Complexity Theory

Part Two

Recap from Last Time

The Complexity Class **P**

- The complexity class **P** (***polynomial time***) is defined as

  $$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$$

- Intuitively, **P** contains all decision problems that can be solved efficiently.
- This is like class **P**, except with "efficiently" tacked onto the end.

The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

$$\textbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$

- Intuitively, **NP** is the set of problems where "yes" answers can be checked efficiently.
- This is like the class **RE**, but with "efficiently" tacked on to the definition.

The Biggest Unsolved Problem in Theoretical Computer Science:

$\mathbf{P \stackrel{?}{=} NP}$

Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to reason about **P** and **NP**?

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

Intuition:

Problem *A* can't be "harder" than problem *B*, because solving problem *B* lets us solve problem *A*.

```
bool solveProblemA(string input) {
    return solveProblemB(transform(input));
}
```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

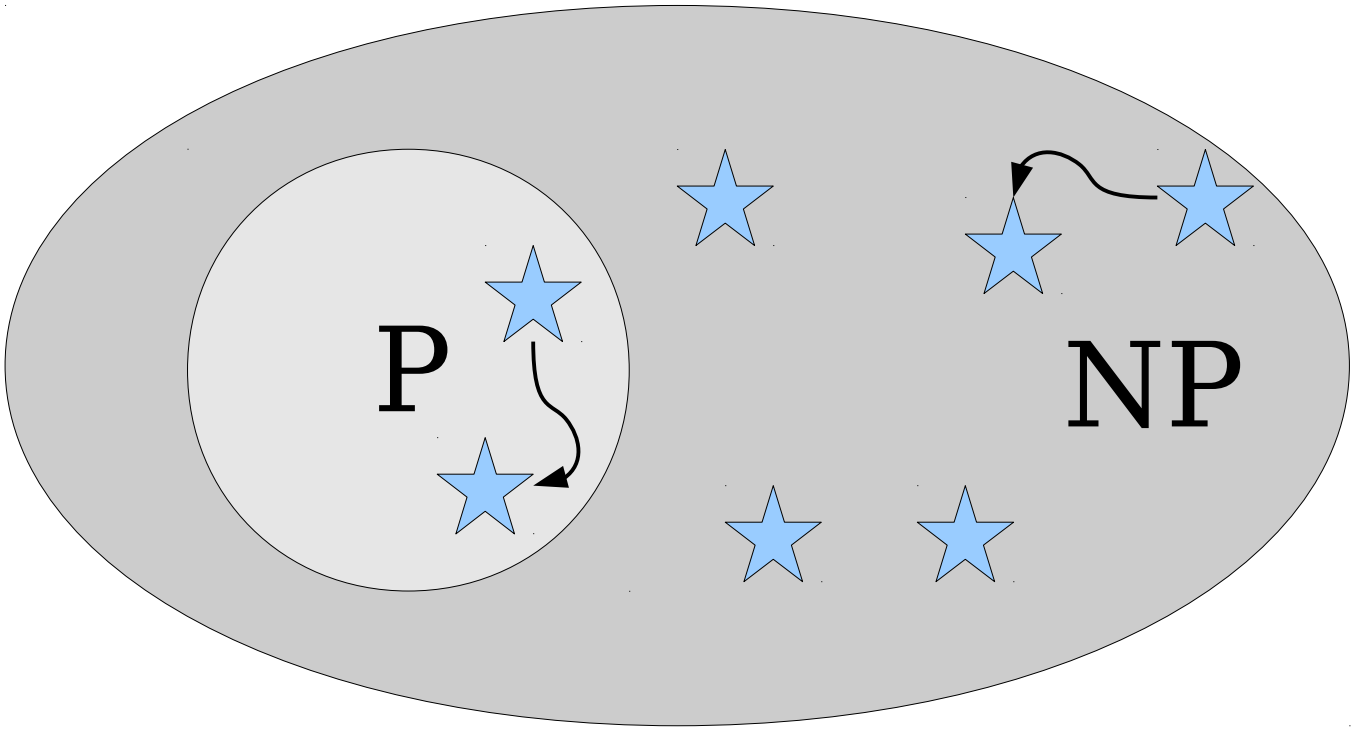
$$A \leq_{\mathrm{p}} B.$$

- We say that ***A is polynomial-time reducible to B***.

* Assuming that `transform` runs in polynomial time.

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

New Stuff!

Satisfiability

- A propositional logic formula φ is called ***satisfiable*** if there is some assignment to its variables that makes it evaluate to true.
  - $p \wedge q$ is satisfiable.
  - $p \wedge \neg p$ is unsatisfiable.
  - $p \rightarrow (q \wedge \neg q)$ is satisfiable.
- An assignment of true and false to the variables of φ that makes it evaluate to true is called a ***satisfying assignment***.

SAT

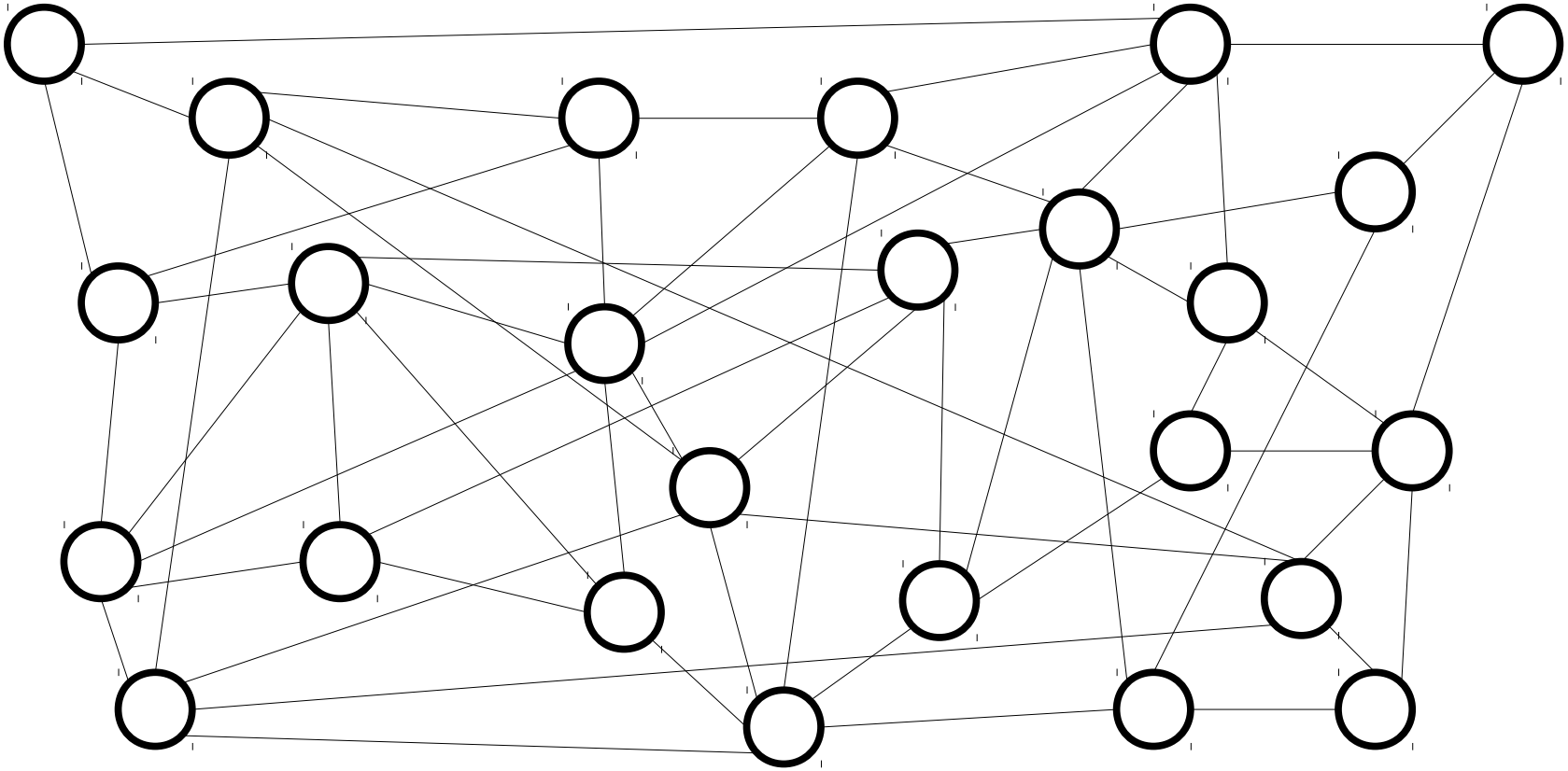
- The ***boolean satisfiability problem*** (***SAT***) is the following:

  Given a propositional logic formula φ, is φ satisfiable?

- Formally:

  SAT **= { ⟨φ⟩ | φ is a satisfiable PL formula }**

Finding Cliques

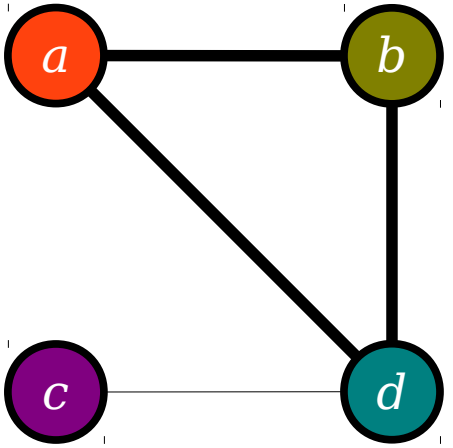


Does this graph contain a k-clique?

Finding Cliques

What is a k-clique?

- A set of k nodes
- Such that there's an edge between every pair of nodes



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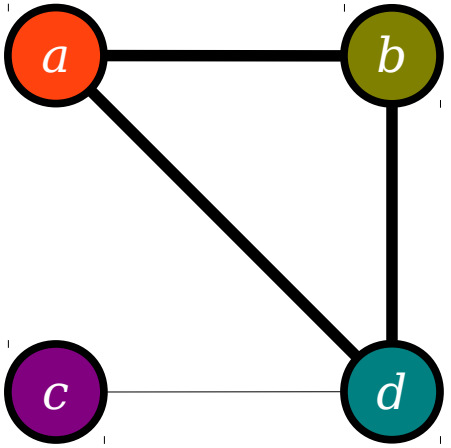

Could we somehow take these rules and encode them as a propositional logic formula?

Finding Cliques

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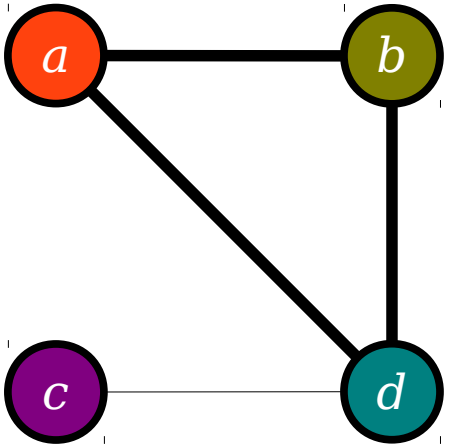

a_1 a_2 a_3 a_4

b_1 b_2 b_3 b_4

c_1 c_2 c_3 c_4

d_1 d_2 d_3 d_4

Finding Cliques

What is a k-clique?

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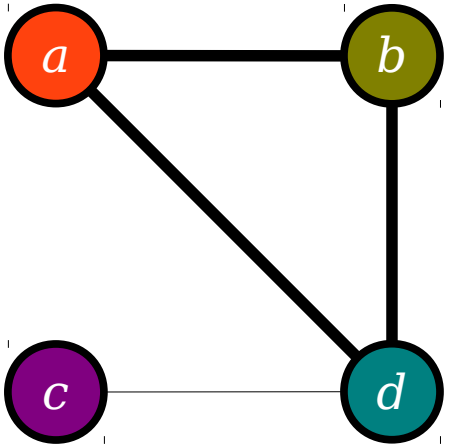

Take your graph and define the following propositional variables.

The variable c_3 represents choosing node c as the 3rd node of your clique.

a_1 a_2 a_3 a_4

b_1 b_2 b_3 b_4

c_1 c_2 c_3 c_4

d_1 d_2 d_3 d_4

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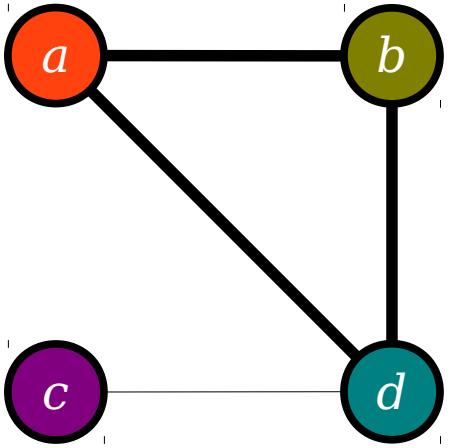

a_1 a_2 a_3 a_4

b_1 b_2 b_3 b_4

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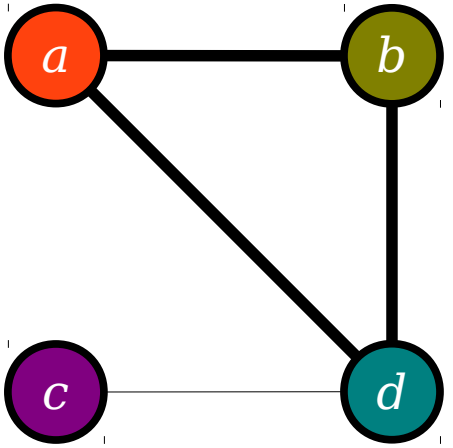


Imagine we're looking for a 3-clique. That means we need one variable in each of these groups to be true.

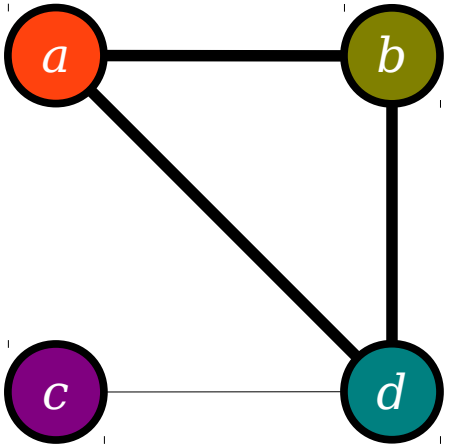
a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

Finding Cliques

What is a k-clique?

- **A set of k nodes**
- Such that there's an edge between every pair of nodes

$(a_1 \vee b_1 \vee c_1 \vee d_1)$



a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

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What is a k-clique?

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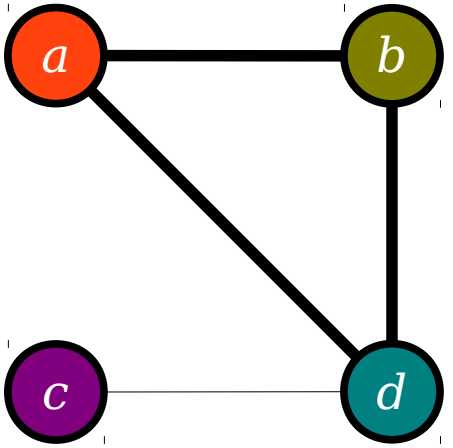

$$(a_1 \vee b_1 \vee c_1 \vee d_1)$$

The first node of your clique is either a or b or c or d.

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

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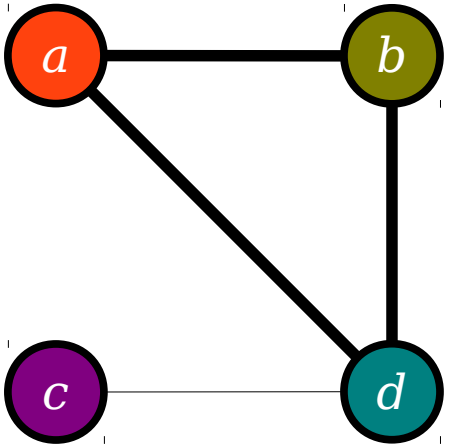
$(a_1 \vee b_1 \vee c_1 \vee d_1)$



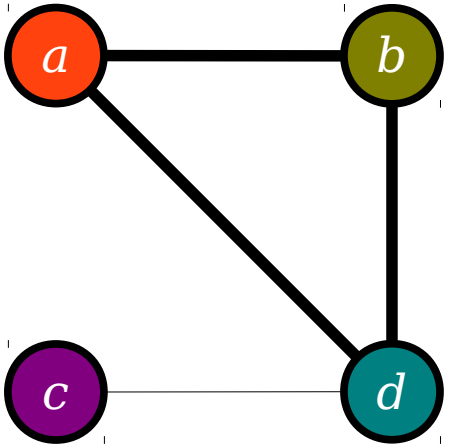
a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

Finding Cliques

What is a k-clique?

- **A set of k nodes**
- Such that there's an edge between every pair of nodes

$$(a_1 \vee b_1 \vee c_1 \vee d_1)$$
$$\wedge$$
$$(a_2 \vee b_2 \vee c_2 \vee d_2)$$



a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

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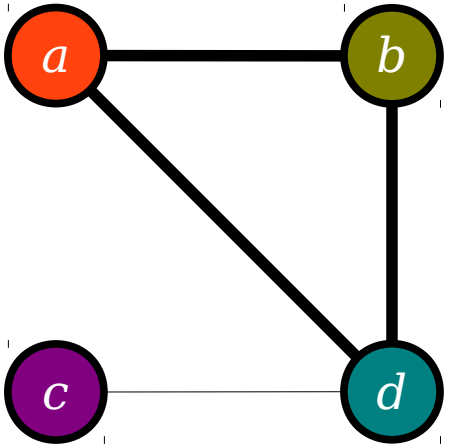

$$(a_1 \vee b_1 \vee c_1 \vee d_1)$$
$$\wedge$$
$$(a_2 \vee b_2 \vee c_2 \vee d_2)$$
$$\wedge$$
$$(a_3 \vee b_3 \vee c_3 \vee d_3)$$

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

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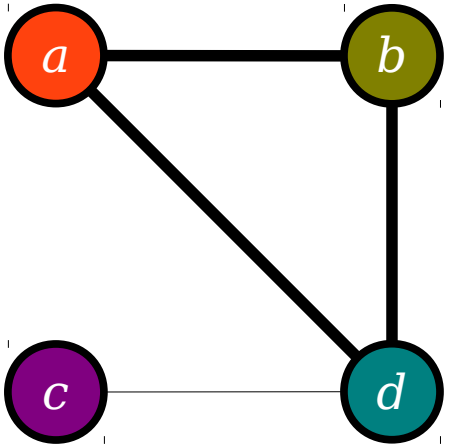

$$(a_1 \vee b_1 \vee c_1 \vee d_1)$$
$$\wedge$$
$$(a_2 \vee b_2 \vee c_2 \vee d_2)$$
$$\wedge$$
$$(a_3 \vee b_3 \vee c_3 \vee d_3)$$

$$a_1 \quad a_2 \quad a_3 \quad a_4$$
$$b_1 \quad b_2 \quad b_3 \quad b_4$$
$$c_1 \quad c_2 \quad c_3 \quad c_4$$
$$d_1 \quad d_2 \quad d_3 \quad d_4$$

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What is a k-clique?

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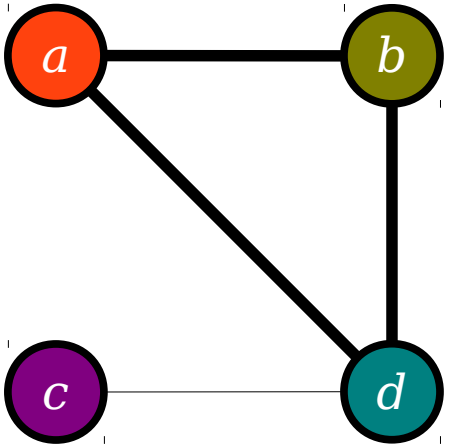

a_1 a_2 a_3 a_4

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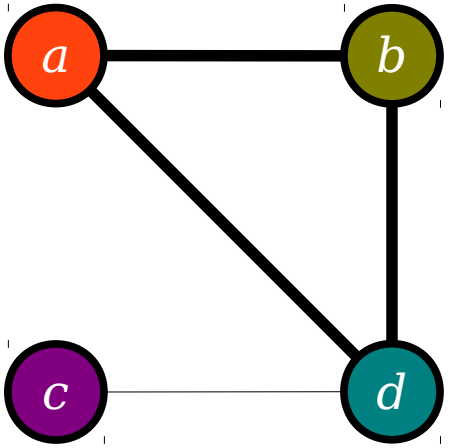


We need to ensure we don't pick a pair of nodes that don't have an edge between them. In this graph, the missing edges are $\{a, c\}$ and $\{b, c\}$

a_1 a_2 a_3 a_4

b_1 b_2 b_3 b_4

c_1 c_2 c_3 c_4

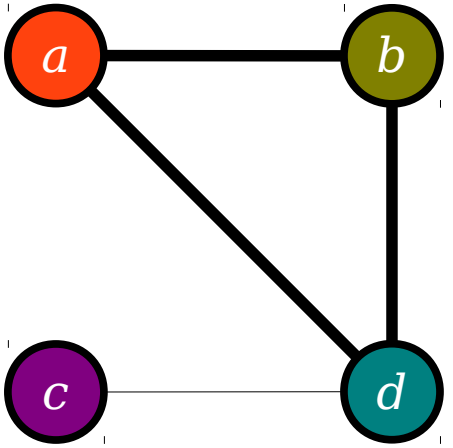
d_1 d_2 d_3 d_4

Finding Cliques

What is a k-clique?

- A set of k nodes
- **Such that there's an edge between every pair of nodes**

for all i, j:
$(\neg a_i \vee \neg c_j)$



a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

We can't choose both a and c because there's no edge between them.

Finding Cliques

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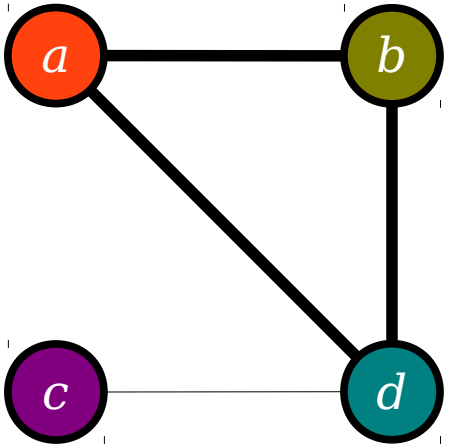

for all i, j:
$(\neg a_i \vee \neg c_j)$

for all i, j:
$(\neg b_i \vee \neg c_j)$

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

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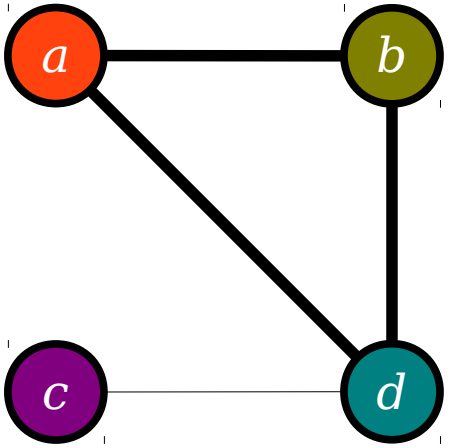

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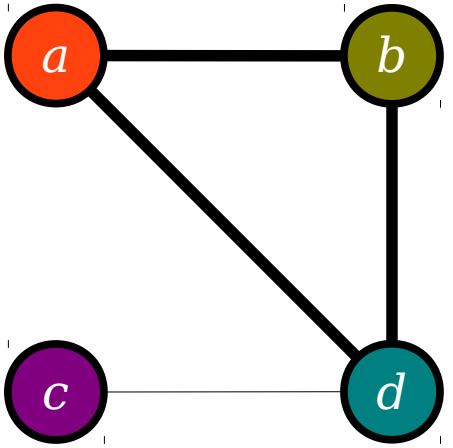

Altogether, finding an assignment of true false values to these variables that satisfies these constraints would amount to finding a clique of desired size in our graph.

a_1 a_2 a_3 a_4

b_1 b_2 b_3 b_4

c_1 c_2 c_3 c_4

d_1 d_2 d_3 d_4

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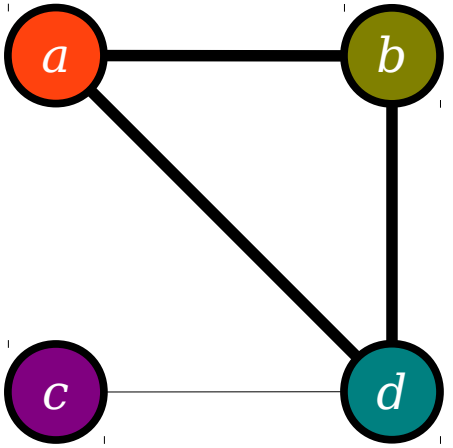

One example that works is to assign a_1, b_2, and d_3 to be true and all other variables to be false.

a_1	a_2	a_3	a_4
b_1	b_2	b_3	b_4
c_1	c_2	c_3	c_4
d_1	d_2	d_3	d_4

Intuition:

Finding a k-clique can't be any harder than solving *SAT*, because we can take any graph and encode it as a propositional logic formula where finding a satisfying assignment corresponds to finding a k-clique.

Solving Sudoku

		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

Does this Sudoku puzzle
have a solution?

Solving Sudoku

What is a Sudoku solution?

- An assignment of numbers 1 through 9 to a 9×9 grid
- Such that each number appears exactly once in each row, column, and 3×3 square
- Subject to some existing constraints (numbers that have been filled in already)

		7		6		1		
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1		3	2		7			8
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Take your puzzle and let the variable $x_{i,j,k}$ represent filling in cell (i, j) with value k.

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8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

for all i, j:

$$x_{i,j,1} \vee \ldots \vee x_{i,j,9}$$

All cells (i, j) should have some value between 1 and 9.

Solving Sudoku

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	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

for all i, j:

$$x_{i,j,1} \vee \ldots \vee x_{i,j,9}$$

$$\wedge$$

for all $k \neq l$:

$$(\neg x_{i,j,k} \vee \neg x_{i,j,l})$$

And each cell should only be assigned one value.

Solving Sudoku

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5	7		4					
		4		8		7		

for all k in [1, 9]:
$(x_{1,1,k} \vee \ldots \vee x_{1,9,k})$

In the first row, every value k has to be assigned to one of the cells.

Solving Sudoku

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1		3	2		7			8
5	7		4					
		4		8		7		

Add similar constraints for the other rows, columns, and 3x3 squares!

Solving Sudoku

What is a Sudoku solution?

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$(x_{1,3,7}$

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5	7		4					
		4		8		7		

$$(X_{1,3,7} \wedge X_{1,5,6}$$

Solving Sudoku

What is a Sudoku solution?

- An assignment of numbers 1 through 9 to a 9×9 grid
- Such that each number appears exactly once in each row, column, and 3×3 square
- **Subject to some existing constraints (numbers that have been filled in already)**

		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

$$(X_{1,3,7} \wedge X_{1,5,6} \wedge X_{1,6,1} \wedge \ldots)$$

Solving Sudoku

What is a Sudoku solution?

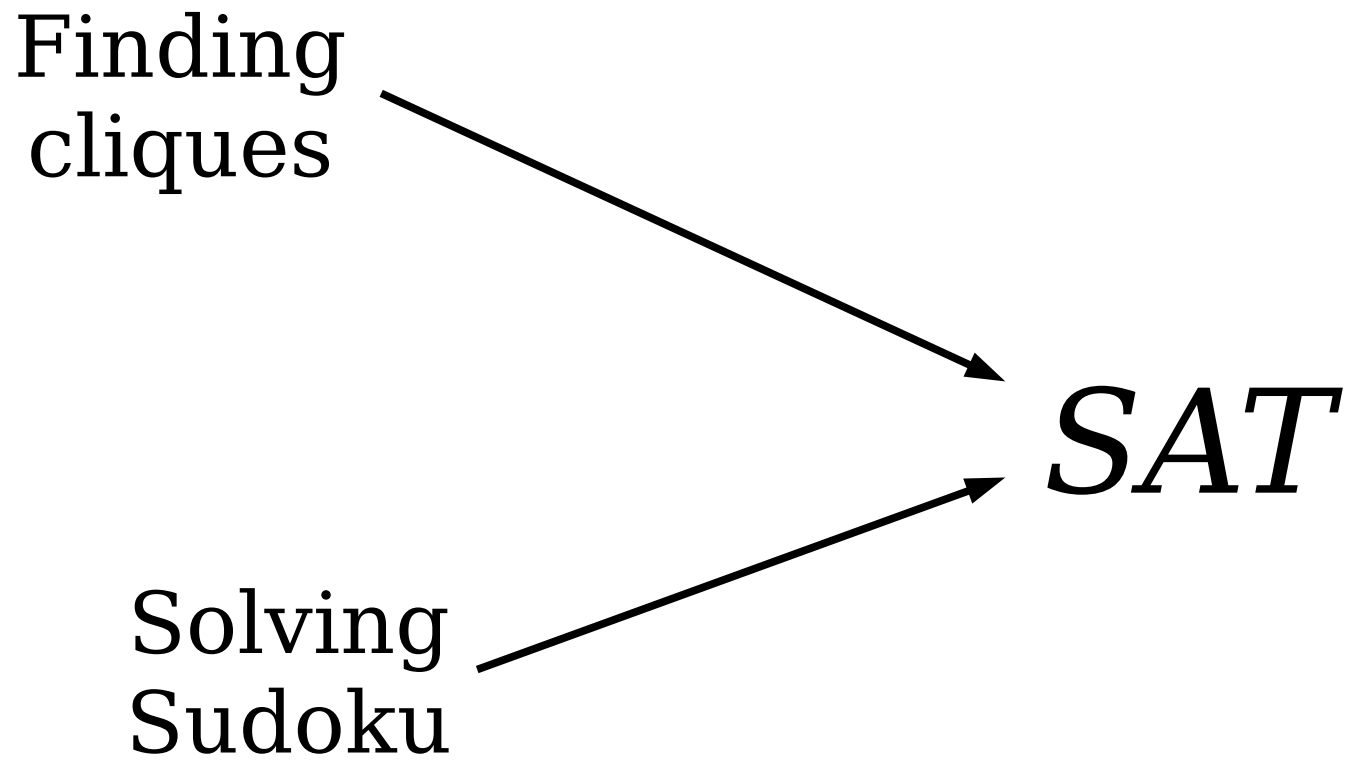
- An assignment of numbers 1 through 9 to a 9×9 grid
- Such that each number appears exactly once in each row, column, and 3×3 square
- Subject to some existing constraints (numbers that have been filled in already)

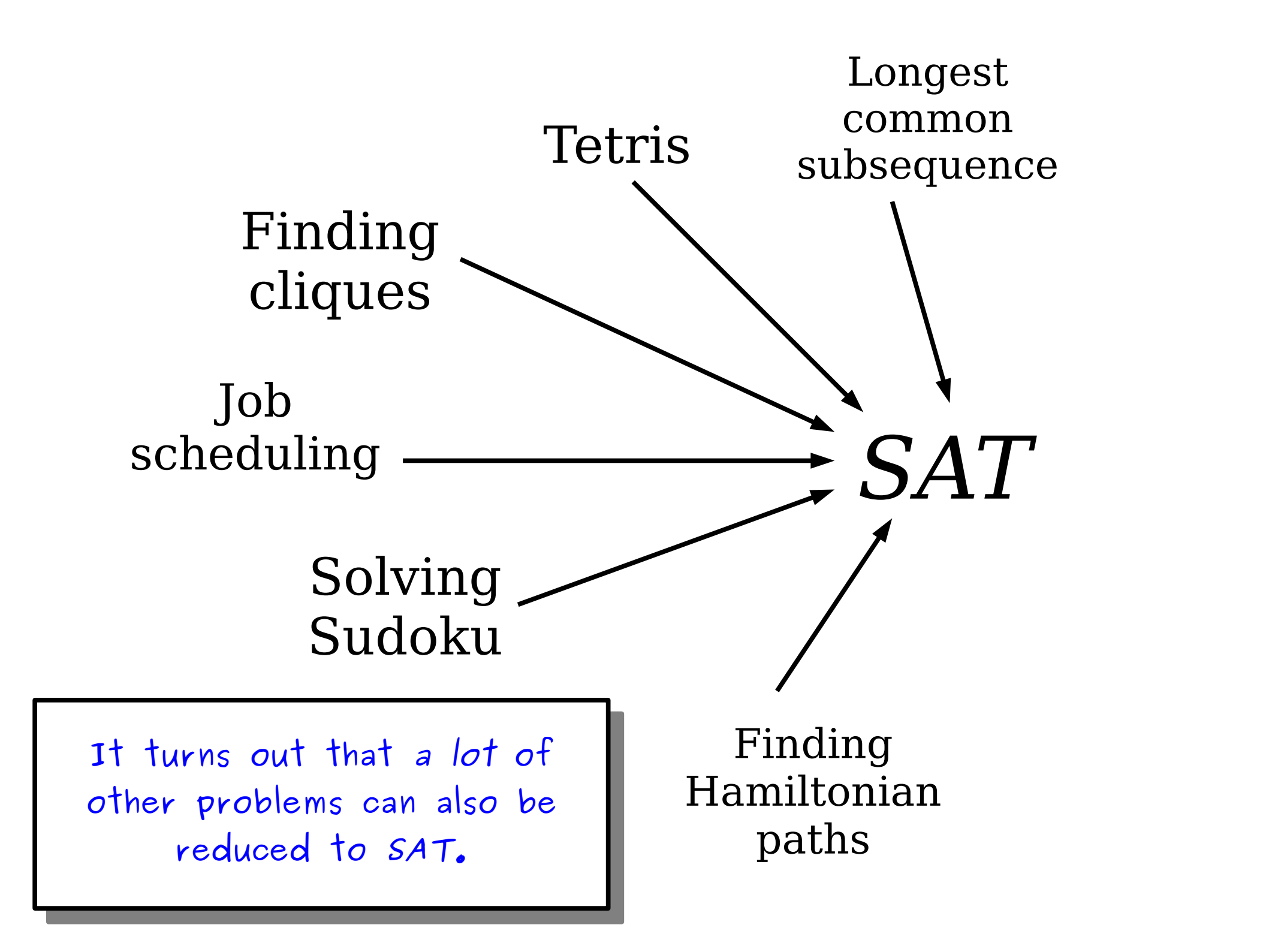
		7		6		1		
					3		5	2
3			1		5	9		7
6		5		3		8		9
	1						2	
8		2		1		5		4
1		3	2		7			8
5	7		4					
		4		8		7		

Altogether, finding an assignment of true false values to these variables that satisfies these constraints would amount to finding a solution to our Sudoku puzzle.

Intuition:

Solving Sudoku can’t be any harder than solving *SAT*, because we can take any Sudoku puzzle and encode it as a propositional logic formula where finding a satisfying assignment corresponds to finding a puzzle solution.


Finding
cliques
SAT
Solving
Sudoku

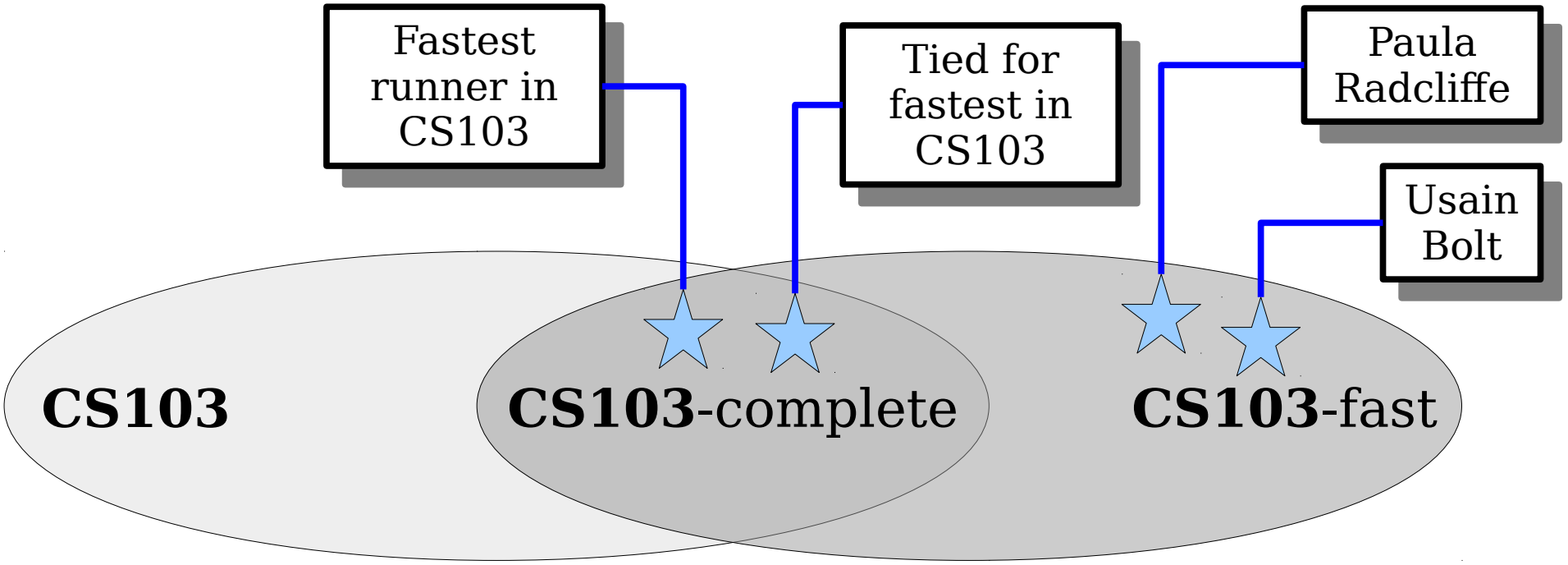

Longest common subsequence
Tetris
Finding cliques
Job scheduling
SAT
Solving Sudoku
Finding Hamiltonian paths
It turns out that a lot of other problems can also be reduced to SAT.

Key Observations:

(1) *SAT* is ***versatile*** – being able to solve *SAT* allows us to solve many other problems.

(2) The fact that lots of problems reduce to *SAT* suggests we can gauge the difficulty of these problems by looking at the difficulty of *SAT*.

An Analogy: Running Really Fast



For people A and B, we say $A \leq_r B$ if A's top running speed is at most B's top speed.

(Intuitively: B can run at least as fast as A.)

We say that person P is ***CS103-fast*** if

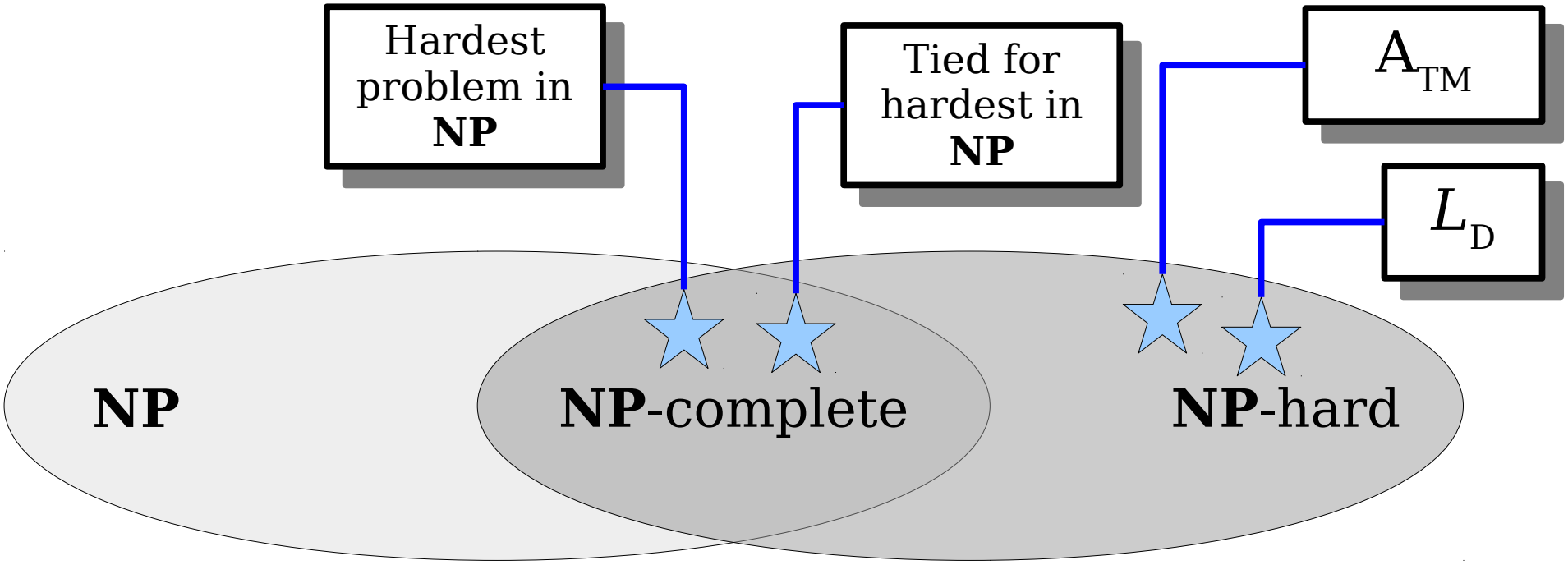
$\forall A \in \textbf{CS103}.\ A \leq_r P.$

(How fast are you if you're CS103-fast?)

We say that person P is ***CS103-complete*** if

$P \in \textbf{CS103}$ and P is **CS103**-fast.

(How fast are you if you're CS103-complete?)



For languages A and B, we say $A \leq_p B$ if A reduces to B in polynomial time.

(Intuitively: B is at least as hard as A.)

We say that a language L is ***NP-hard*** if

$\forall A \in \mathbf{NP}.\ A \leq_p L.$

(How hard is a problem that's NP-hard?)

We say that a language L is ***NP-complete*** if

$L \in \mathbf{NP}$ and L is **NP**-hard.

(How hard is a problem that's NP-complete?)

Intuition: The **NP**-complete problems are the hardest problems in **NP**.

If we can determine how hard those problems are, it would tell us a lot about the **P** $\stackrel{?}{=}$ **NP** question.

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Intuition: This means the hardest problems in **NP** aren't actually that hard. We can solve them in polynomial time. So that means we can solve all problems in **NP** in polynomial time.

The Tantalizing Truth

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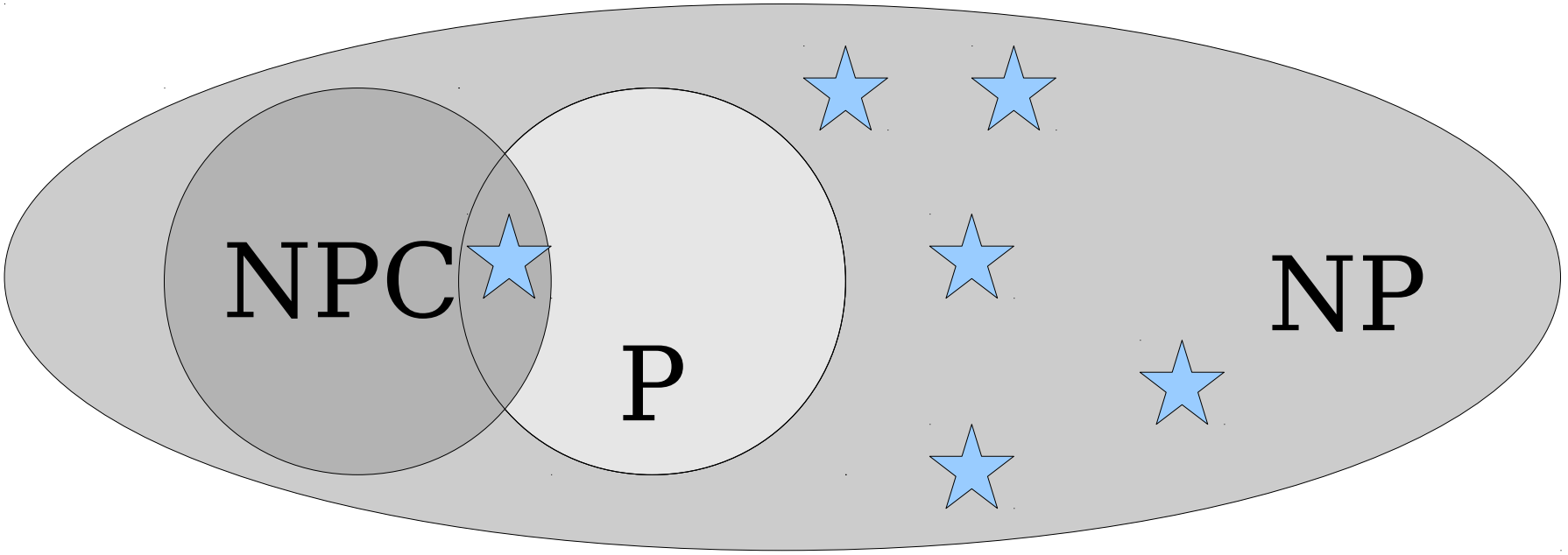

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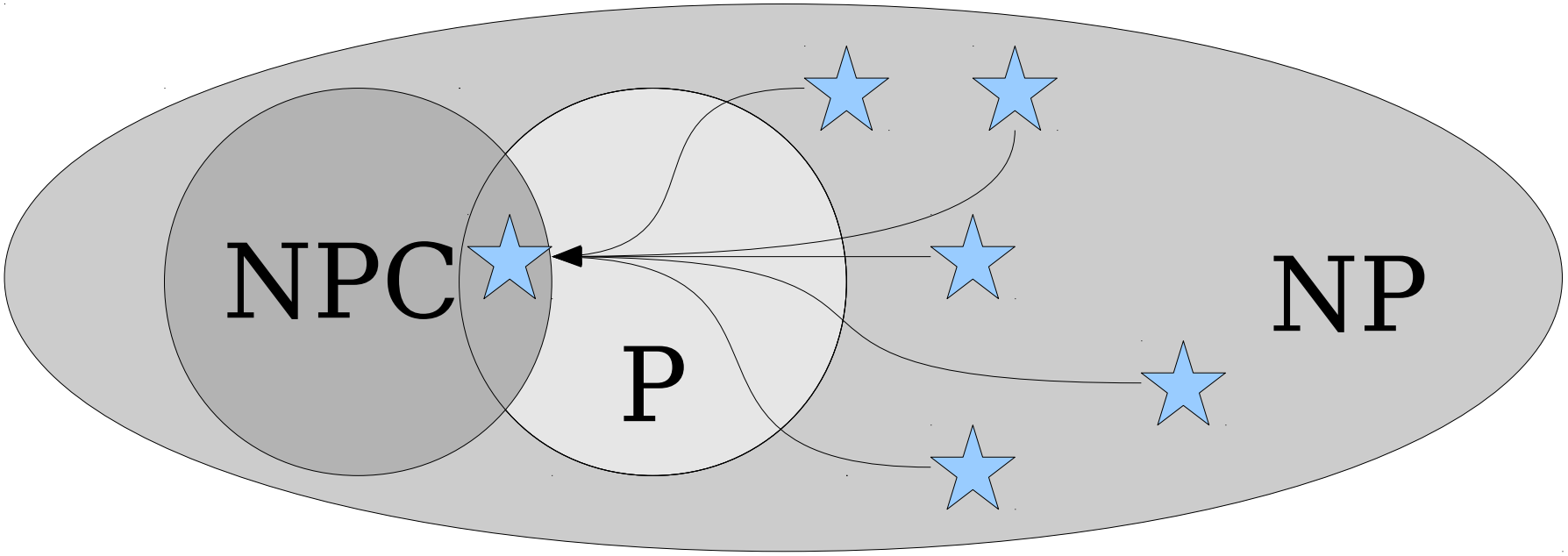

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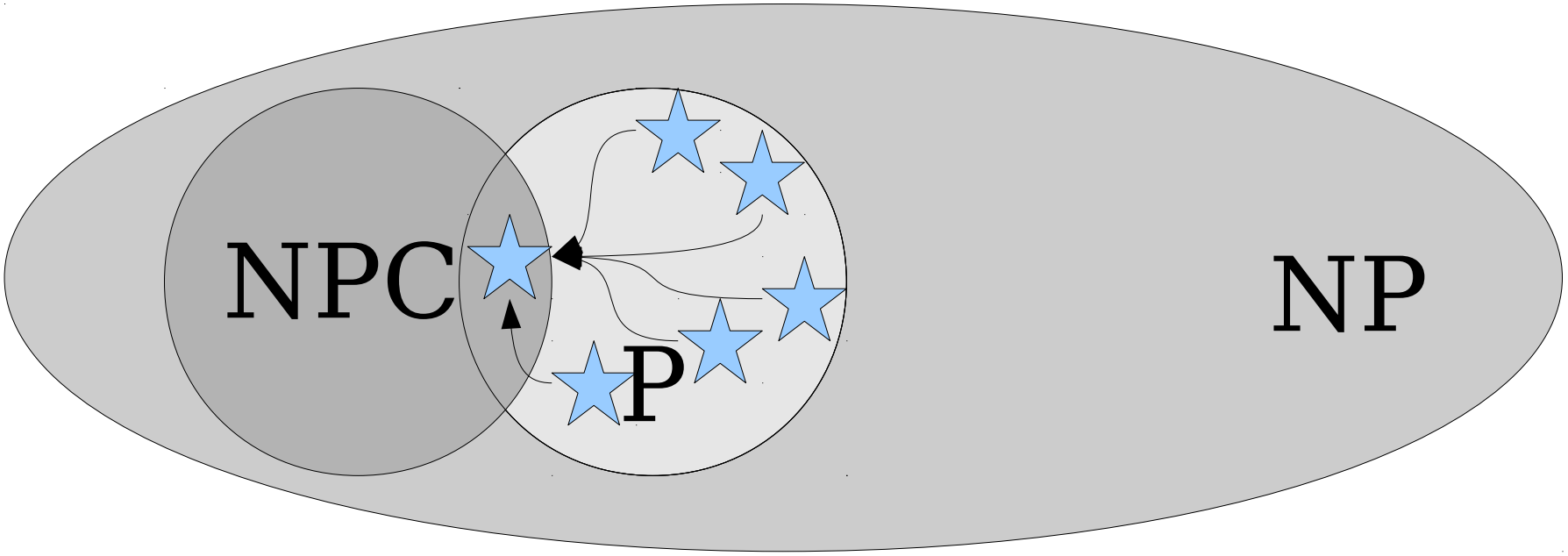

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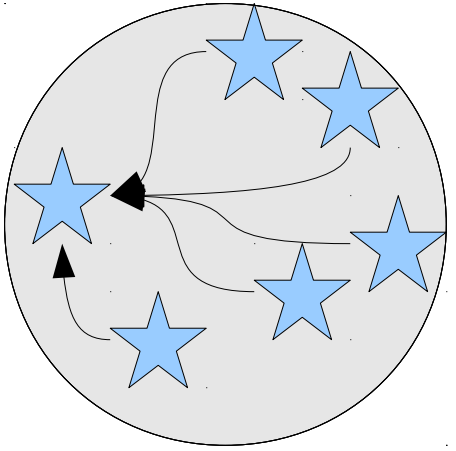
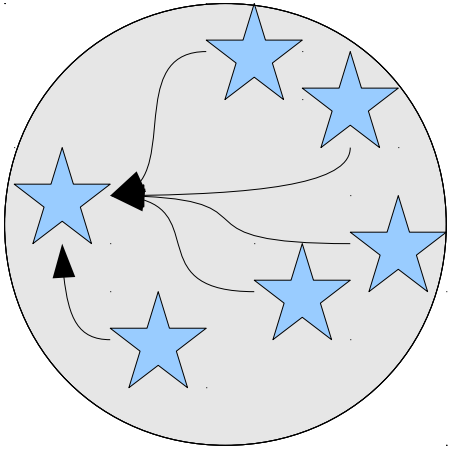


P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Proof: Suppose that L is **NP**-complete and $L \in \mathbf{P}$. Now consider any arbitrary **NP** problem A. Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in \mathbf{P}$ and $A \leq_p L$, we see that $A \in \mathbf{P}$. Since our choice of A was arbitrary, this means that $\mathbf{NP} \subseteq \mathbf{P}$, so $\mathbf{P} = \mathbf{NP}$. ■



P = NP

The Tantalizing Truth

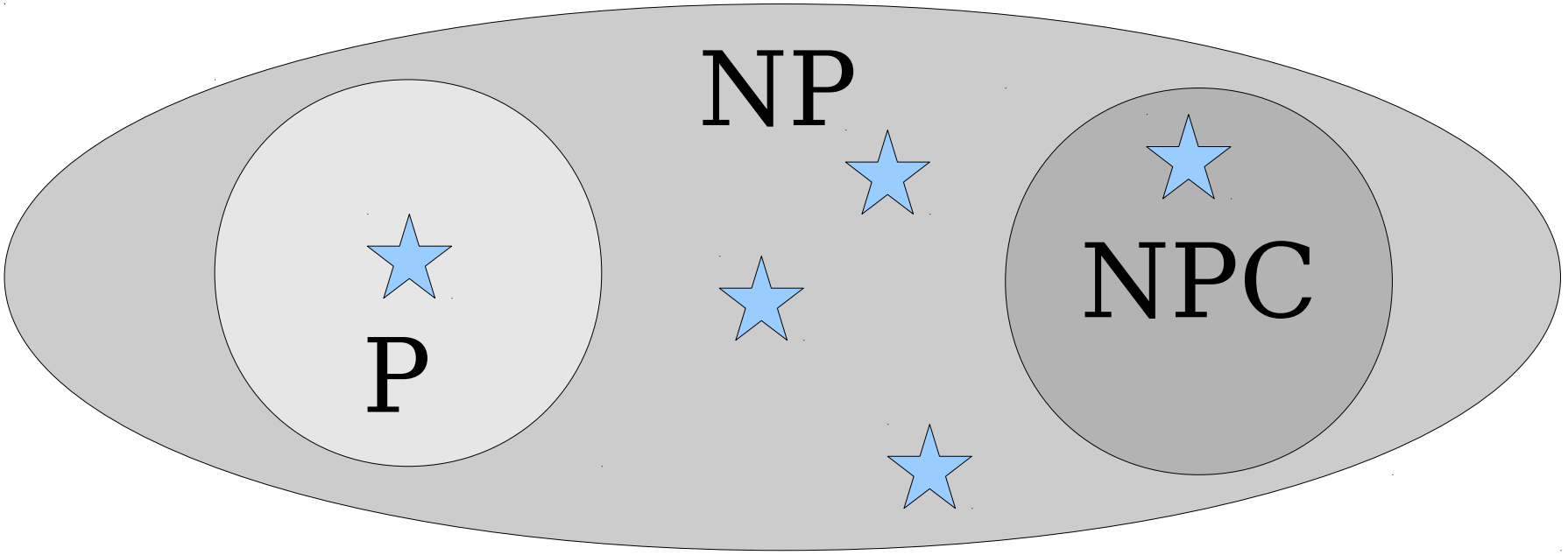
Theorem: If *any* **NP**-complete language is not in **P**, then **P ≠ NP**.

Intuition: This means the hardest problems in **NP** are so hard that they can't be solved in polynomial time. So the hardest problems in **NP** aren't in **P**, meaning **P ≠ NP**.

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in$ **NP**. Therefore, we know that $L \in$ **NP** and $L \notin$ **P**, so **P** ≠ **NP**. ■



How do we even know NP-complete problems exist in the first place?

Theorem (Cook-Levin): SAT is **NP**-complete.

Proof Idea: To see that **SAT** ∈ **NP**, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that **SAT** is **NP**-hard, given a polymomial-time verifier V for an arbitrary **NP** language L, for any string w you can construct a polynomially-sized formula $\varphi(w)$ that says "there is a certificate c where V accepts $\langle w, c \rangle$." This formula is satisfiable if and only if $w \in L$, so deciding whether the formula is satisfiable decides whether w is in L. ■-*ish*

Proof: Take CS154!

Why All This Matters

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.

$$\text{SAT} \in \mathbf{P} \quad \leftrightarrow \quad \mathbf{P = NP}$$

- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.
- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!

Why All This Matters

- You will almost certainly encounter **NP**-hard problems in practice – they're everywhere!
- If a problem is **NP**-hard, then there is no known algorithm for that problem that
  - is efficient on all inputs,
  - always gives back the right answer, and
  - runs deterministically.
- ***Useful intuition:*** If you need to solve an **NP**-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.

Sample **NP**-Hard Problems

- ***Computational biology:*** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? *(Maximum parsimony problem)*
- ***Game theory:*** Given an arbitrary perfect-information, finite, two-player game, who wins? *(Generalized geography problem)*
- ***Operations research:*** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? *(Job scheduling problem)*
- ***Machine learning:*** Given a set of data, find the simplest way of modeling the statistical patterns in that data *(Bayesian network inference problem)*
- ***Medicine:*** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who survive. *(Cycle cover problem)*
- ***Systems:*** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible *(Processor scheduling problem)*

Coda: What if $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is resolved?

Time-Out for Announcements!

Problem Sets

- Problem Set 7 was due thirty minutes ago. Solutions are available here and also on the course website.

Congratulations – you're done with CS103 problem sets!

Please evaluate this course on Axess.

Your feedback really makes a difference.

Final Exam Logistics

- Our final exam is Friday, August 16th from 7PM – 10PM in ***Bishop Auditorium***.
- The exam is cumulative. You're responsible for topics from PS0 – PS7 and all of the lectures up through and including Unsolvable Problems.
- The exam is closed-book, closed-computer, and limited-note. You can bring one double-sided sheet of 8.5" × 11" notes with you to the exam, decorated any way you'd like.
- Students with OAE accommodations: if we don't yet have your OAE letter, please send it to us ASAP.

Preparing for the Exam

- We've posted two practice final exams, with solutions, to the course website. They're on the *Extra Practice* page under *Resources*.
  - The practice exam we'll be using during the practice final will be released today as well.
- ***Practice Final*** Today from 5:30-8:30 PM upstairs in Gates 104.

The Big Picture

Take a minute to reflect on your journey.

Set Theory
Power Sets
Cantor’s Theorem
Direct Proofs
Parity
Proof by Contrapositive
Proof by Contradiction
Modular Congruence
Number Theory
Propositional Logic
First-Order Logic
Logic Translations
Logical Negations
Propositional Completeness
Vacuous Truths
Tournament Graphs
Binary Relations
Equivalence Relations
Equivalence Classes
Systems of Representatives
Strict Orders
Functions
Injections
Surjections
Bijections
Inverse Functions
Permutations
Graphs
Connectivity
Graph Automorphisms
Vertex Covers
Bipartite Graphs
Mathematical Induction
Loop Invariants
Complete Induction
Tiling Problems
Bezout's Identity
Euclid's Algorithm
Hypercubes
Formal Languages
DFAs
Regular Languages
Closure Properties
NFAs
Subset Construction
Kleene Closures
Monoids
5-Tuples
Regular Expressions
State Elimination
Distinguishability
Myhill-Nerode Theorem
Nonregular Languages
Extended Transition Functions
Equivalence Relation Indices
Axiom of Choice
Context-Free Grammars
Turing Machines
Church-Turing Thesis
TM Encodings
Universal Turing Machines
Self-Reference
Decidability
Recognizability
Self-Defeating Objects
Undecidable Problems
Halting Problem
Verifiers
Diagonalization Language
Complexity Class **P**
Complexity Class **NP**
P $\stackrel{?}{=}$ **NP** Problem
Polynomial-Time Reducibility
NP-Completeness

You’ve done more than just check
a bunch of boxes off a list.

You’ve given yourself the foundation to tackle problems from all over computer science.

From CS255

A **Shannon cipher** is a pair $\mathcal{E} = (E, D)$ of functions.

- The function E (the **encryption function**) takes as input a **key** k and a **message** m (also called a **plaintext**), and produces as output a **ciphertext** c. That is,

$$c = E(k, m),$$

  and we say that c is the **encryption of** m **under** k.

- The function D (the **decryption function**) takes as input a key k and a ciphertext c, and produces a message m. That is,

$$m = D(k, c),$$

  and we say that m is the **decryption of** c **under** k.

- We require that decryption "undoes" encryption; that is, the cipher [illegible] **correctness property**: for all keys k and all messages m, we have

$$D(k,\ E(k,\ m)\ ) = m.$$

Kinda sorta like a left inverse!

To be slightly more formal, let us assume that $\mathcal{K}$ is the set of all keys (the **key space**), $\mathcal{M}$ is the set of all messages (the **message space**), and that $\mathcal{C}$ is the set of all ciphertexts (the **ciphertext space**). With this notation, we can write:

$$E : \mathcal{K} \times \mathcal{M} \to \mathcal{C},$$
$$D : \mathcal{K} \times \mathcal{C} \to \mathcal{M}.$$

Hey, you've seen this before!

Also, we shall say that $\mathcal{E}$ is **defined over** $(\mathcal{K}, \mathcal{M}, \mathcal{C})$.

From CS145

Semantics of JOINs (2 tables)

```
SELECT R.A
FROM   R, S
WHERE  R.A = S.B
```

Cartesian products!

1. Take **cross product**:
   $X = R \times S$

Recall: Cross product (A X B) is the set of all unique tuples in A,B
Ex: {a,b,c} X {1,2}
= {(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)}

Set-builder notation!

2. Apply **selections / conditions**:
   $Y = \{(r,s) \in X \mid r.A == r.B\}$

= Filtering!

3. Apply **projections** to get final output:
   $Z = (y.A,) \text{ for } y \in Y$

= Returning only *some* attributes

Remembering this order is critical to understanding the output of certain queries (see later on…)

From CS251

CS251: Cryptocurrencies and Blockchain Technologies **Fall 2018**

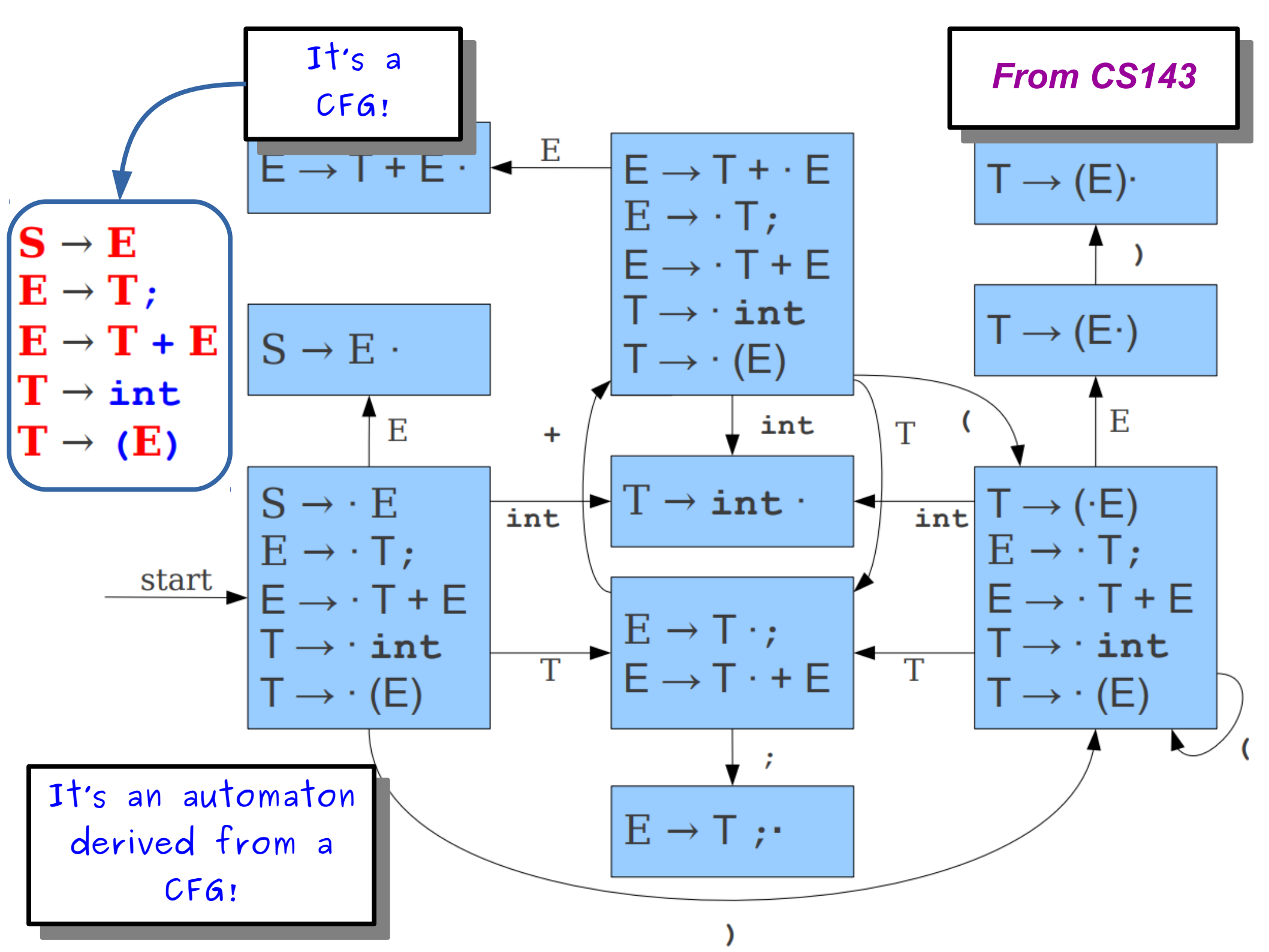
Assignment #1

Due: 11:59pm on Mon., **Oct. 8, 2018**
Submit via Gradescope (each answer on a separate page) code: **9RZGVZ**

Problem 1. Hash functions and proofs of work. In class we defined two security properties for a hash function, one called collision resistance and the other called proof-of-work security. Show that a collision-resistant hash function may not be proof-of-work secure.

Hint: let $H : X \times Y \to \{0, 1, \ldots, 2^n - 1\}$ be a collision-resistant hash function. Construct a new hash function $H' : X \times Y \to \{0, 1, \ldots, 2^m - 1\}$ (where m may be greater than n) that is also collision resistant, but for a fixed difficulty D (say, $D = 2^{32}$) is not proof-of-work secure with difficulty D. That is, for every puzzle $x \in X$ it should be trivial to find a solution $y \in Y$ such that $H'(x, y) < 2^m/D$. This is despite H' being collision resistant. Remember to explain why your H' is collision resistant, that is, explain why a collision on H' would yield a collision on H.

Whoa, it's a function!


It's a CFG!
From CS143
S → E
E → T;
E → T + E
T → int
T → (E)
E → T + E ·
E → T + · E
E → · T;
E → · T + E
T → · int
T → · (E)
T → (E)·
T → (E·)
S → E ·
S → · E
E → · T;
E → · T + E
T → · int
T → · (E)
T → int ·
E → T ·;
E → T · + E
T → (·E)
E → · T;
E → · T + E
T → · int
T → · (E)
E → T ;·
start
E
+
int
T
(
)
;
It's an automaton derived from a CFG!

Search problems

From CS221

Definition: search problem

States: the set of states

$s_{\text{start}} \in \text{States}$: starting state

$\text{Actions}(s)$: possible actions from state s

$\text{Succ}(s, a)$: where we end up if take action a in state s

$\text{Cost}(s, a)$: cost for taking action a in state s

$\text{IsEnd}(s)$: whether at end

- $\text{Succ}(s, a) \Rightarrow T(s, a, s')$
- $\text{Cost}(s, a) \Rightarrow \text{Reward}(s, a, s')$

It's a DFA!

pronounced "big-oh of …" or sometimes "oh of …"

From CS161

O(…) means an upper bound

- Let T(n), g(n) be functions of positive integers.
  - Think of T(n) as being a runtime: positive and increasing in n.

- We say "T(n) is O(g(n))" if g(n) grows at least as fast as T(n) as n gets large.

- Formally,

$$T(n) = O(g(n))$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot g(n)$$

It's FOL and functions!

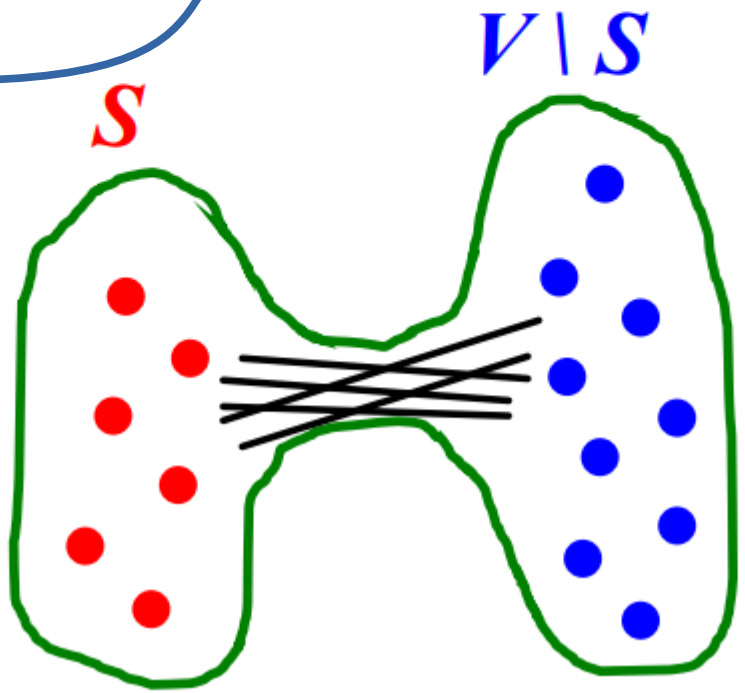
From CS224W

- Graph $G(V, E)$ has **expansion α**: if $\forall S \subseteq V$:
  $\#$ of edges leaving $S \geq \alpha \cdot \min(|S|, |V \setminus S|)$
- **Or equivalently:**

$$\alpha = \min_{S \subseteq V} \frac{\#\,edges\ leaving\ S}{\min(|S|, |V \setminus S|)}$$

Set difference and cardinality!

First-order definitions on graphs!



From CS242

Typed lambda calculus

To understand the formal concept of a type system, we're going to extend our lambda calculus from last week (henceforth the "untyped" lambda calculus) with a notion of types (the "simply typed" lambda calculus). Here's the essentials of the language:

Type τ ::=	int	integer
	$\tau_1 \to \tau_2$	function
Expression e ::=	x	variable
	n	integer
	$e_1 \oplus e_2$	binary operation
	$\lambda\ (x : \tau)\ .\ e$	function
	$e_1\ e_2$	application
Binop $\oplus$ ::=	$+ \mid - \mid * \mid /$	

It's a CFG!

First, we introduce a language of types, indicated by the variable tau (τ). A type is either an integer, or a function from an input type τ_1 to an output type τ_2. Then we extend our untyped lambda calculus with the same arithmetic language from the first lecture (numbers and binary operators)[4]. Usage of the language looks similar to before:

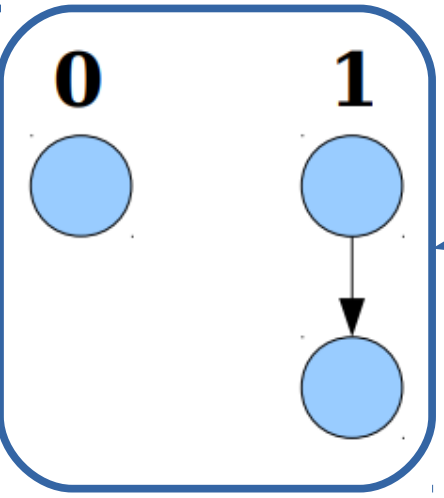
From CS166

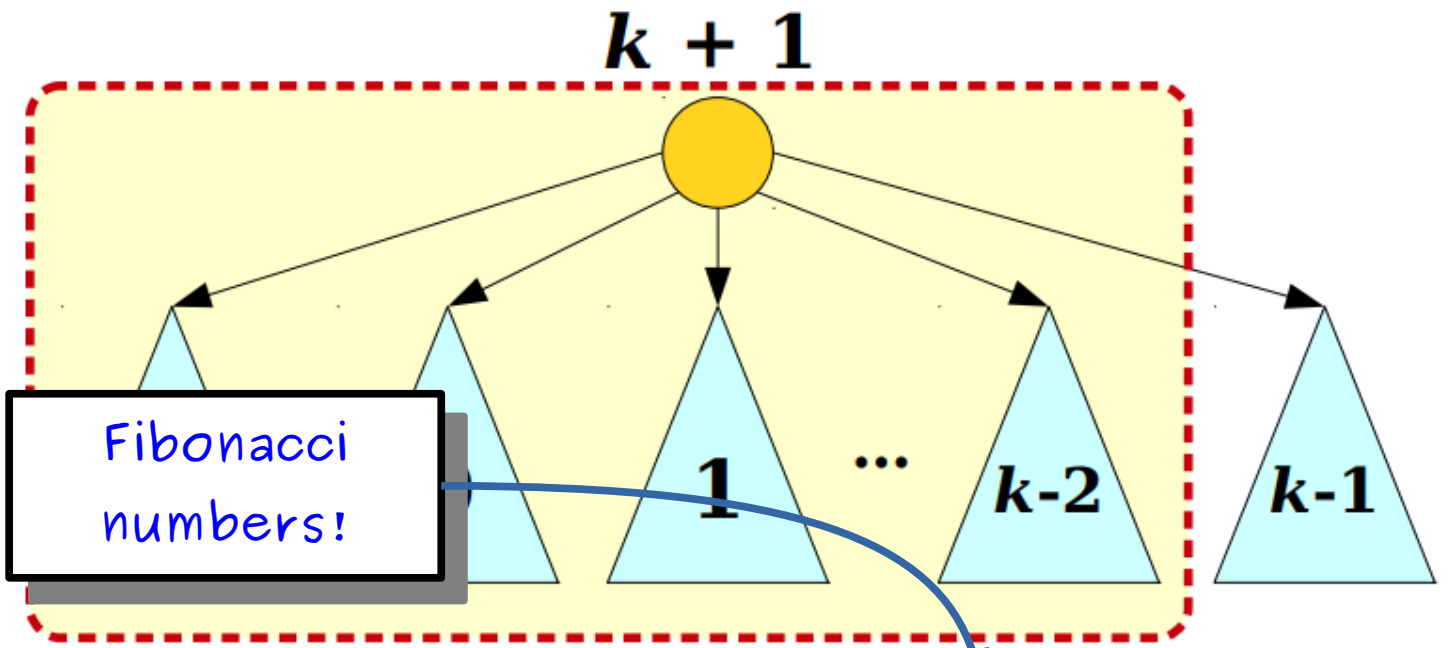
- ***Theorem:*** The number of nodes in a maximally-damaged tree of order k is F_{k+2}.

- ***Proof:*** Induction.

Formal proofs!

Trees!





Fibonacci numbers!

F_2 F_3 F_{k+2} + F_{k+1}

From CS238

3.1.1 Constraints on Rational Preferences

Just as we imposed a set of constraints on beliefs, we will impose some constraints on preferences. These constraints are sometimes called the *von Neumann-Morgenstern axioms*, named after John von Neumann and Oskar Morgenstern, who formulated a variation of these axioms in the 1940s.

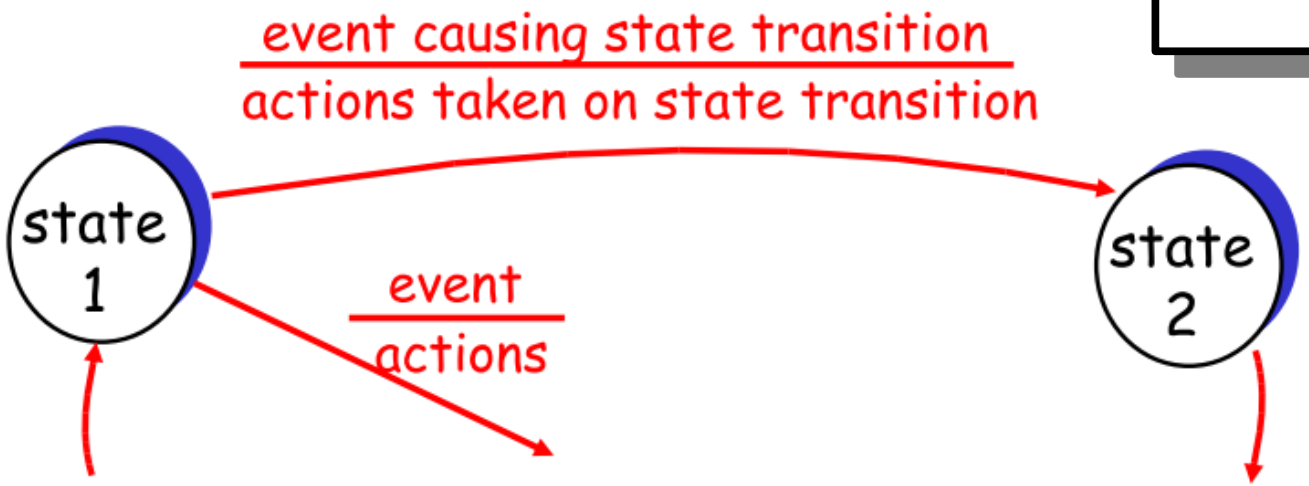
- *Completeness.* Exactly one of the following hold: $A \succ B$, $B \succ A$, or $A \sim B$.
- *Transitivity.* If $A \succeq B$ and $B \succeq C$, then $A \succeq C$.
- *Continuity.* If $A \succeq C \succeq B$, then there exists a probability p such that $[A : p; B : 1 - p] \sim C$.
- *Independence.* If $A \succ B$, then for any C and probability p, $[A : p; C : 1 - p] \succeq [B : p; C : 1 - p]$.

These are constraints on *rational preferences*. They say nothing about the preferences of actu fact, there is strong evidence that humans are not very rational (Sec jective in this book is to understand rational decision making from perspective so that we can build useful systems. The possible extension of this theory to understanding human decision making is of only secondary interest.

Hey, we know that one!

Finite State Machines

From CS144



- **Represent protocols using state machines**
  - Sender and receiver each have a state
  - Start in some initial state
  - Events cause each side to select a state

It's a generalization of DFAs!

- **Transition specifies action taken**
  - Specified as events/actions
  - E.g., software calls `send`/put packet on network
  - E.g., packet arrives/send acknowledgment

From CS168

Reducibility!

By definition, we need to output y if and only if $y \in S$. That is, *answering membership queries reduces to solving the Heavy Hitters problem.* By the "membership problem," we mean the task of preprocessing a set S to answer queries of the form "is $y \in S$"? (A hash table is the most common solution to this problem.) It is intuitive that you cannot correctly answer all membership queries for a set S without storing S (thereby using linear, rather than constant, space) — if you throw some of S out, you might get a query asking about the part you threw out, and you won't know the answer. It's not too hard to make this idea precise using the Pigeonhole Principle.[5]

A Myhill–Nerode–style argument!

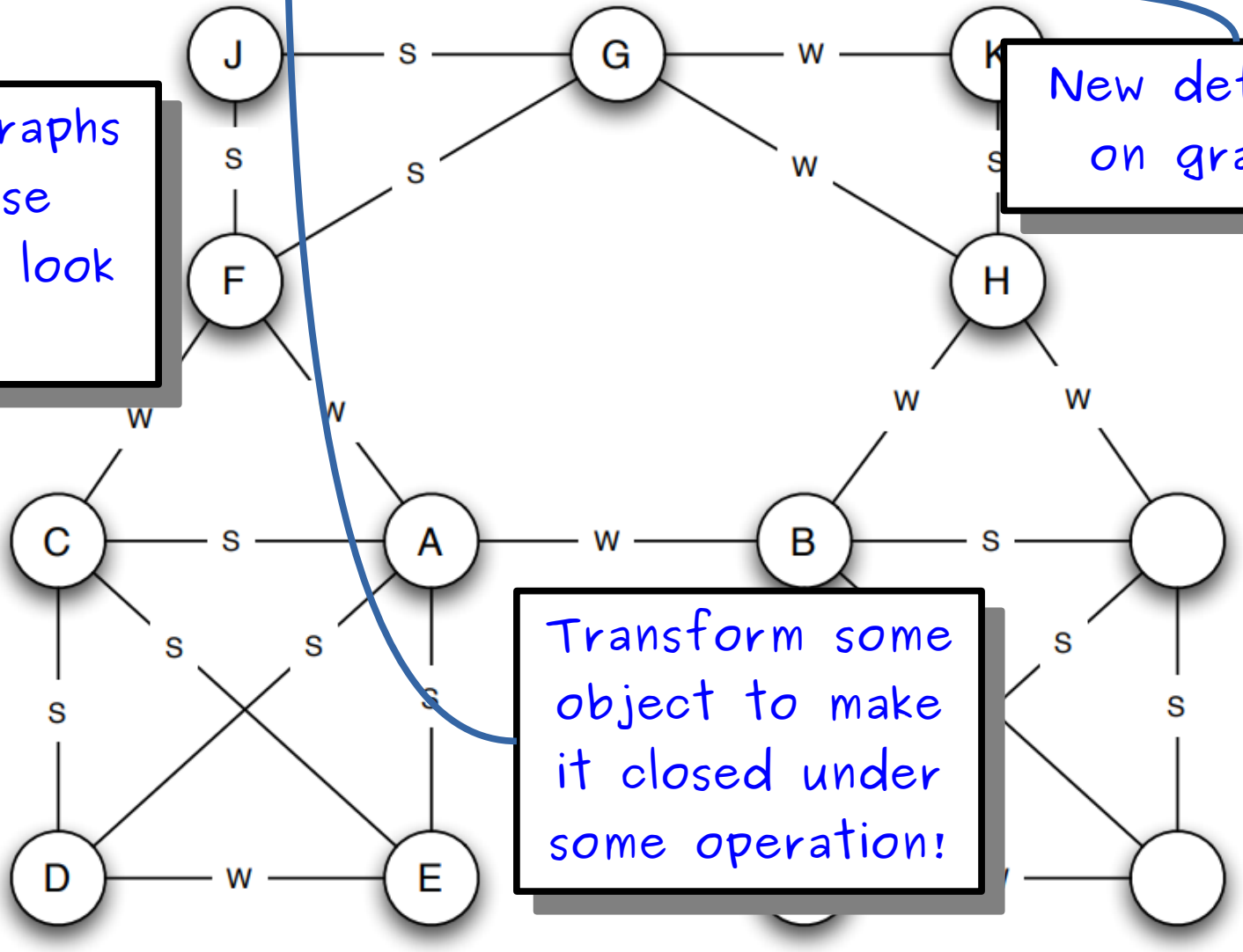
Strong triadic closure

From CS124

If a node Q has two strong ties to nodes Y and Z, there is an edge between Y and Z

What do graphs with these properties look like?

New definitions on graphs!



Transform some object to make it closed under some operation!

From CS154

Kolmogorov Complexity (1960's)

Definition: The *shortest description of* x, denoted as $d(x)$, is the lexicographically shortest string $\langle M,w \rangle$ such that $M(w)$ halts with only x on its tape.

Definition: The *Kolmogorov complexity of* x, denoted as $K(x)$, is $|d(x)|$.

Using Turing machines to define intrinsic information content!

From CS246

- **Suppose we are given a set of documents D**
  - Each document **d** covers a set X_d of words/topics/named entities **W**
- **For a set of documents $A \subseteq D$ we define**

$$F(A) = \left| \bigcup_{d \in A} X_d \right|$$

Functions, set union, and set cardinality!

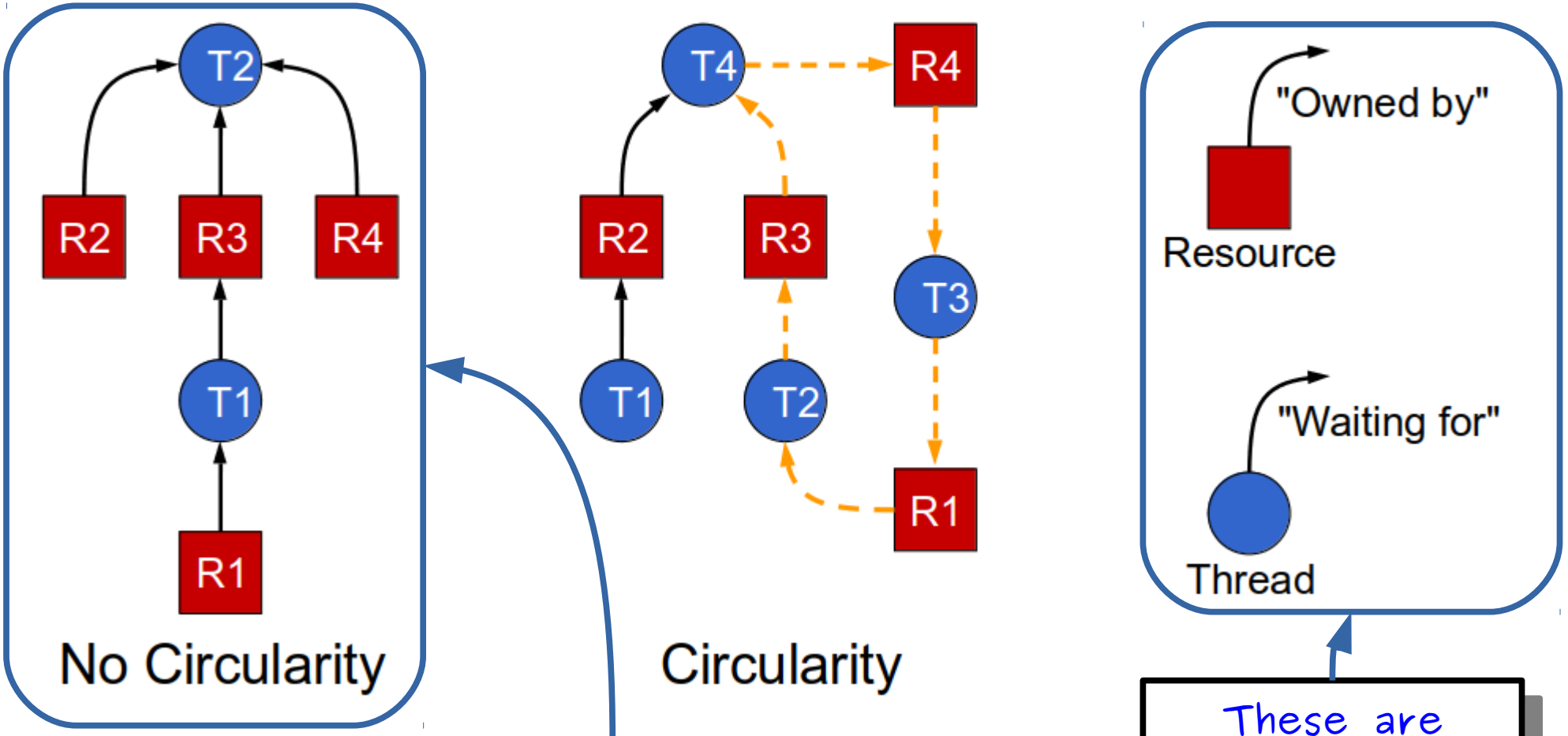
- **Goal: We want to**

$$\max_{|A| \leq k} F(A)$$

- **Note: *F(A)* is a set function: $F(A)$: Sets $\rightarrow \mathbb{N}$**

From CS140

Circular Requests



This is a strict order!

These are binary relations!

You’ve given yourself the foundation to tackle problems from all over computer science.

There’s so much more to explore.
Where should you go next?

Course Recommendations

Theoryland

- CS154
- Phil 151
- Phil 152

Computability

- Math 107
- Math 108

Graphs

- Math 120

Symmetries

- Math 113

Functions

- Math 161

Set Theory

- Math 152

Number Theory

Applications

- CS124
- CS143

Languages / Automata

- CS161

Graphs

- CS224W
- CS243

Relations

- CS246

Functions

- CS242
- CS251
- CS255

Your Questions

"As computers become more ubiquitous, it seems that knowledge about computers and how they work does not. How can we address this? Does LinkedIn have a perspective on this? Your personal thoughts?"

Though there's still a lot we don't know about computation, the amount that we do know is increasing every day! CS research (theoretical and applied) is constantly discovering new things about what's possible with a computer and new algorithms, new applications of existing techniques, etc.

“If I get an awful grade on the final exam, is it possible to still do okay-ish in the class (if I've been keeping up with the problem sets)?”

I don't think this is the right question to be asking right now: 1) If you've been keeping up with the problem sets, there's no reason to expect that you'll do poorly on the exam, and 2) the final hasn't happened yet so the outcome is still very much in your hands! Instead of thinking of what could happen if the exam goes badly, focus your energy on actively targeting your weak spots and working to make the exam go well.

"What is the high-level knowledge map for math used in CS? PSets do give a glimpse into how much math is involved in many areas of CS but what is the big picture?"

We saw some specific examples earlier today, here's a broader overview*

* Huge caveat: this is largely based on which areas of CS I've personally been exposed to! There's lots that I don't have much experience in.

AI

Networking

NLP

Compilers

Databases

Crypto

Information Theory

Set theory

Logic

Probability

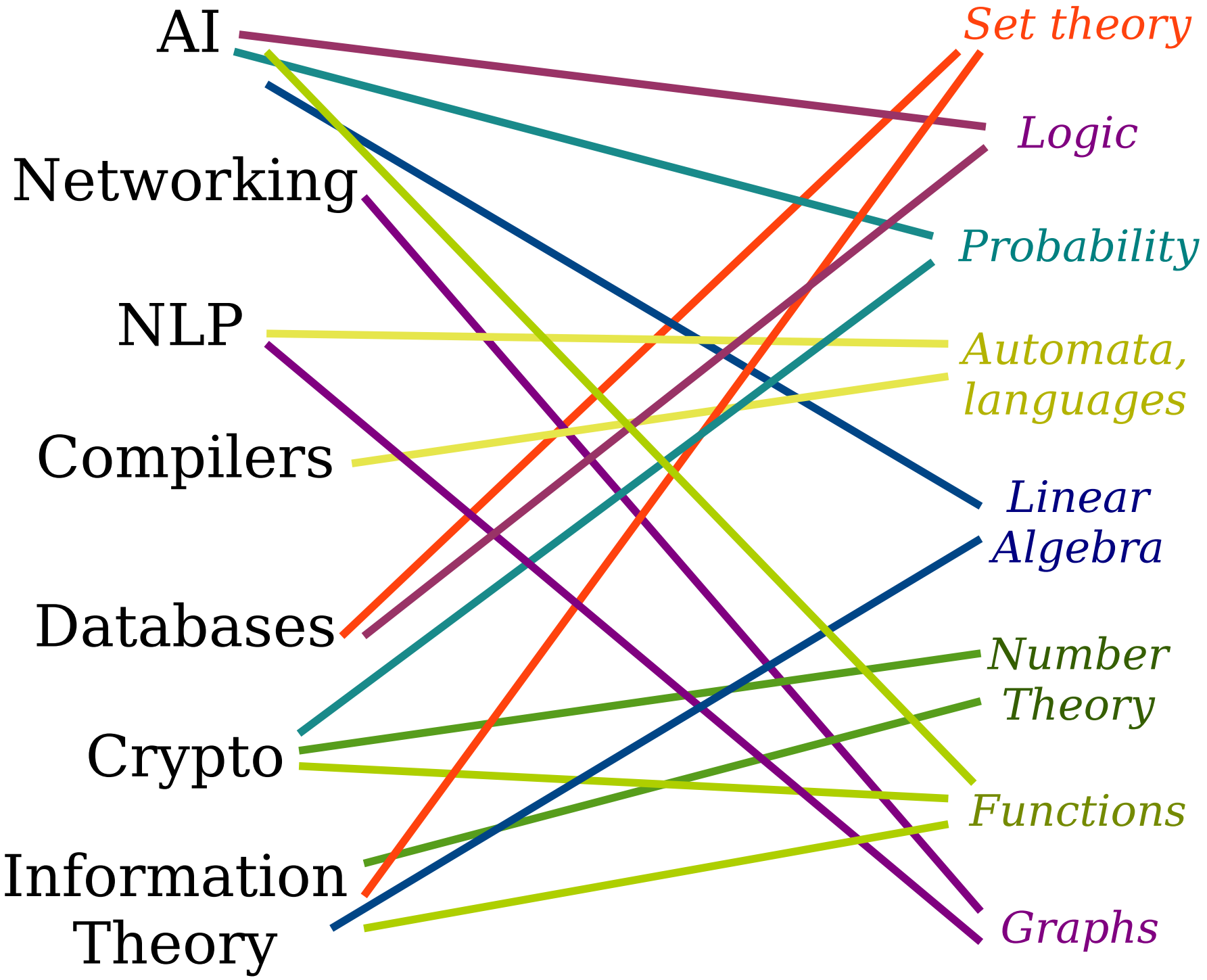
Automata, languages

Linear Algebra

Number Theory

Functions

Graphs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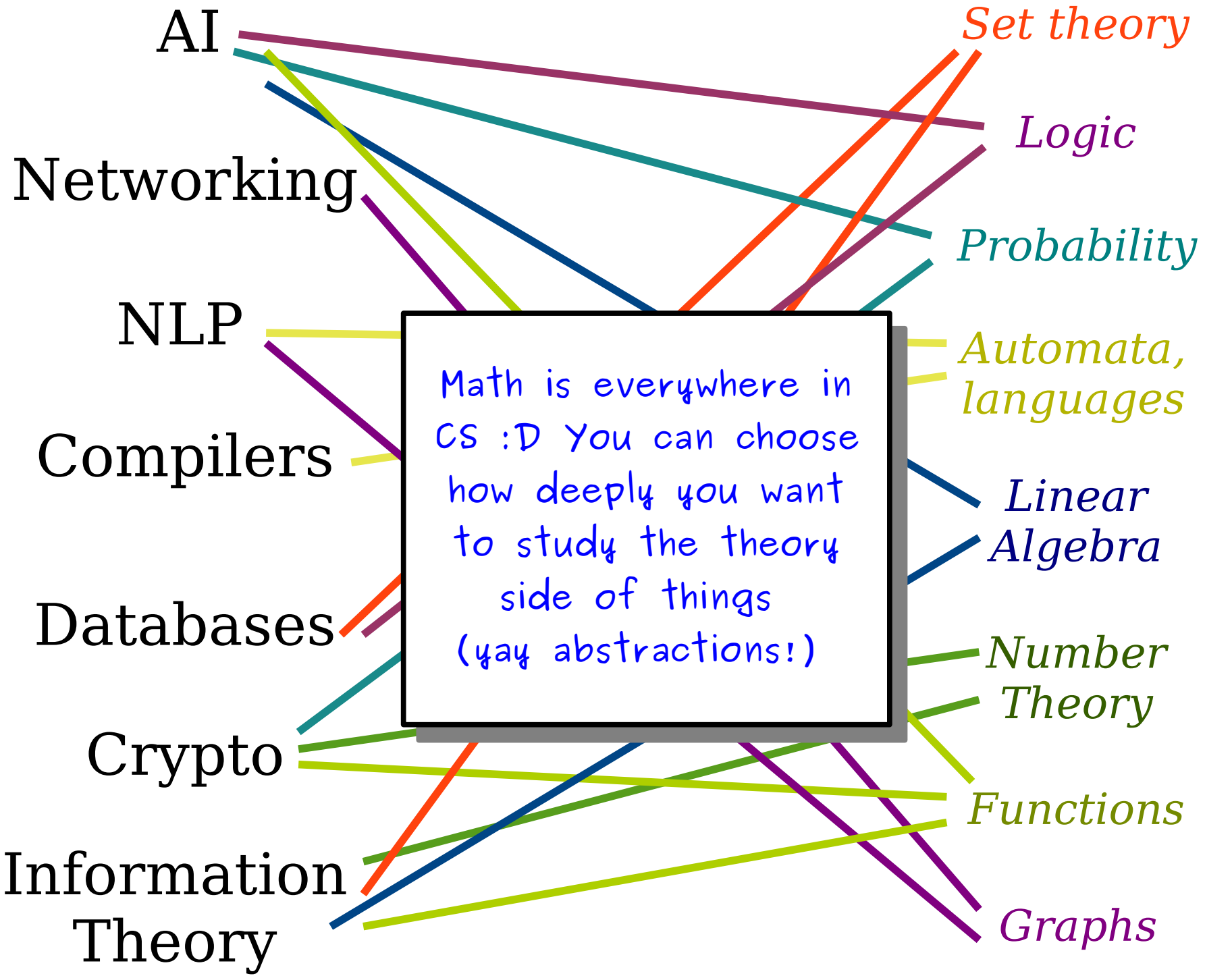
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Graphs
Math is everywhere in CS :D You can choose how deeply you want to study the theory side of things (yay abstractions!)

“What kind of music do you listen to?”

I like a mix of stuff! Here’s a sample:

- Kishi Bashi
- Death Cab for Cutie
- Tom Misch
- Radiohead
- The National Parks
- Rebelution
- Sleeping At Last

“Now that I'm armed with all this discrete math and computability theory, how can I better explain to my (non-academic) family and friends that computers/robots/AI won't take over the world?”

Though computers are powerful, you've seen in this class that there are problems that are fundamentally impossible for computers to solve. Ideas like diagonalization actually don't require too much technical background to understand and explain!

“What does doing research in CS look like? Like for bench science, we will conduct experiments to test an idea. It's hard to imagine CS researches as sitting in an office and coding all day long.”

Similar process, just different tools! CS research can take many forms: coming up with new ways of modeling and predicting phenomena, designing new algorithms and proving that they meet certain runtime/space constraints, finding new ways of applying CS to other fields (education, healthcare, transportation, …)

Would be happy to connect you to folks here in the department who are doing CS research!

“Out of all the CS courses offered at Stanford…why CS103? What makes this course more compelling to you than others?”

:)

We get to take these abstract, complex, philosophical ideas (the nature of computation, infinity, truth) and make them accessible and tangible.

As a class in the CS core @ Stanford, CS103 presents the unique opportunity to get people excited about computer science from a completely different perspective.

Final Thoughts

A Huge Round of Thanks!

There are more problems to solve than there are programs capable of solving them.

There is so much more to explore and so many big questions to ask – ***many of which haven't been asked yet!***


Theory
Practice

You now know what problems we can solve, what problems we can't solve, and what problems we believe we can't solve efficiently.

Our questions to you:

What problems will you ***choose*** to solve?
Why do those problems matter to you?
And how are you going to solve them?