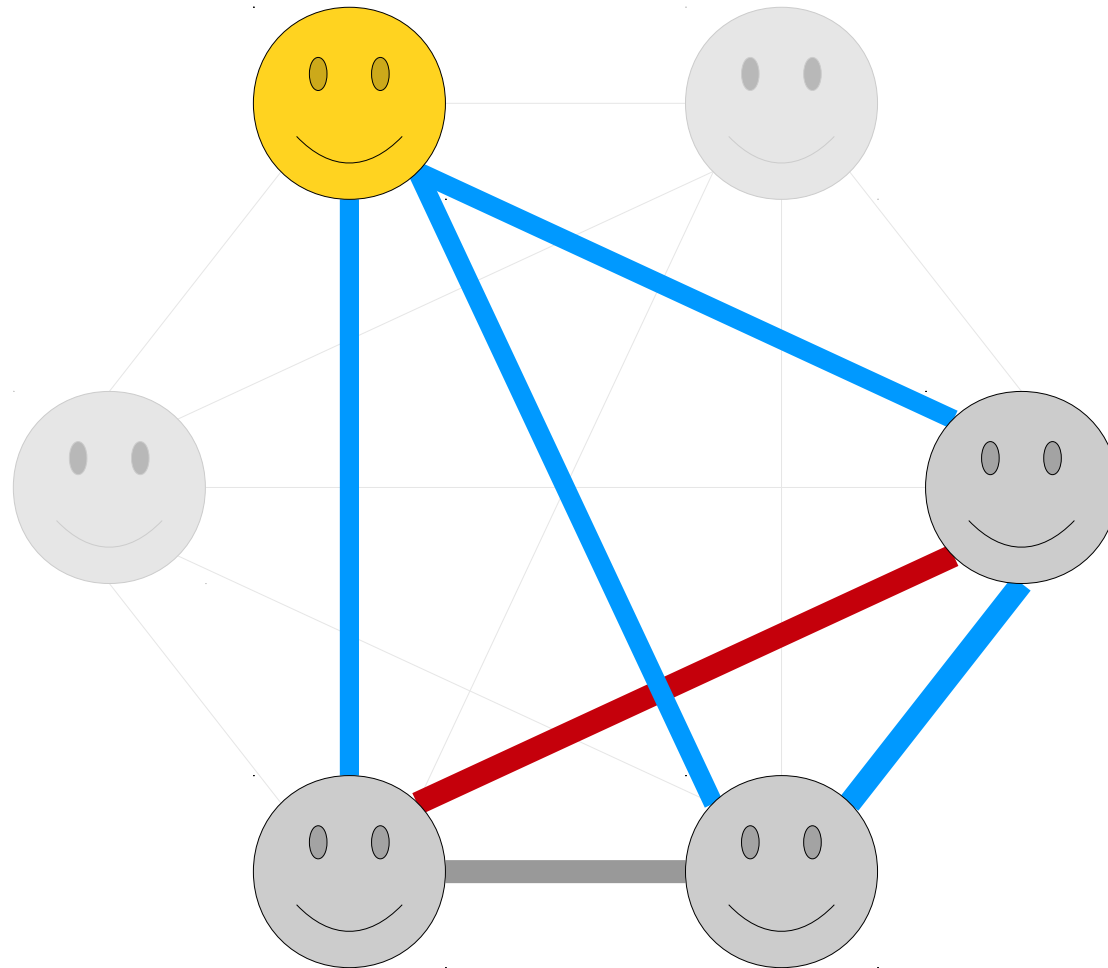
Observation 1: If we pick any node in the graph, that node will have at least $\lceil 5/2 \rceil = 3$ edges of the same color incident to it.

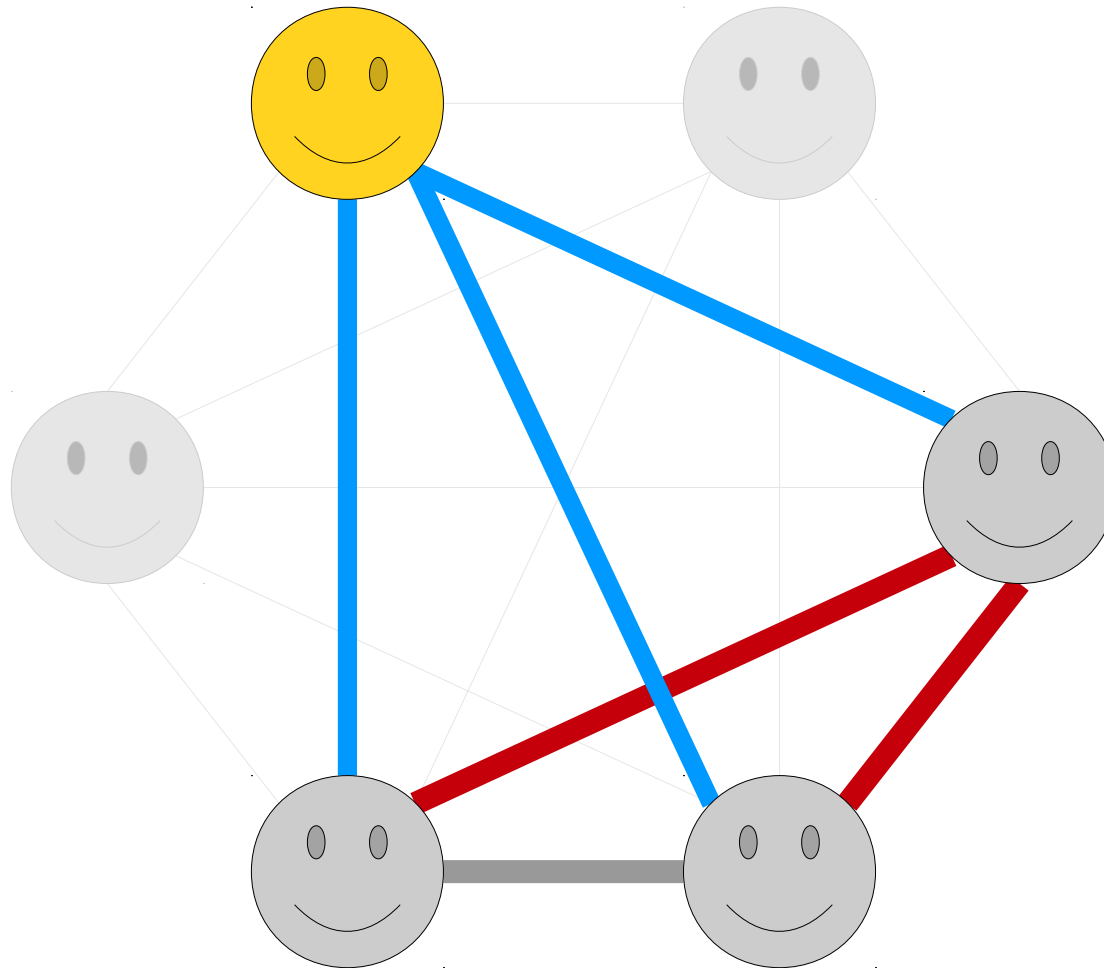


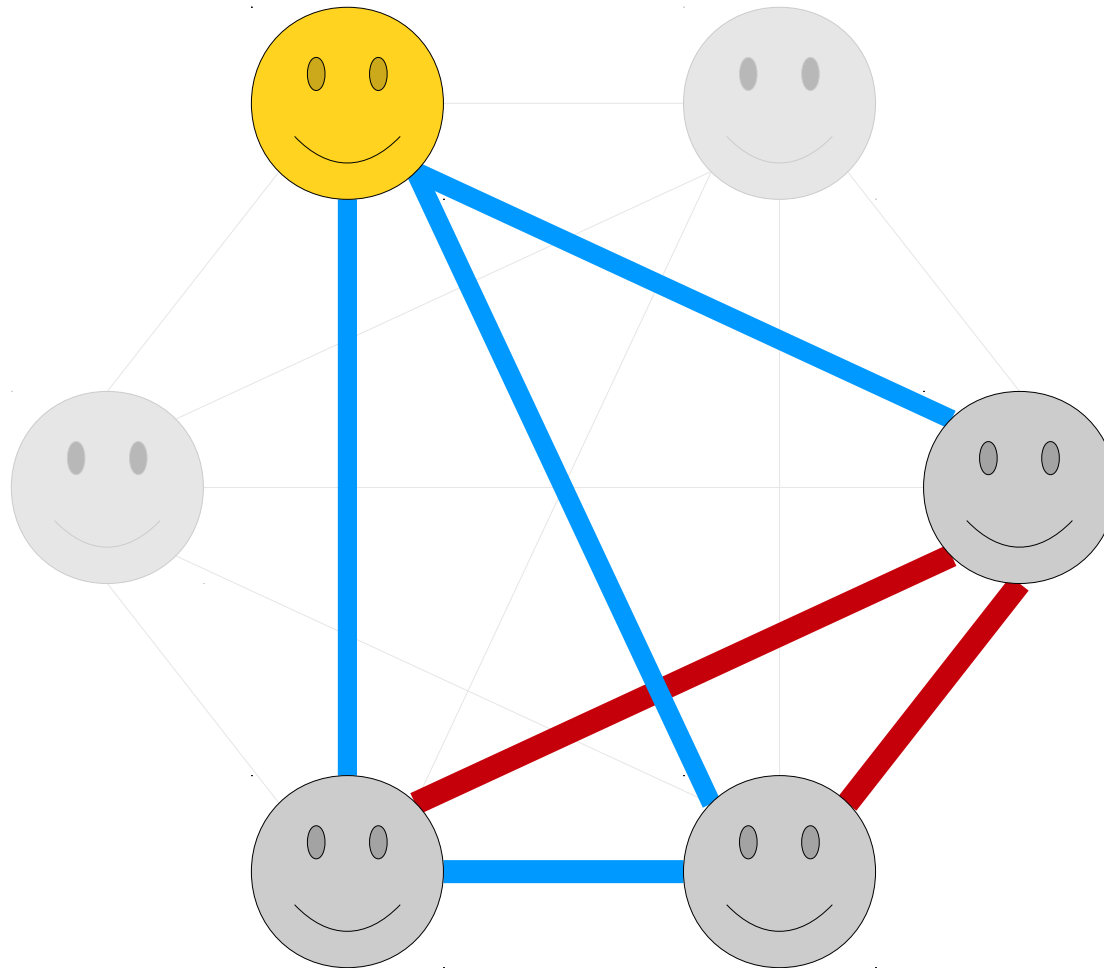


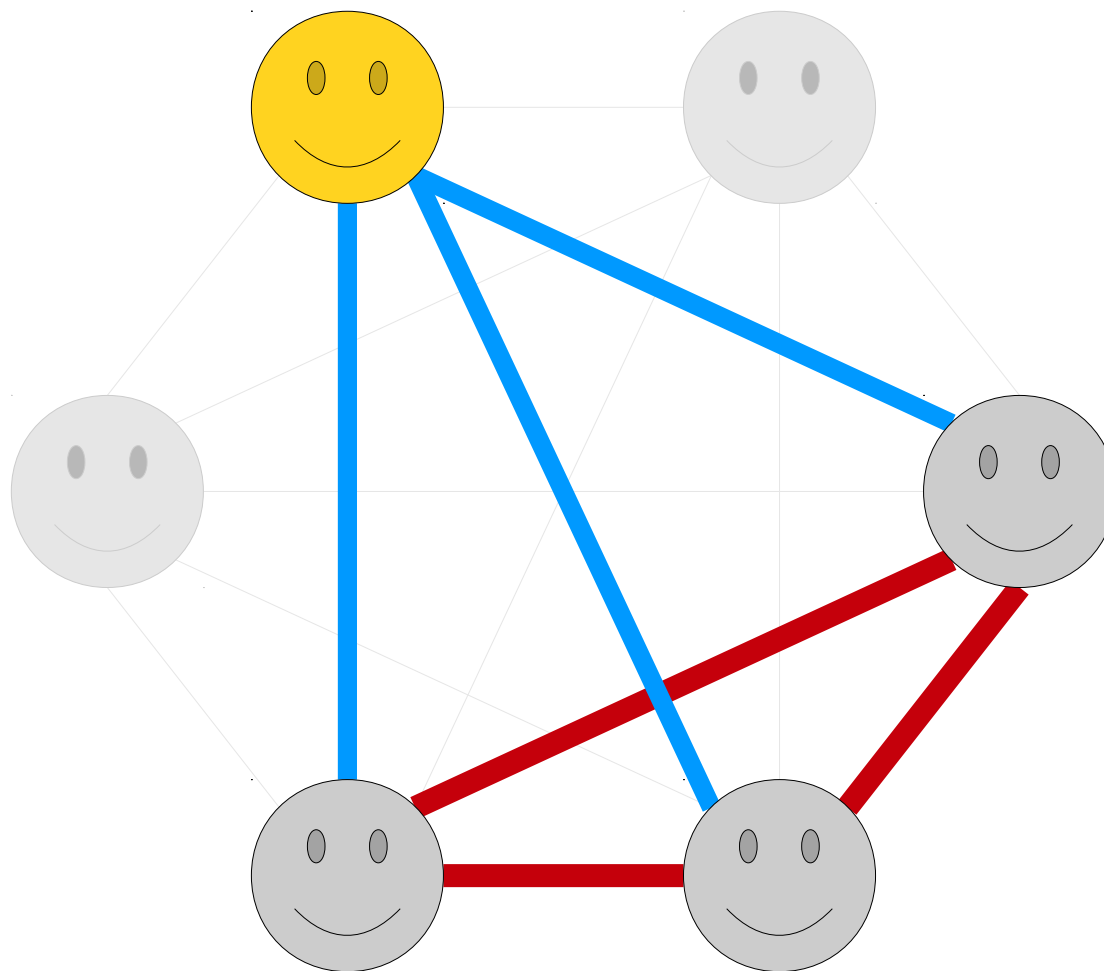












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: Color the edges of the 6-clique either red or blue arbitrarily. Let x be any node in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Call that color c_1 and let the other color be c_2 .

Let r , s , and t be three of the nodes adjacent to node x along an edge of color c_1 . If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are of color c_1 , then one of those edges plus the two edges connecting back to node x form a triangle of color c_1 . Otherwise, all three of those edges are of color c_2 , and they form a triangle of color c_2 . Overall, this gives a red triangle or a blue triangle, as required. ■

Ramsey Theory

- The proof we did is a special case of a broader result.
- ***Theorem (Ramsey's Theorem):*** For any natural number n , there is a smallest natural number $R(n)$ such that if the edges of an $R(n)$ -clique are colored red or blue, the resulting graph will contain either a red n -clique or a blue n -clique.
 - Our proof was that $R(3) \leq 6$.
- A more philosophical take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.

Time-Out for Announcements!



CS + SOCIAL GOOD

★ APPLY FOR STUDIO



The Studio program is a two unit class winter and spring quarter, where you will identify a need within a problem space you're interested in, work closely with a community and a partner organization to design and build a solution for the need, and generate real social impact.



Technical and non-technical folks from all majors and backgrounds are welcome to apply!

Learn more at: cs51.stanford.edu

Apply at <https://tinyurl.com/cssgstudio1819>

Applications due Sunday, 10/28 at 11.59pm

In partnership with:

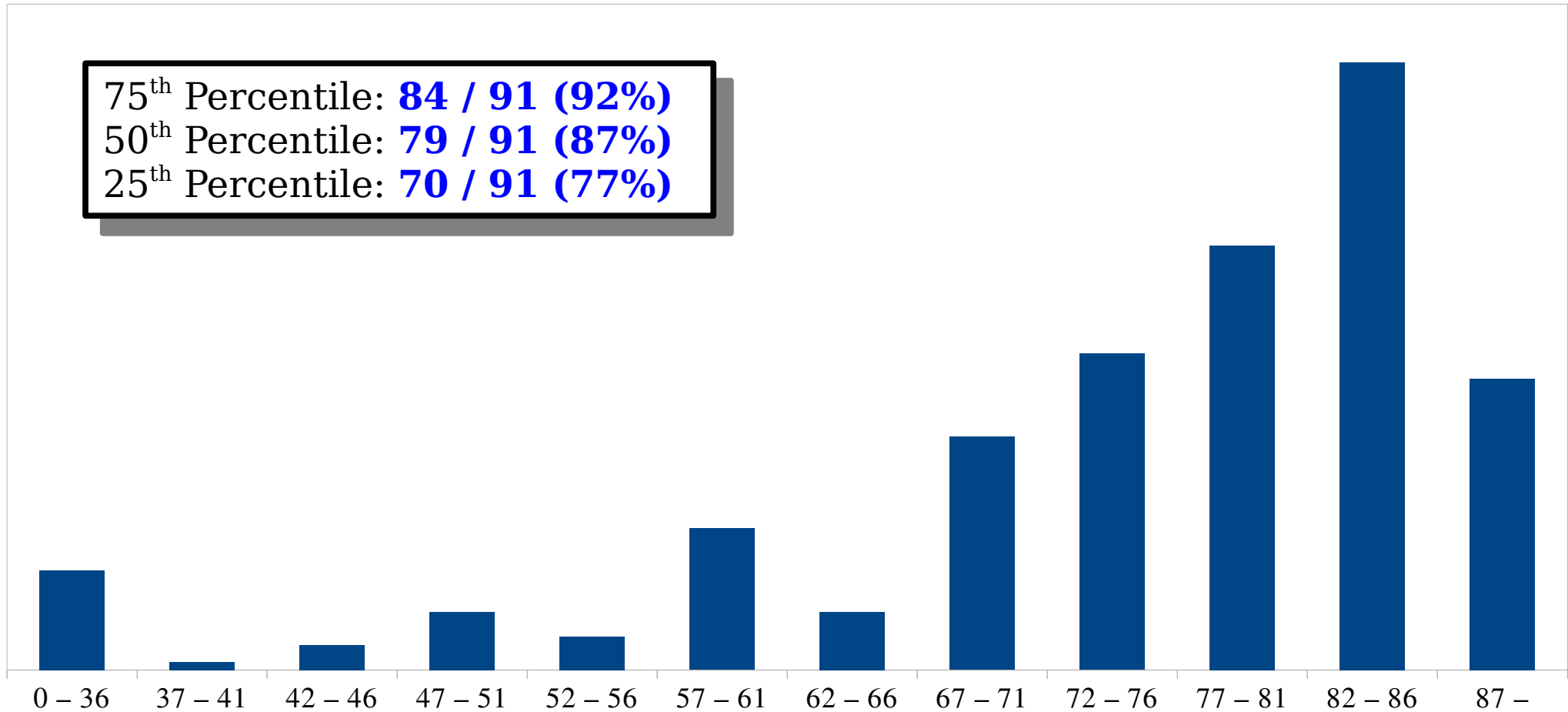


Problem Set Two

75th Percentile: **84 / 91 (92%)**

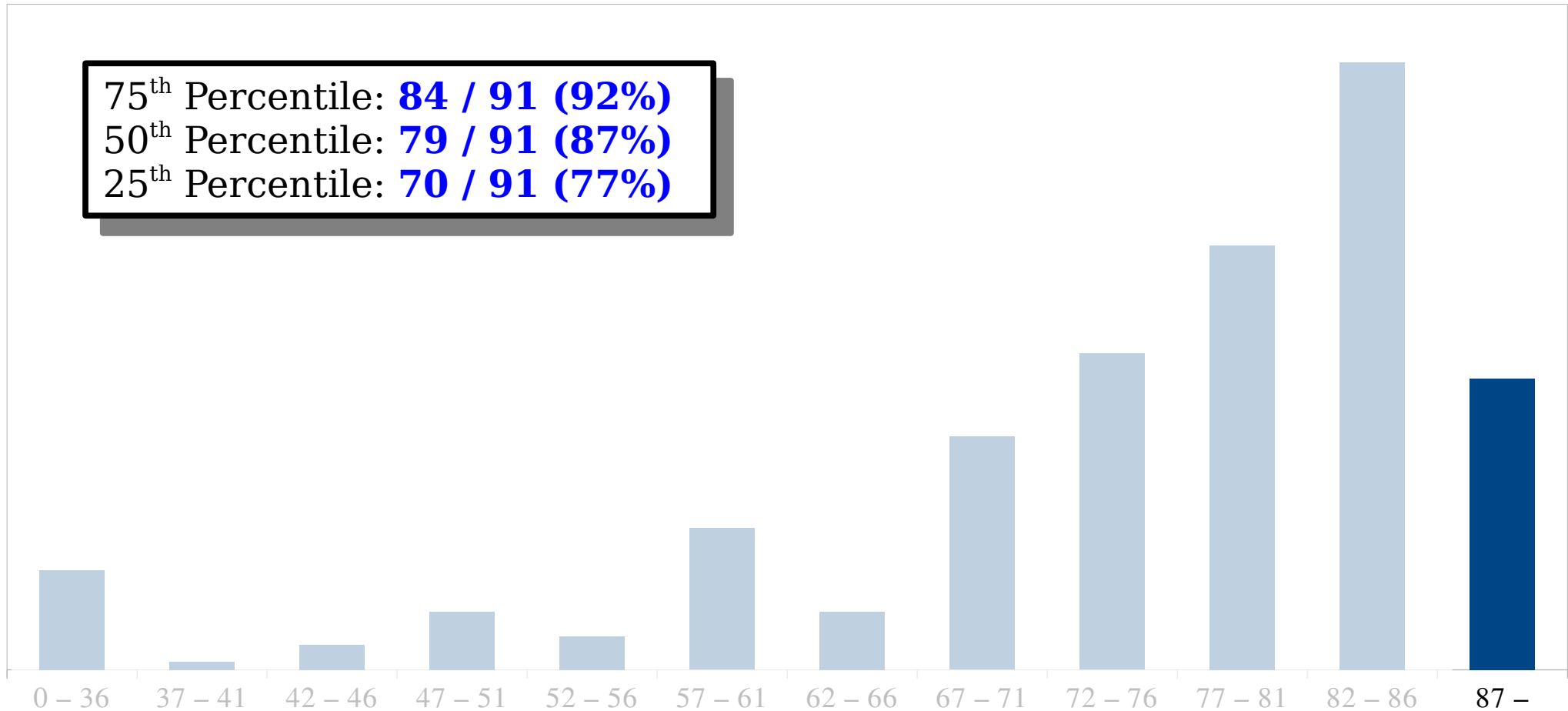
50th Percentile: **79 / 91 (87%)**

25th Percentile: **70 / 91 (77%)**



Problem Set Two

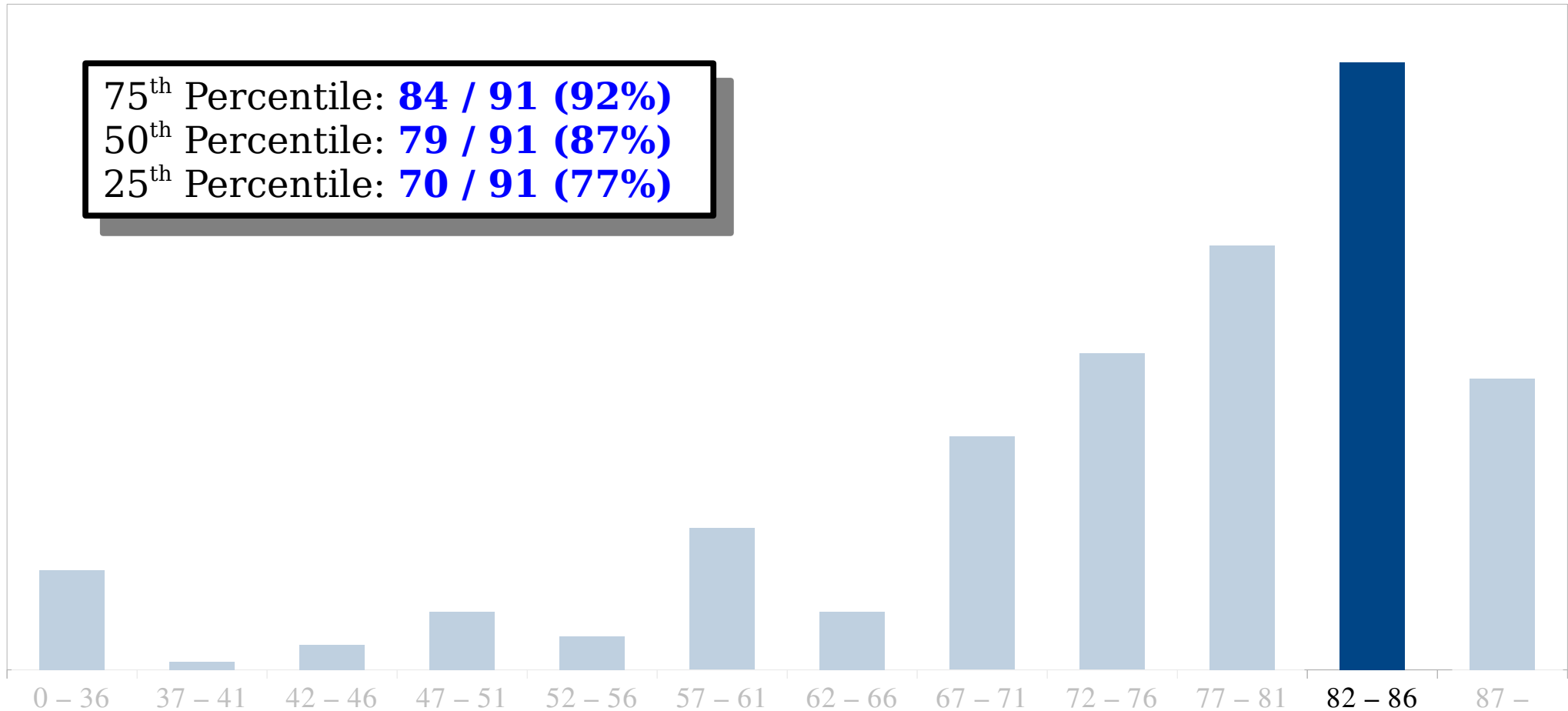
75th Percentile: **84 / 91 (92%)**
50th Percentile: **79 / 91 (87%)**
25th Percentile: **70 / 91 (77%)**



Very nicely done! Make sure to look at your feedback for those last few bits.

Problem Set Two

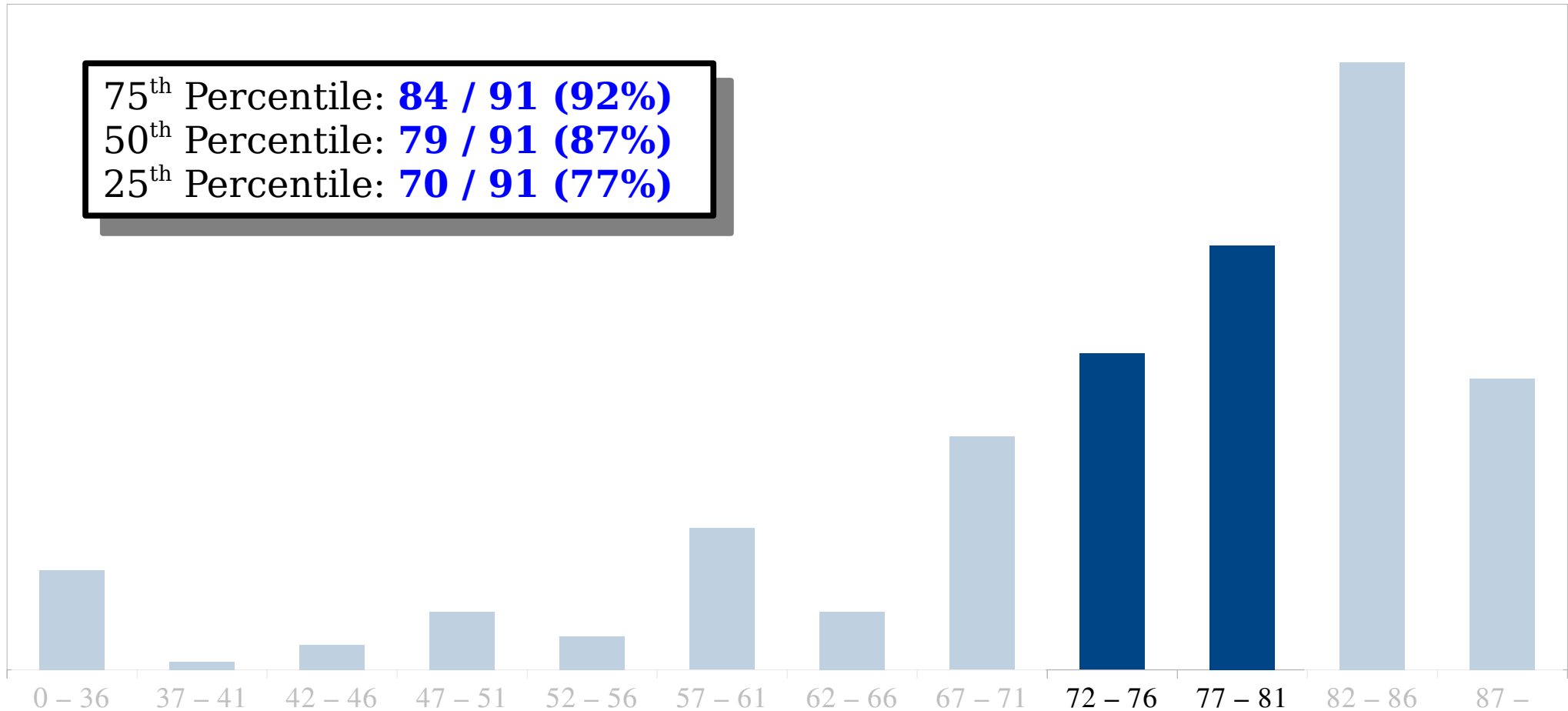
75th Percentile: **84 / 91 (92%)**
50th Percentile: **79 / 91 (87%)**
25th Percentile: **70 / 91 (77%)**



Great job! You're getting the hang of this. Check out your feedback to see where you need to improve.

Problem Set Two

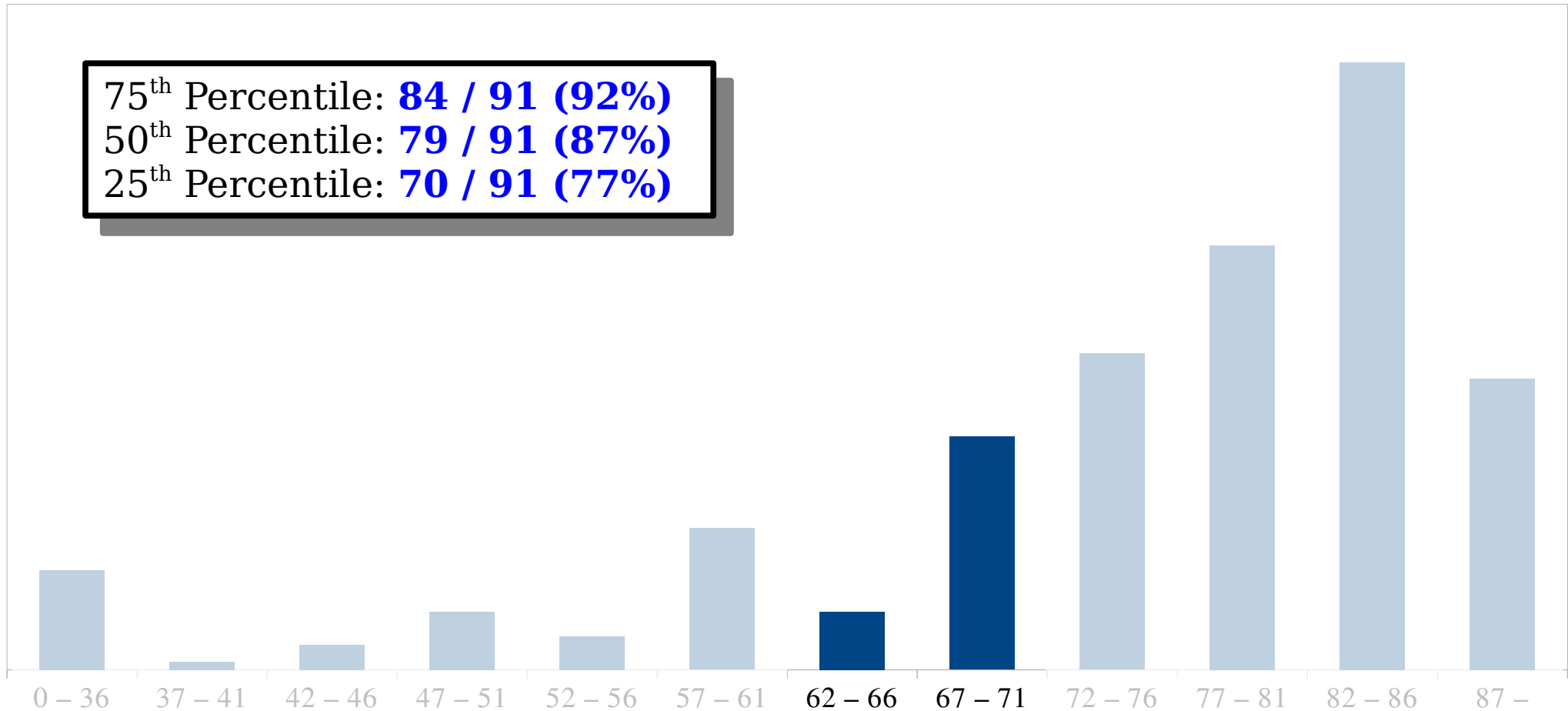
75th Percentile: **84 / 91 (92%)**
50th Percentile: **79 / 91 (87%)**
25th Percentile: **70 / 91 (77%)**



Nice work! You've got a good handle on things and have some room to grow. See what skills you need to practice, get in some more reps, and you'll be in good shape.

Problem Set Two

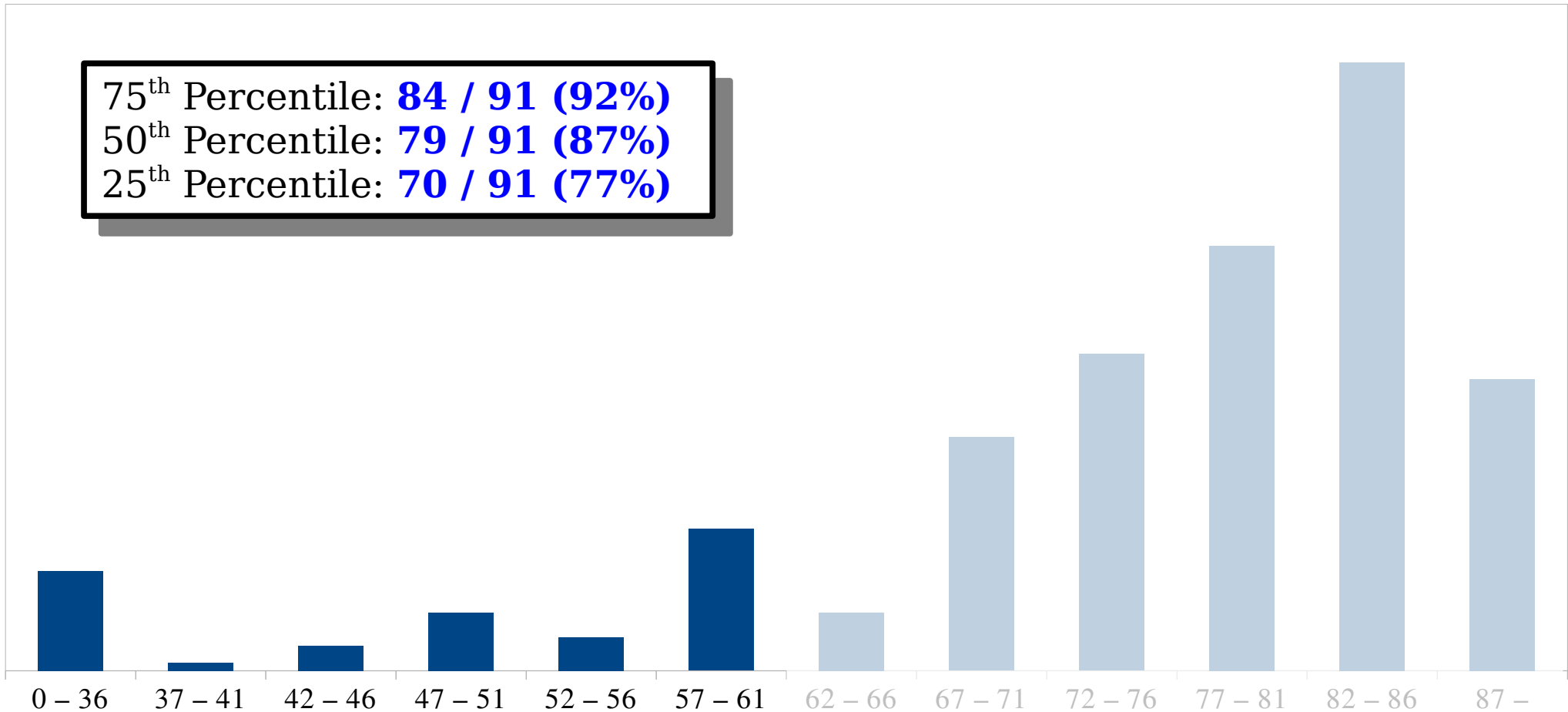
75th Percentile: **84 / 91 (92%)**
50th Percentile: **79 / 91 (87%)**
25th Percentile: **70 / 91 (77%)**



Looks like there are some skills that haven't solidified yet, but you'll get there. Check your feedback to see what worked. Keep doing that. See what you need to focus on, and let us know how we can help.

Problem Set Two

75th Percentile: **84 / 91 (92%)**
50th Percentile: **79 / 91 (87%)**
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You aren't quite where you need to be, but you have plenty of time to get more practice. Reach out with questions. Look over your feedback, work through these problems again, and stay in touch. You can do this.

Midterm Exam Logistics

- Our first midterm exam is next ***Monday, October 22nd***, from ***7:00PM - 10:00PM***. Locations are divvied up by last (family) name:
 - A - L: Go to Bishop Auditorium.
 - M - Z: Go to Cubberley Auditorium.
- You're responsible for Lectures 00 - 05 and topics covered in PS0 - PS2. Later lectures (relations forward) and problem sets (PS3 onward) won't be tested here.
- The exam is closed-book, closed-computer, and limited-note. You can bring a double-sided, 8.5" × 11" sheet of notes with you to the exam, decorated however you'd like.

More Practice

- Want more practice? We've posted online
 - Extra Practice Problems 1;
 - four practice midterms, with solutions; and
 - the CS103A materials for the past few weeks.
- Please feel free to ask questions on Piazza over the weekend - we're happy to help out!

Problem Sets

- Problem Set Three was due today at 2:30PM.
 - You can use late days to extend that to Sunday at 2:30PM if you'd like.
- Problem Set Four goes out today.
 - ***There is no checkpoint.*** The remaining problems are due on Friday at 2:30PM.
 - Play around with the finite, the infinite, and everything in between!

Your Questions

“I genuinely like CS, but I’ve struggled with/not enjoyed all the CS courses I’ve taken. Can CS really be for me even though I struggle with it every quarter?”

It’s perfectly normal to struggle with this material – it’s legitimately tricky! We’re asking a lot from you, but that’s because we know that you can do it.

I’d like to dig deeper as to what it is about CS that you like and why that isn’t matching your experiences with classes here. There are lots of reasons to enjoy this material and lots of ways classes can be un-fun. How to interpret those signals will depend on what the underlying issue is. Feel free to come talk to me in office hours!

Back to CS103!

A Little Math Puzzle

“In a group of $n > 0$ people ...

- 90% of those people enjoyed *Get Out*,
- 80% of those people enjoyed *Lady Bird*,
- 70% of those people enjoyed *Arrival*, and
- 60% of those people enjoyed *Zootopia*.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Other Pigeonhole-Type Results

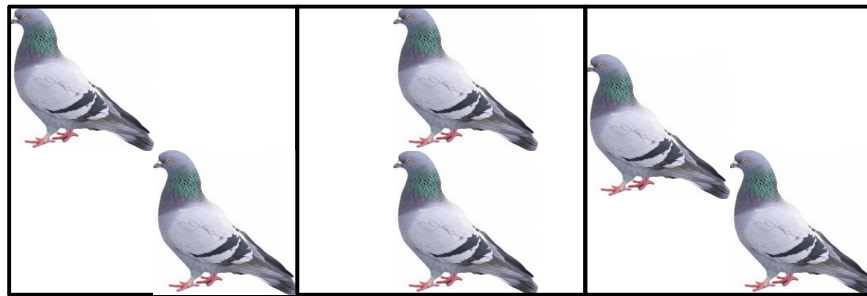
*If m objects are distributed into n boxes, then **[condition]** holds.*

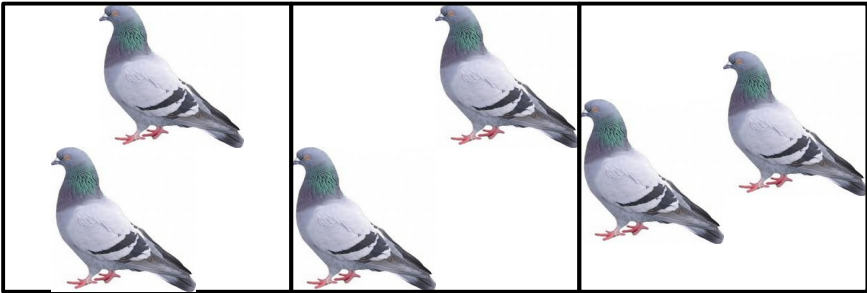
*If m objects are distributed into n boxes, then **some box is loaded to at least the average m/n , and some box is loaded to at most the average m/n .***

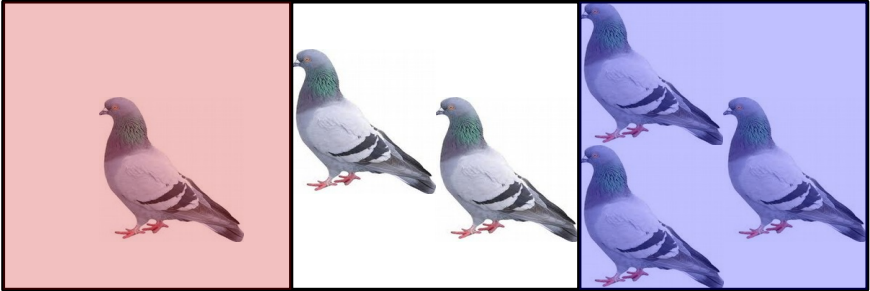
*If m objects are distributed into n boxes, then **[condition]** holds.*



--	--	--









Theorem: If m objects are distributed into n bins, then there is a bin containing more than m/n objects if and only if there is a bin containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than $\lceil m/n \rceil$ objects, then there are no bins containing fewer than $\lfloor m/n \rfloor$ objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than $\lceil m/n \rceil$ objects, then there are no bins containing fewer than $\lfloor m/n \rfloor$ objects.

Proof: Assume for the sake of contradiction that m objects are distributed into n bins such that no bin contains more than $\lceil m/n \rceil$ objects, yet some bin has fewer than $\lfloor m/n \rfloor$ objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

Proof: Assume for the sake of contradiction that m objects are distributed into n bins such that no bin contains more than m/n objects, yet some bin has fewer than m/n objects.

For simplicity, denote by x_i the number of objects in bin i .

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Proof: Assume for the sake of contradiction that m objects are distributed into n bins such that no bin contains more than m/n objects, yet some bin has fewer than m/n objects.

For simplicity, denote by x_i the number of objects in bin i . Without loss of generality, assume that bin 1 has fewer than m/n objects, meaning that $x_1 < m/n$.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

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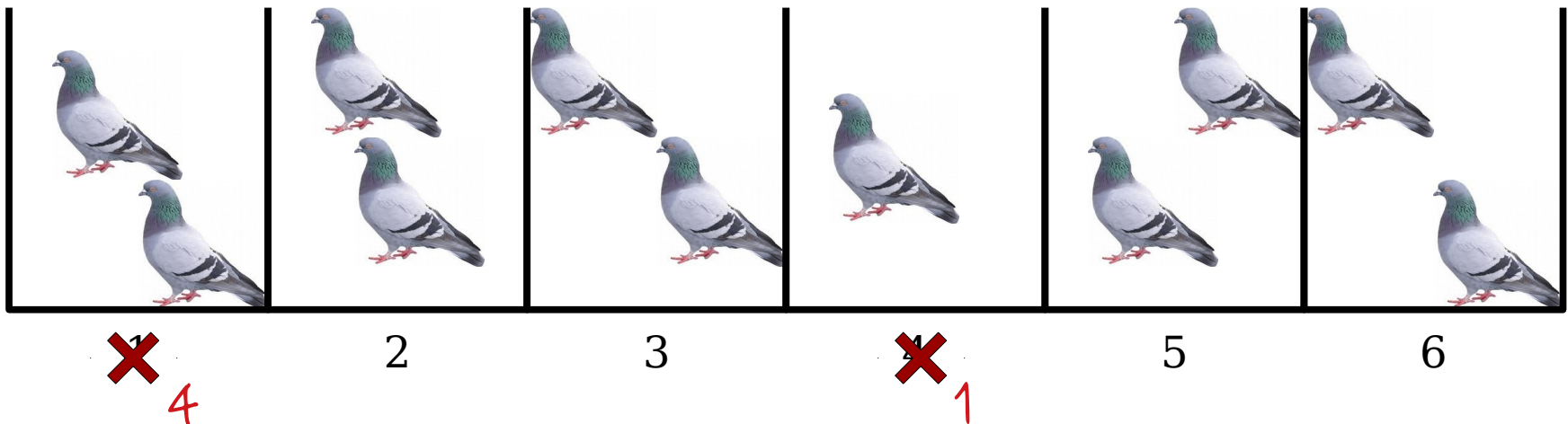
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For simplicity, denote by x_i the number of objects in bin i . Without loss of generality, assume that bin 1 has fewer than m/n objects, meaning that $x_1 < m/n$.

This magic phrase means "we get to pick how we're labeling things anyway, so if it doesn't work out, just relabel things."



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For simplicity, denote by x_i the number of objects in bin i . Without loss of generality, assume that bin 1 has fewer than $\lfloor m/n \rfloor$ objects, meaning that $x_1 < \lfloor m/n \rfloor$. Adding up the number of objects in each bin tells us that

$$m = x_1 + x_2 + x_3 + \dots + x_n$$

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$$\begin{aligned} m &= x_1 + x_2 + x_3 + \dots + x_n \\ &< m/n + x_2 + x_3 + \dots + x_n \\ &\leq m/n + m/n + m/n + \dots + m/n. \end{aligned}$$

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This third step follows because each remaining bin has at most m/n objects.

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This third step follows because each remaining bin has at most m/n objects. Grouping the n copies of the m/n term here tells us that

$$\begin{aligned} m &< m/n + m/n + m/n + \dots + m/n \\ &= m. \end{aligned}$$

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But this means $m < m$, which is impossible.

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But this means $m < m$, which is impossible. We've reached a contradiction, so our assumption was wrong, so if m objects are distributed into n bins and no bin has more than m/n objects, no bin has fewer than m/n objects either.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

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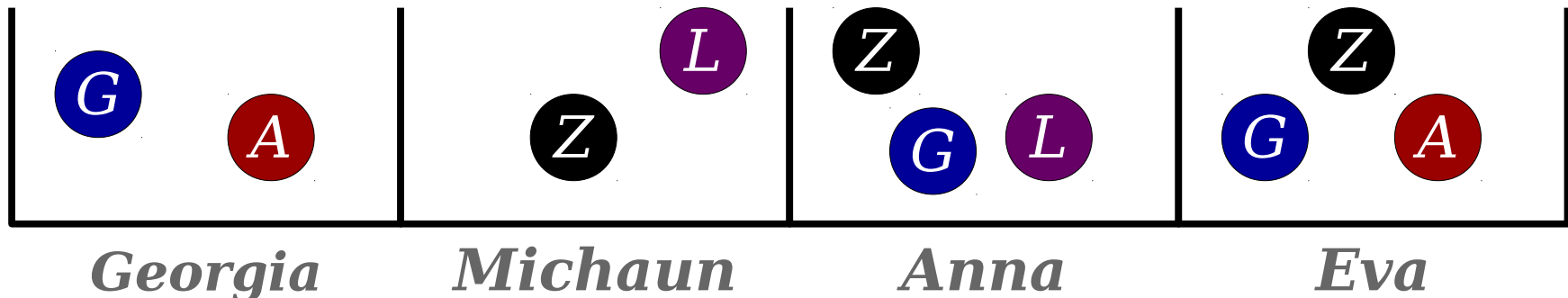
“In a group of $n > 0$ people ...

- 90% of those people enjoyed **Get Out**,
- 80% of those people enjoyed **Lady Bird**,
- 70% of those people enjoyed **Arrival**, and
- 60% of those people enjoyed **Zootopia**.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Insight 1: Model movie preferences as balls (movies) in bins (people).

Insight 2: There are n total bins, one for each person.



“In a group of $n > 0$ people ...

- 90% of those people enjoyed **Get Out**,
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No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

$$\begin{aligned} & .9n + .8n + .7n + .6n \\ & = 3n \end{aligned}$$

Insight 3: There are $3n$ balls being distributed into n bins.

Insight 4: The average number of balls in each bin is 3.

“In a group of $n > 0$ people ...

- 90% of those people enjoyed *Get Out*,
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No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Insight 5: No one enjoyed more than three movies...

Insight 6: ... so no one enjoyed fewer than three movies ...

Insight 7: ... so everyone enjoyed exactly three movies.

“In a group of $n > 0$ people ...

- 90% of those people enjoyed *Get Out*,
- 80% of those people enjoyed *Lady Bird*,
- 70% of those people enjoyed *Arrival*, and
- 60% of those people enjoyed *Zootopia*.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Insight 8: You have to enjoy at least one of these movies to enjoy three of the four movies.

Conclusion: Everyone liked at least one of these two movies!

Theorem: In the scenario described here, all n people enjoyed at least one of *Get Out* and *Arrival*.

“In a group of $n > 0$ people ...

- 90% of those people enjoyed ***Get Out***,
- 80% of those people enjoyed ***Lady Bird***,
- 70% of those people enjoyed ***Arrival***, and
- 60% of those people enjoyed ***Zootopia***.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Theorem: In the scenario described here, all n people enjoyed at least one of *Get Out* and *Arrival*.

Proof: Suppose there is a group of n people meeting these criteria.

“In a group of $n > 0$ people ...

- 90% of those people enjoyed ***Get Out***,
- 80% of those people enjoyed ***Lady Bird***,
- 70% of those people enjoyed ***Arrival***, and
- 60% of those people enjoyed ***Zootopia***.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?”

Theorem: In the scenario described here, all n people enjoyed at least one of *Get Out* and *Arrival*.

Proof: Suppose there is a group of n people meeting these criteria. We can model this problem by representing each person as a bin and each time a person enjoys a movie as a ball.

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and since there are n people, there are n bins.

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We've reached a contradiction, so our assumption was wrong and each person enjoyed at least one of *Get Out* and *Arrival*. ■

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Going Further

- The pigeonhole principle can be used to prove a *ton* of amazing theorems. Here's a sampler:
 - There is always a way to fairly split rent among multiple people, even if different people want different rooms. (*Sperner's lemma*)
 - You and a friend can climb any mountain from two different starting points so that the two of you maintain the same altitude at each point in time. (*Mountain-climbing theorem*)
 - If you model coffee in a cup as a collection of infinitely many points and then stir the coffee, some point is always where it initially started. (*Brouwer's fixed-point theorem*)
 - A complex process that doesn't parallelize well must contain a large serial subprocess. (*Mirksy's theorem*)
 - Any positive integer n has a nonzero multiple that can be written purely using the digits 1 and 0. (*Doesn't have a name, but still cool!*)

Next Time

- ***No class on Monday - you have a midterm!***
- ***Then, when we get back:***
 - ***Mathematical Induction***
 - Reasoning about stepwise processes!
 - ***Applications of Induction***
 - To numbers!
 - To anticounterfeiting!
 - To puzzles!