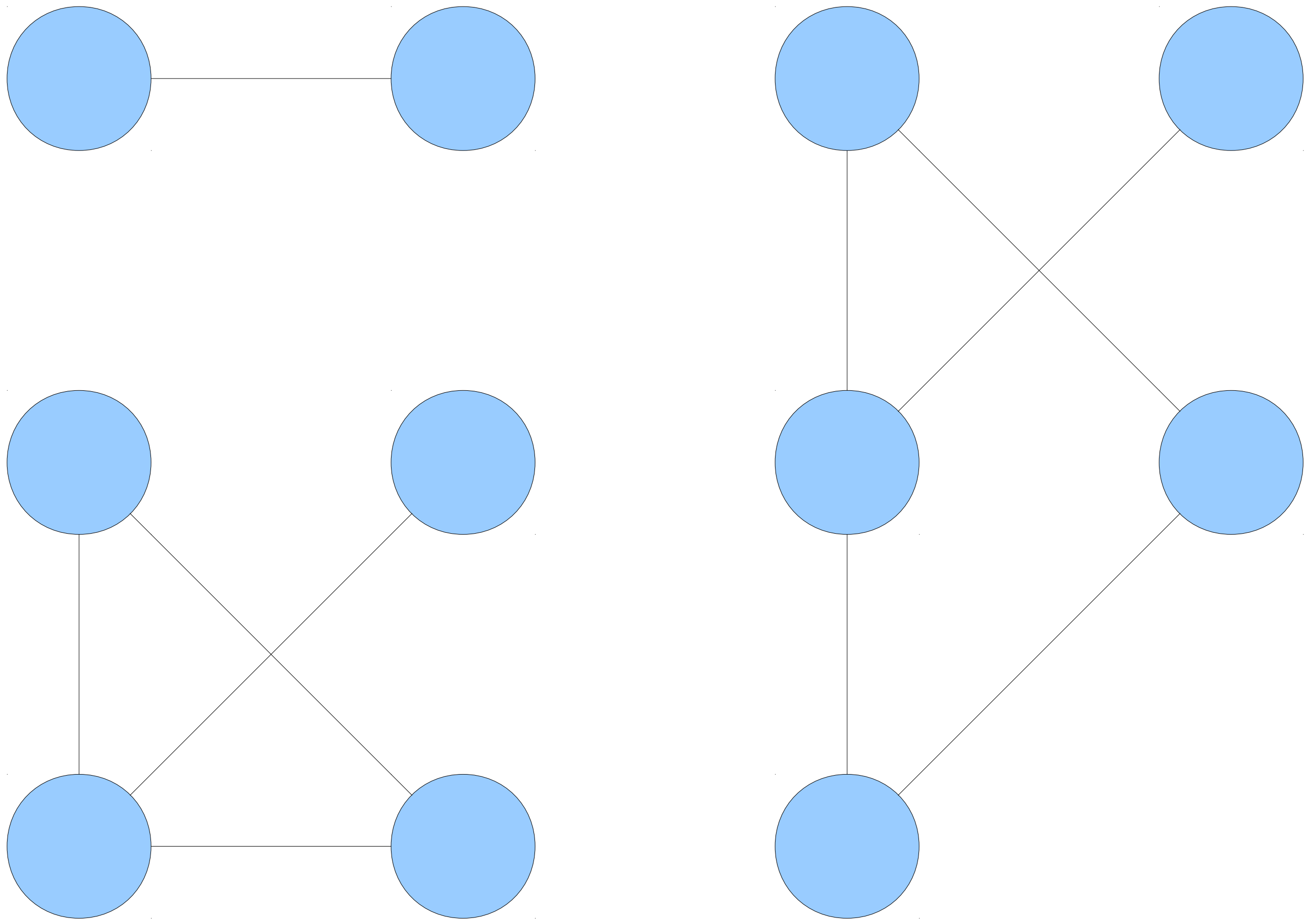
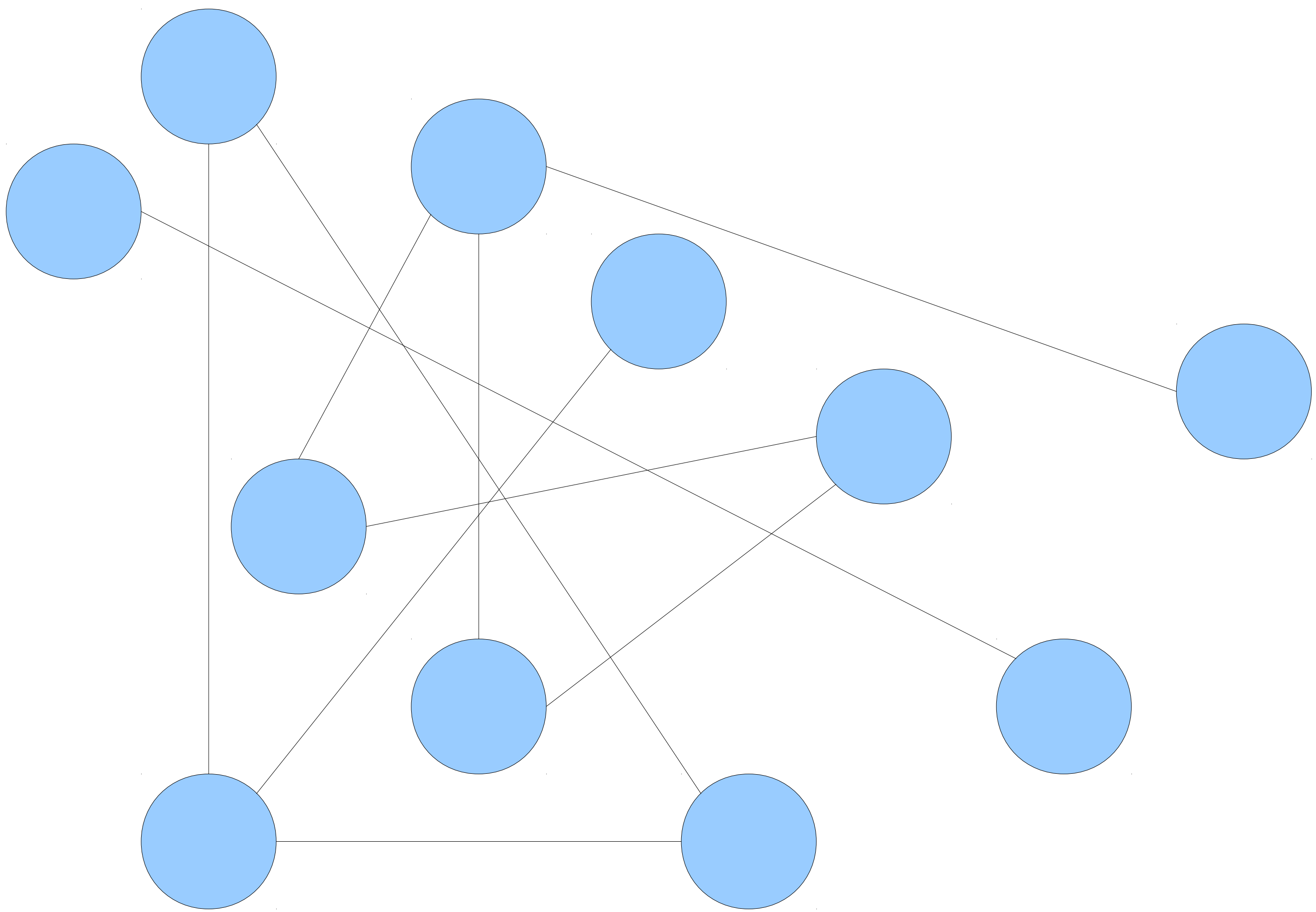
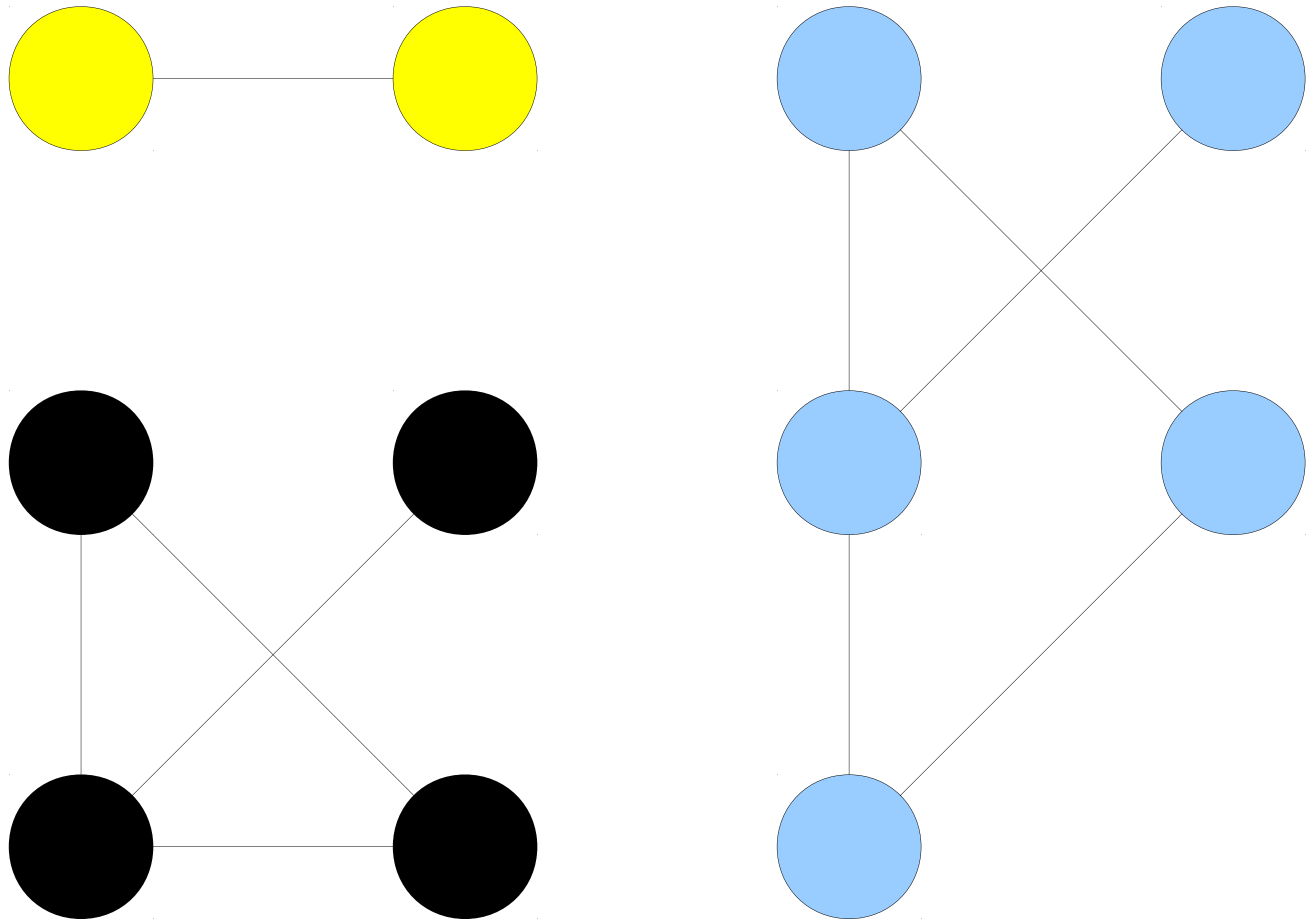
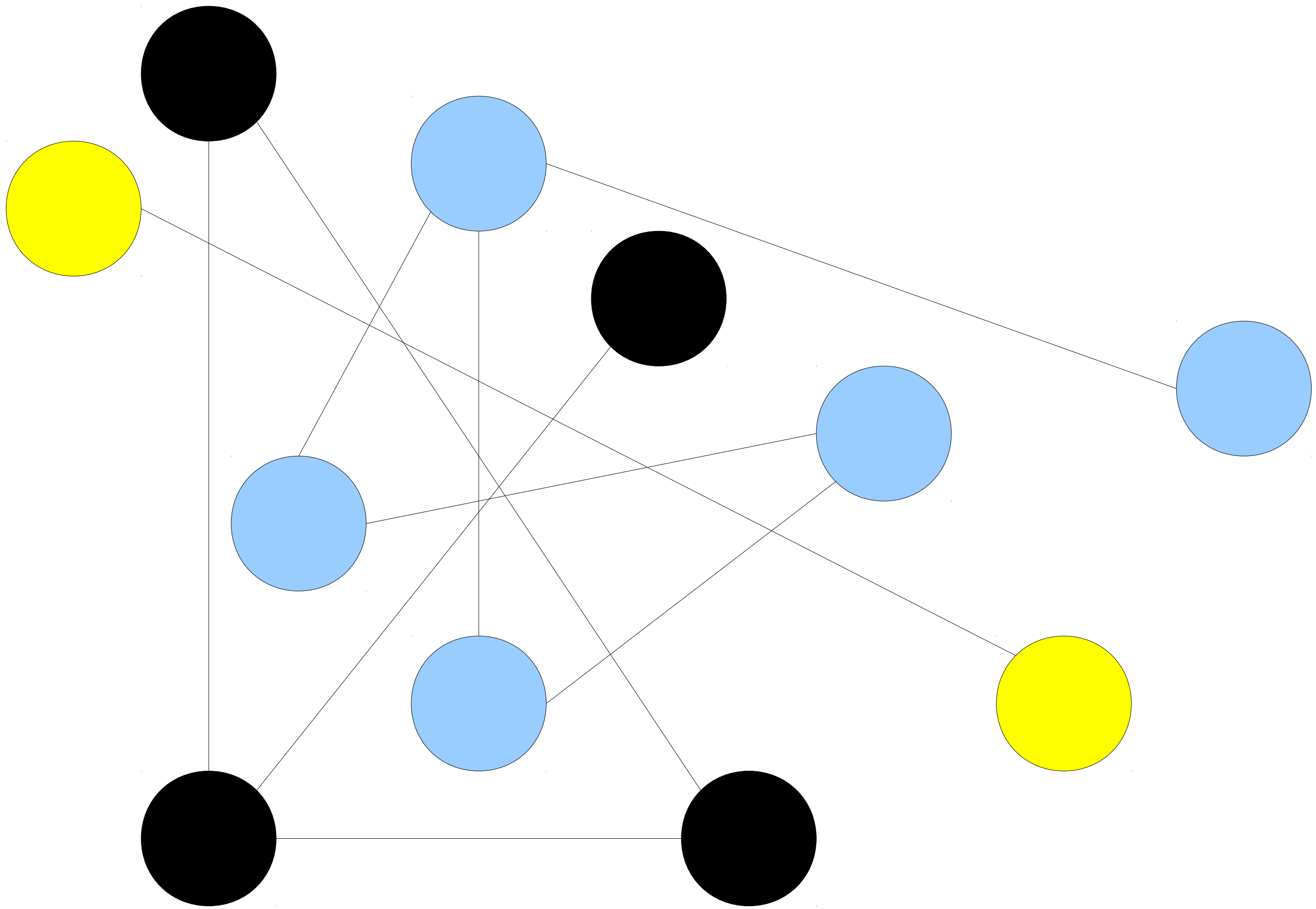
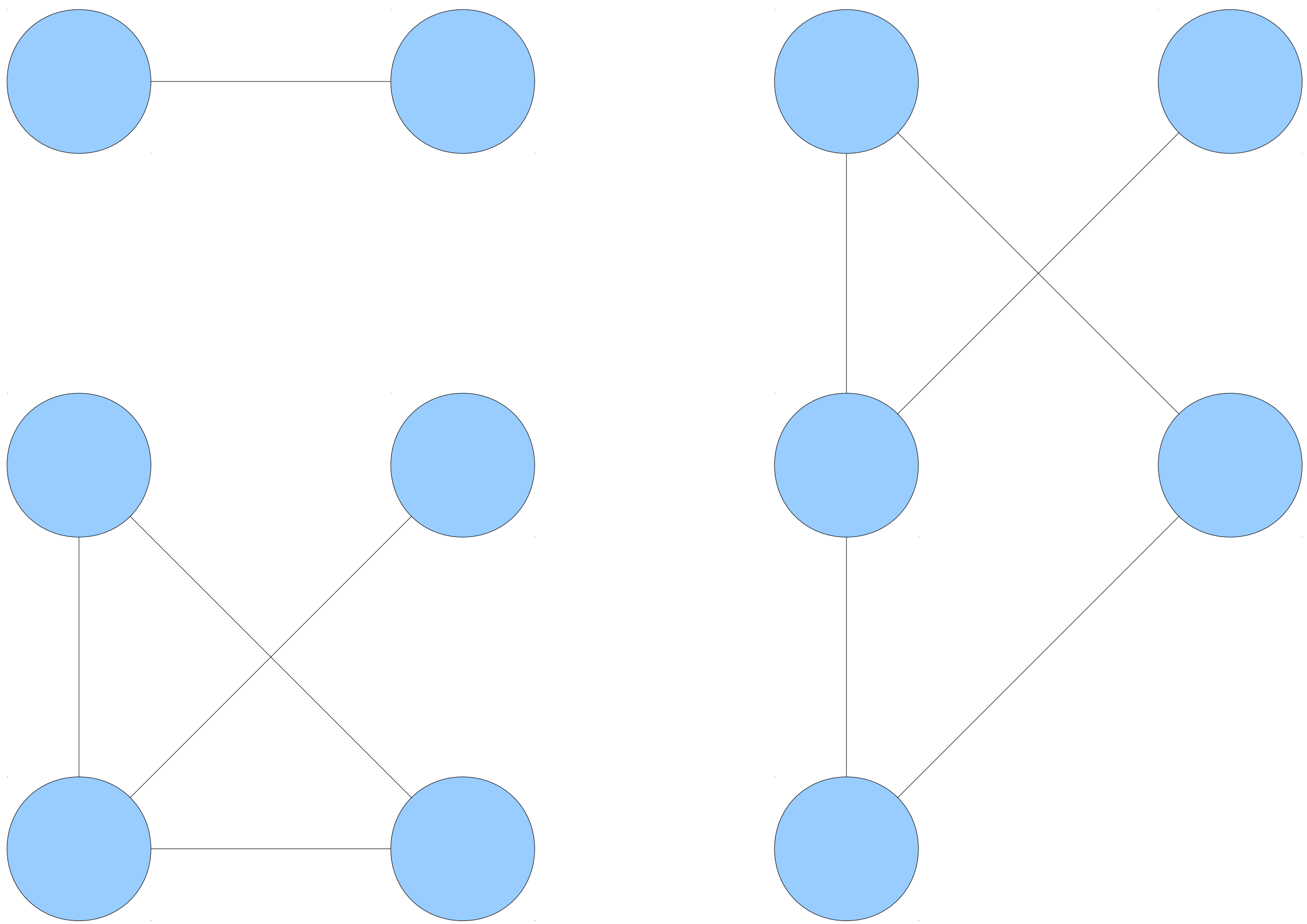
Connected Components

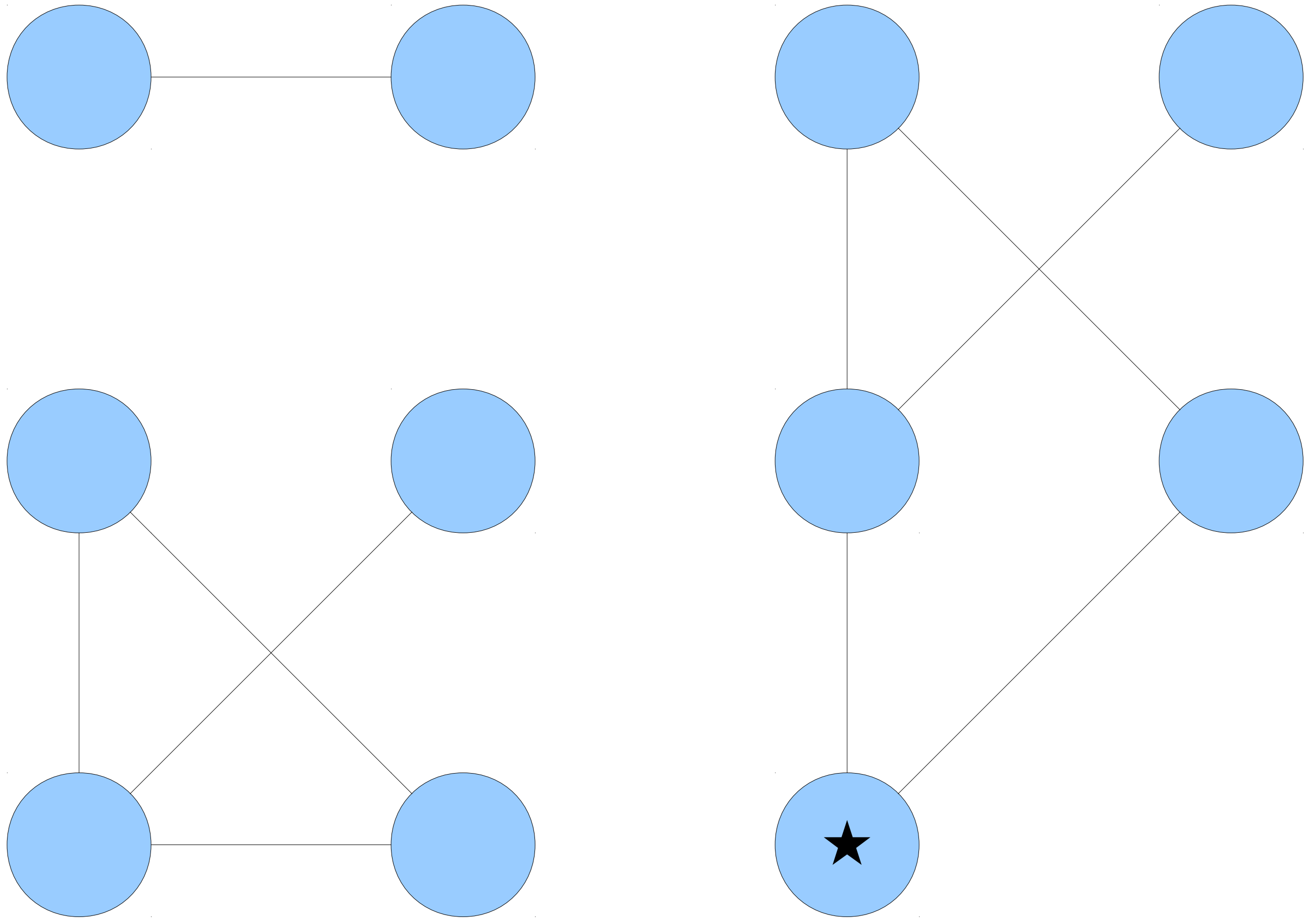


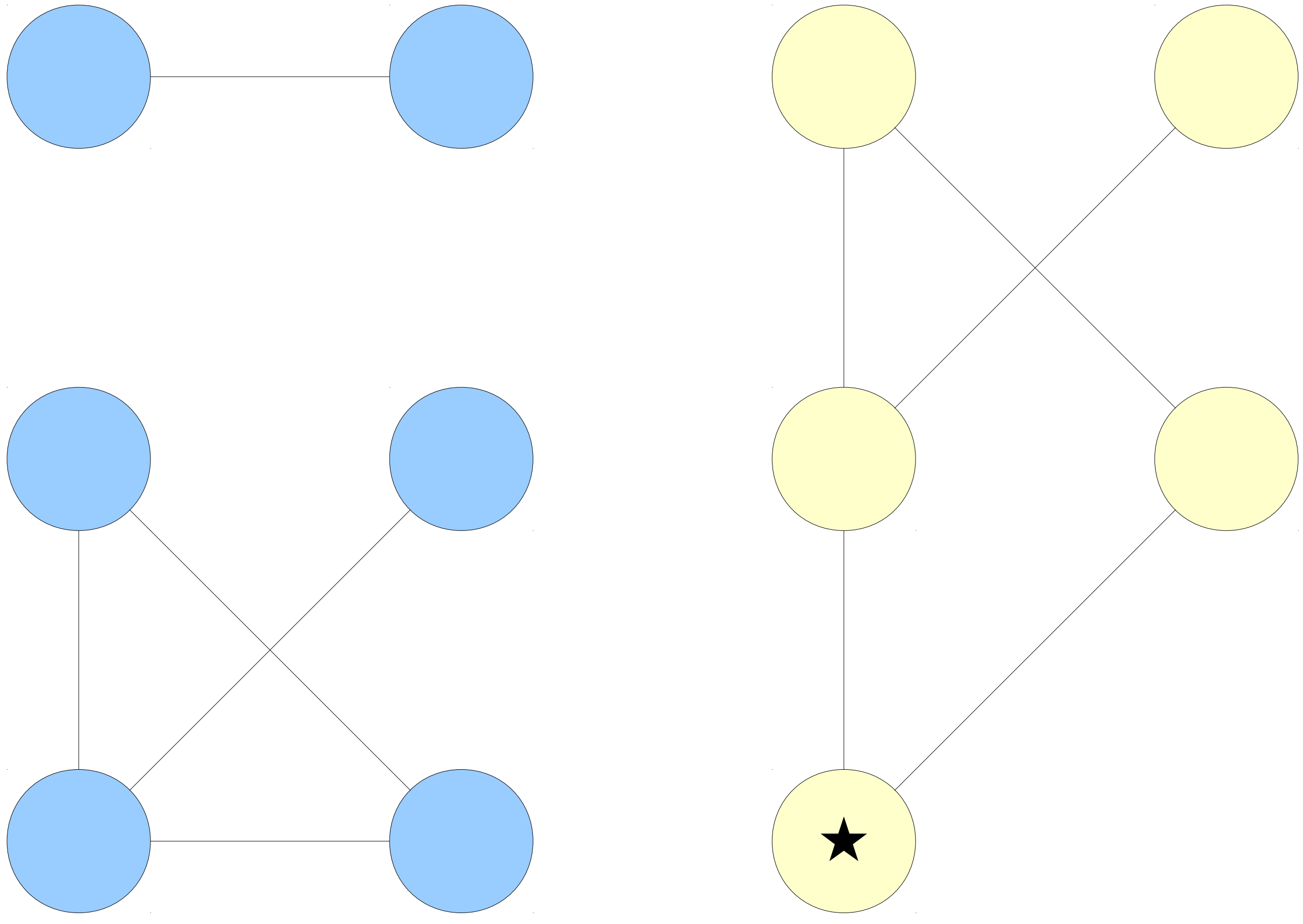












Connected Components

- Let $G = (V, E)$ be a graph. For each $v \in V$, the ***connected component*** containing v is the set

$$[v] = \{ x \in V \mid v \text{ is connected to } x \}$$

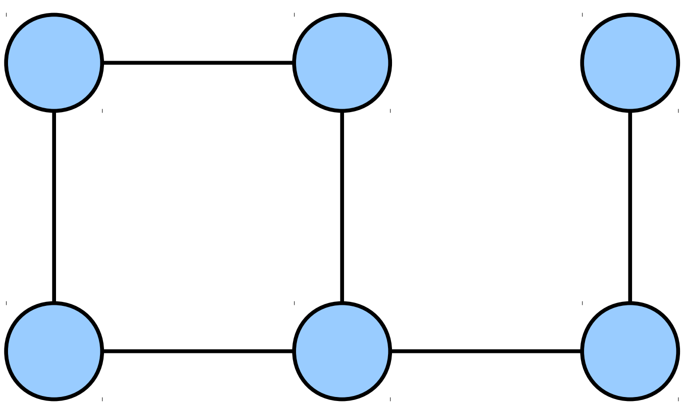
- Intuitively, a connected component is a “piece” of a graph in the sense we just talked about.
- ***Question:*** How do we know that this particular definition of a “piece” of a graph is a good one?
- ***Goal:*** Prove that any graph can be broken apart into different connected components.

We’re trying to reason about some way of partitioning the nodes in a graph into different groups.

What structure have we studied that captures the idea of a partition?

Connectivity

- ***Claim:*** For any graph G, the “is connected to” relation is an equivalence relation.
  - Is it reflexive?
  - Is it symmetric?
  - Is it transitive?



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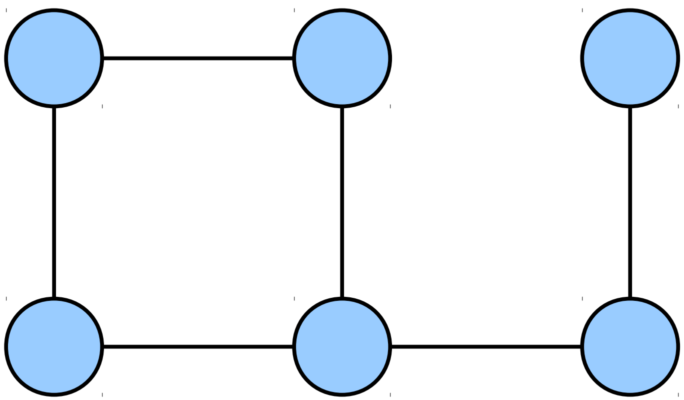
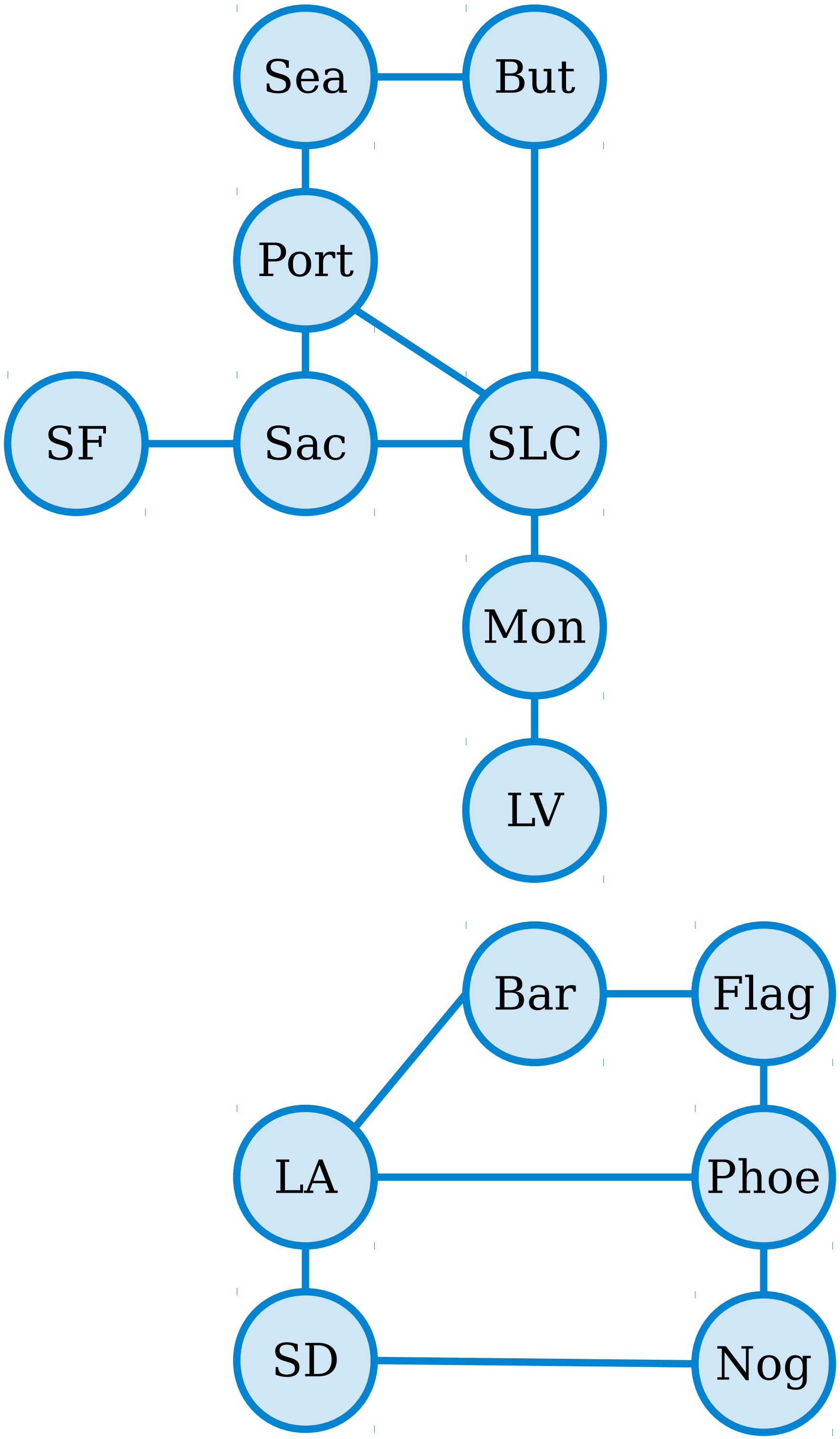
$$\forall v \in V.\ Conn(v, v)$$





A ***path*** in a graph $G = (V, E)$ is a sequence of one or more nodes $v_1, v_2, v_3, \ldots, v_n$ such that any two consecutive nodes in the sequence are adjacent.

Two nodes in a graph are called ***connected*** if there is a path between them

A graph G as a whole is called ***connected*** if all pairs of nodes in G are connected.

(This graph is not connected.)

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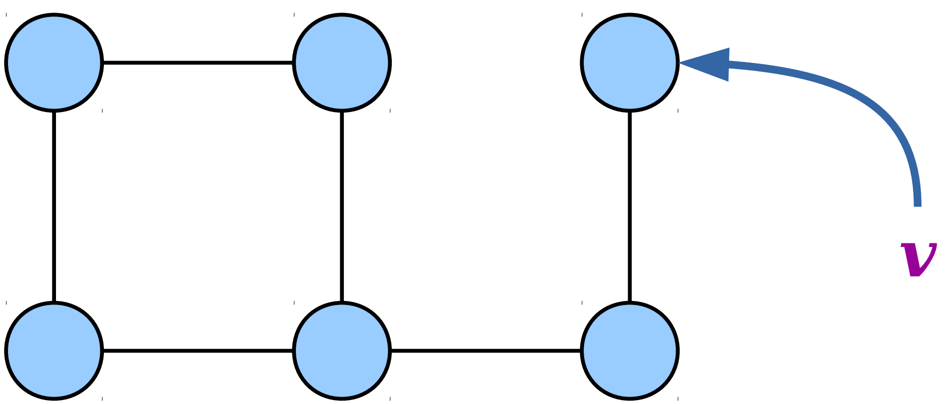

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Connectivity

Claim: E
connecte
relation.

$$\forall x \in V.\ \forall y \in V.\ (Conn(x, y) \to Conn(y, x))$$

Is it reflexive?

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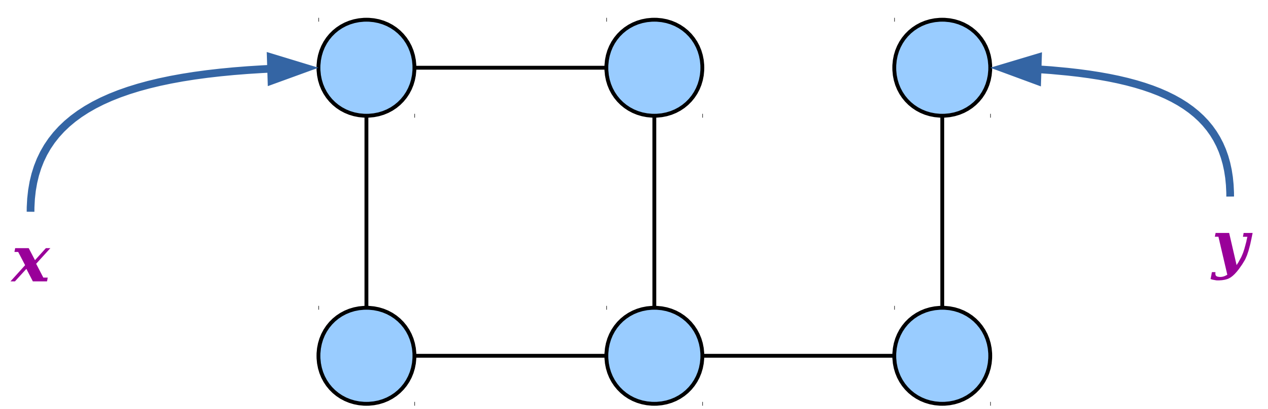

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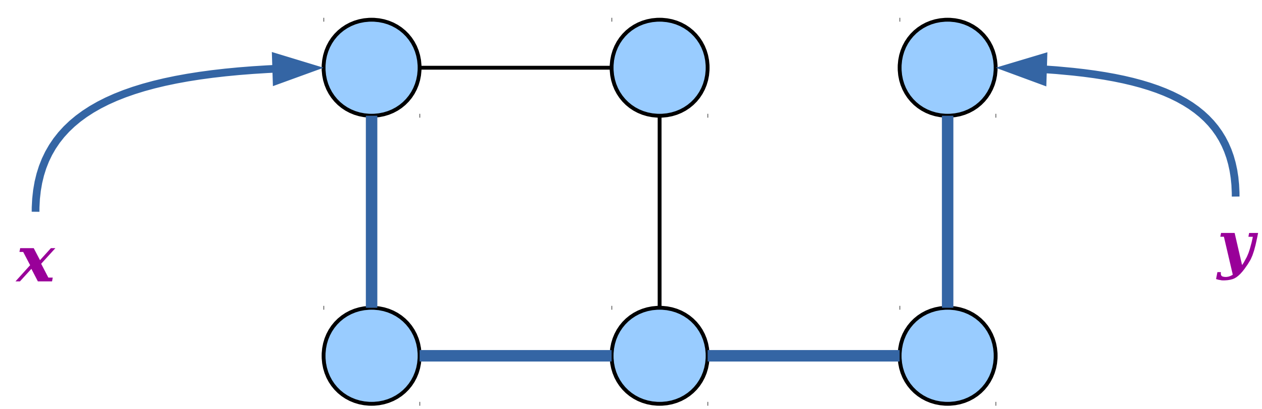

Connectivity

$$\forall x \in V.\ \forall y \in V.\ \forall z \in V.\ (Conn(x, y) \land Conn(y, z) \rightarrow Conn(x, z))$$

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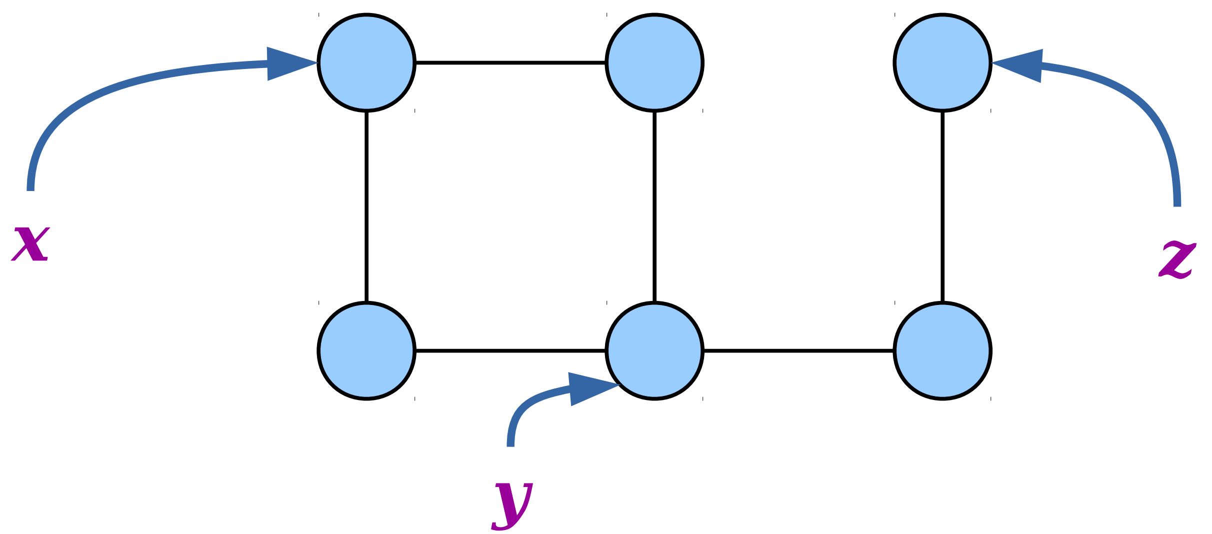

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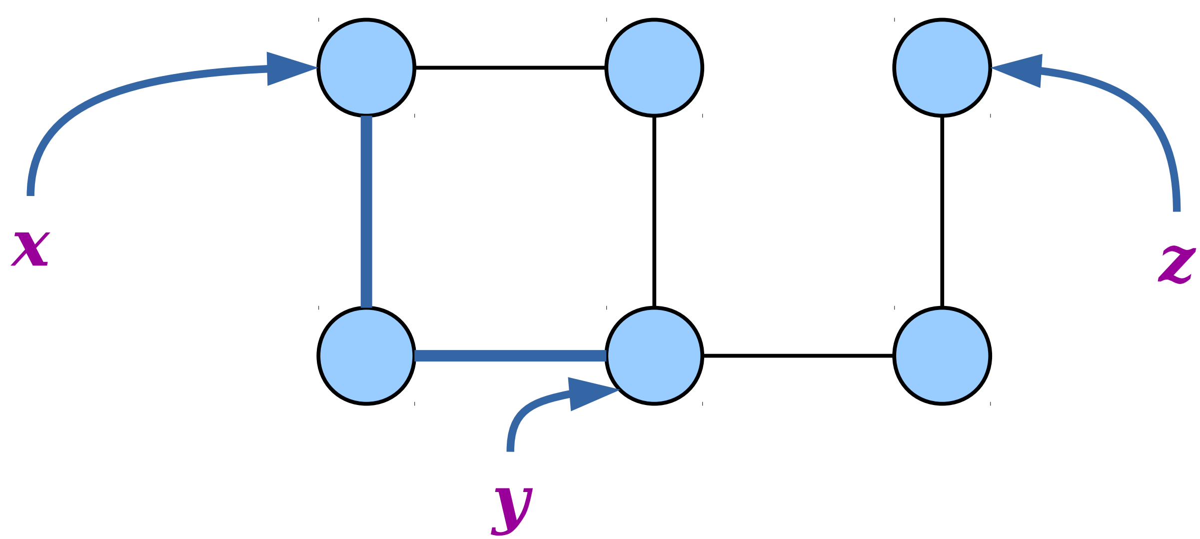

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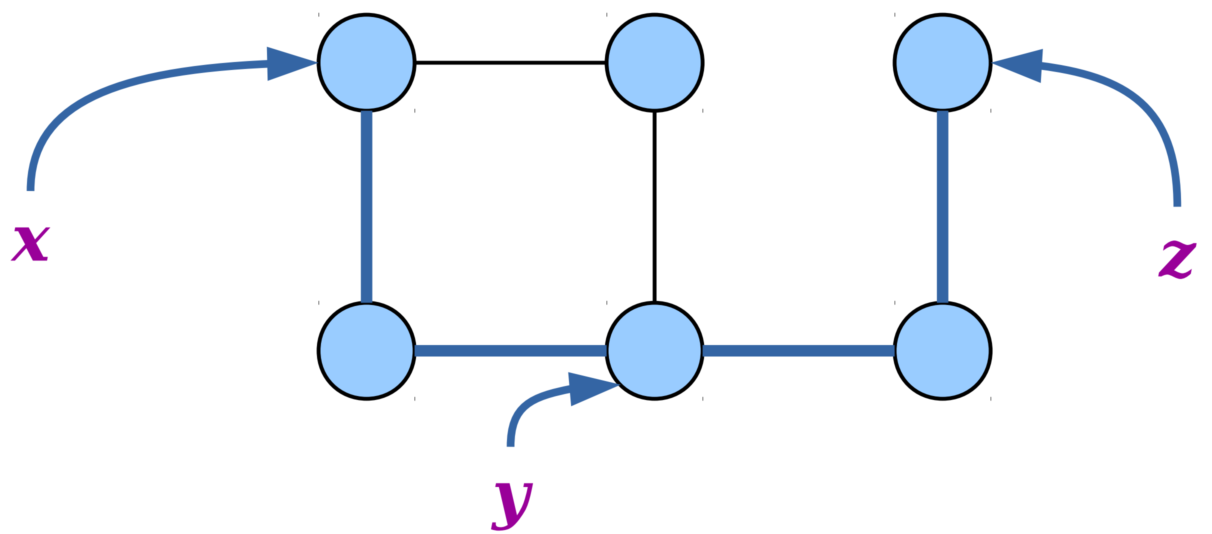


Theorem: Let $G = (V, E)$ be a graph. Then the connectivity relation over V is an equivalence relation.

Proof: Consider an arbitrary graph $G = (V, E)$. We will prove that the connectivity relation over V is reflexive, symmetric, and transitive.

To show that connectivity is reflexive, consider any $v \in V$. Then the singleton path v is a path from v to itself. Therefore, v is connected to itself, as required.

To show that connectivity is symmetric, consider any $x, y \in V$ where x is connected to y. We need to show that y is connected to x. Since x is connected to y, there is some path $x, v_1, \ldots, v_n, y$ from x to y. Then $y, v_n, \ldots, v_1, x$ is a path from y back to x, so y is connected to x.

Finally, to show that connectivity is transitive, let $x, y, z \in V$ be arbitrary nodes where x is connected to y and y is connected to z. We will prove that x is connected to z. Since x is connected to y, there is a path $x, u_1, \ldots, u_n, y$ from x to y. Since y is connected to z, there is a path $y, v_1, \ldots, v_k, z$ from y to z. Then the path $x, u_1, \ldots, u_n, y, v_1, \ldots, v_k, z$ goes from x to z. Thus x is connected to z, as required. ■

Putting Things Together

- Earlier, we defined the connected component of a node v to be

$$[v] = \{\ x \in V \mid v \text{ is connected to } x\ \}$$

- Connectivity is an equivalence relation! So what's the equivalence class of a node v with respect to connectivity?

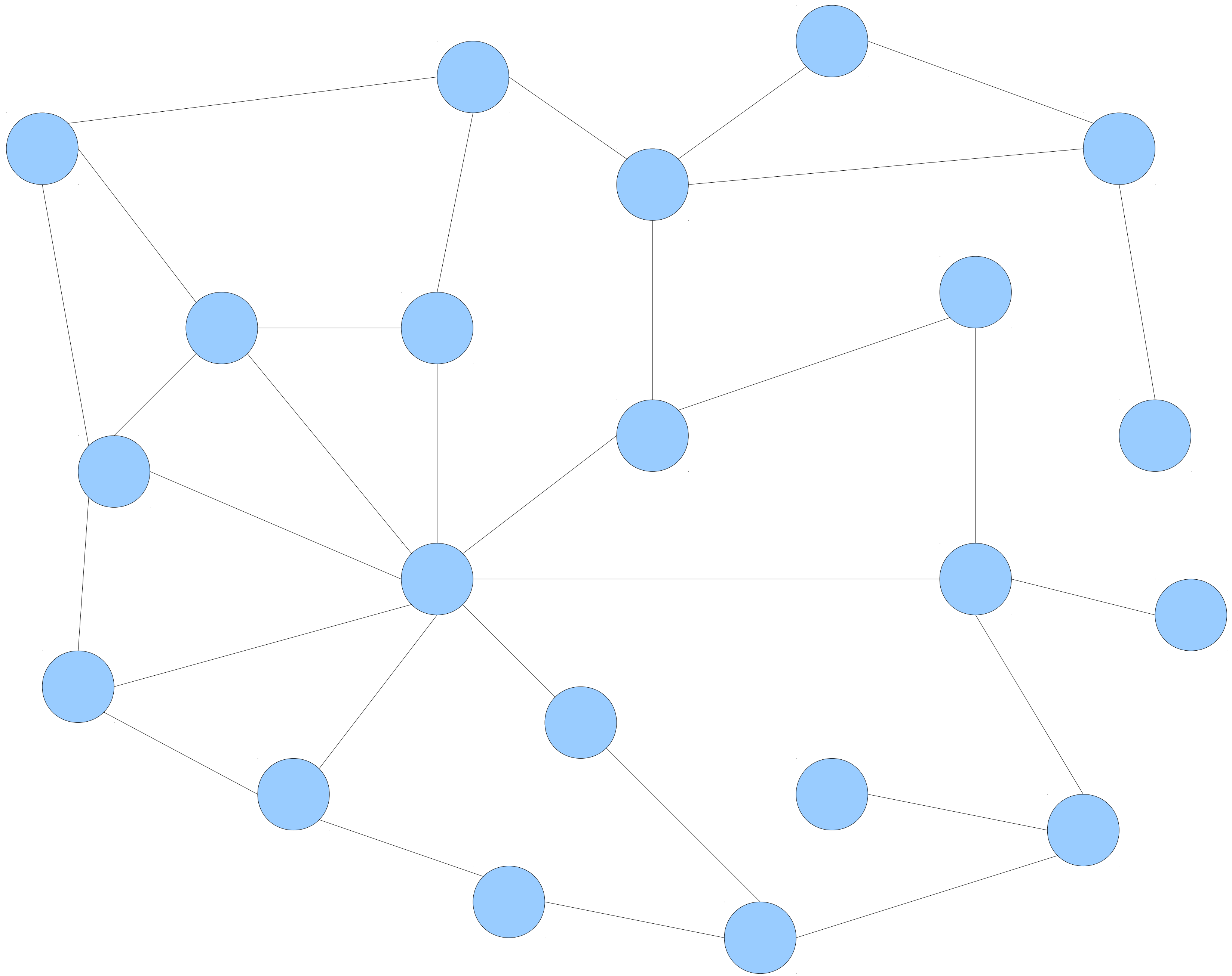
$$[v] = \{\ x \in V \mid v \text{ is connected to } x\ \}$$

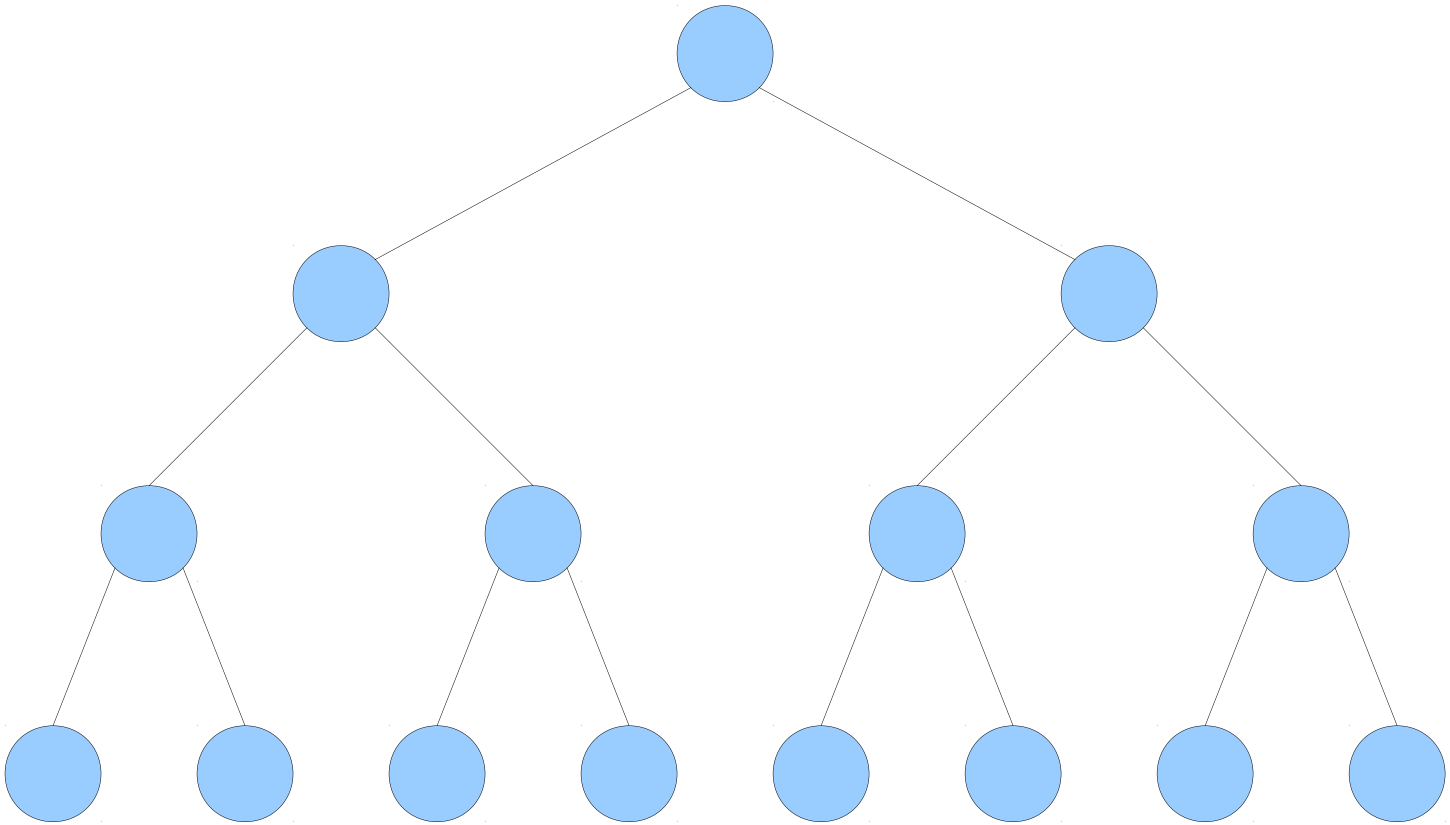
- ***Connected components are equivalence classes of the connectivity relation!***

Theorem: If $G = (V, E)$ is a graph, then every node in G belongs to exactly one connected component of G.

Proof: Let $G = (V, E)$ be an arbitrary graph and let $v \in V$ be any node in G. The connected components of G are just the equivalence classes of the connectivity relation in G. The Fundamental Theorem of Equivalence Relations guarantees that v belongs to exactly one equivalence class of the connectivity relation. Therefore, v belongs to exactly one connected component in G. ■

Planar Graphs





A graph is called a ***planar graph*** if there is some way to draw it in a 2D plane without any of the edges crossing.

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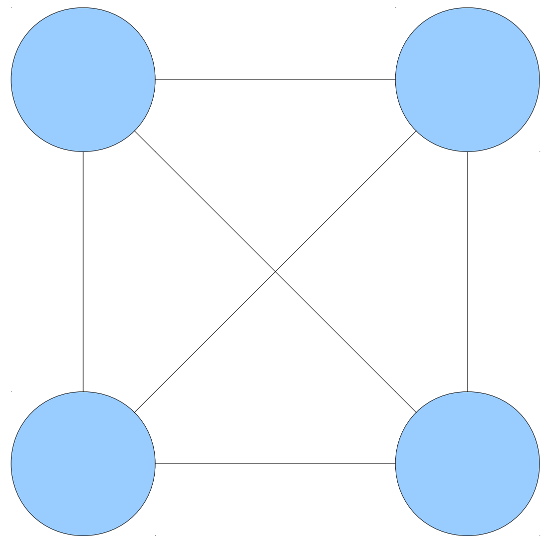
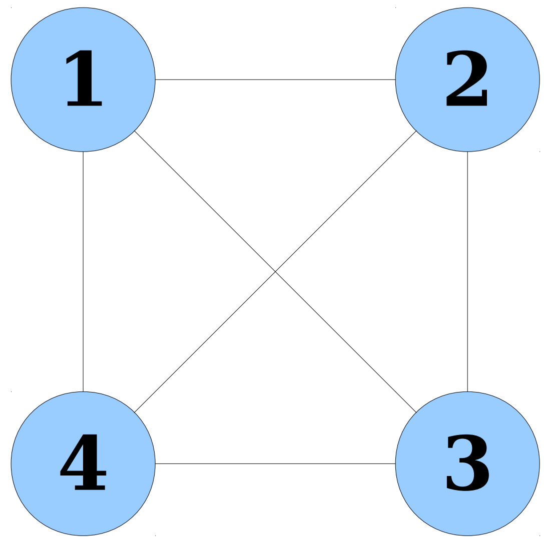
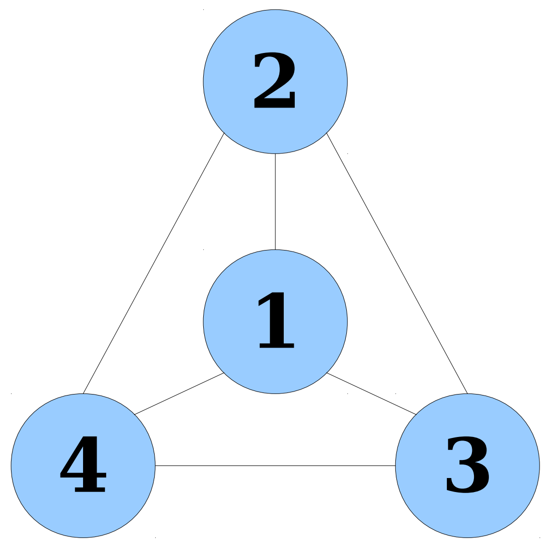


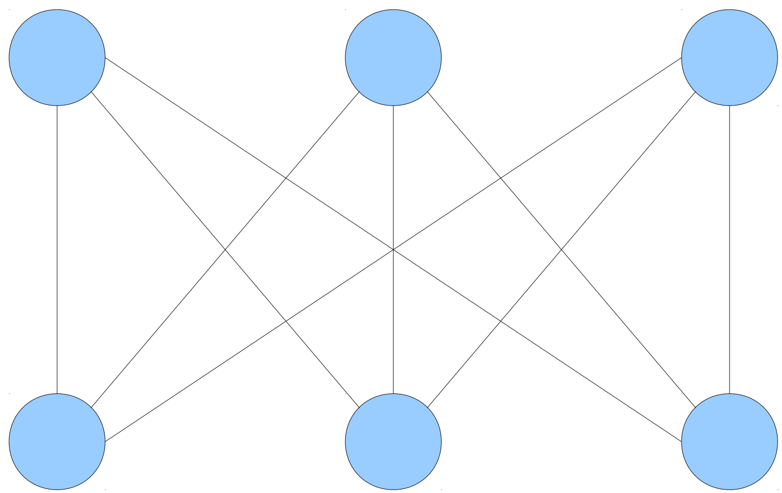
Is this graph planar?

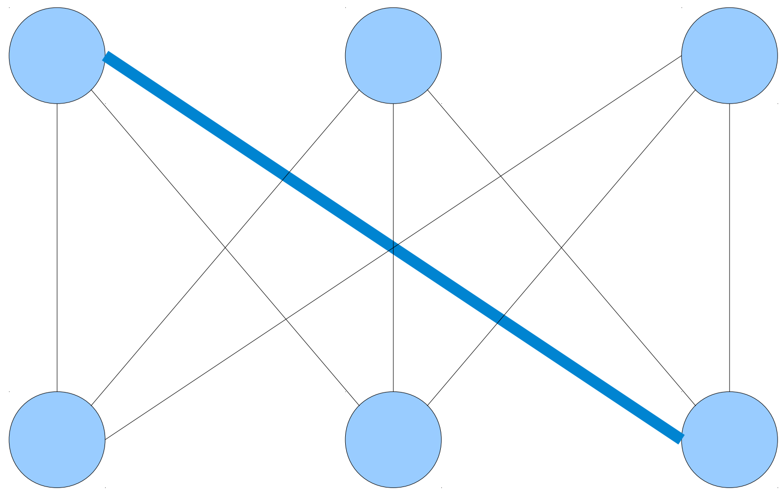
Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **Y** or **N**.

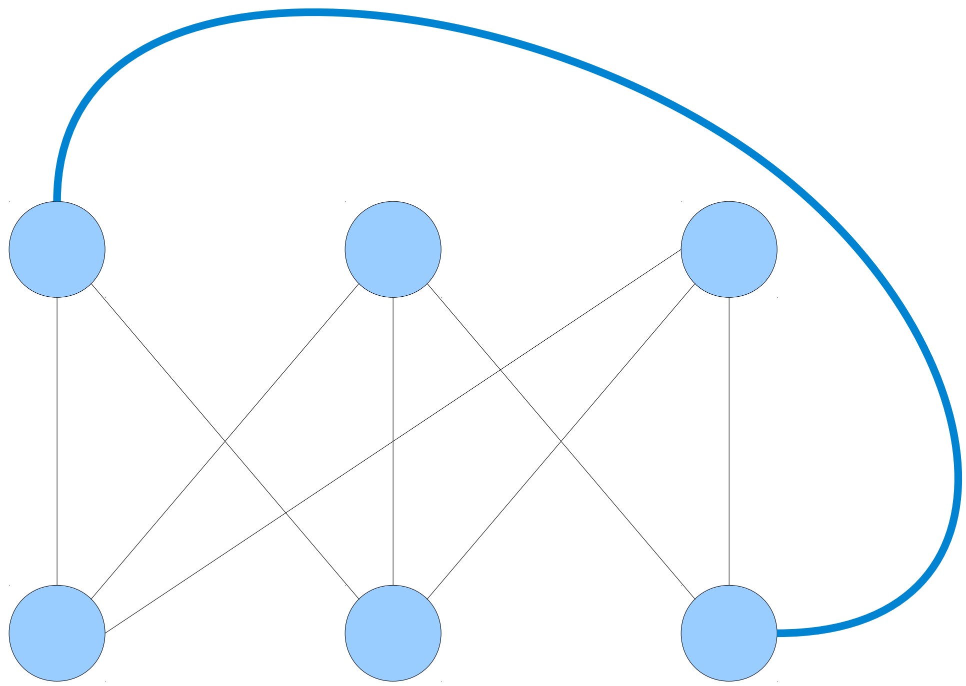


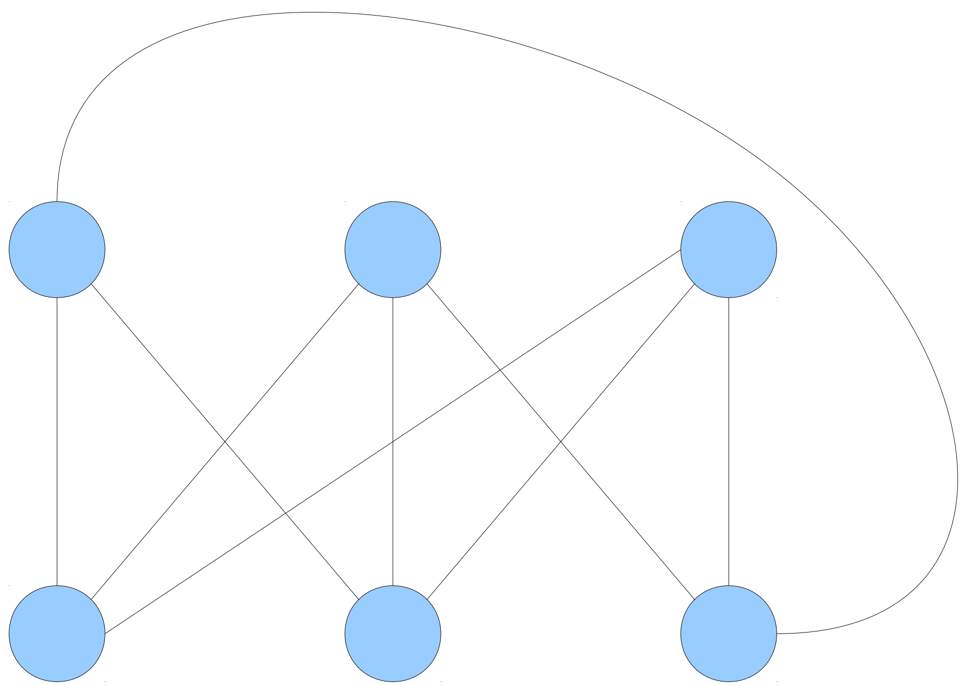
1
2
4
3

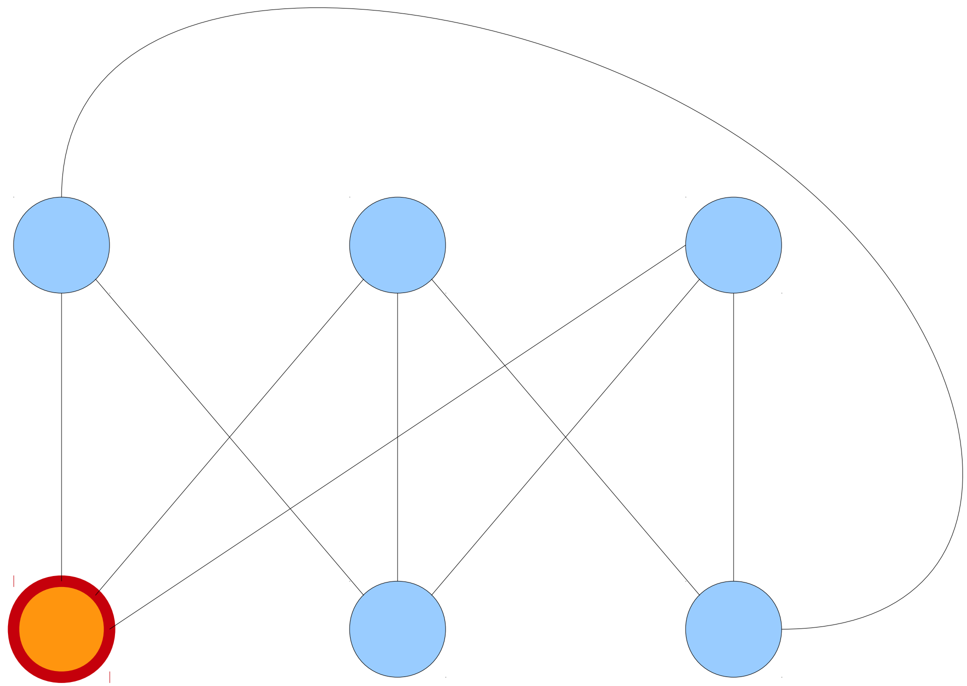

2
1
4
3

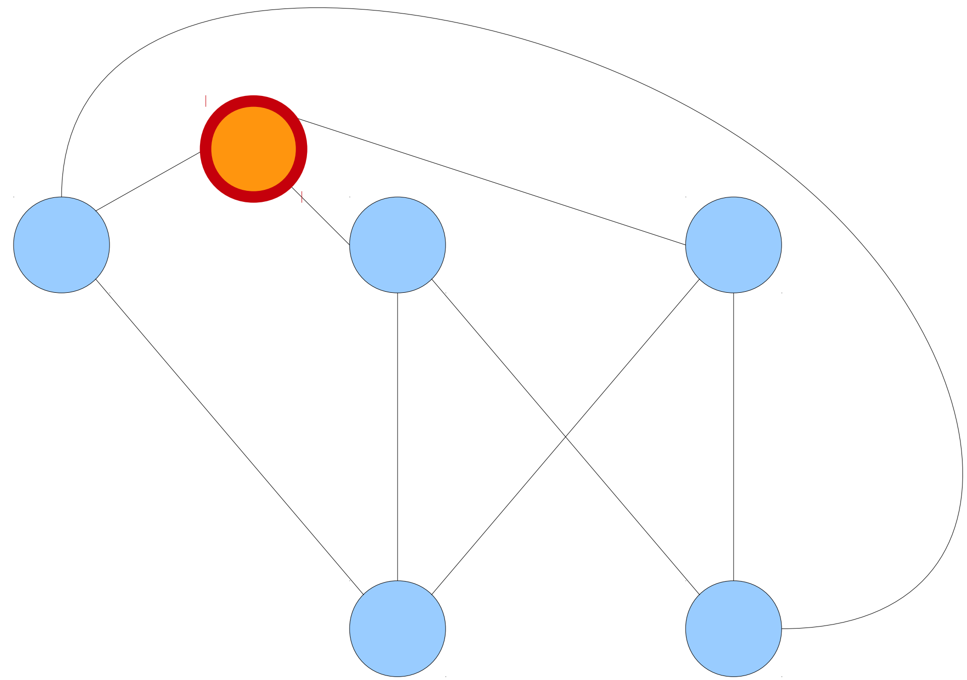


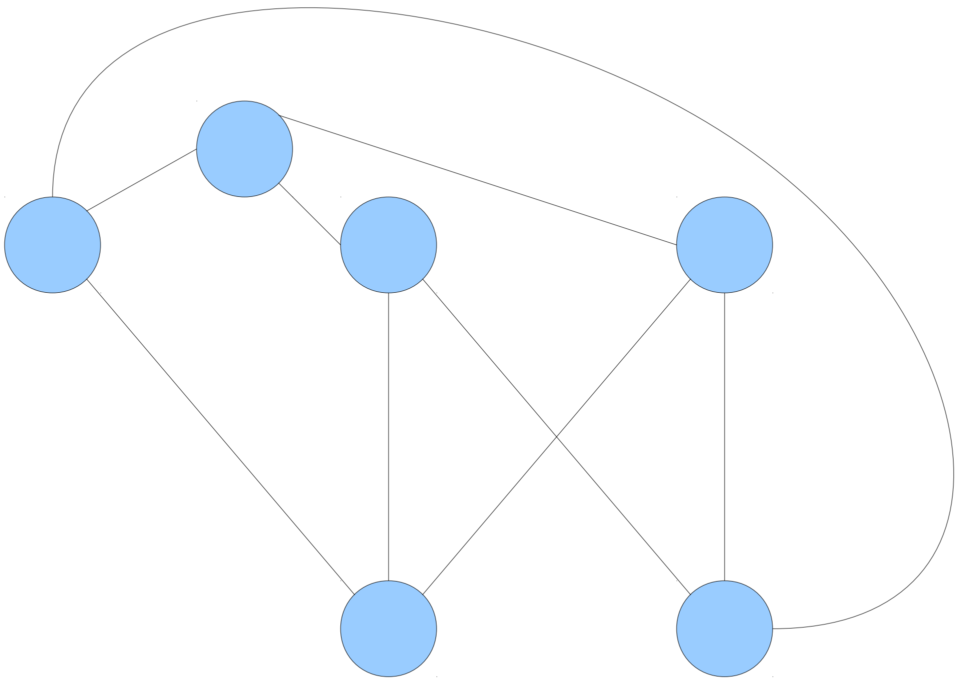


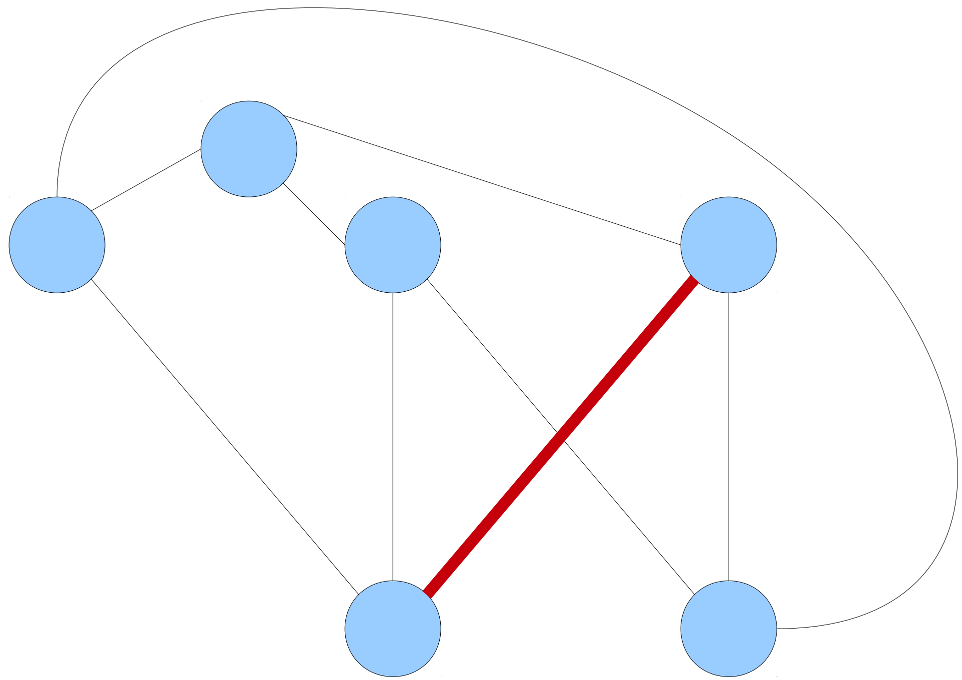


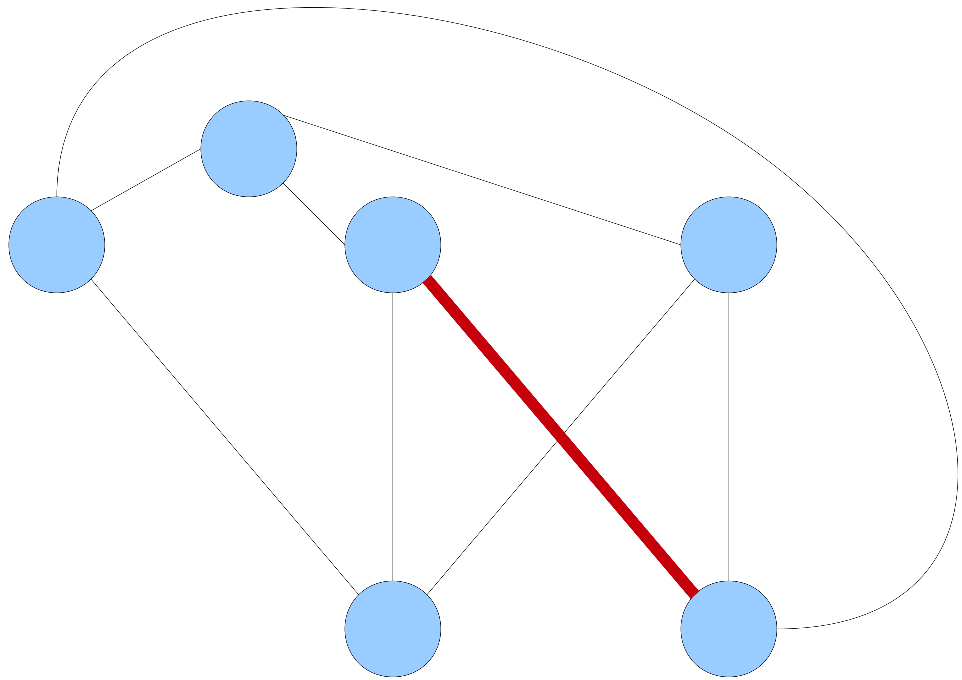


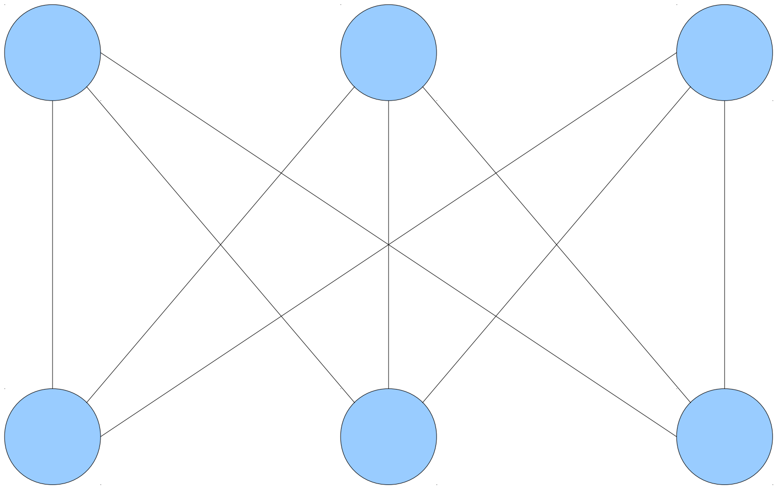








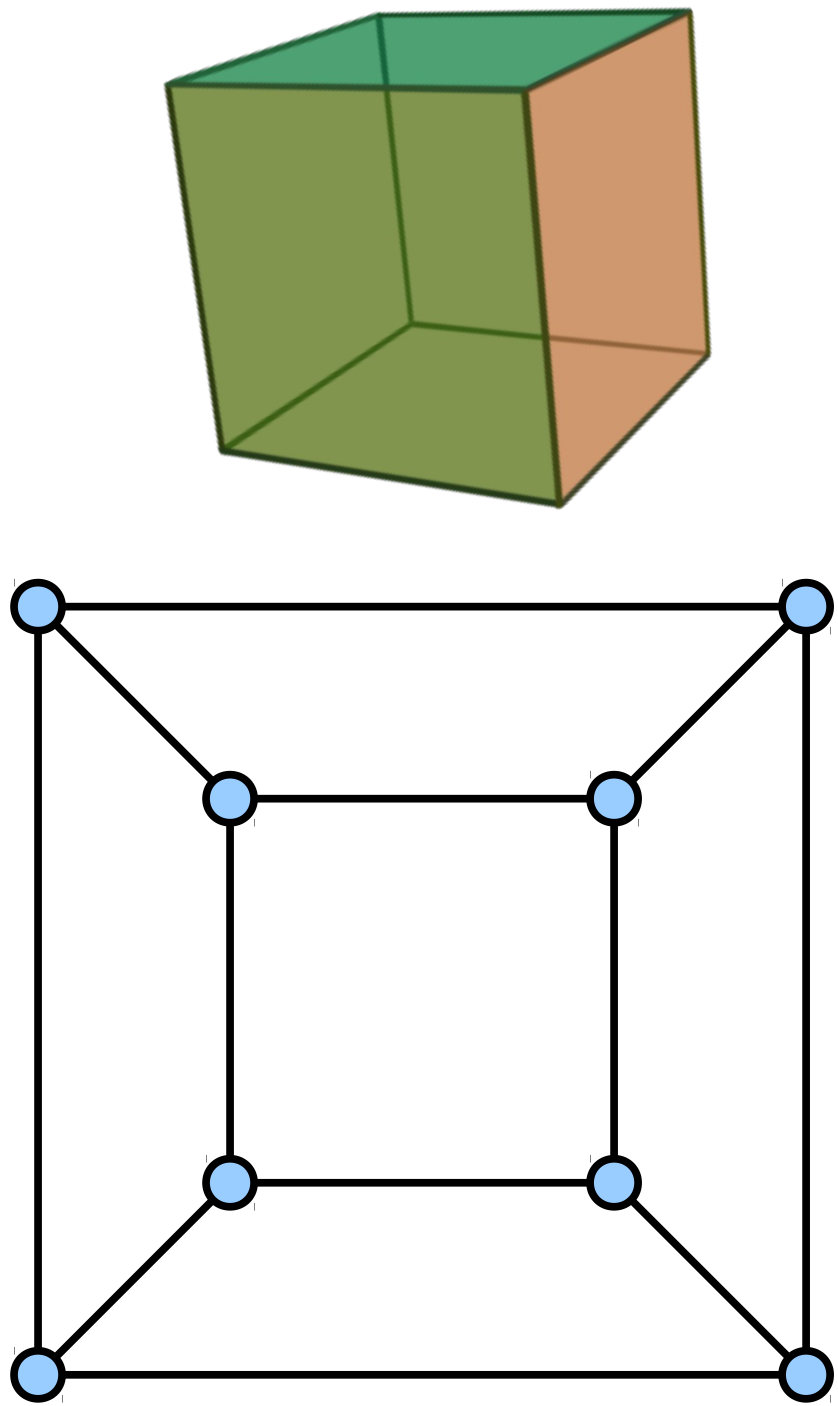


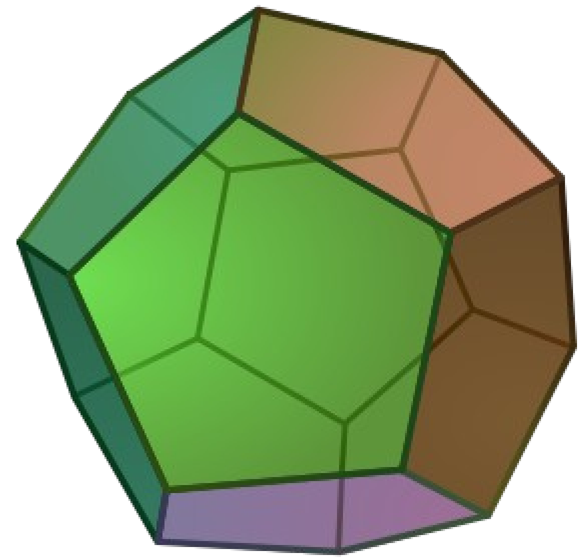


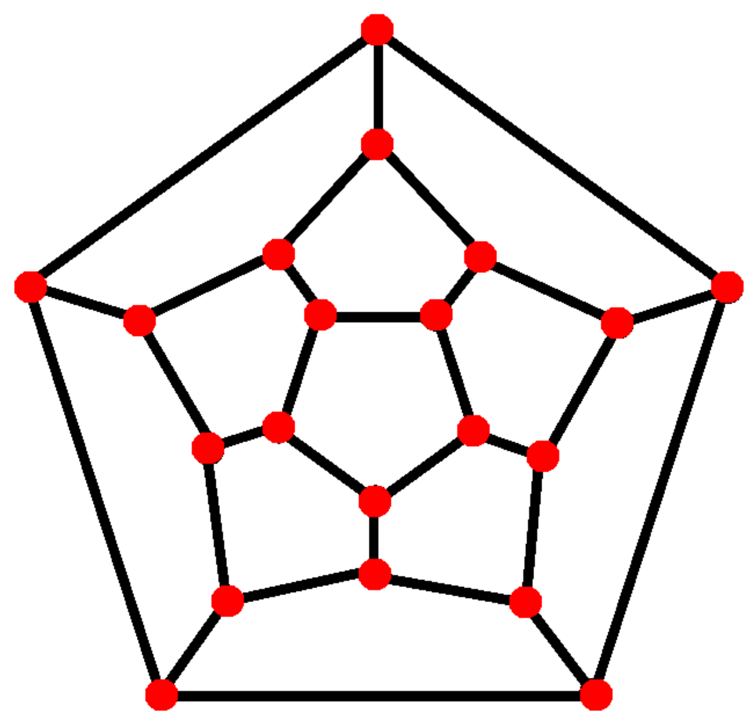
This graph is called the ***utility graph***. There is no way to draw it in the plane without edges crossing. Check out ***this video*** for an explanation!

A fun game by a former CS103er:

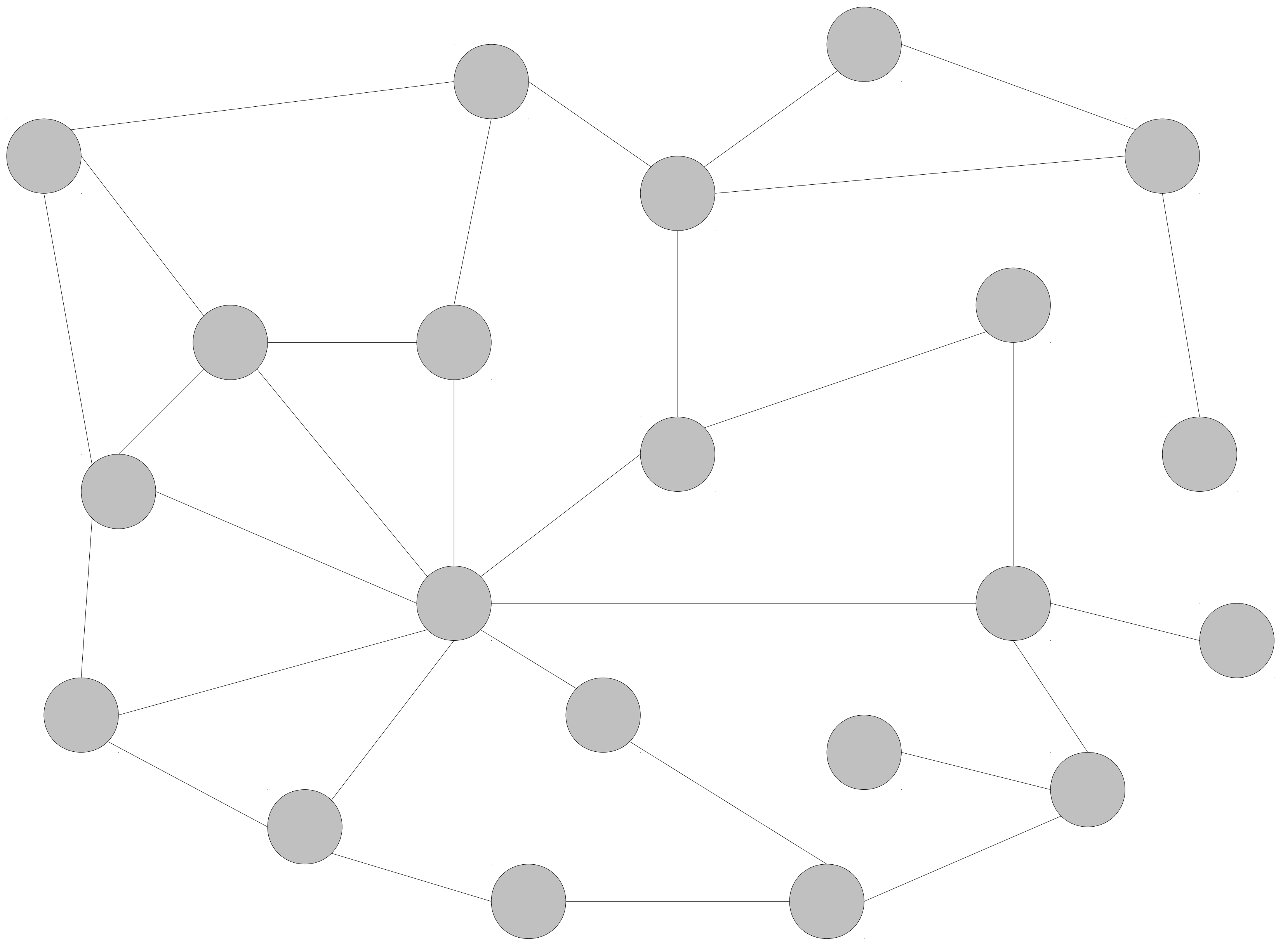
http://www.nkhem.com/planarity-knot/







Graph Coloring



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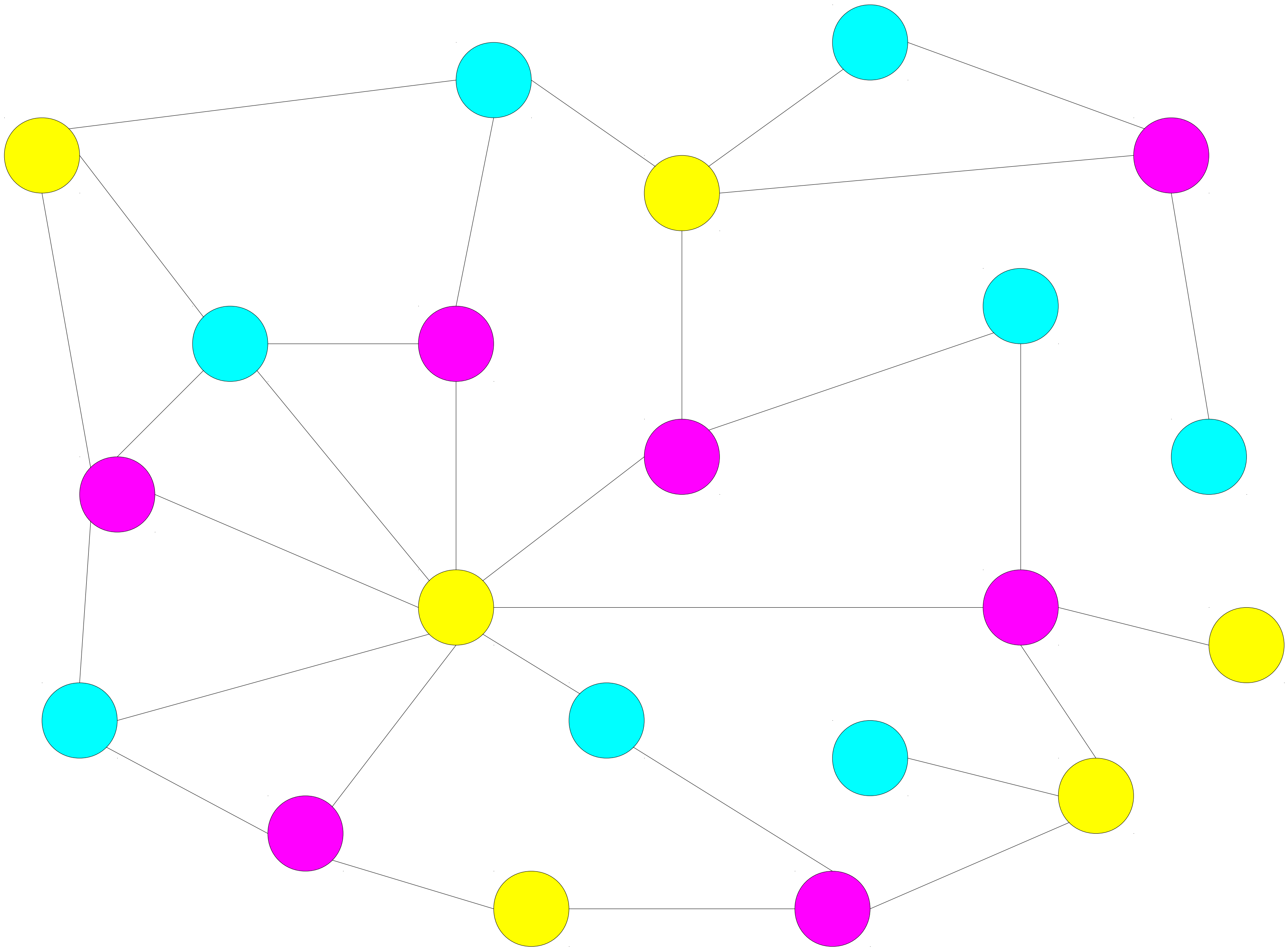

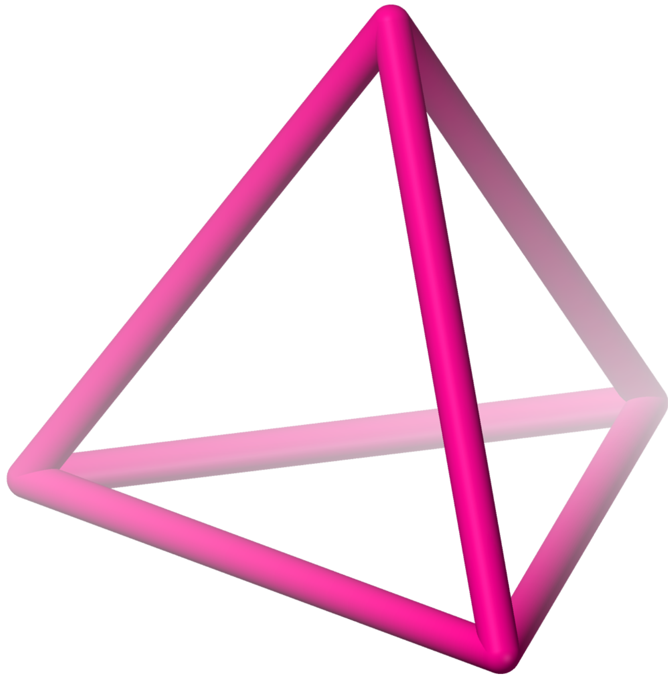

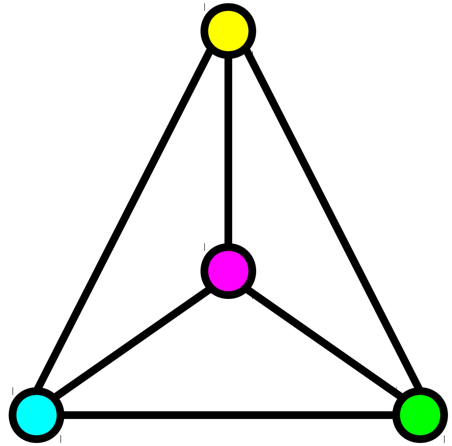

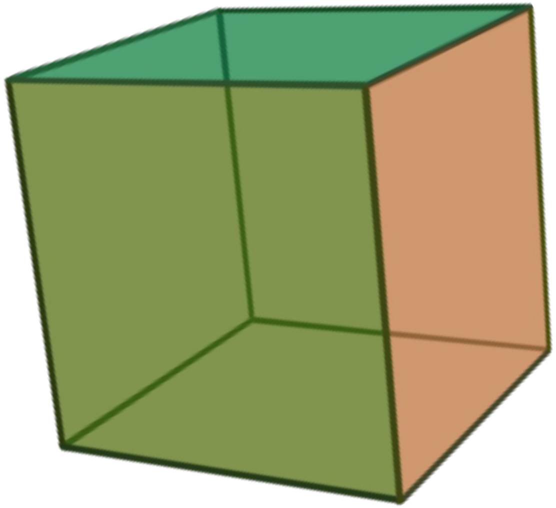

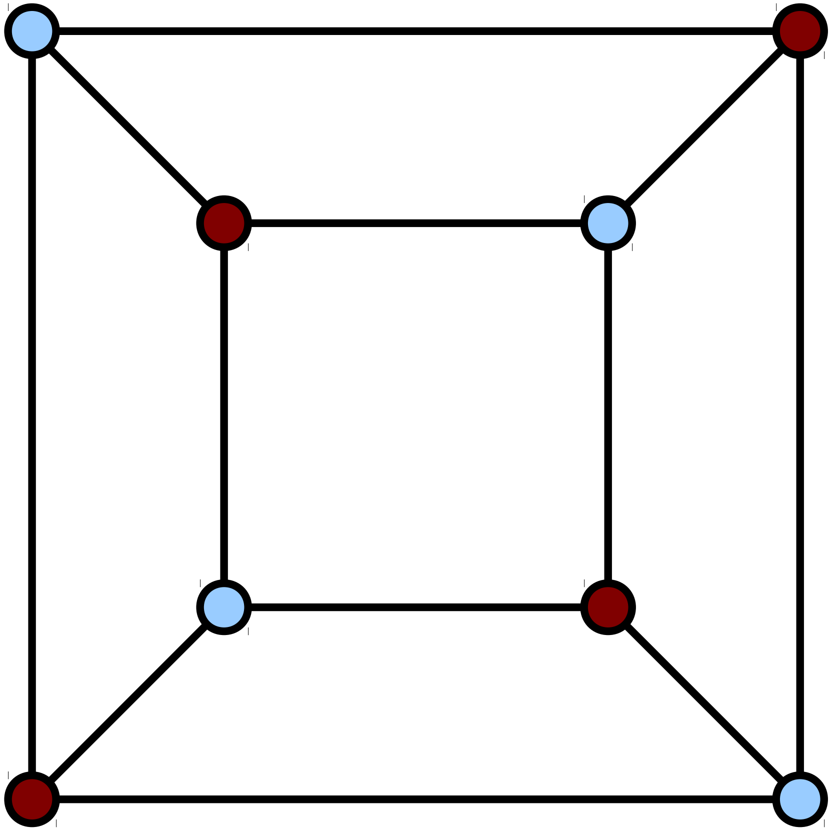

Graph Coloring

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$$f : V \to \{1, 2, \ldots, k\}$$

such that

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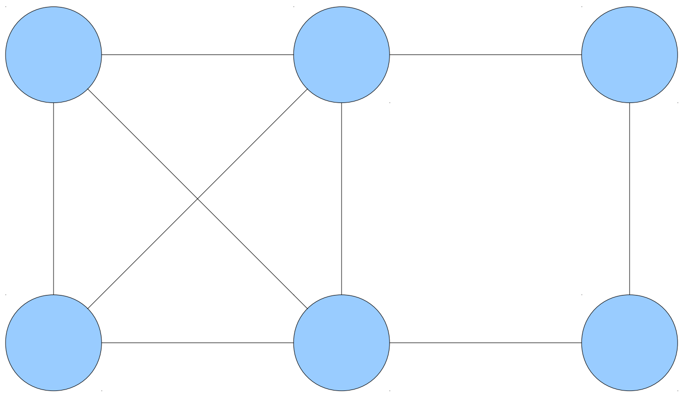

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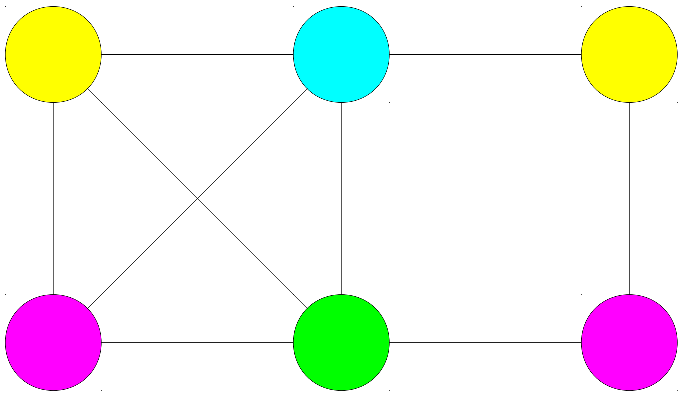

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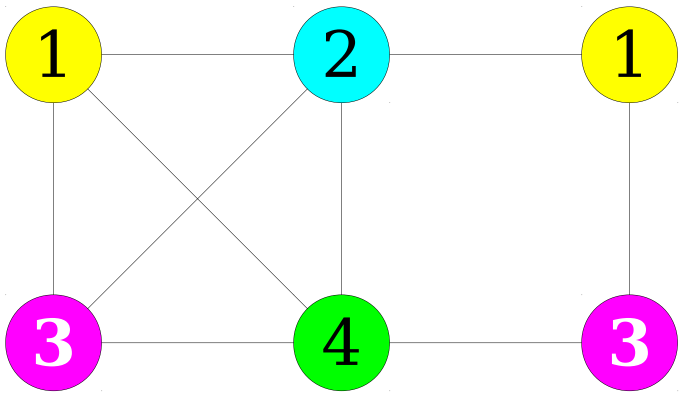

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Although this is the formal definition of a k-vertex-coloring, you rarely see it used in proofs. It's more common to just talk about assigning colors to nodes. However, this definition is super useful if you want to write programs to reason about graph colorings!

Graph Coloring

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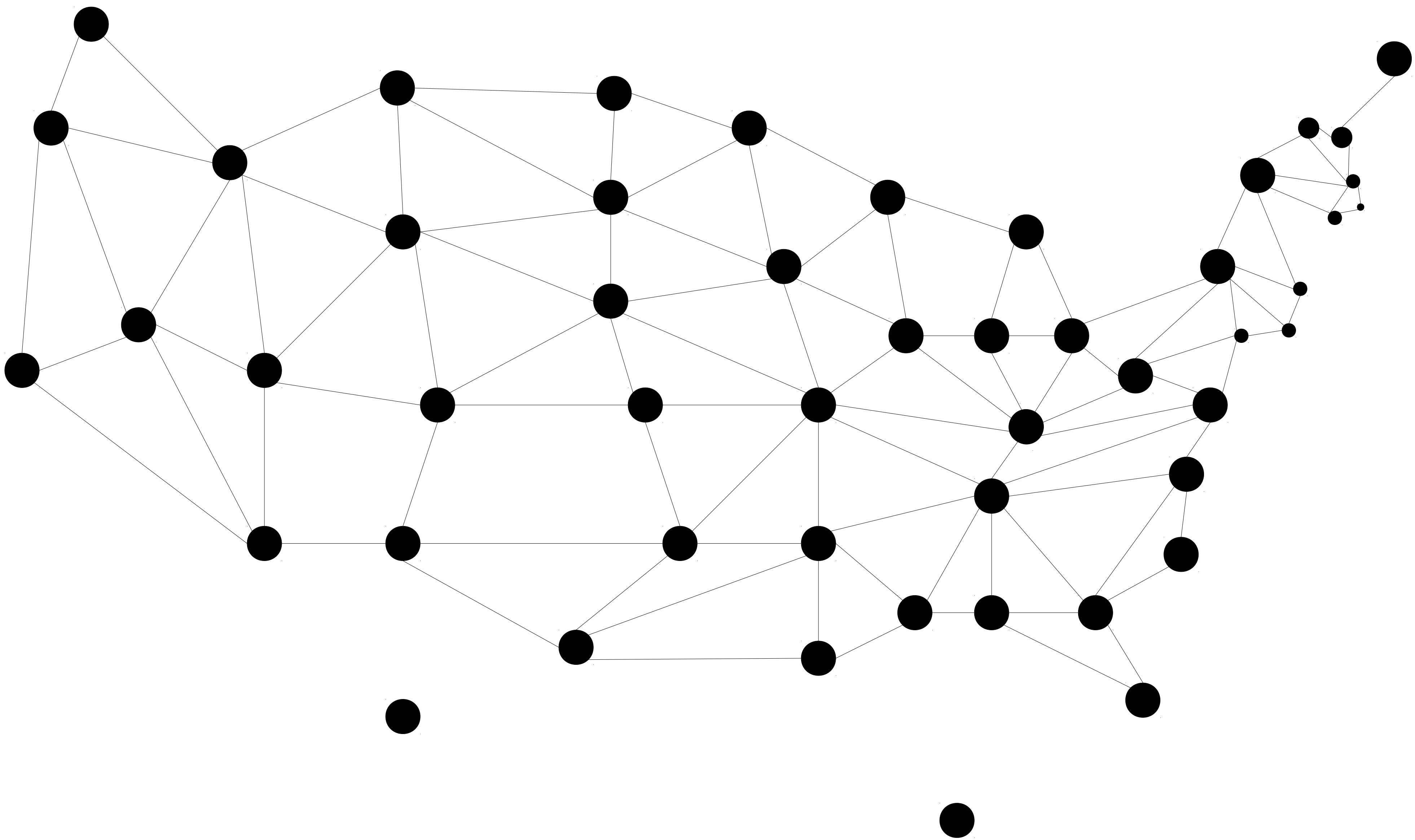
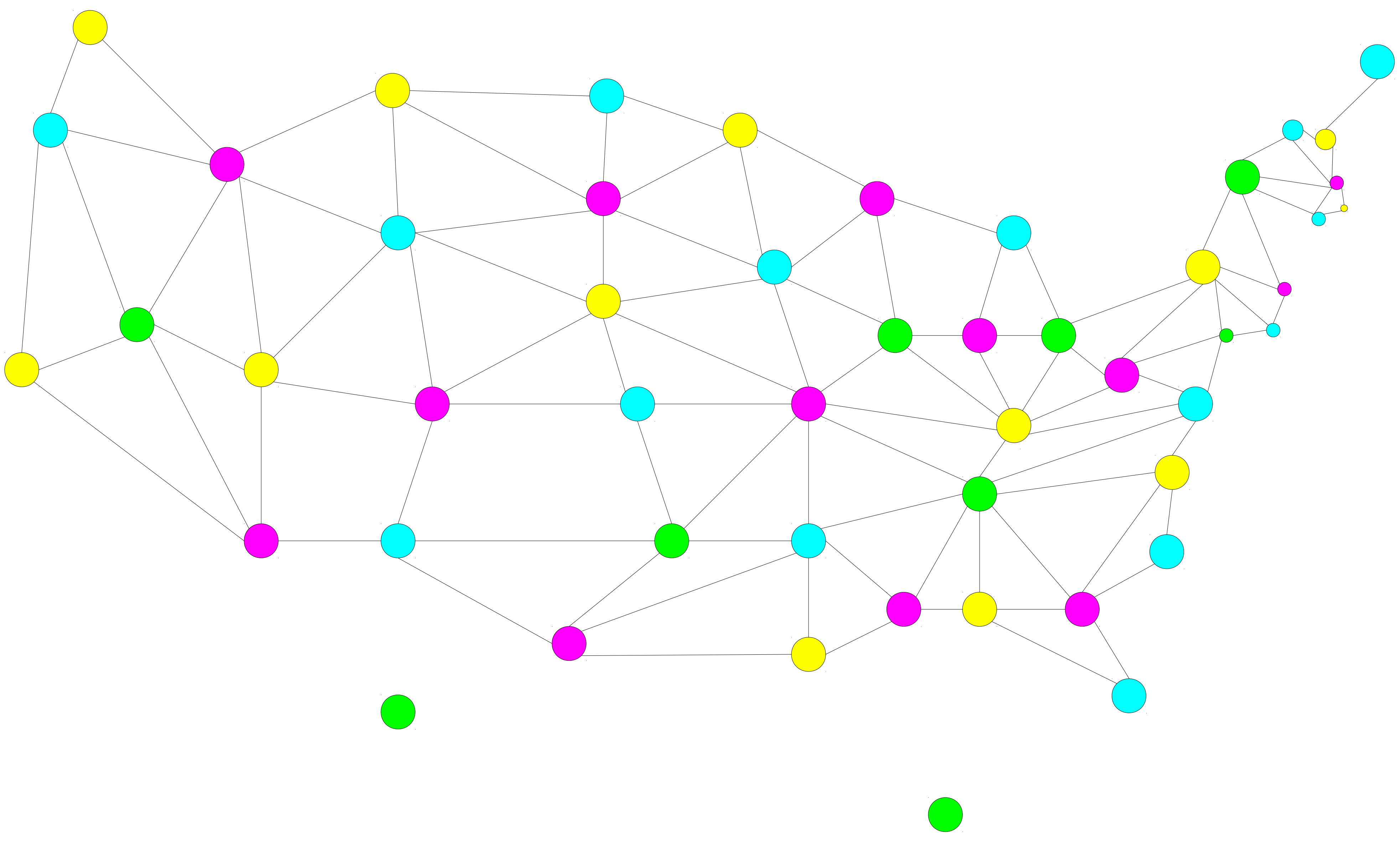
- A graph G is ***k-colorable*** if a k-vertex coloring of G exists.
- The smallest k for which G is k-colorable is its ***chromatic number***.
  - The chromatic number of a graph G is denoted $\chi(G)$, from the Greek *χρώμα*, meaning “color.”

Graph Coloring



Graph Coloring



Theorem (Four-Color Theorem): Every planar graph is 4-colorable.

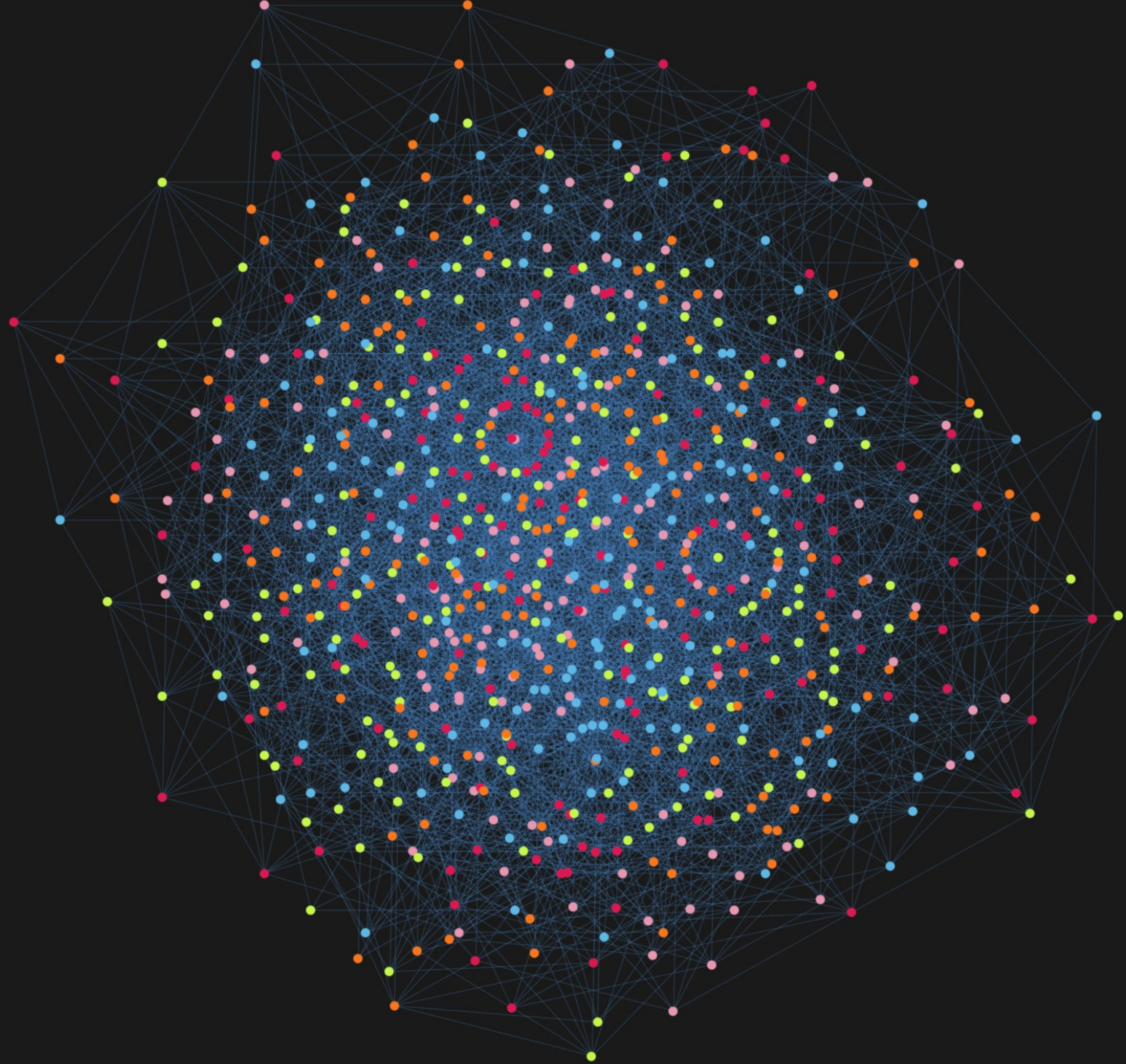
- **1850s:** Four-Color Conjecture posed.
- **1879:** Kempe proves the Four-Color Theorem.
- **1890:** Heawood finds a flaw in Kempe's proof.
- **1976:** Appel and Haken design a computer program that proves the Four-Color Theorem. The program checked 1,936 specific cases that are “minimal counterexamples;” any counterexample to the theorem must contain one of the 1,936 specific cases.
- **1980s:** Doubts rise about the validity of the proof due to errors in the software.
- **1989:** Appel and Haken revise their proof and show it is indeed correct. They publish a book including a 400-page appendix of all the cases to check.
- **1996:** Roberts, Sanders, Seymour, and Thomas reduce the number of cases to check down to 633.
- **2005:** Werner and Gonthier repeat the proof using an established automatic theorem prover (Coq), improving confidence in the truth of the theorem.

Then less than one year ago (!)

- An amateur mathematician *disproved* this conjecture:

  For all graphs where the edges are the same length (length 1 unit), and the points are all on a plane (but the edges can cross), the graph is 4-colorable.

- Q: How do you *disprove* a universal statement? In other words, how do you prove the negation, which is an existential statement?
- A: You demonstrate the thing. Here's the thing:



Next Time

- ***The Pigeonhole Principle***
  - A simple, powerful, versatile theorem.
- ***Graph Theory Party Tricks***
  - Applying math to graphs of people!