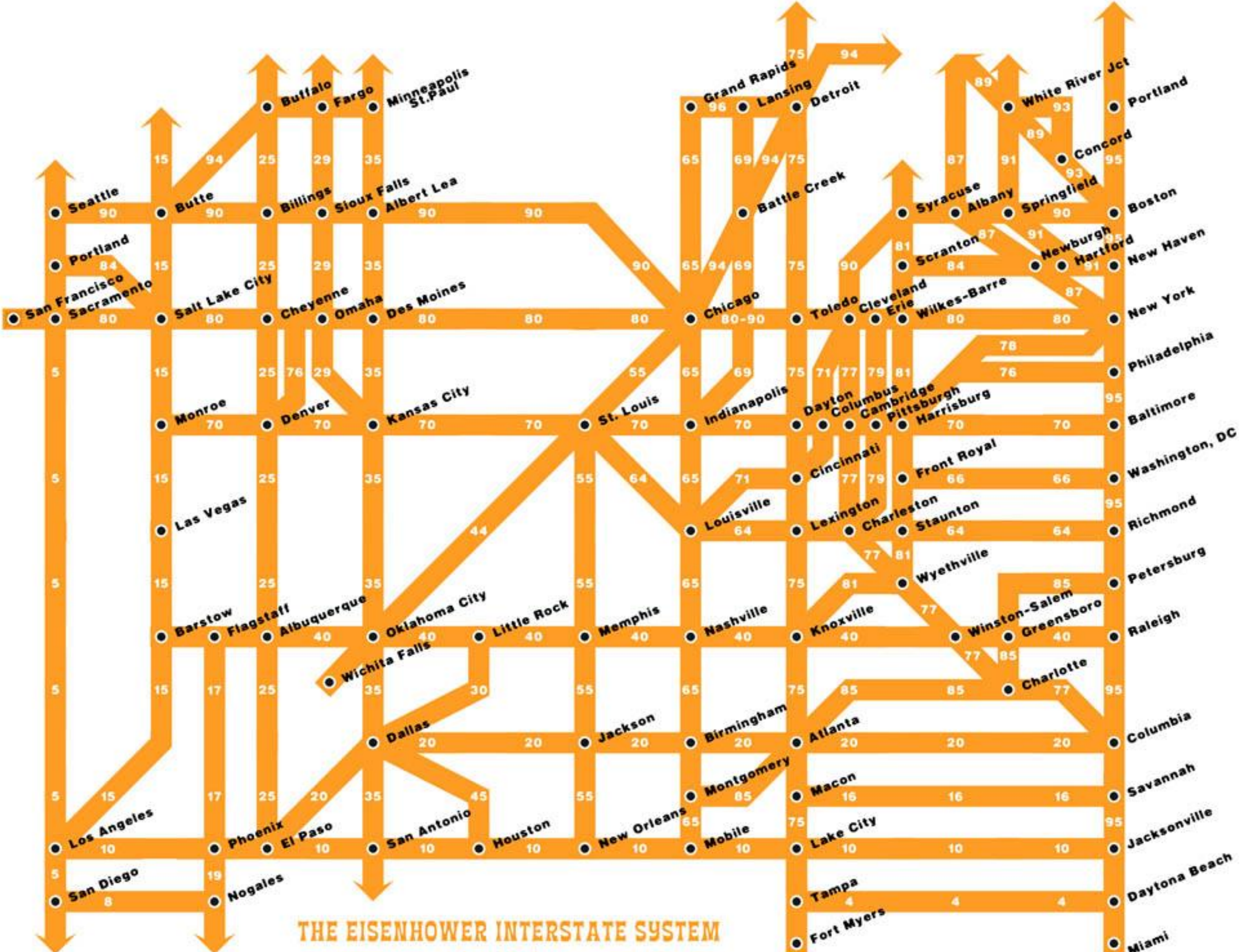
Graph Theory

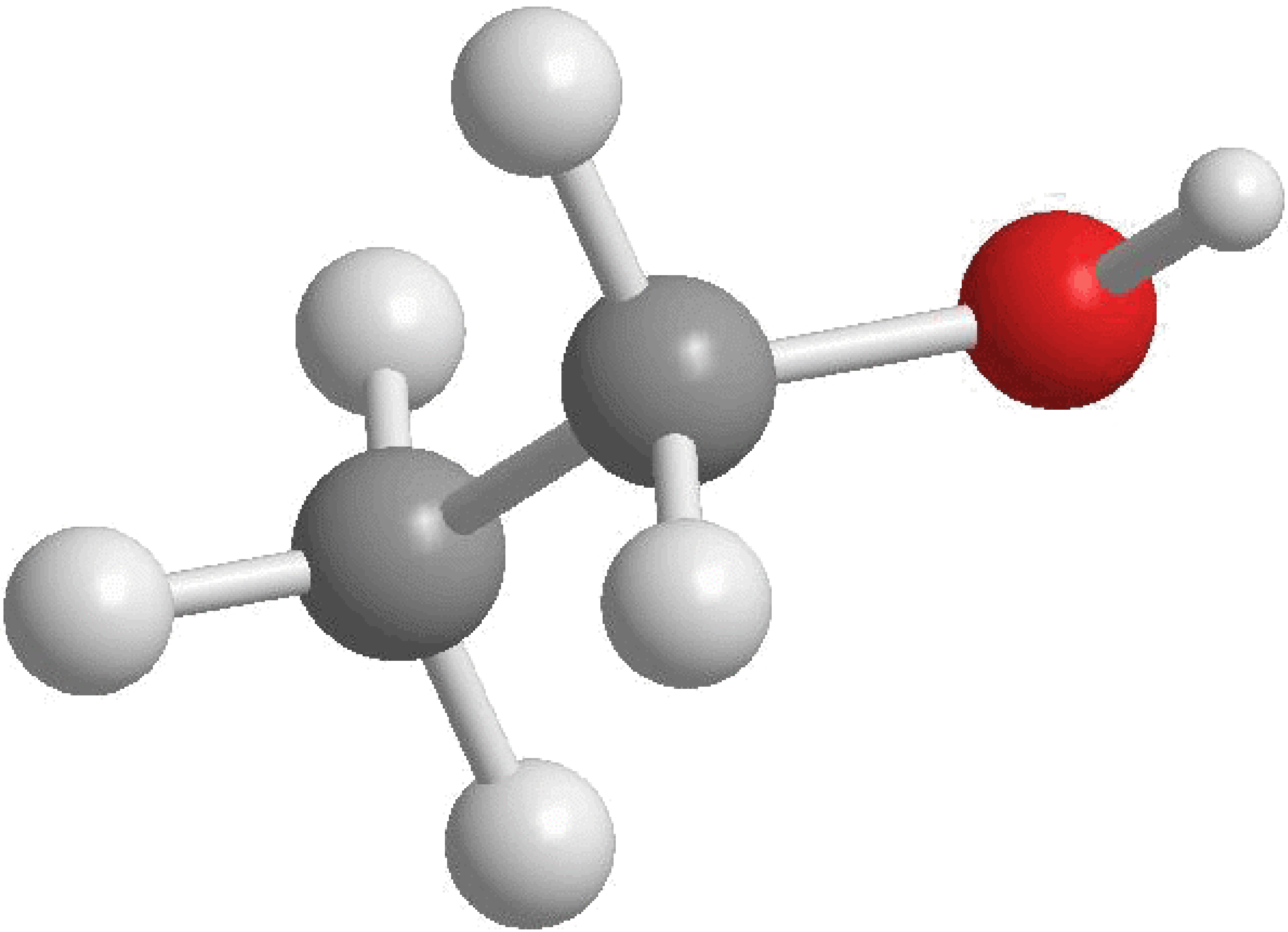


THE EISENHOWER INTERSTATE SYSTEM
(simplified)

CHRIS YATES 2007

http://strangemaps.files.wordpress.com/2007/02/fullinterstatemap-web.jpg

Chemical Bonds



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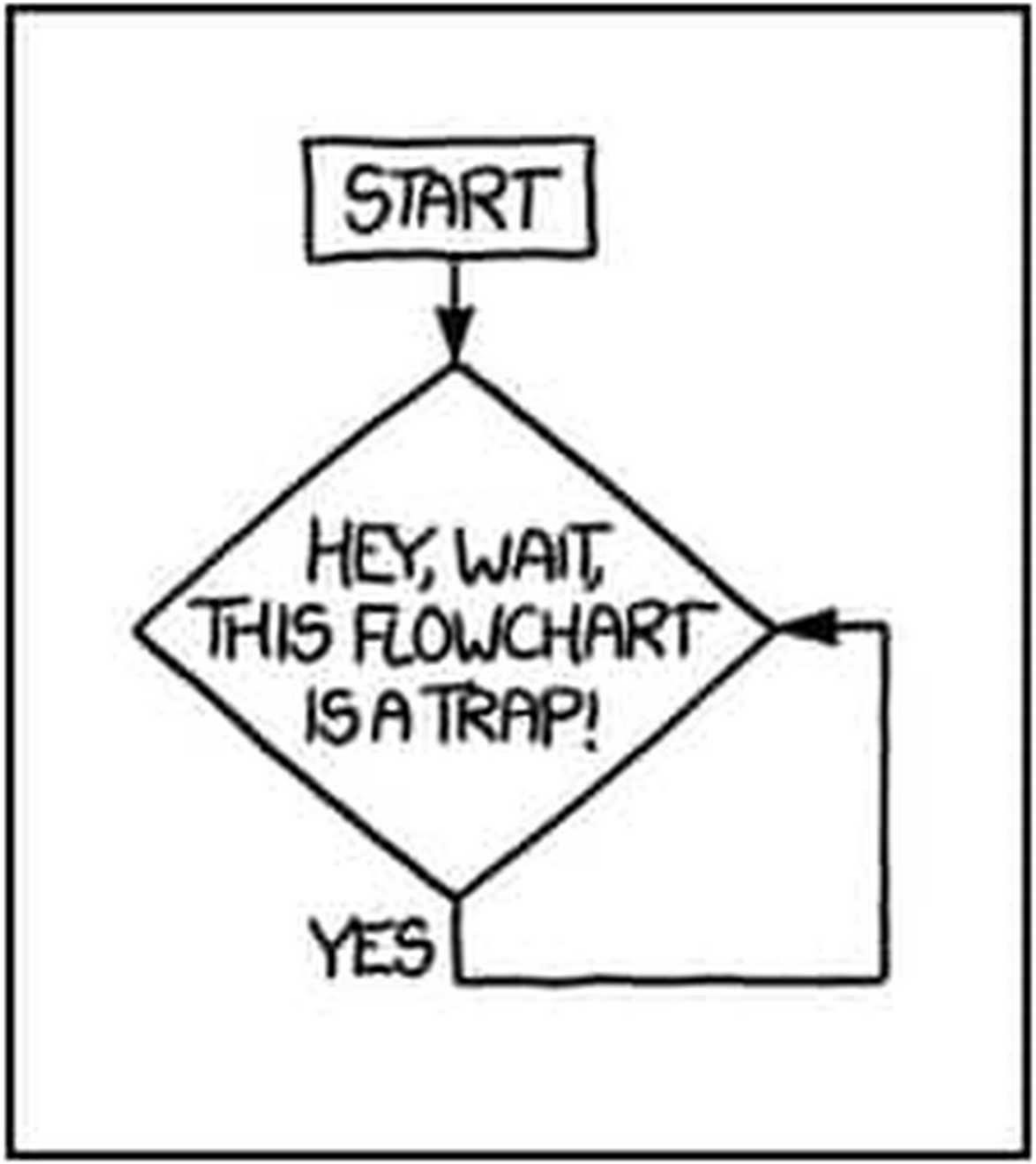
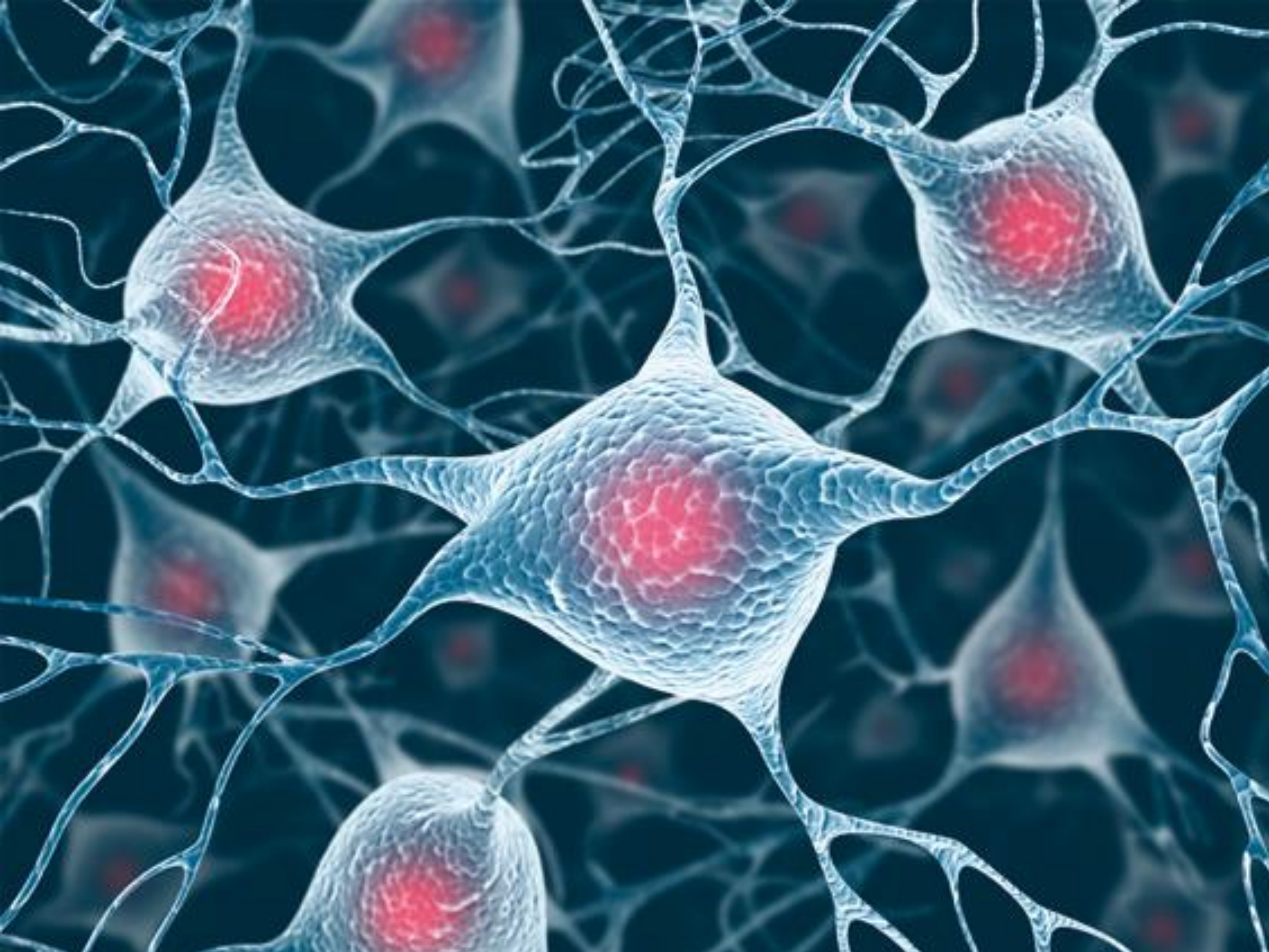

START
HEY, WAIT,
THIS FLOWCHART
IS A TRAP!
YES



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What's in Common

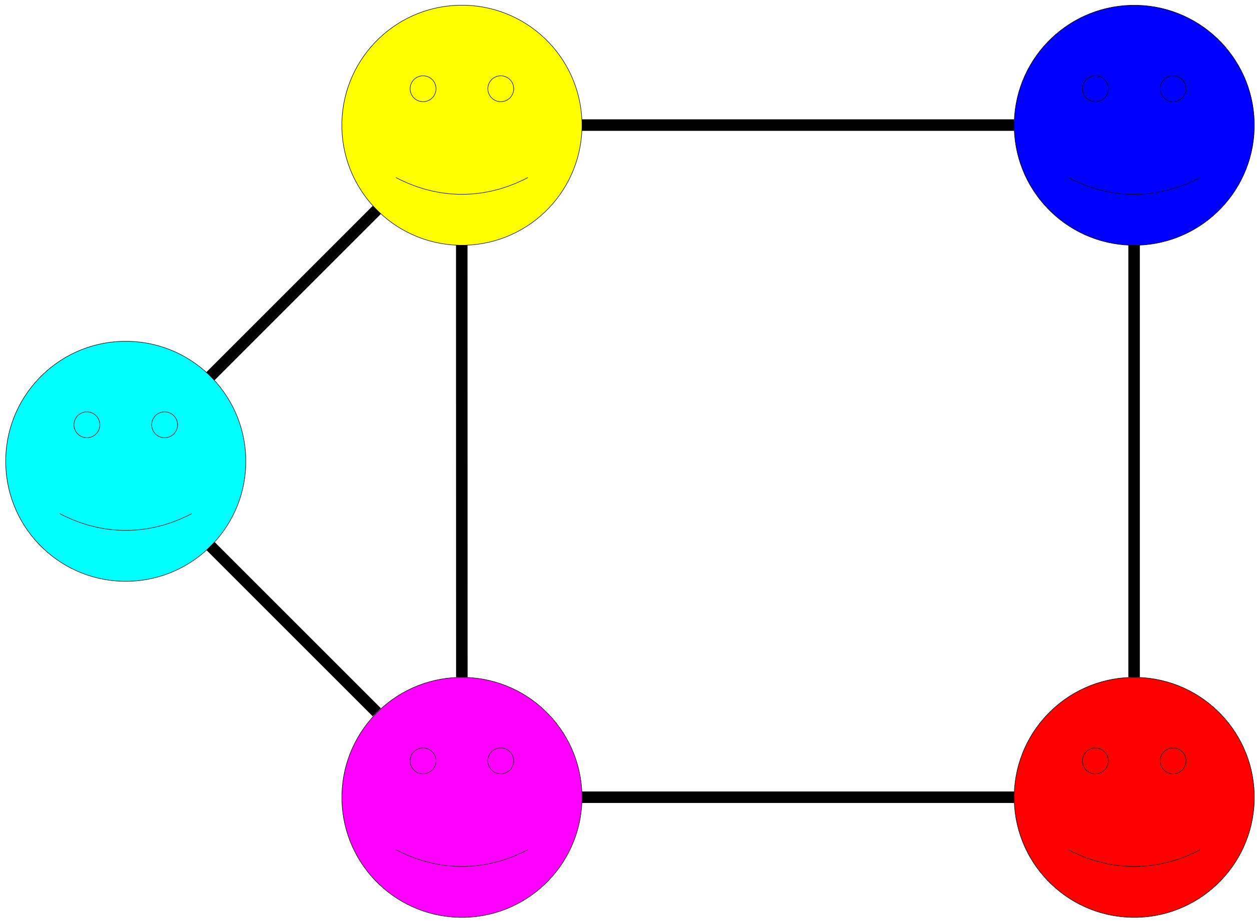
Each of these structures consists of

a collection of objects and

links between those objects.

Goal: find a general framework for describing these objects and their properties.

A ***graph*** is a mathematical structure for representing relationships.



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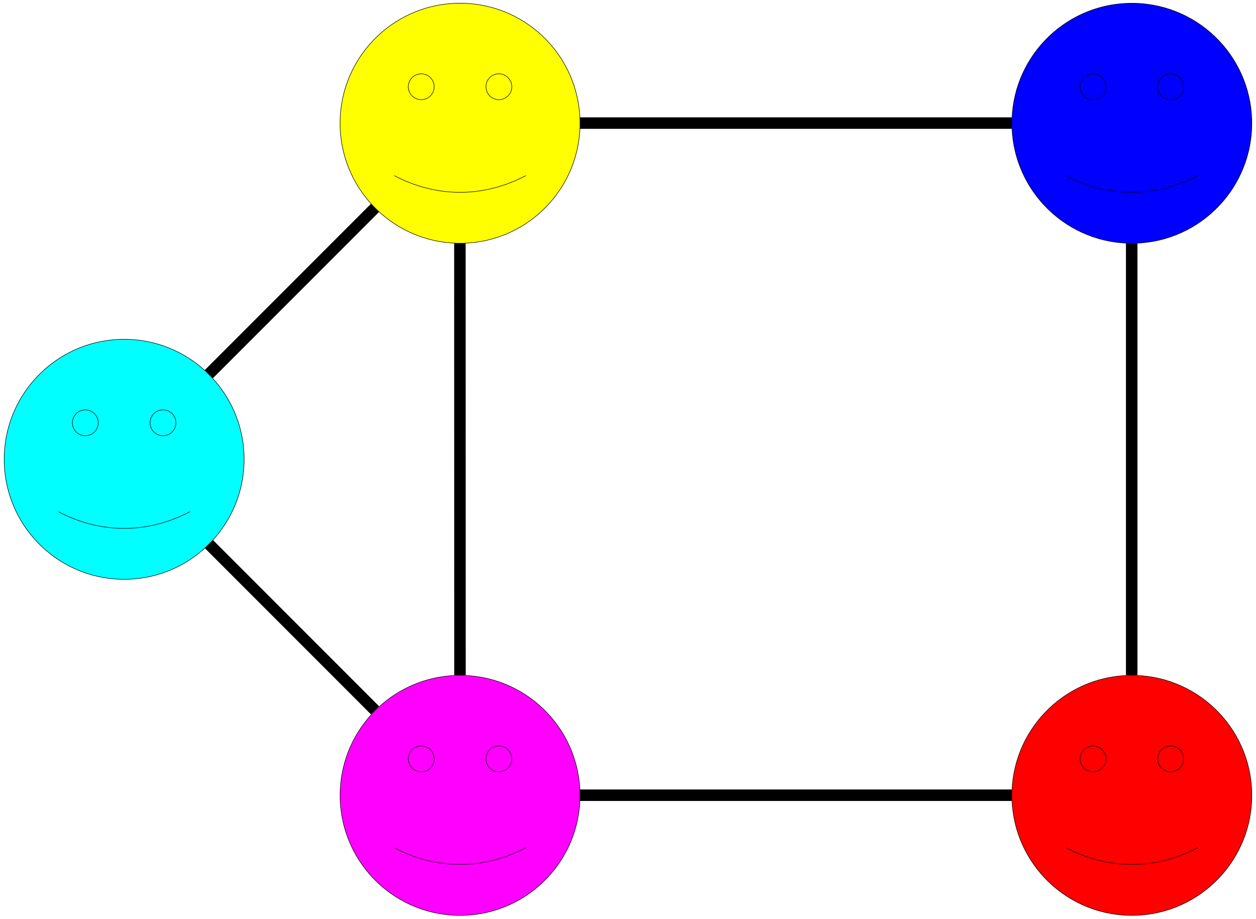

A graph consists of a set of ***nodes*** (or ***vertices***)
connected by ***edges*** (or ***arcs***)

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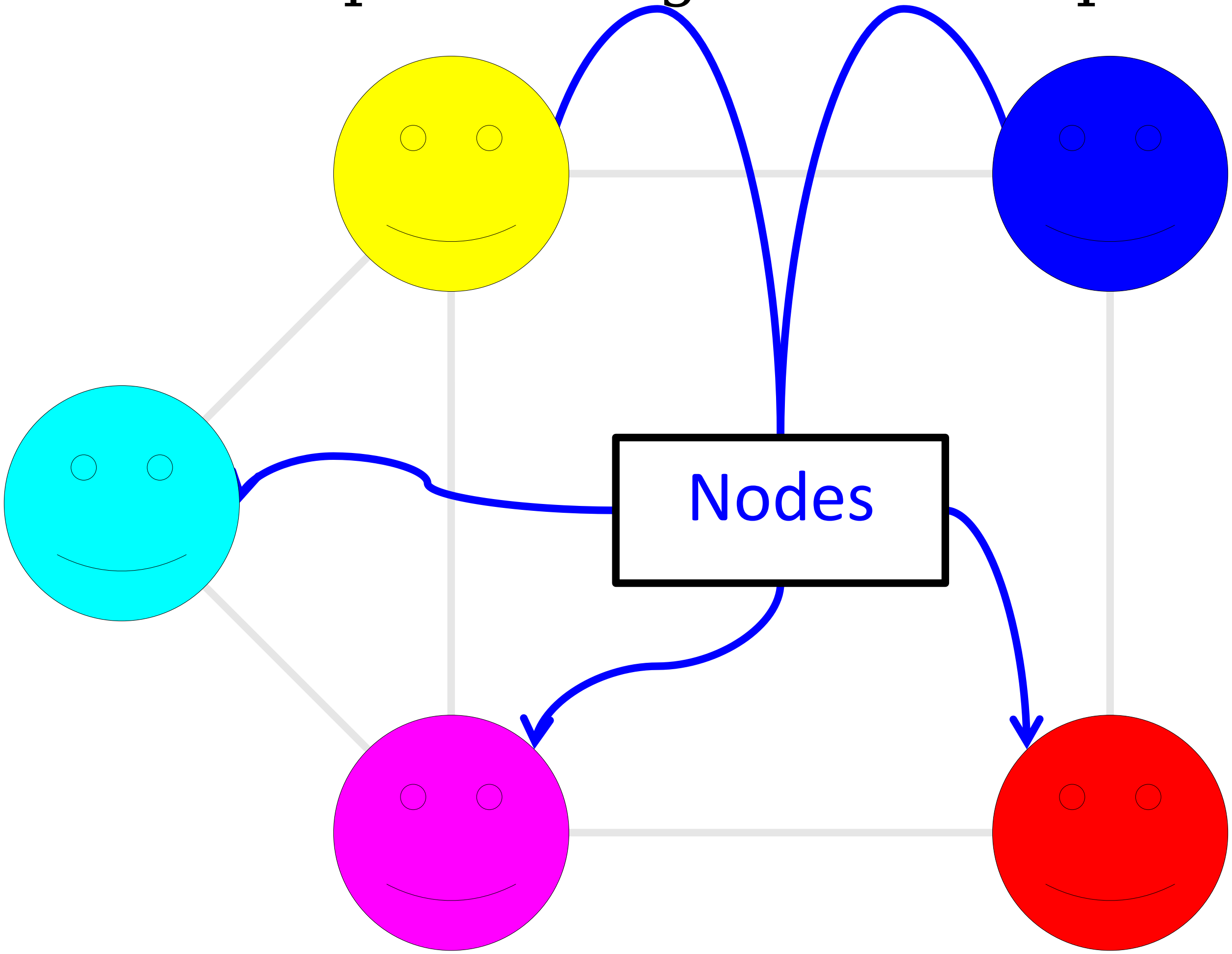

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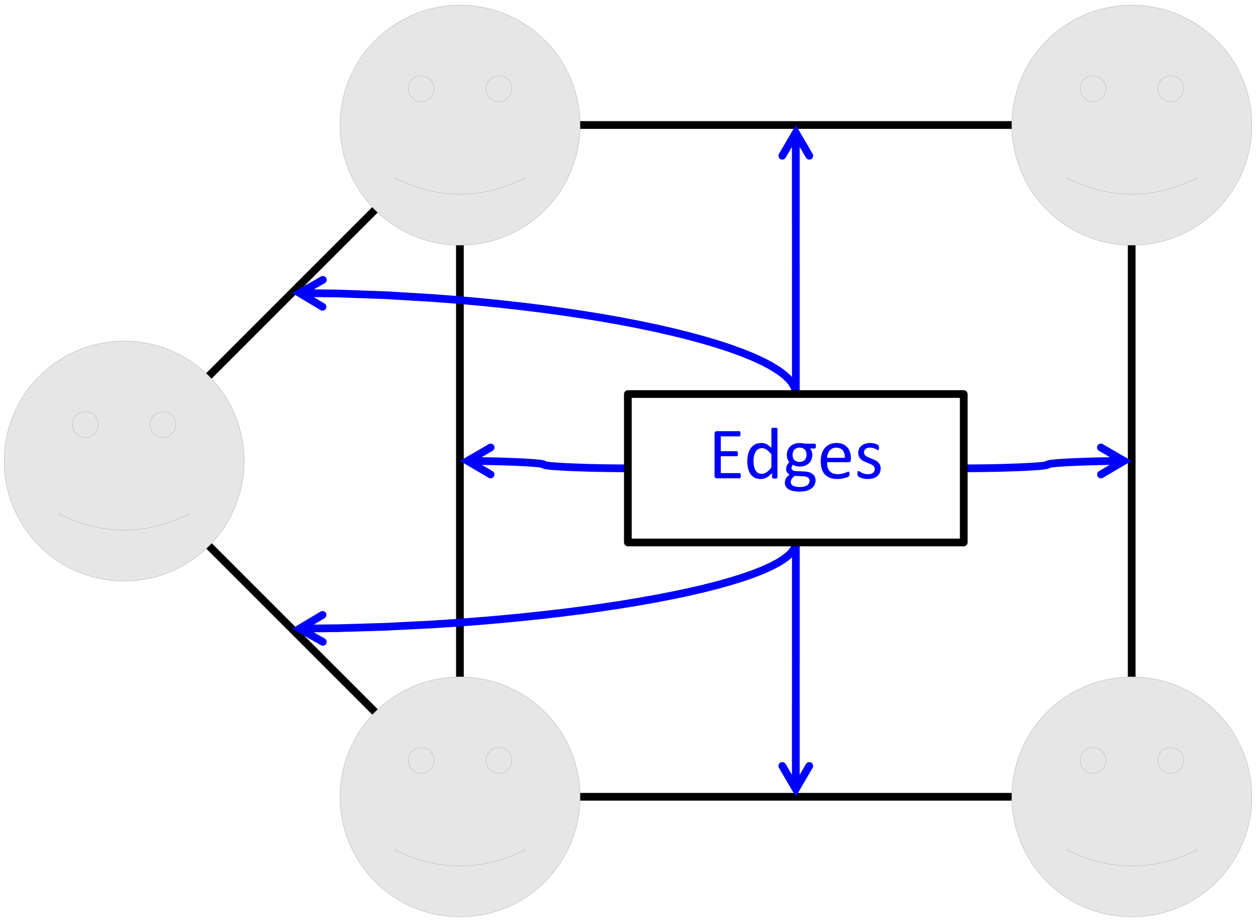

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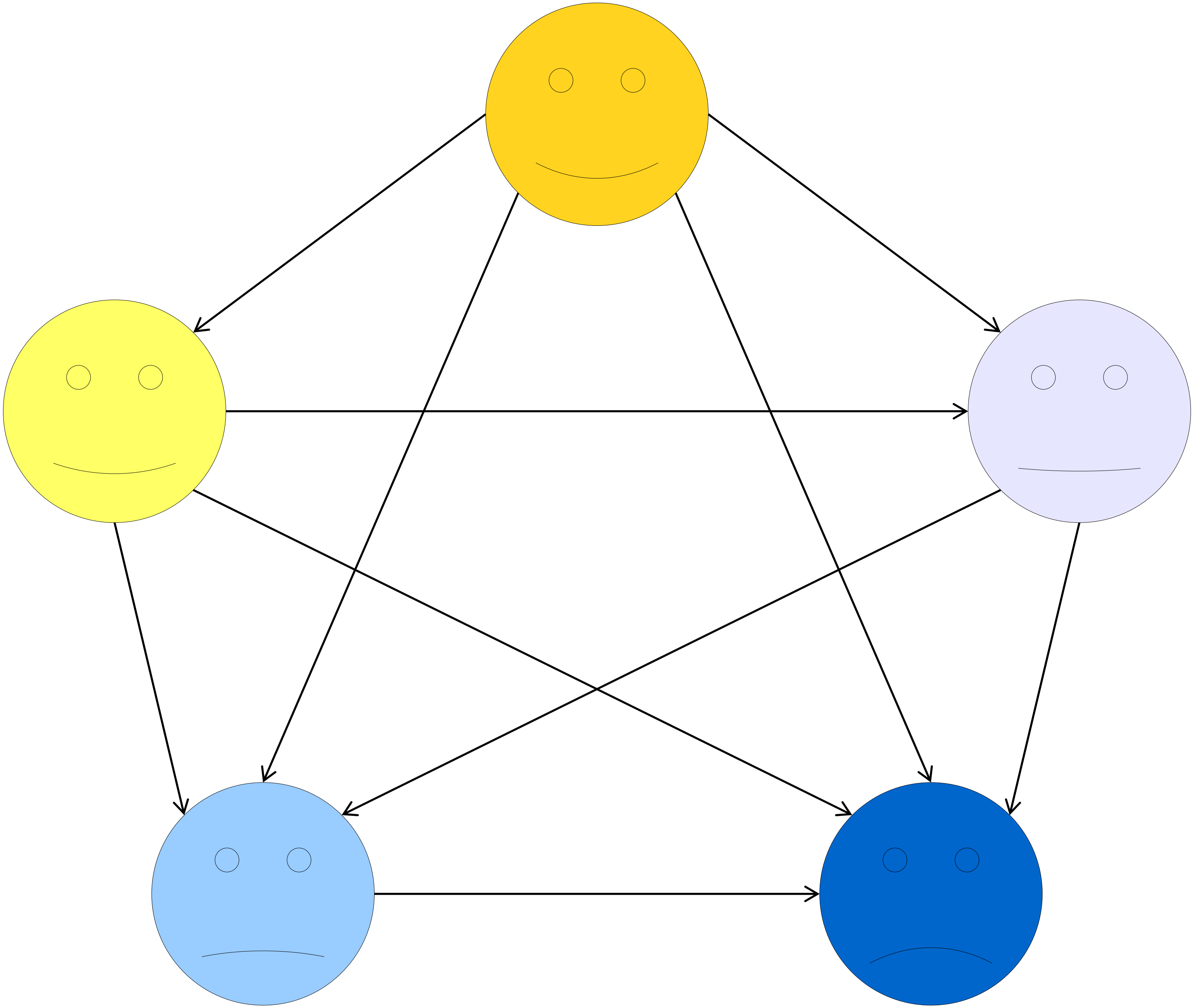
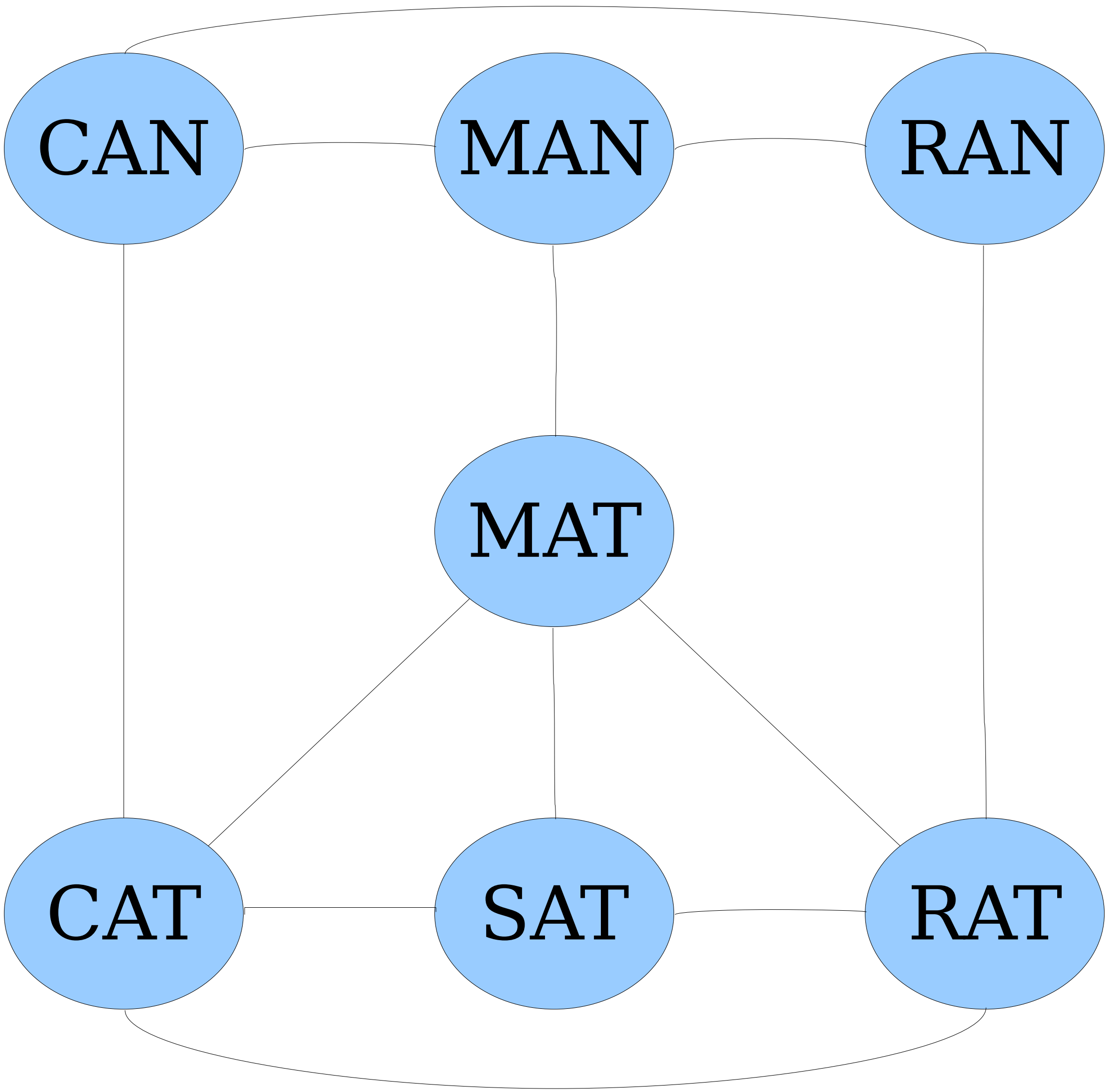
Some graphs are ***directed***.



Some graphs are ***undirected***.



Going forward, we're primarily going to focus on undirected graphs.

The term “graph” generally refers to undirected graphs with a finite number of nodes, unless specified otherwise.

Formalizing Graphs

How might we define a graph mathematically?

We need to specify

- what the nodes in the graph are, and
- which edges are in the graph.

The nodes can be pretty much anything.

What about the edges?

Formalizing Graphs

An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$. (Remember that sets are unordered).

$\{0, 1\} = \{1, 0\}$

An ***undirected graph*** is an ordered pair $G = (V, E)$, where

V is a set of nodes, which can be anything, and

E is a set of edges, which are unordered pairs of nodes drawn from V.

A ***directed graph*** is an ordered pair $G = (V, E)$, where

V is a set of nodes, which can be anything, and

E is a set of edges, which are *ordered* pairs of nodes drawn from V.

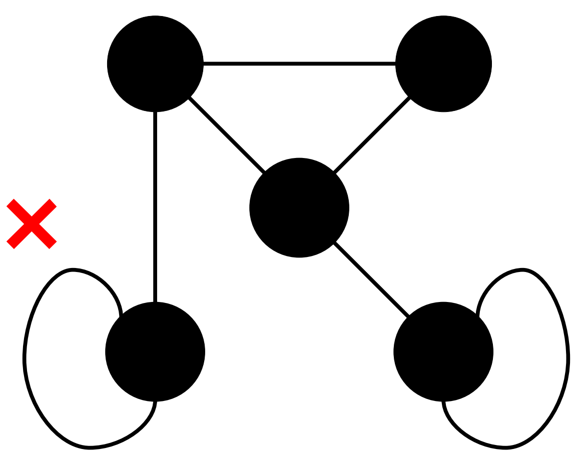
Self-Loops

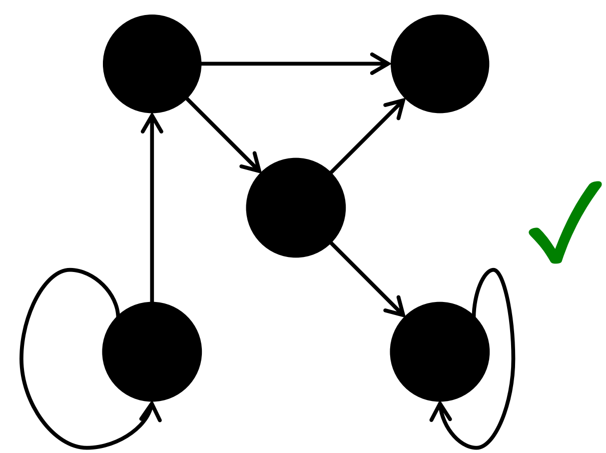
An edge from a node to itself is called a ***self-loop***.

In undirected graphs, self-loops are generally not allowed.

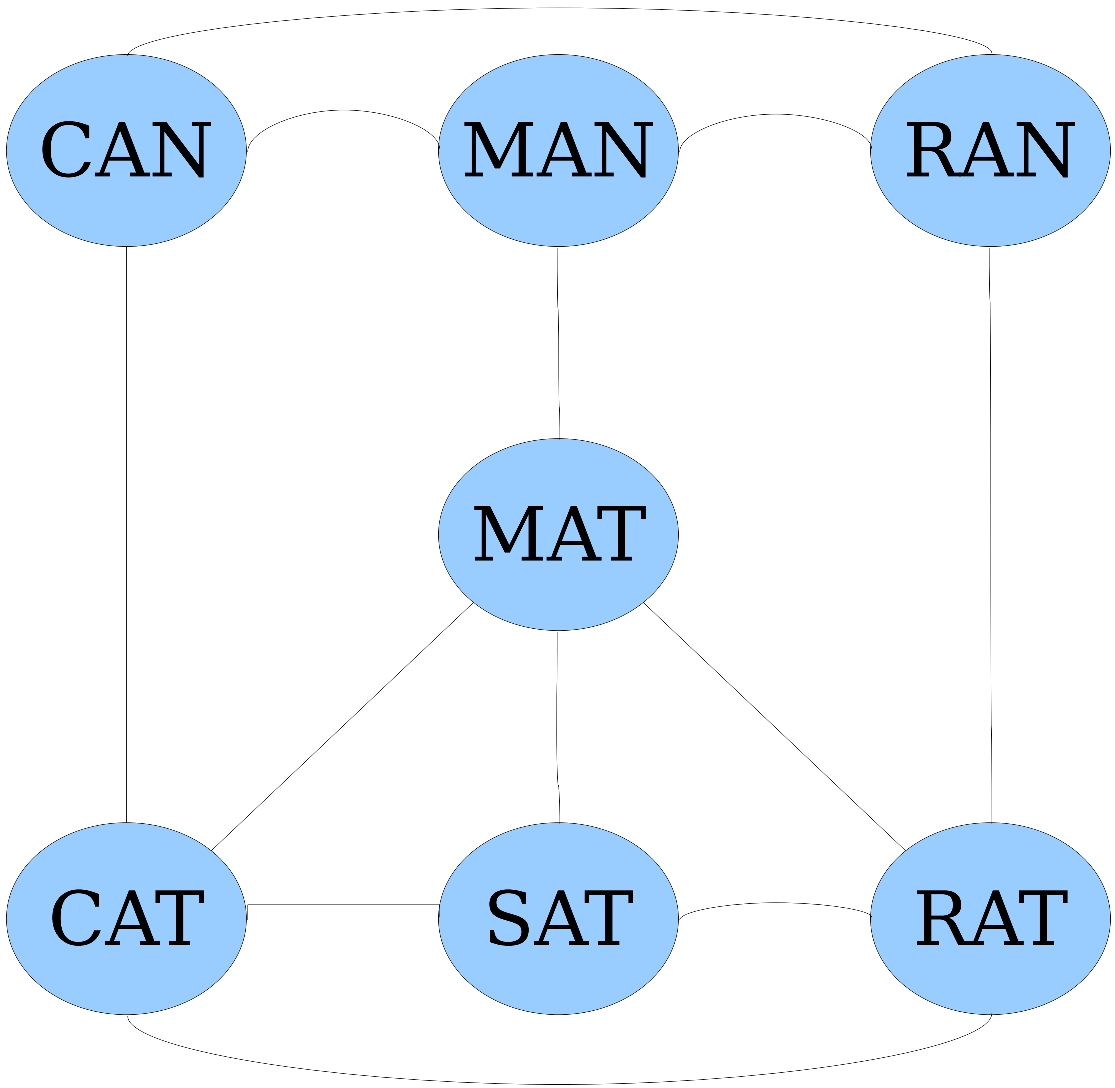
Can you see how this follows from the definition?

In directed graphs, self-loops are generally allowed unless specified otherwise.

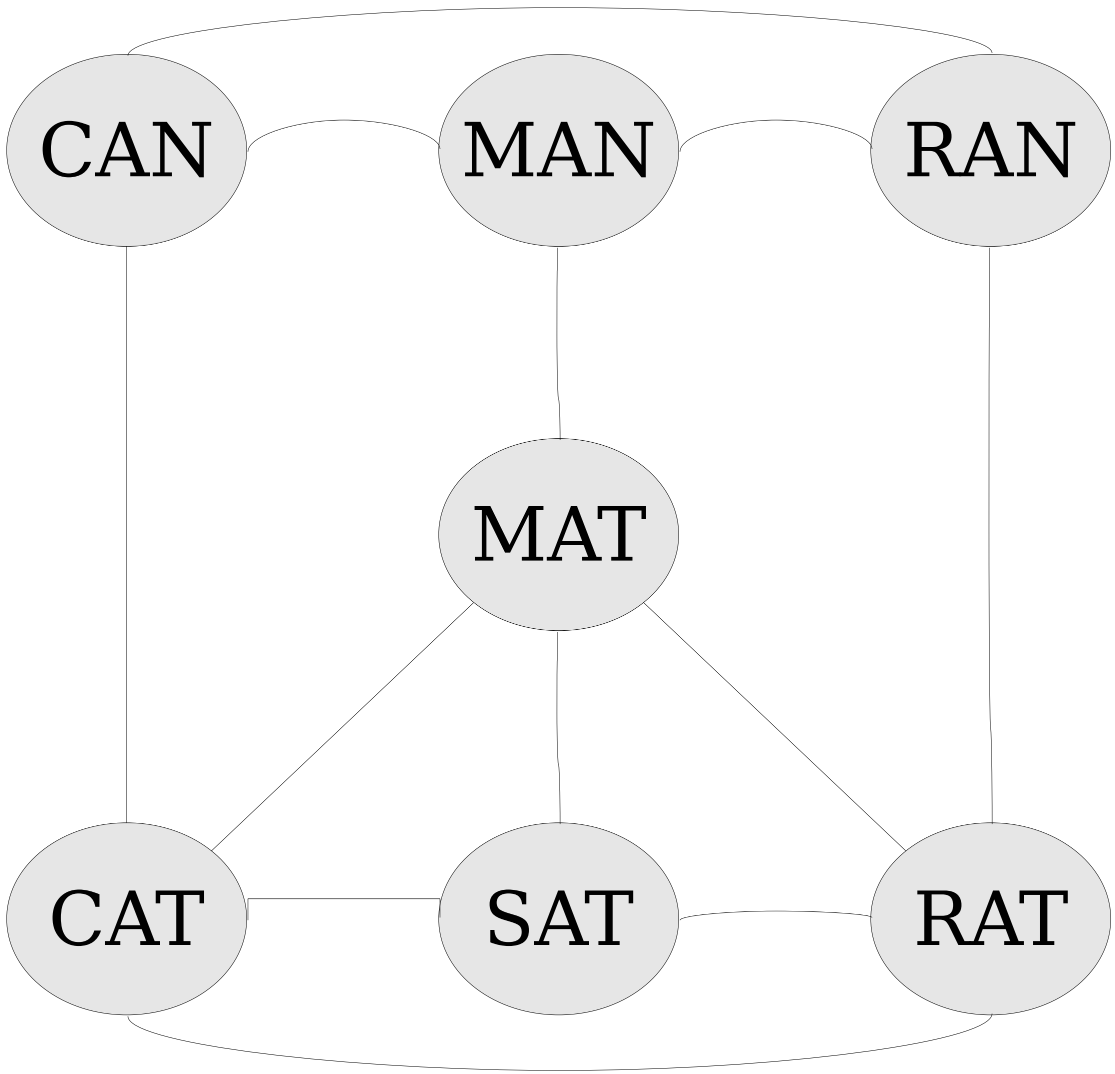




Standard Graph Terminology



Two nodes are called ***adjacent*** if there is an edge between them.



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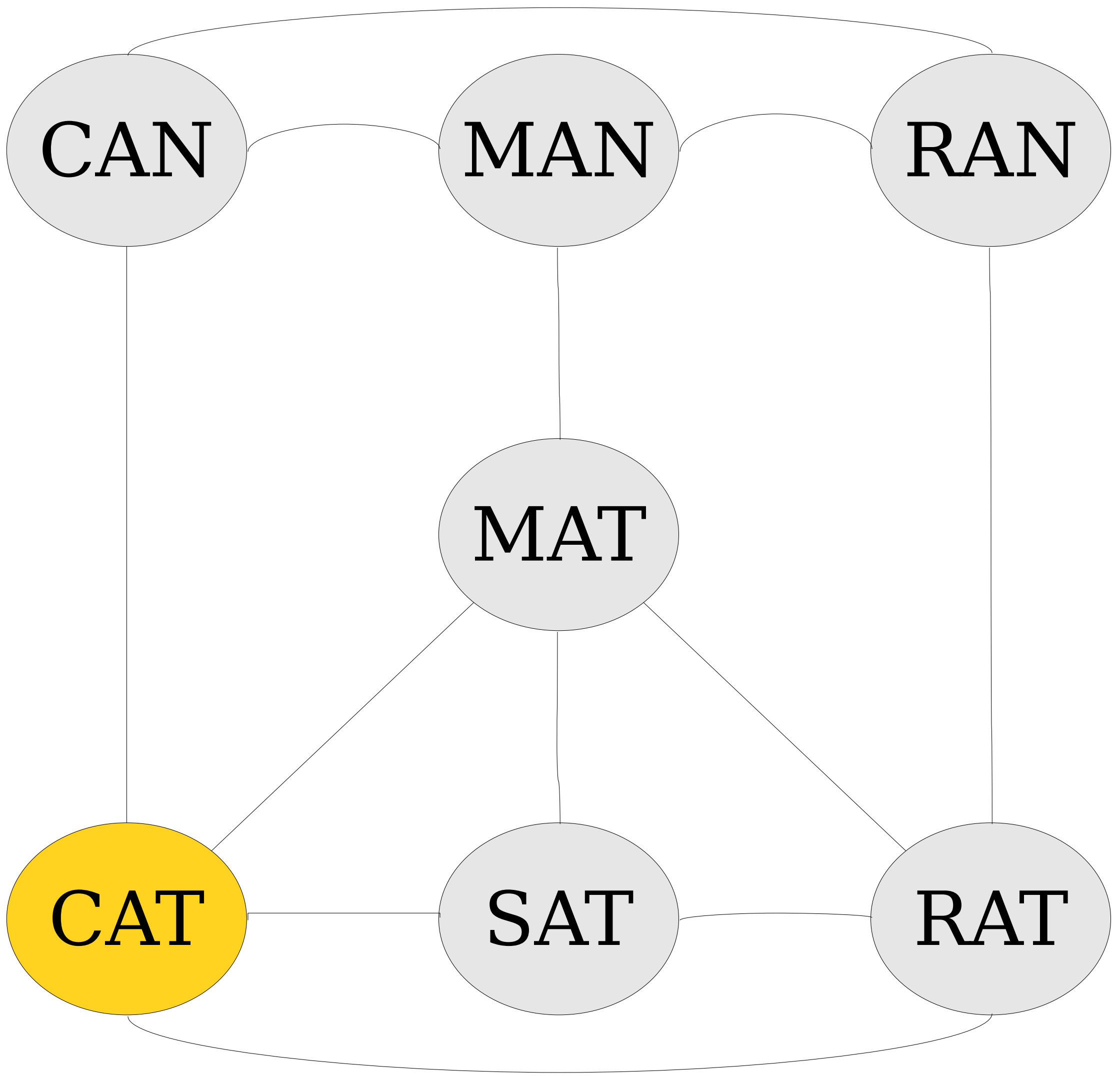

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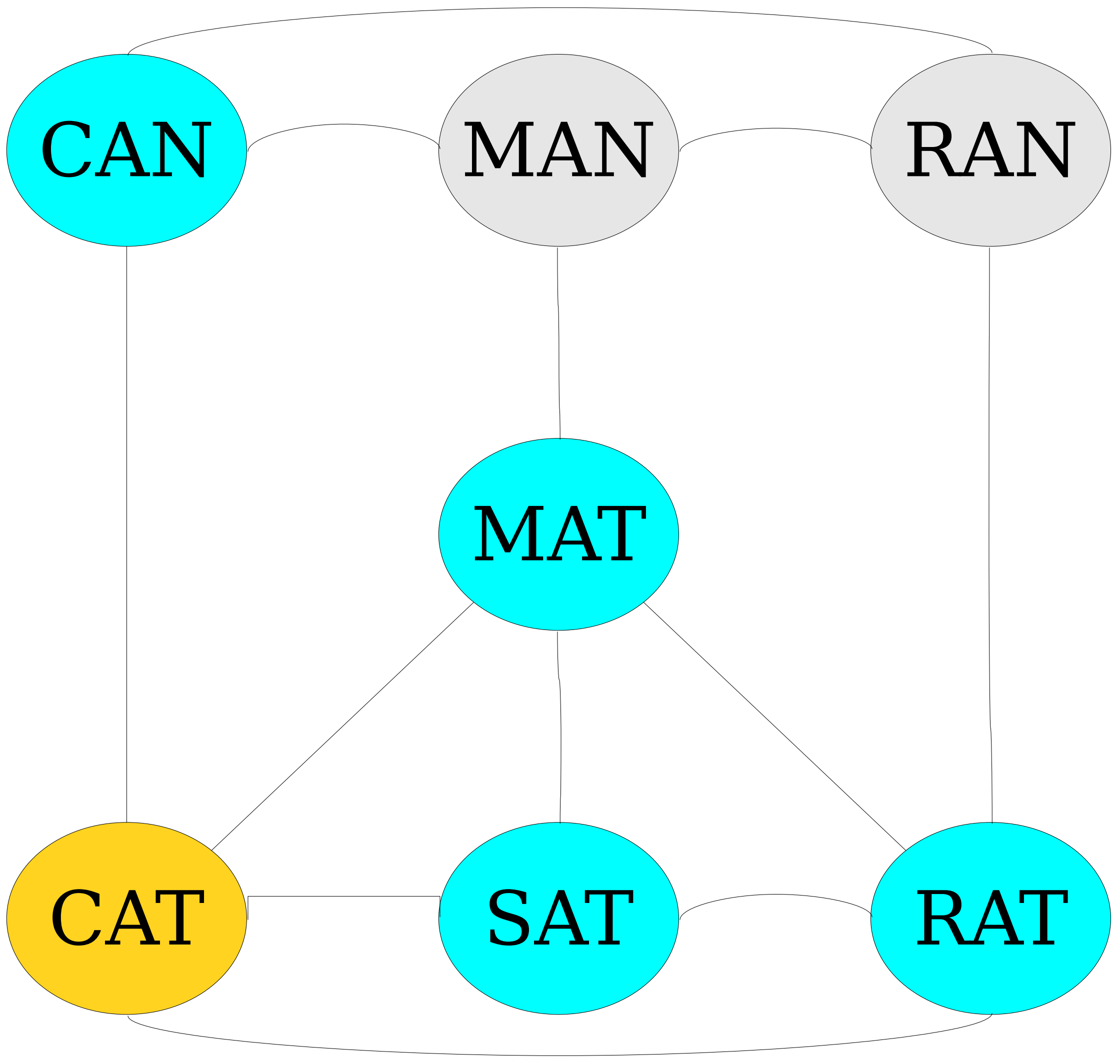

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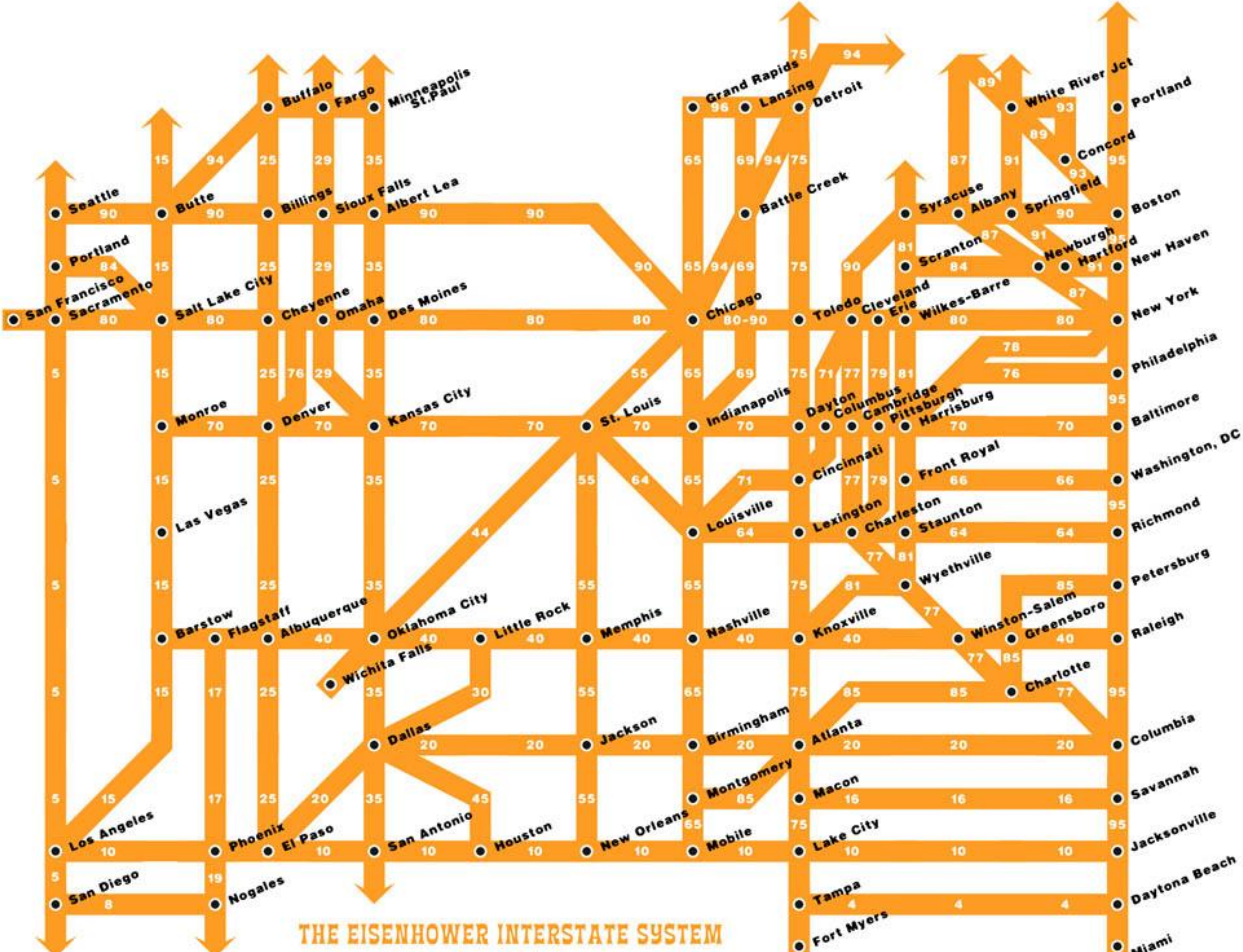
Using our Formalisms

Let $G = (V, E)$ be a graph.

Intuitively, two nodes are adjacent if they're linked by an edge.

Formally speaking, we say that two nodes $u, v \in V$ are ***adjacent*** if $\{u, v\} \in E$.



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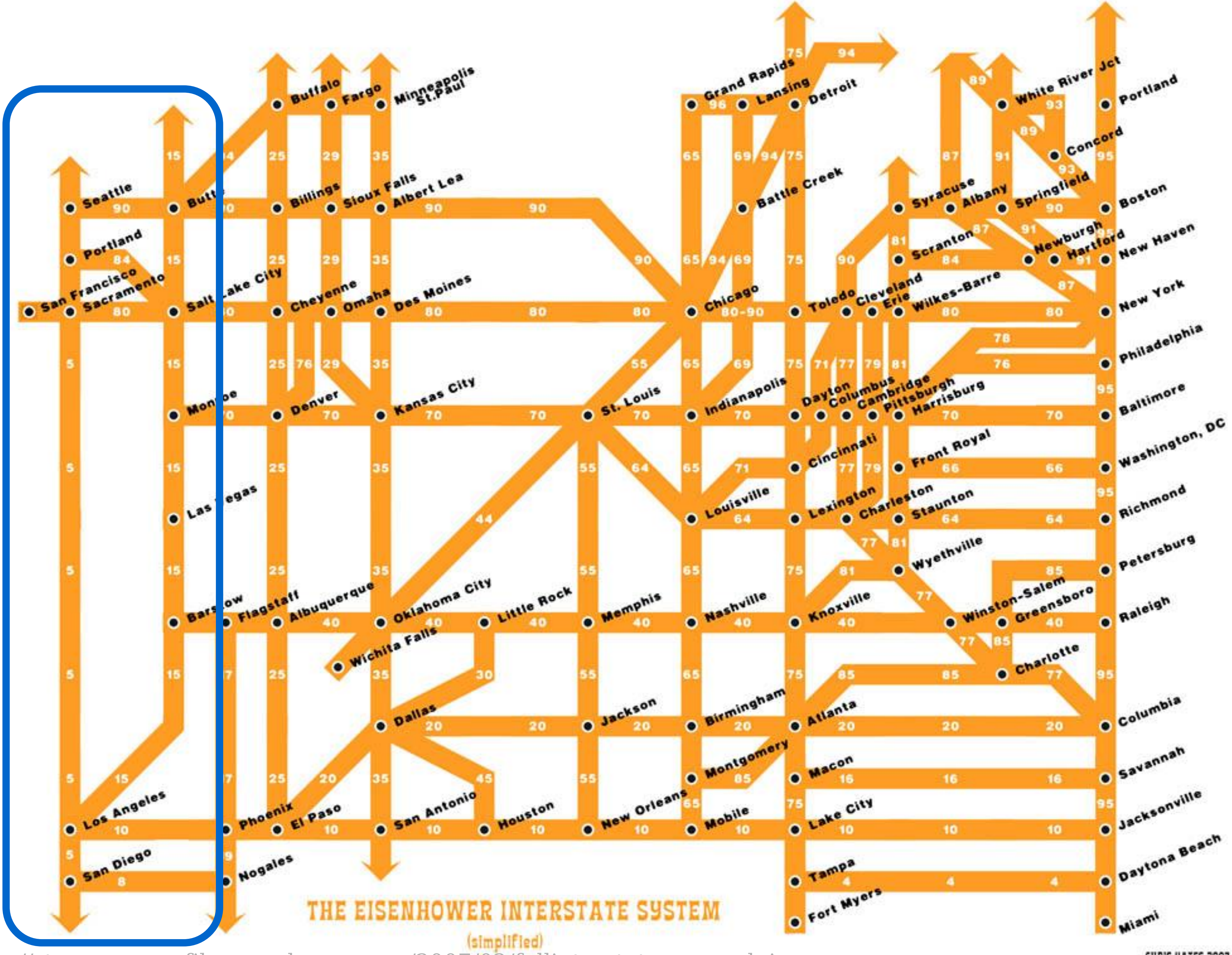

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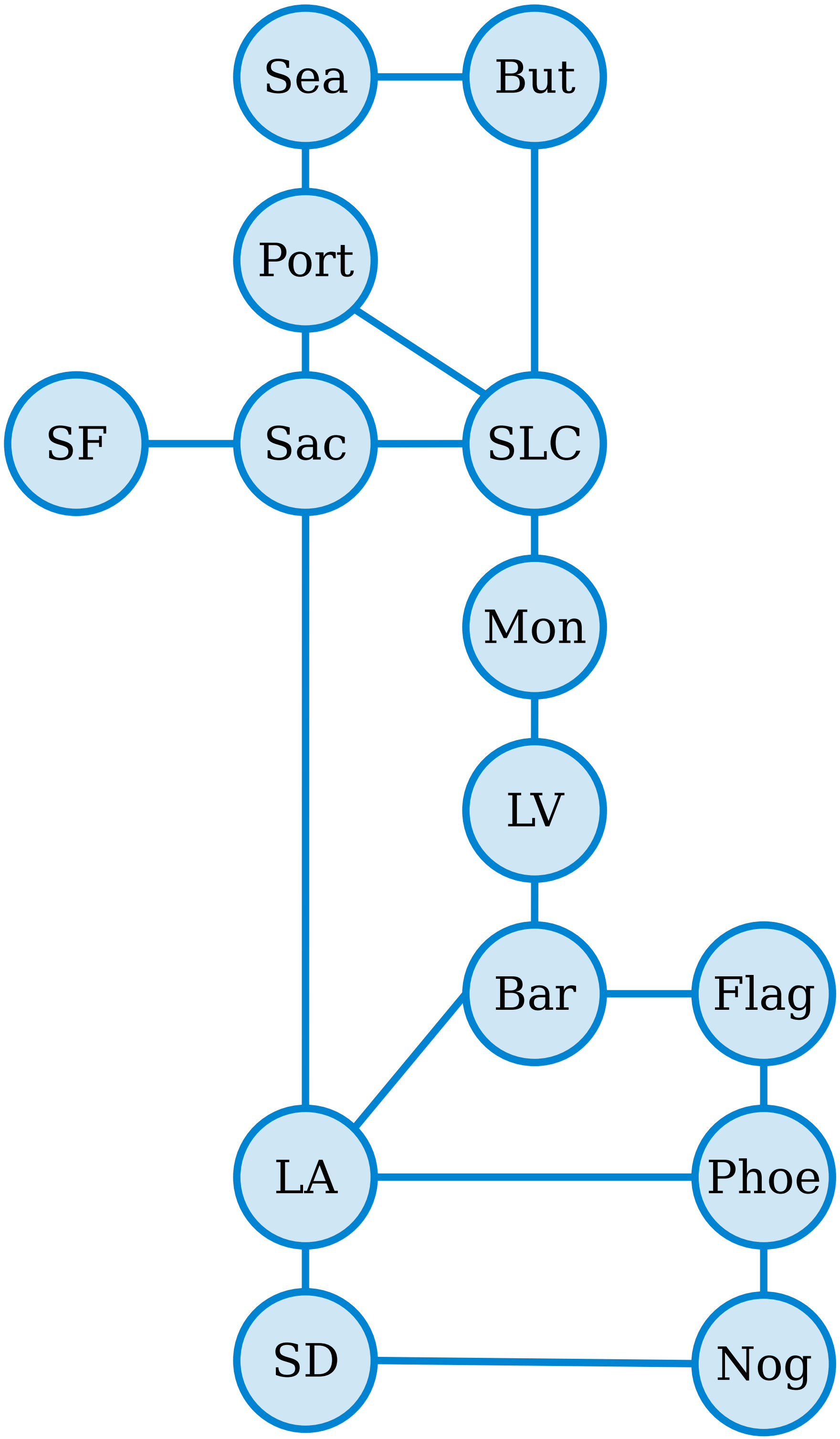

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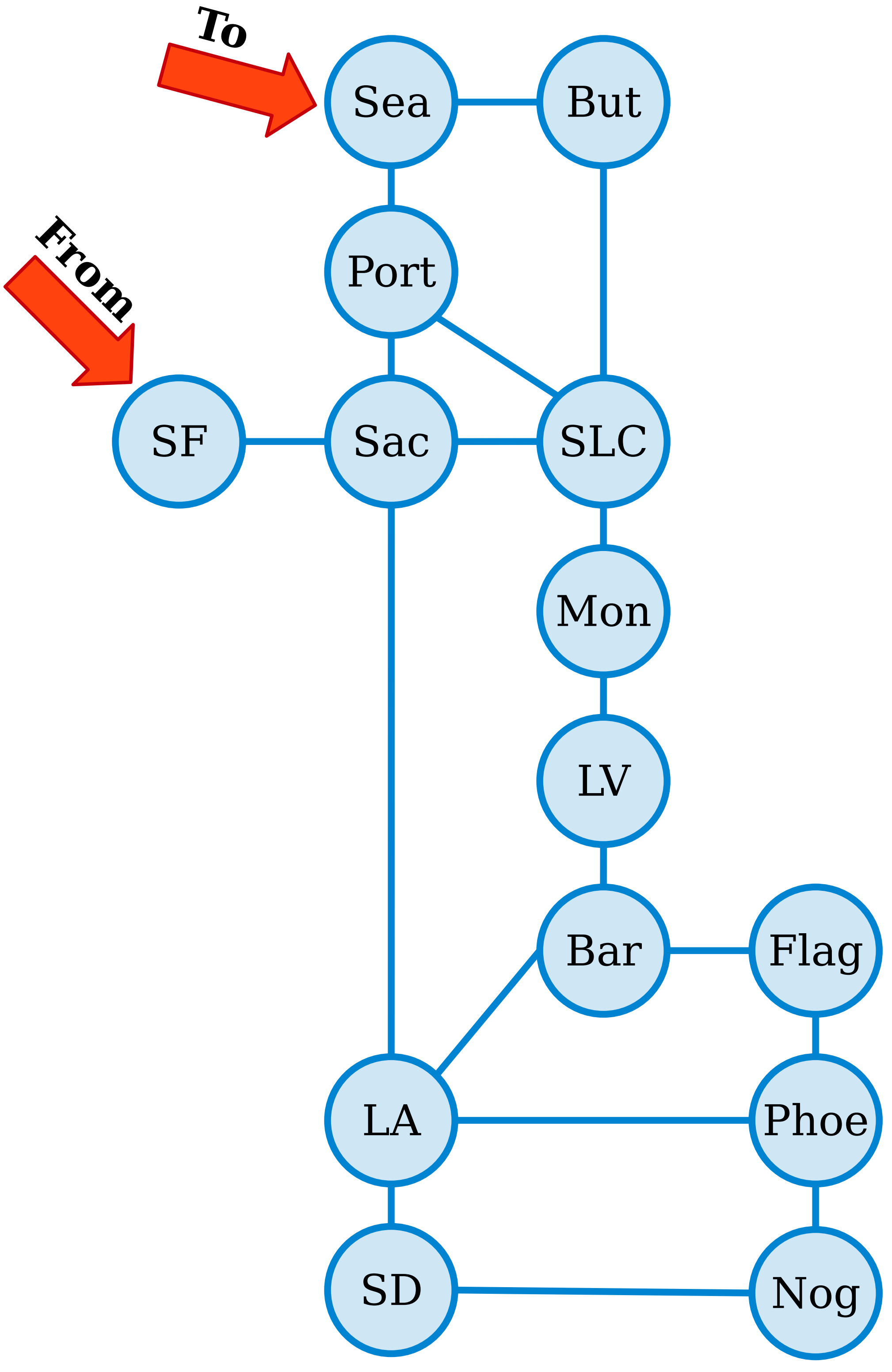
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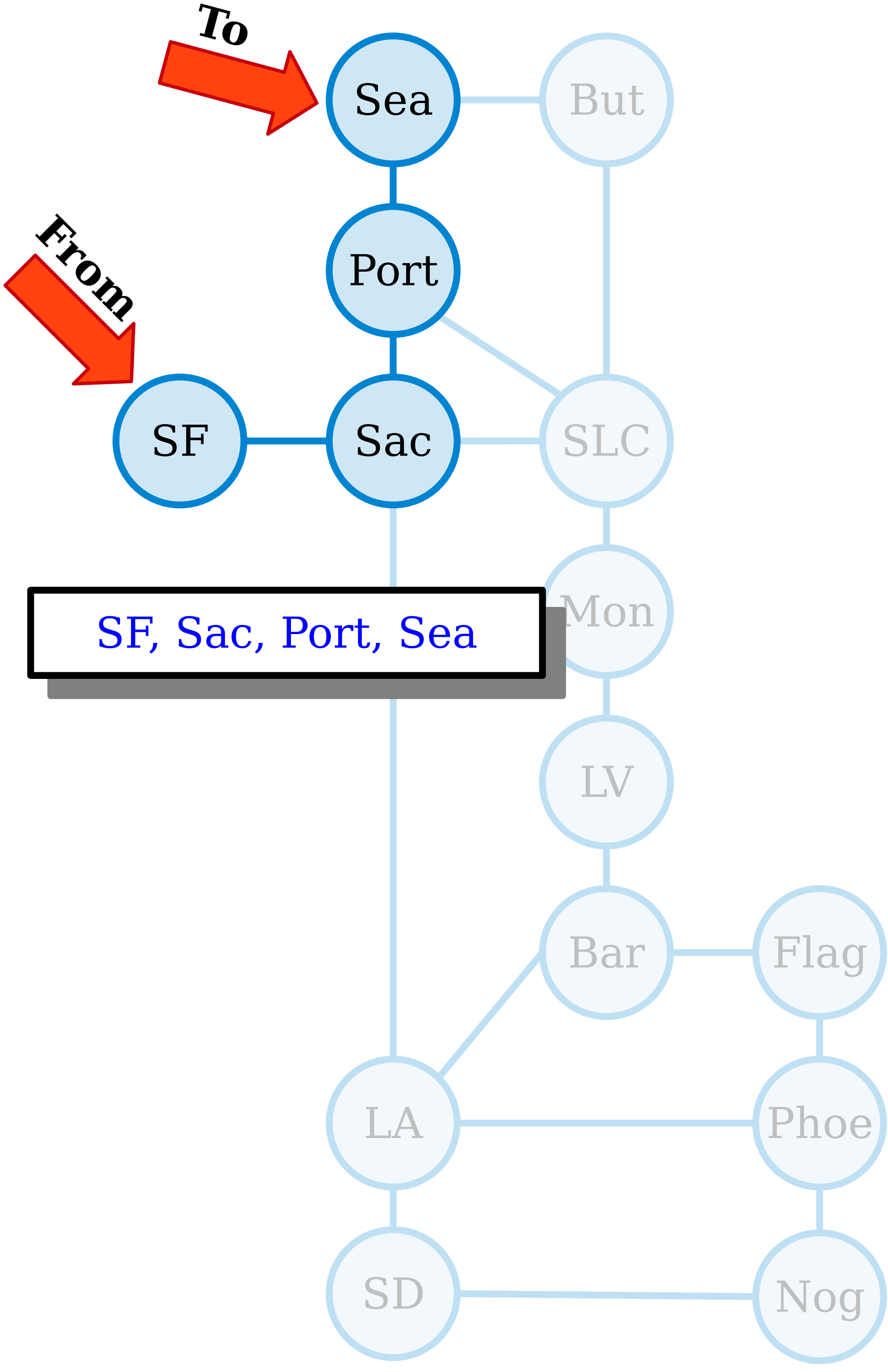
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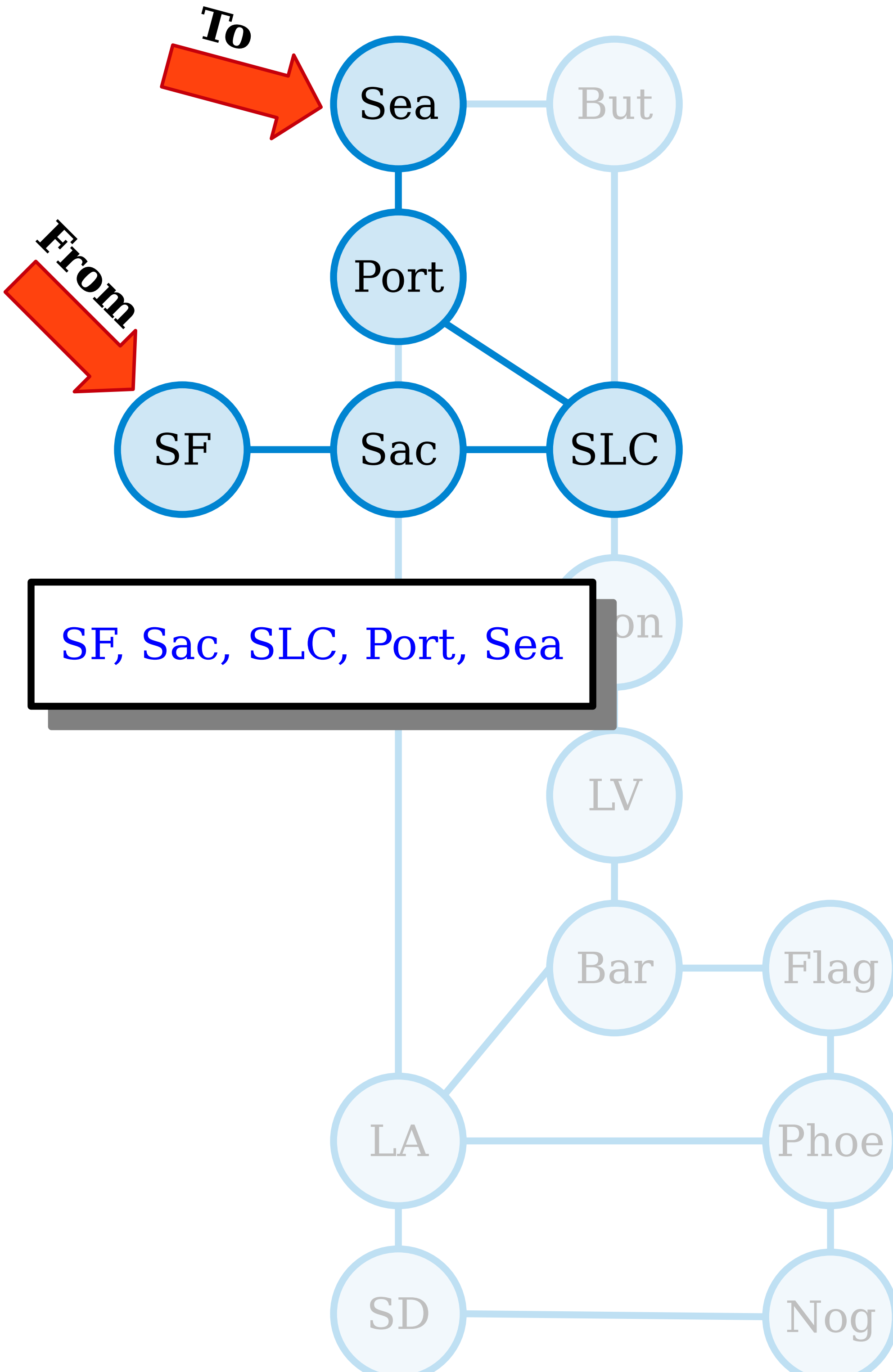
SF, Sac, Port, Sea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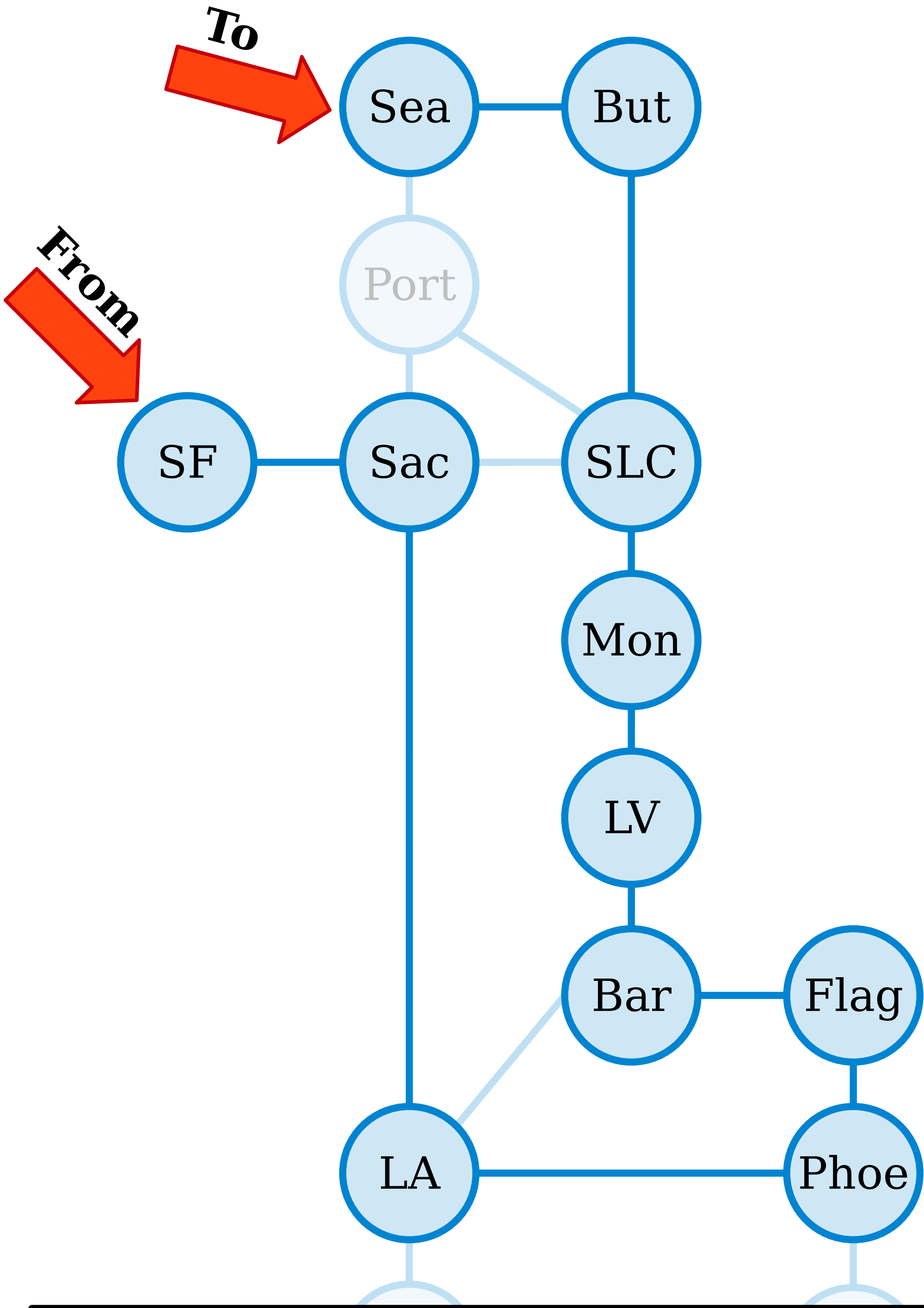
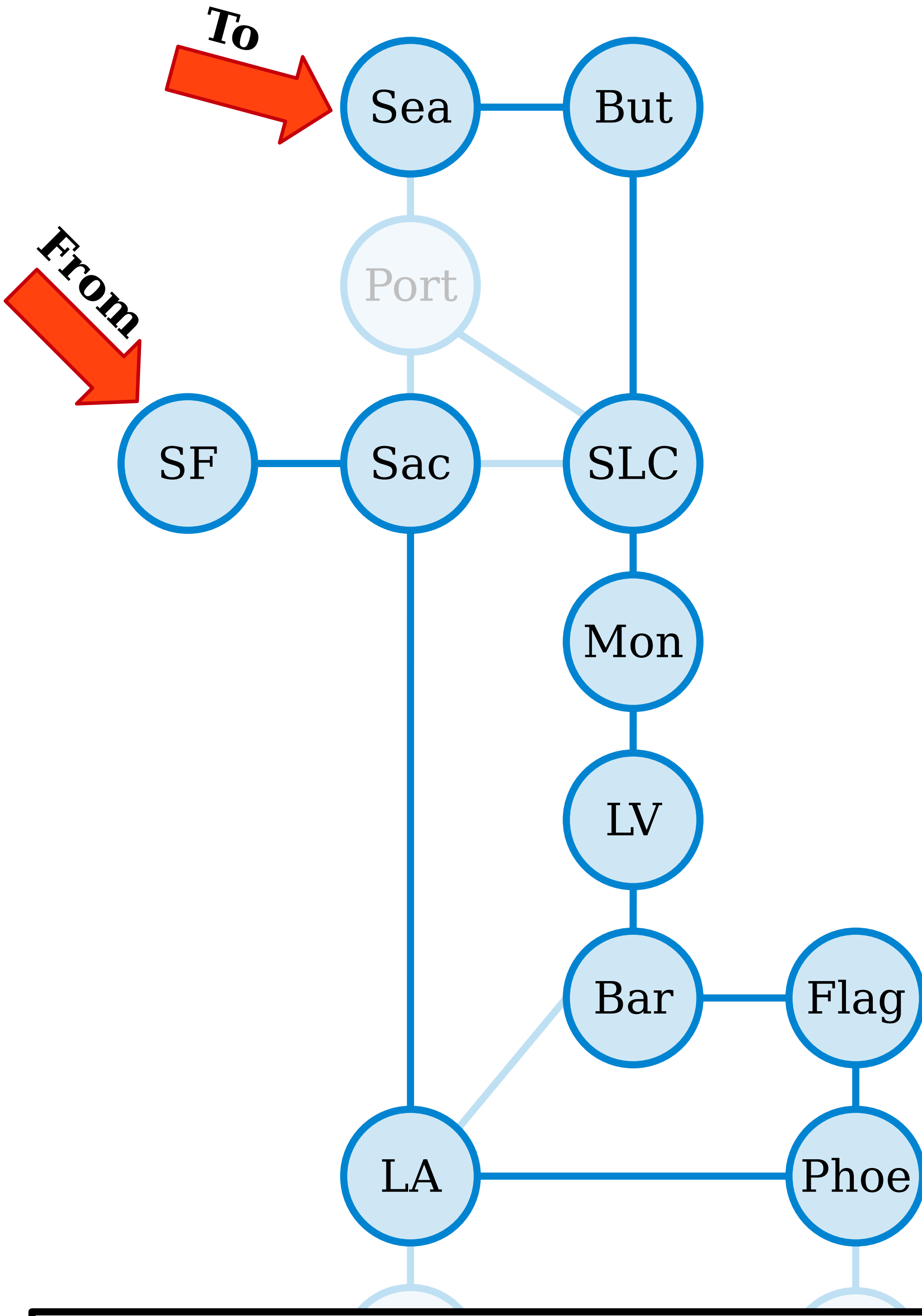
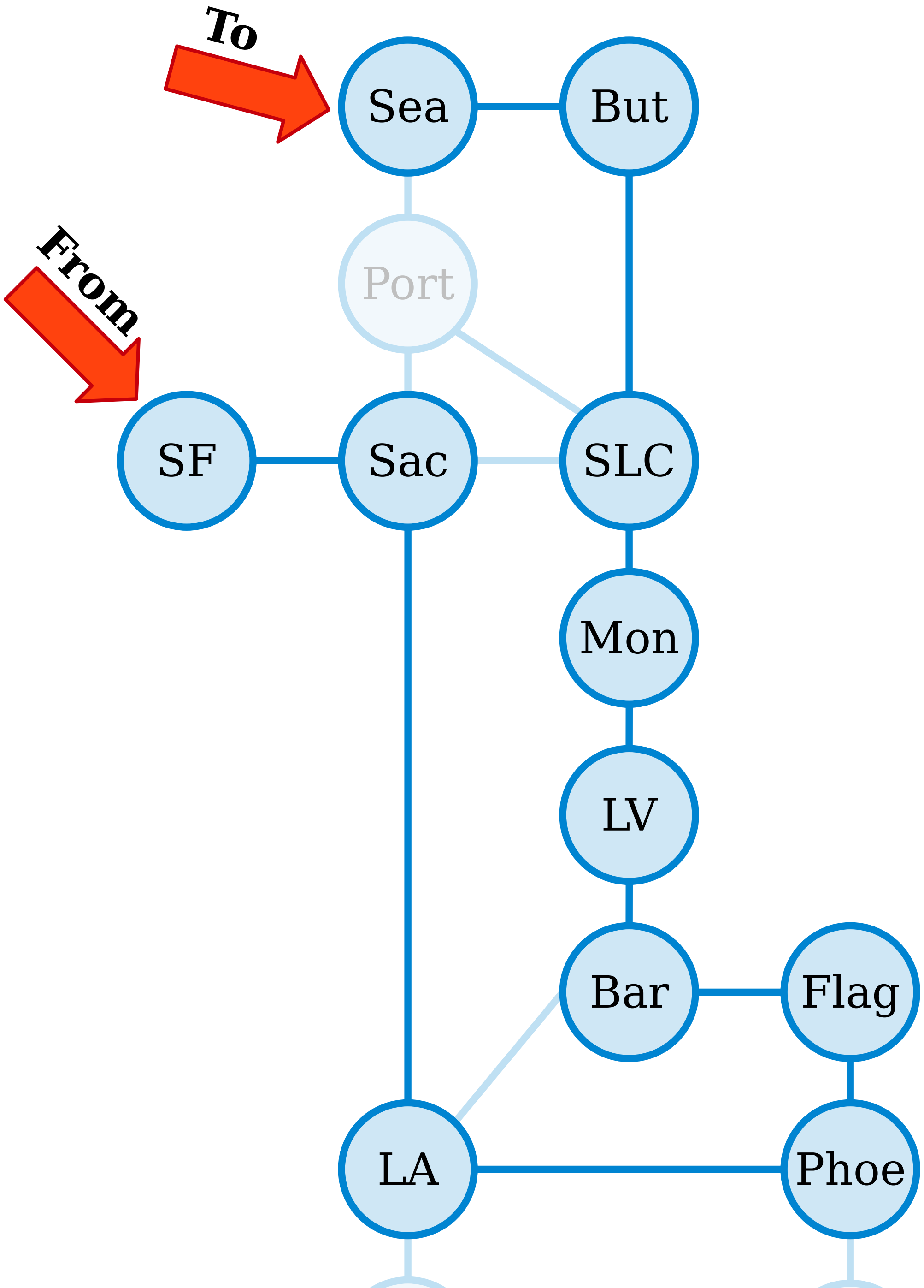


SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



A ***path*** in a graph $G = (V, E)$ is a sequence of one or more nodes $v_1, v_2, v_3, \ldots, v_n$ such that any two consecutive nodes in the sequence are adjacent.

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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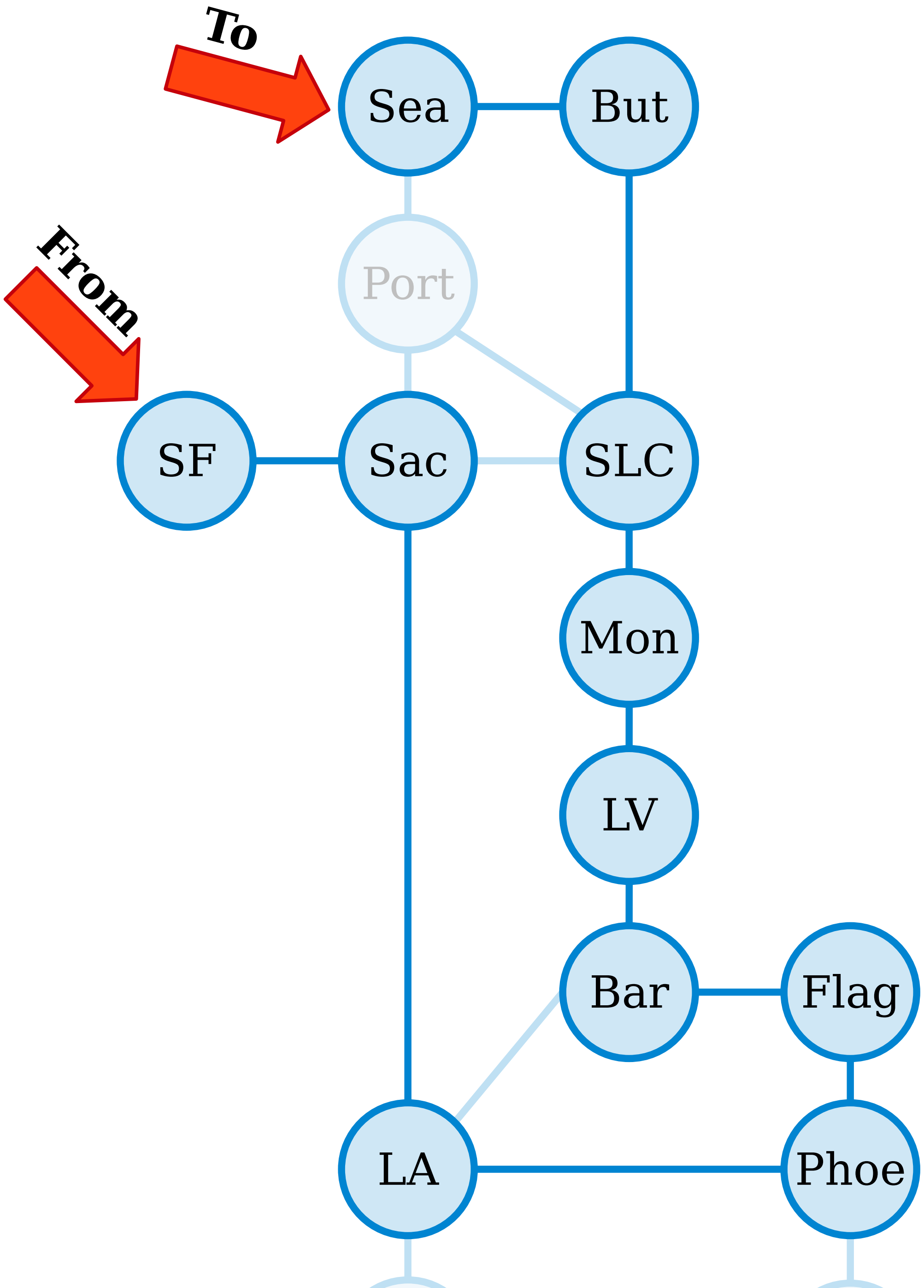
The ***length*** of the path $v_1, \ldots, v_n$ is $n - 1$.

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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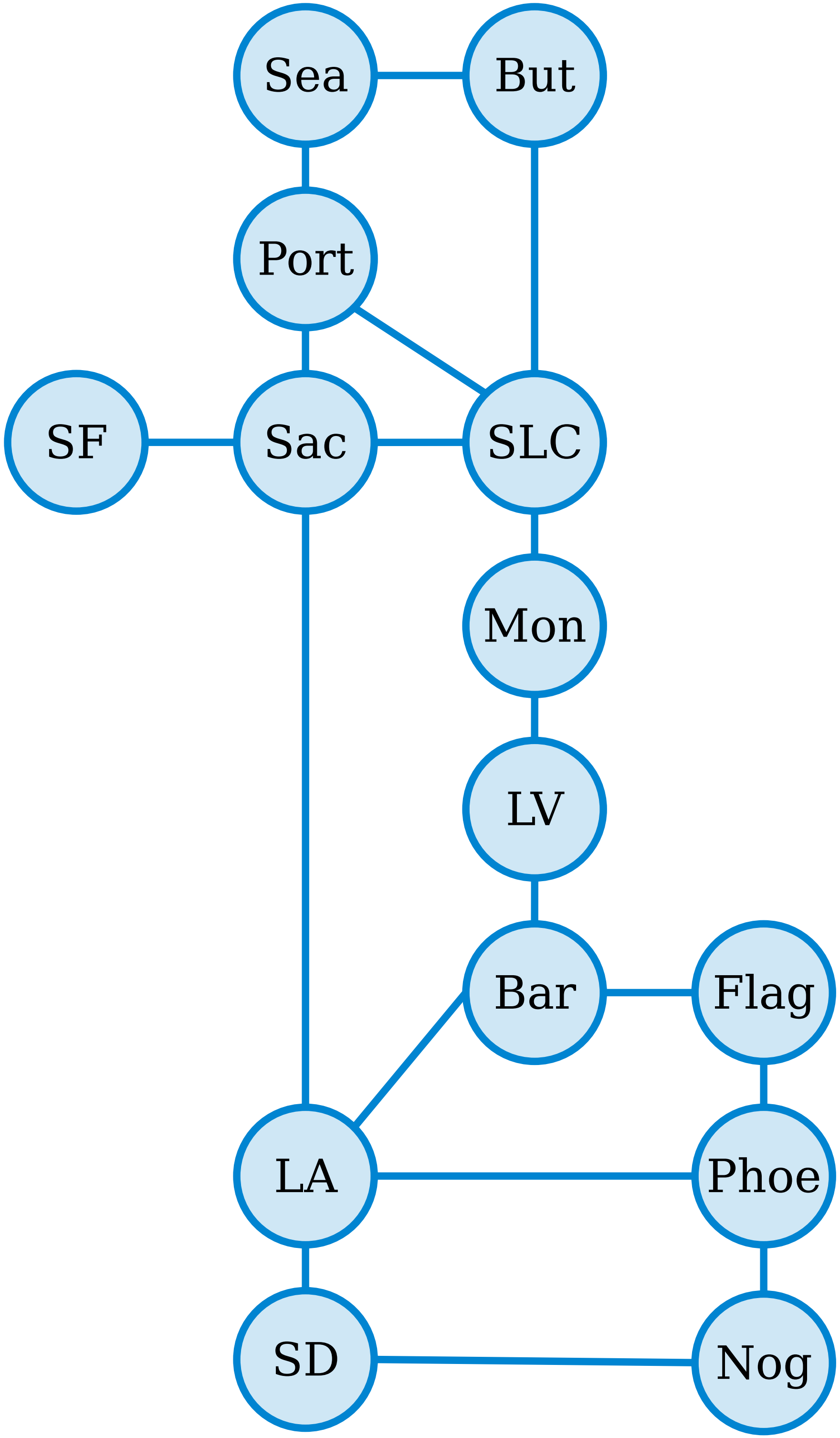
(This path has length 10, but visits 11 cities.)

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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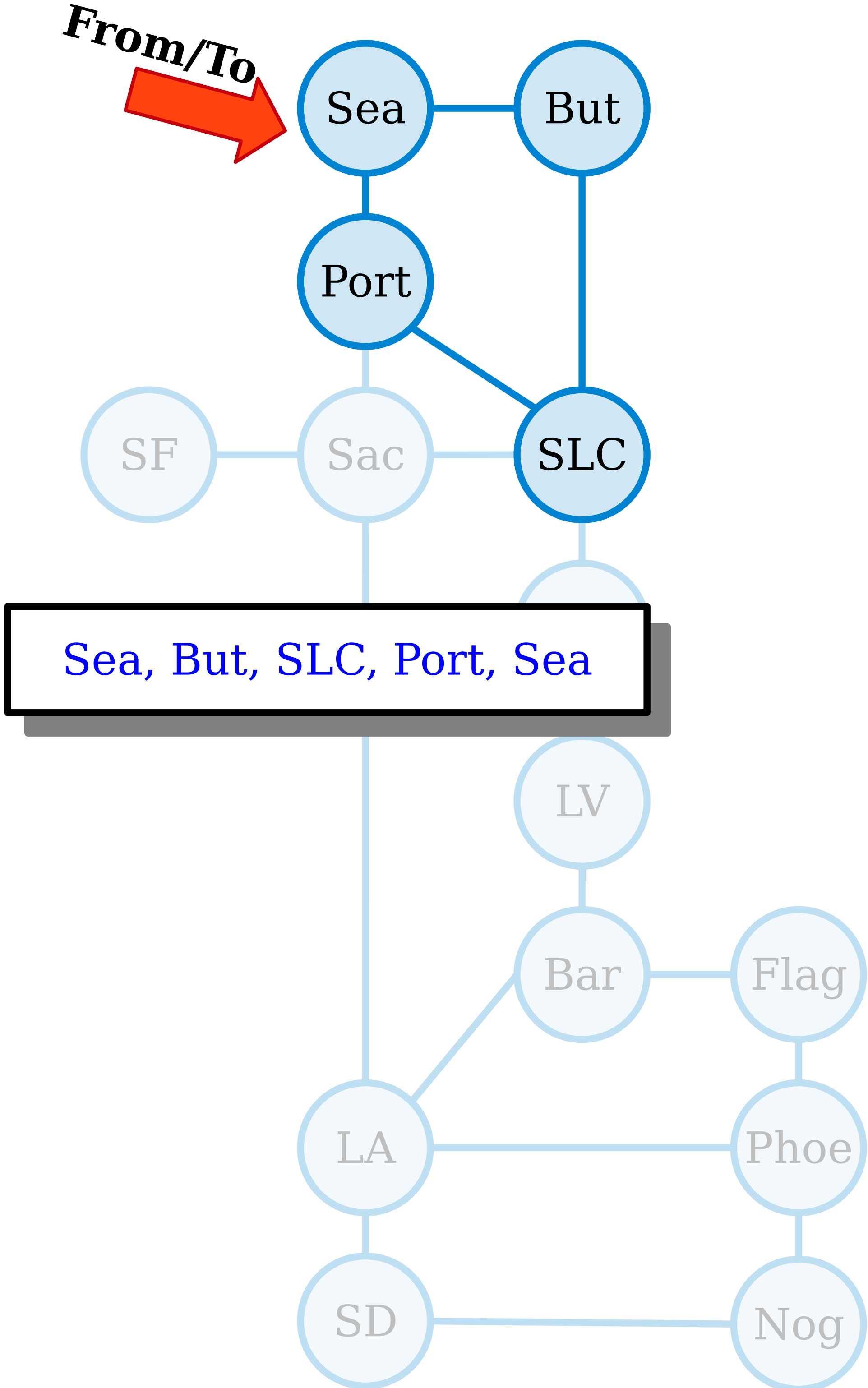

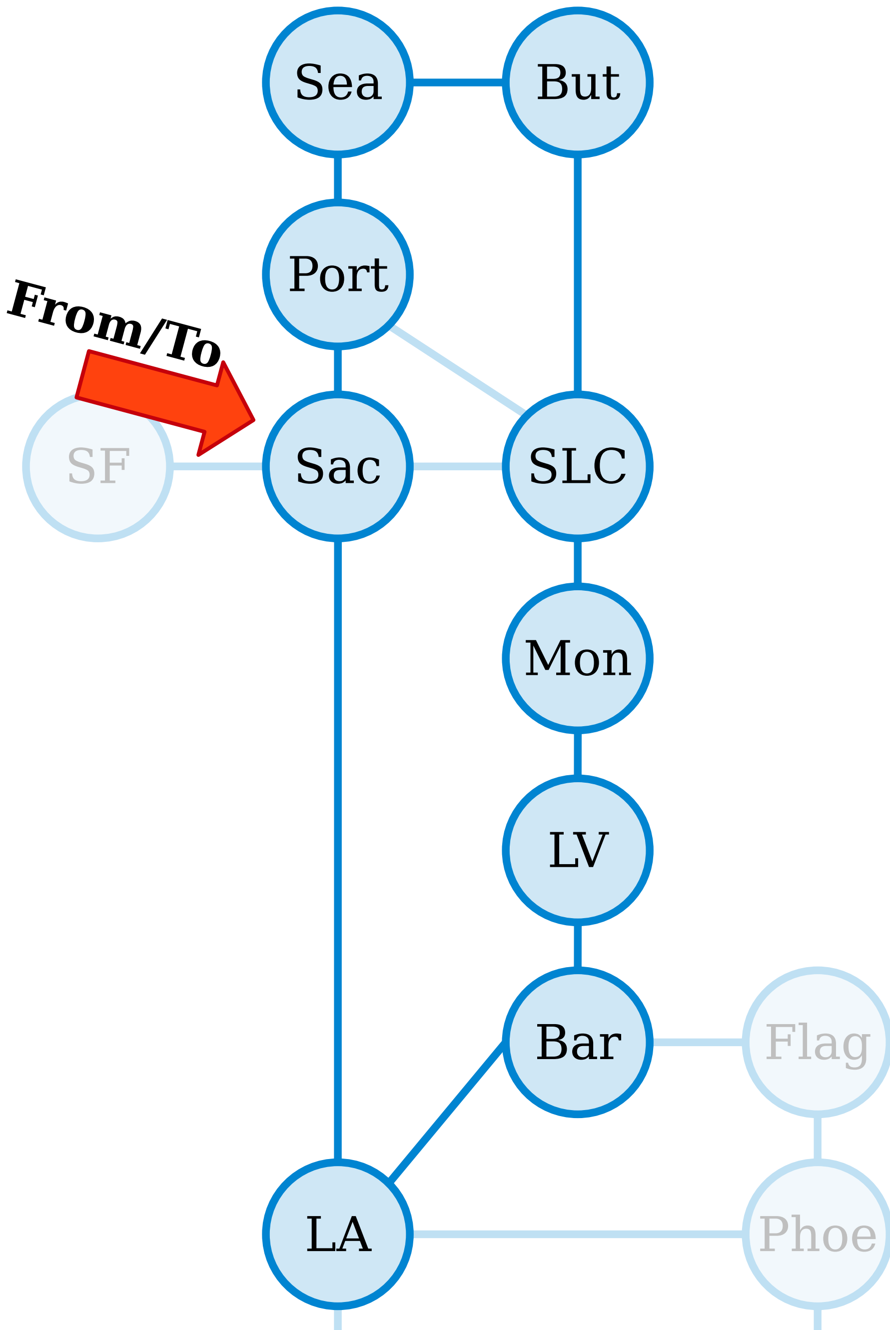

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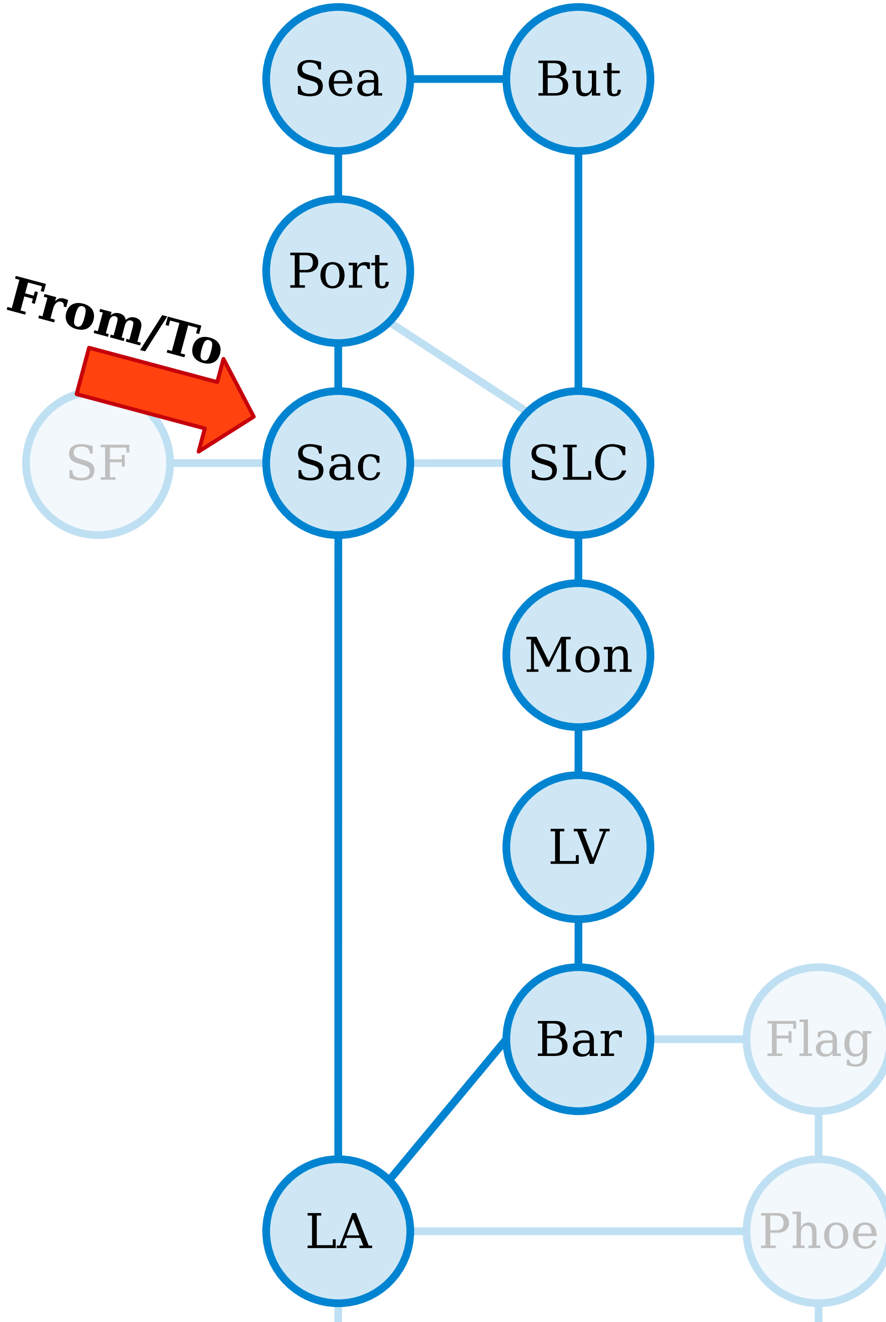
Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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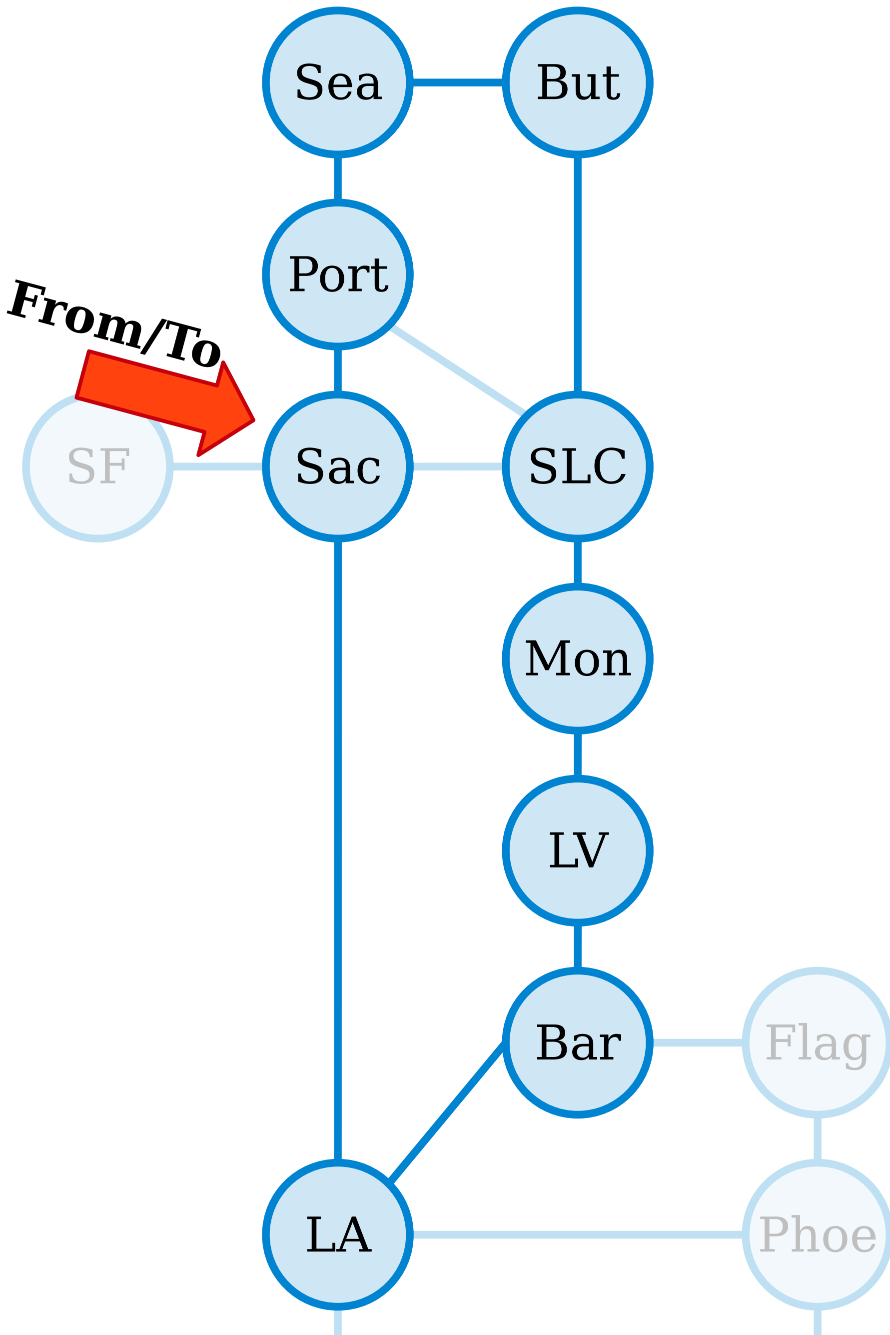
A ***cycle*** in a graph is a path from a node back to itself. (By convention, a cycle cannot have length zero.)

Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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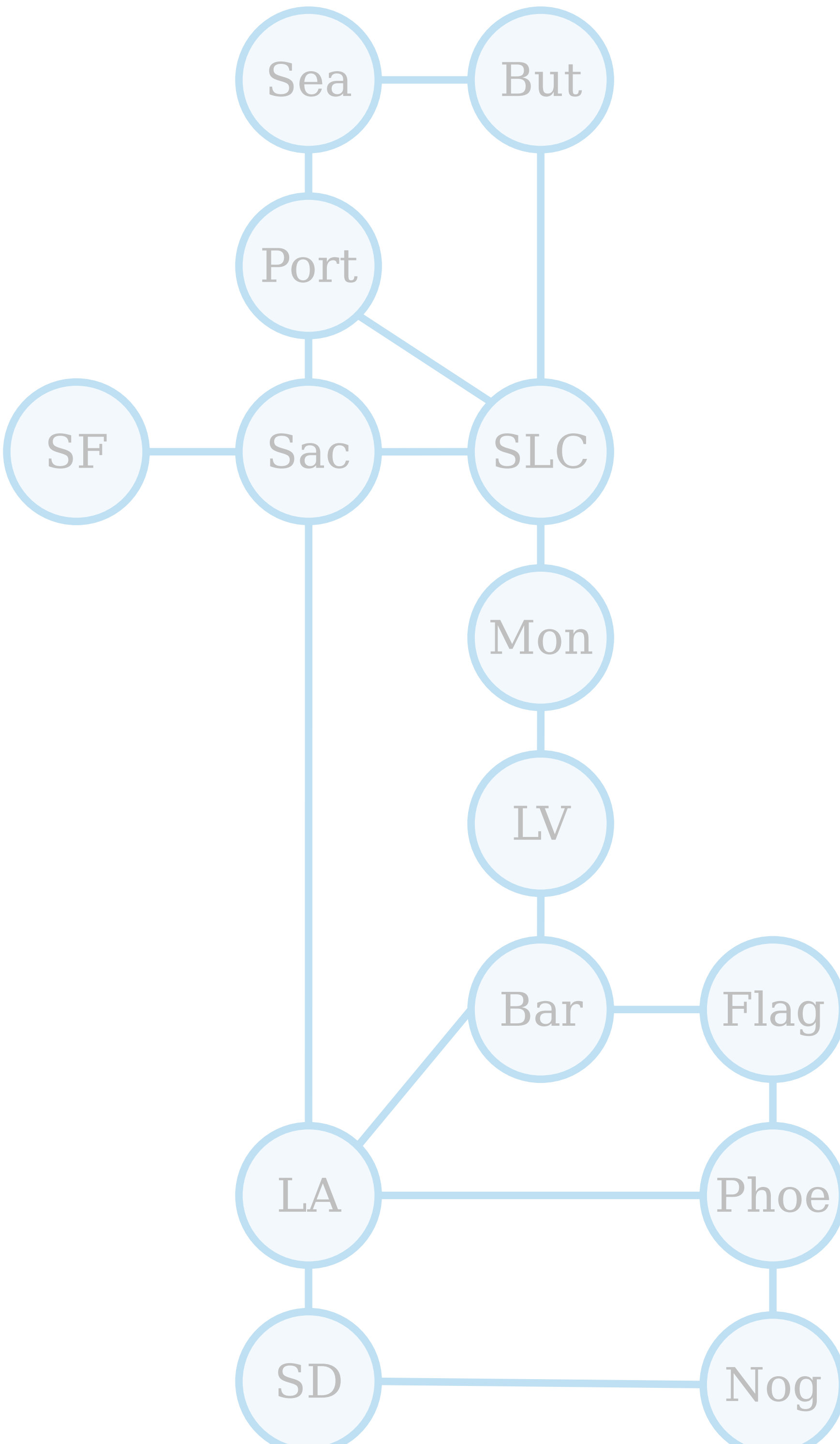
(This cycle has length nine and visits nine different cities.)

Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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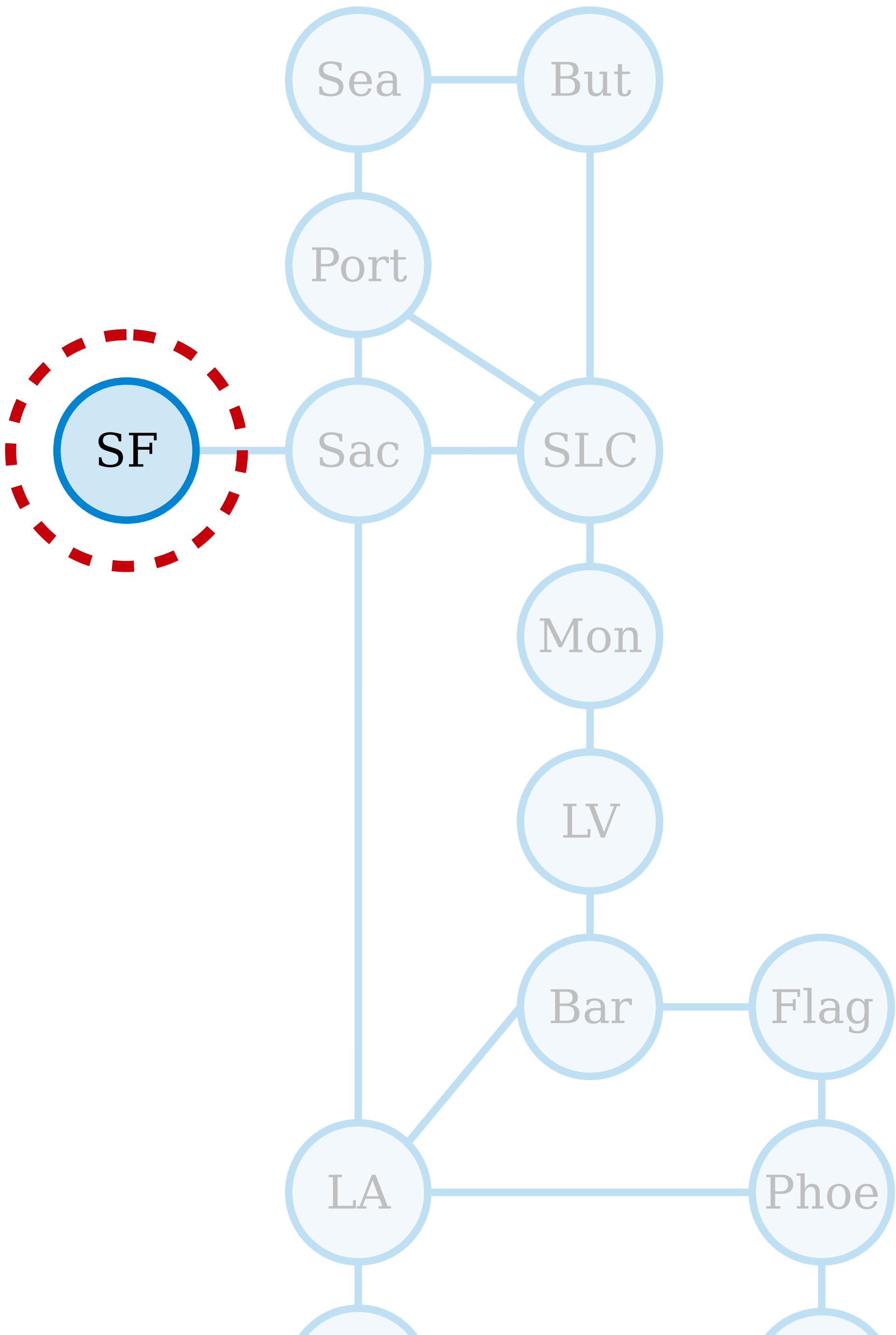

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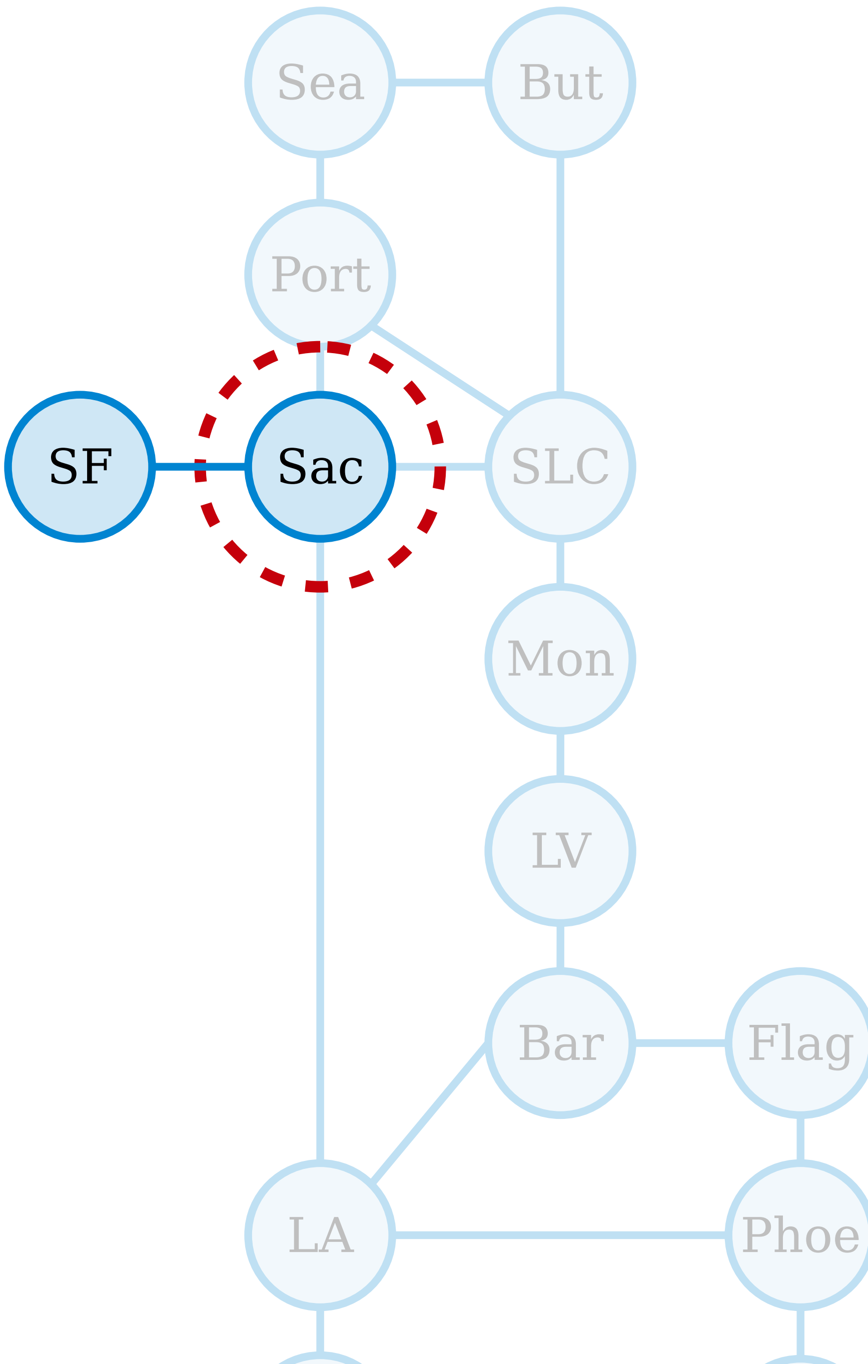
SF



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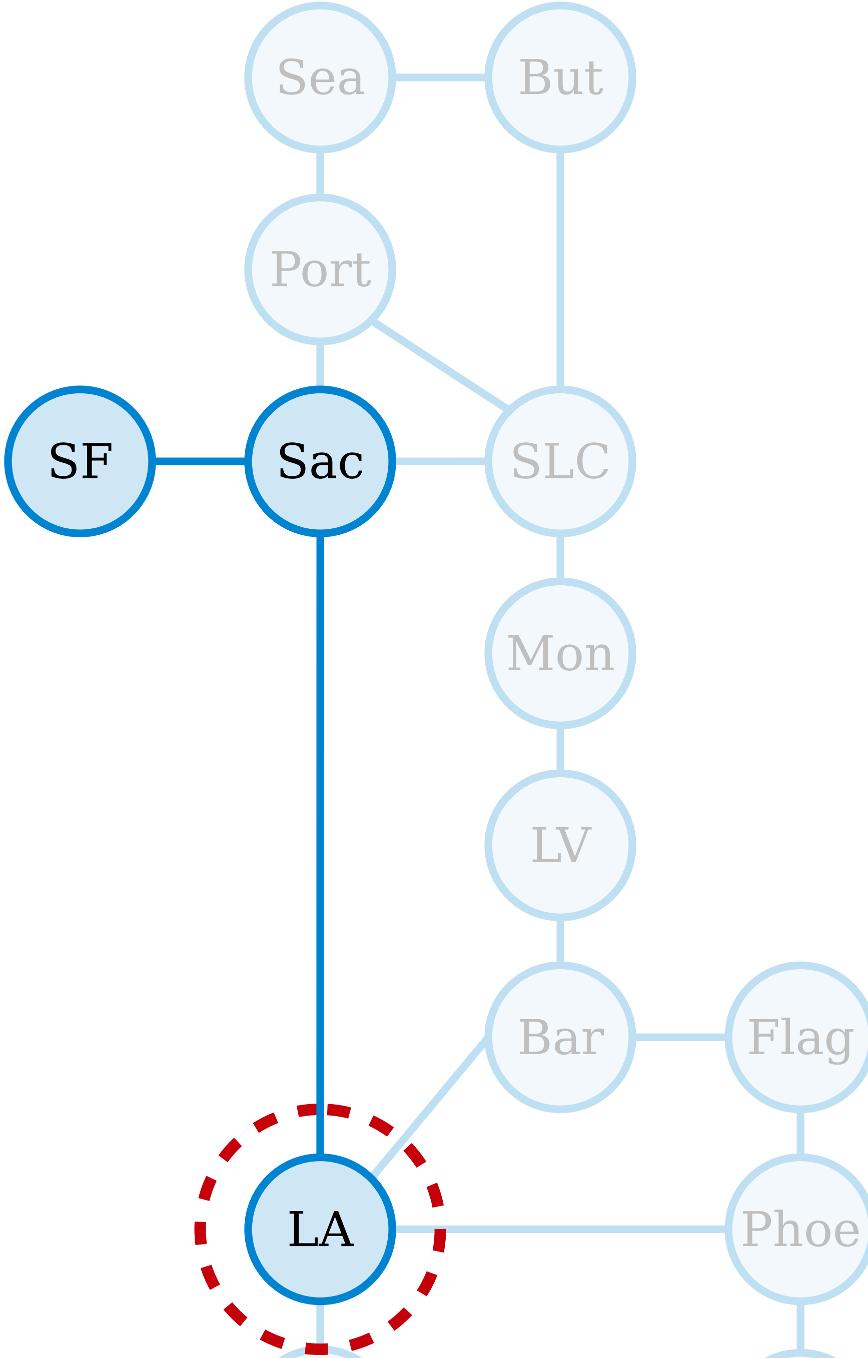
SF, Sac



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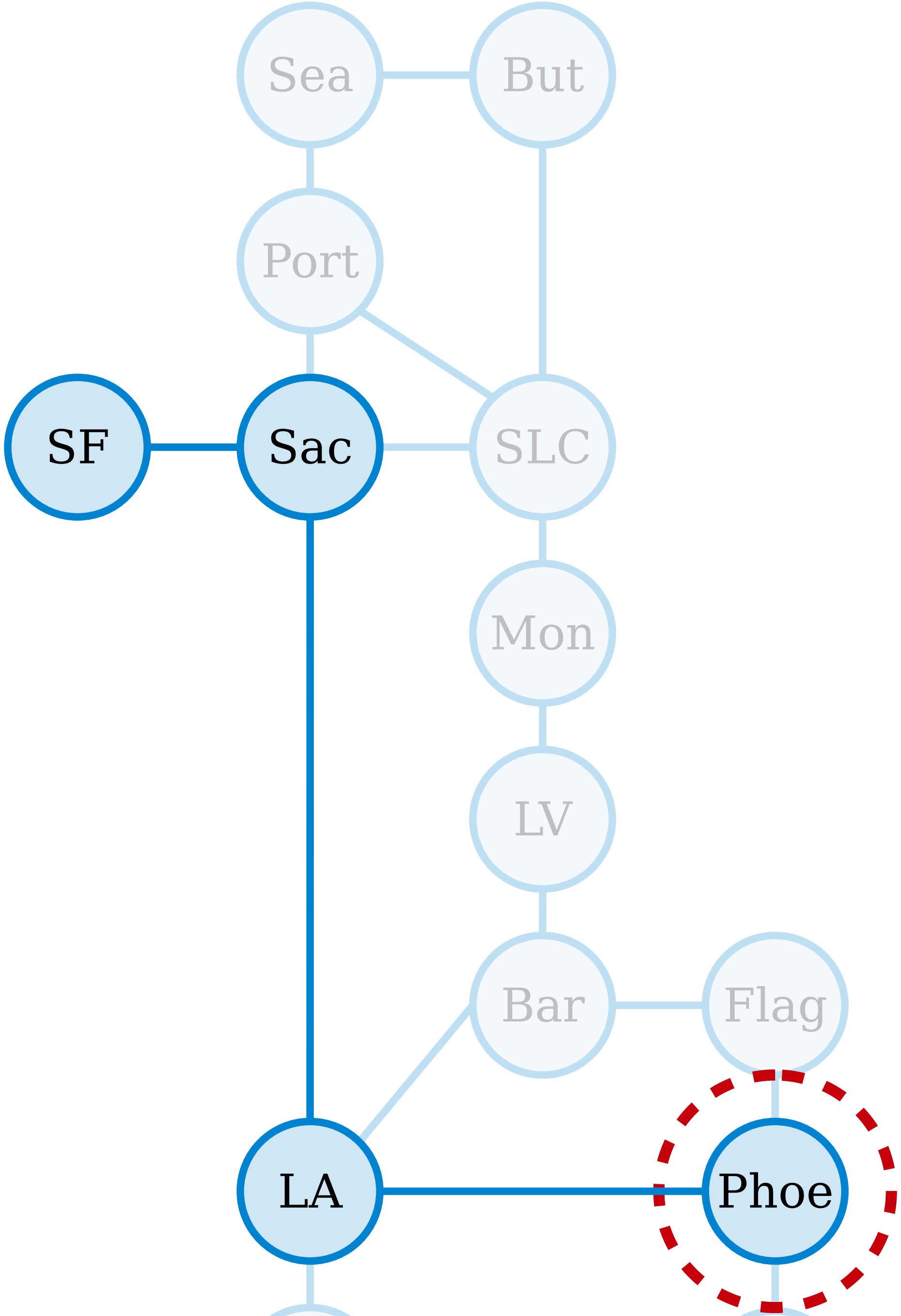
SF, Sac, LA



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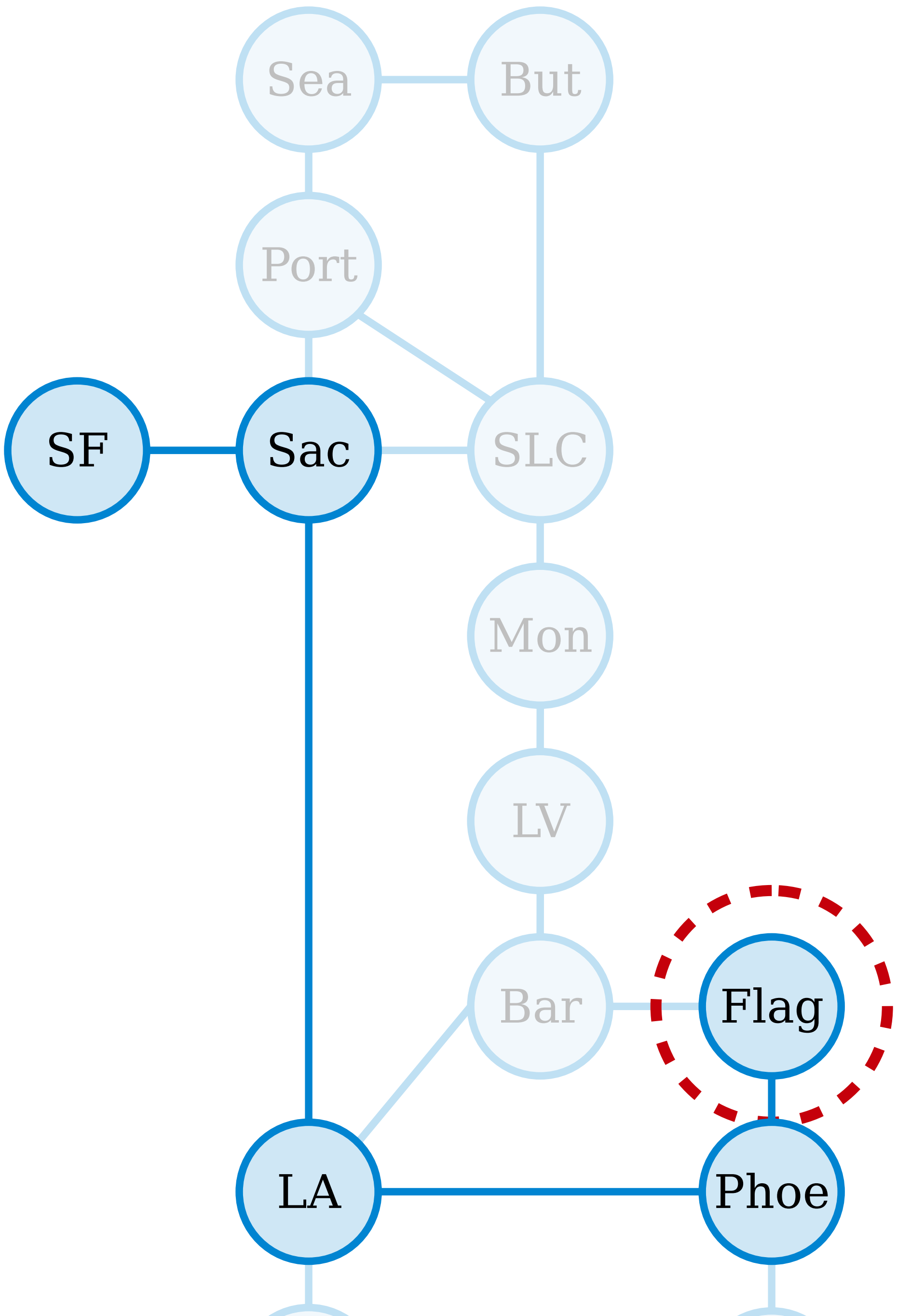
SF, Sac, LA, Phoe



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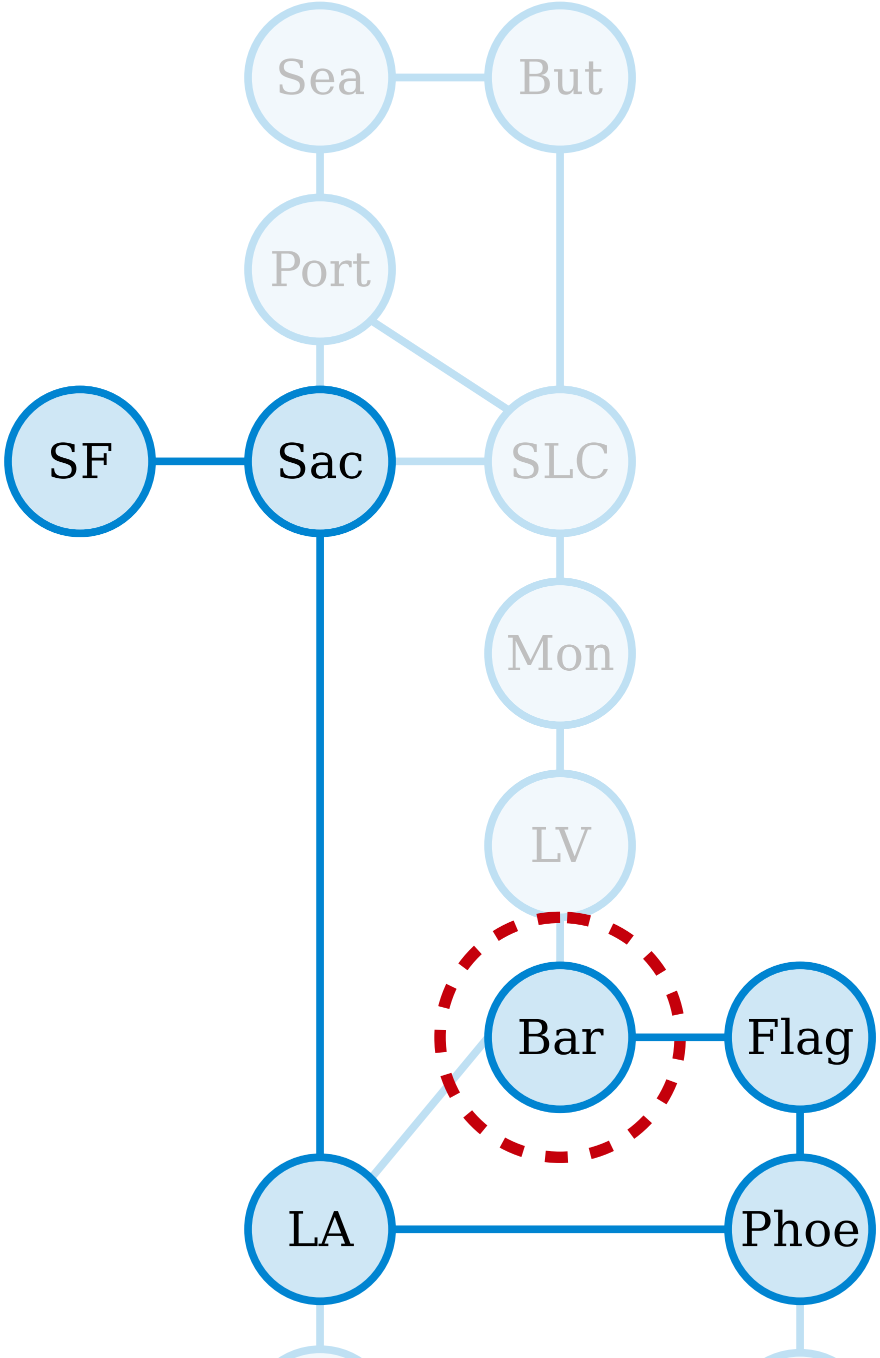
SF, Sac, LA, Phoe, Flag



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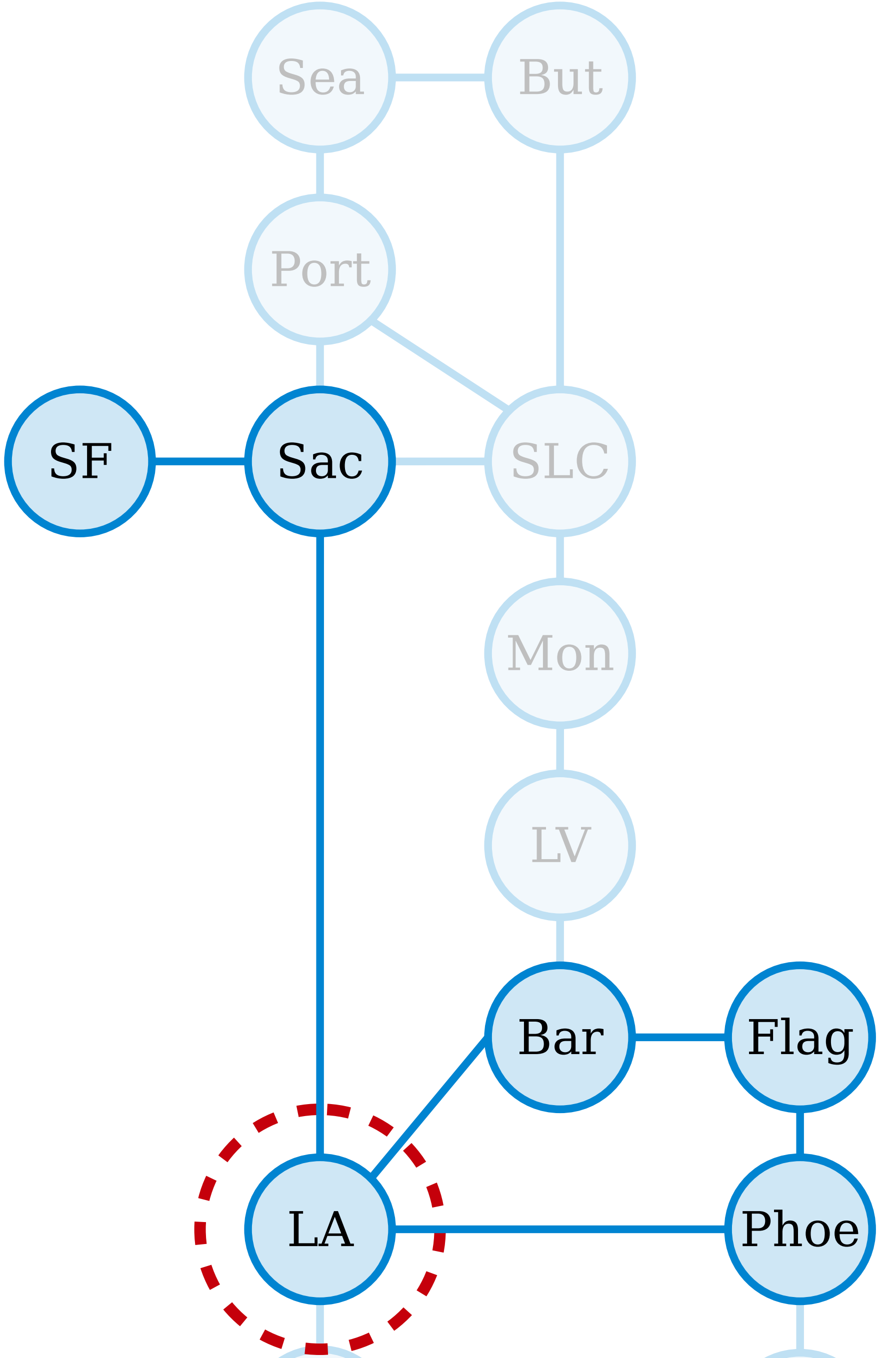
SF, Sac, LA, Phoe, Flag, Bar



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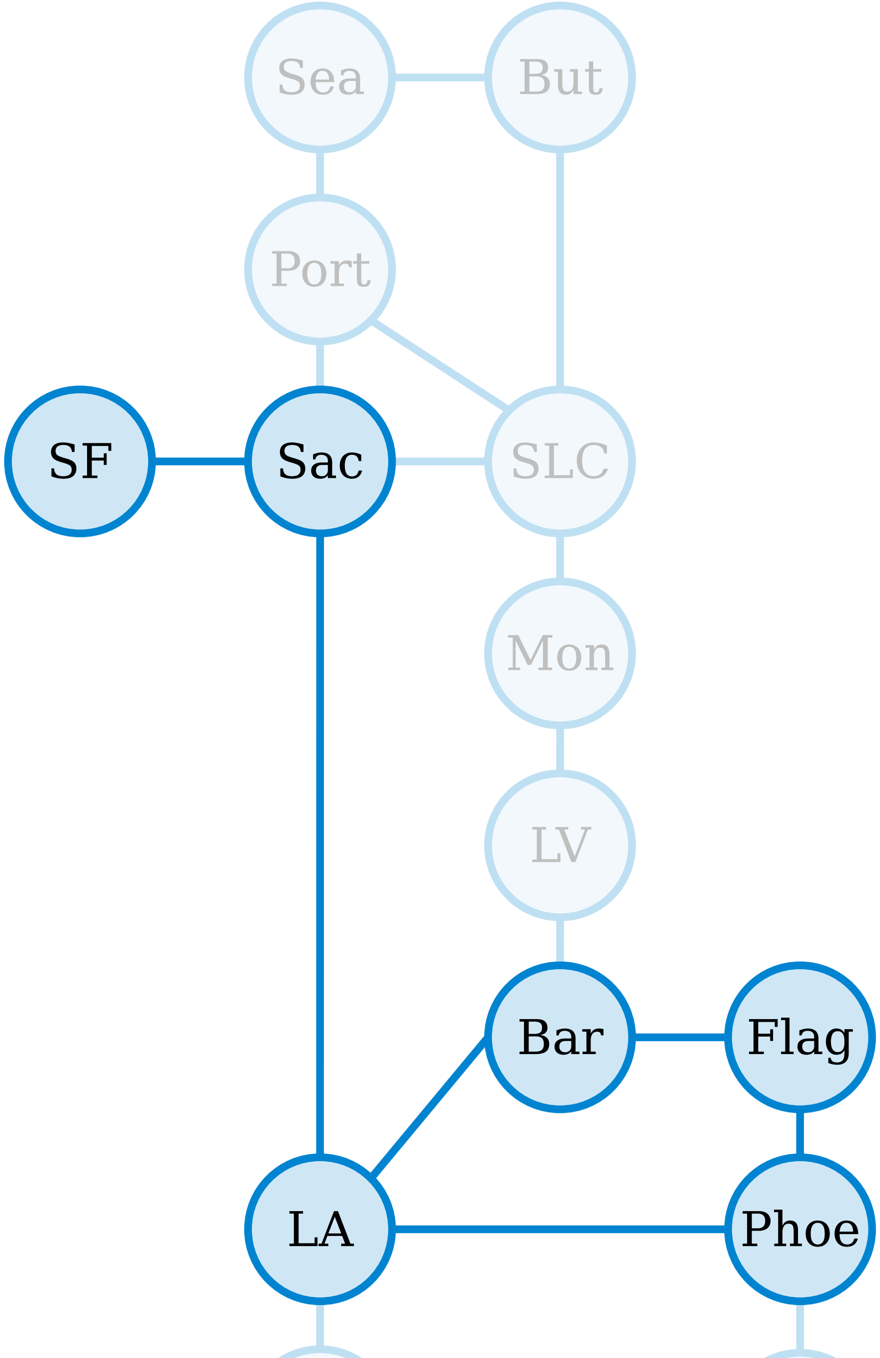
SF, Sac, LA, Phoe, Flag, Bar, LA



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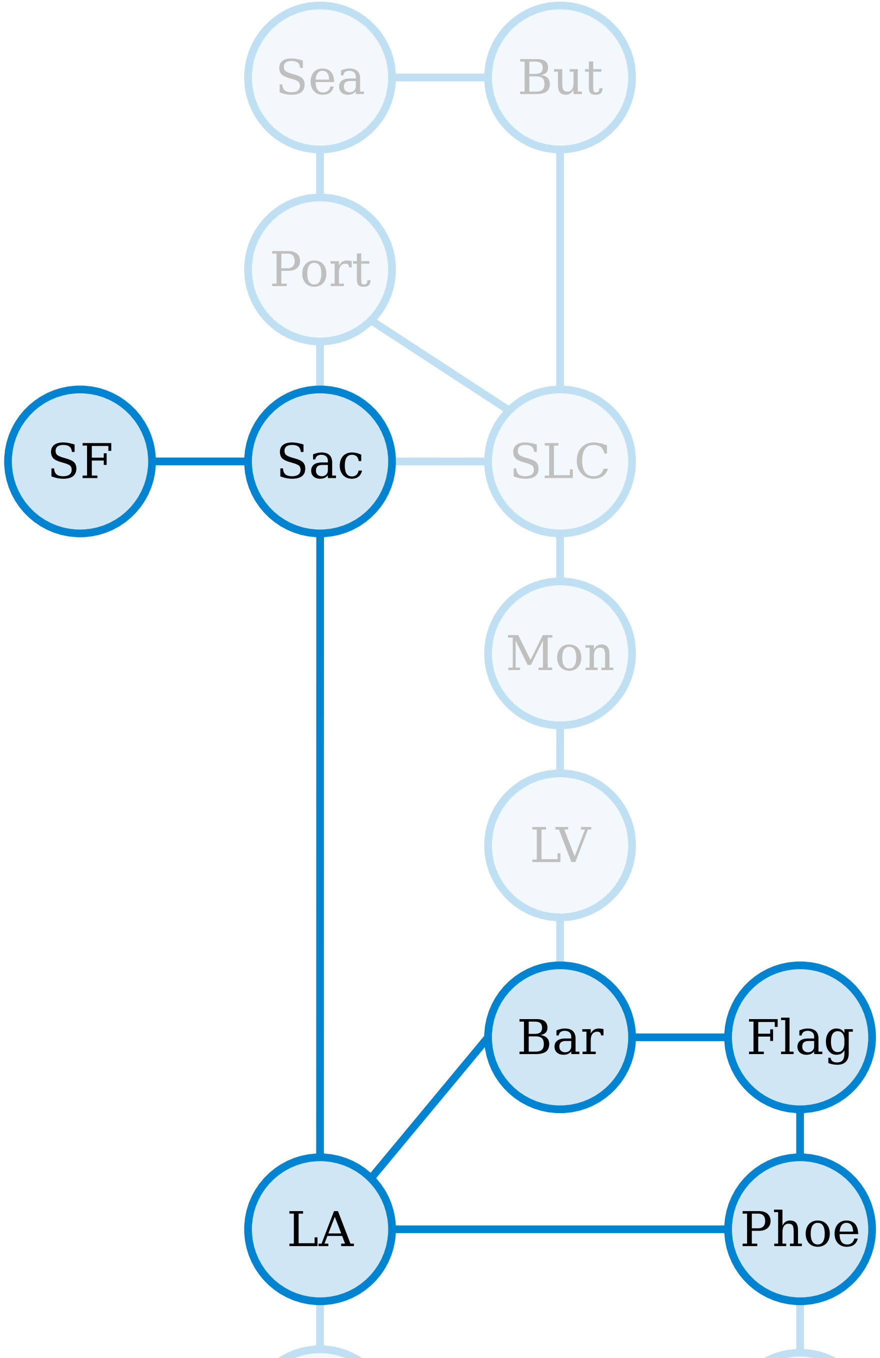
SF, Sac, LA, Phoe, Flag, Bar, LA



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The ***length*** of the path $v_1, \ldots, v_n$ is $n - 1$.

A ***cycle*** in a graph is a path from a node back to itself. (By convention, a cycle cannot have length zero.)

A ***simple path*** in a graph is path that does not repeat any nodes or edges.

SF, Sac, LA, Phoe, Flag, Bar, LA

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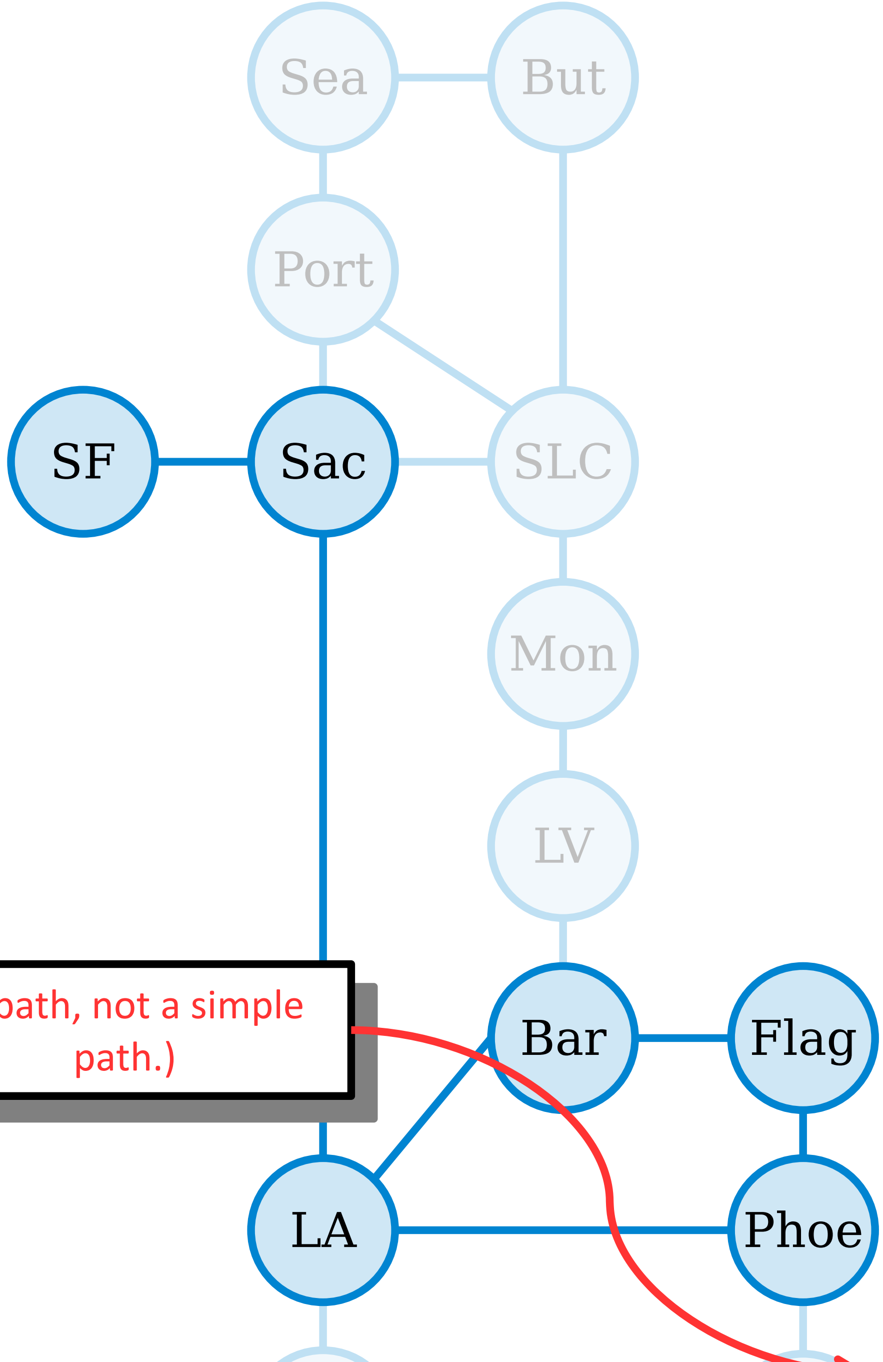
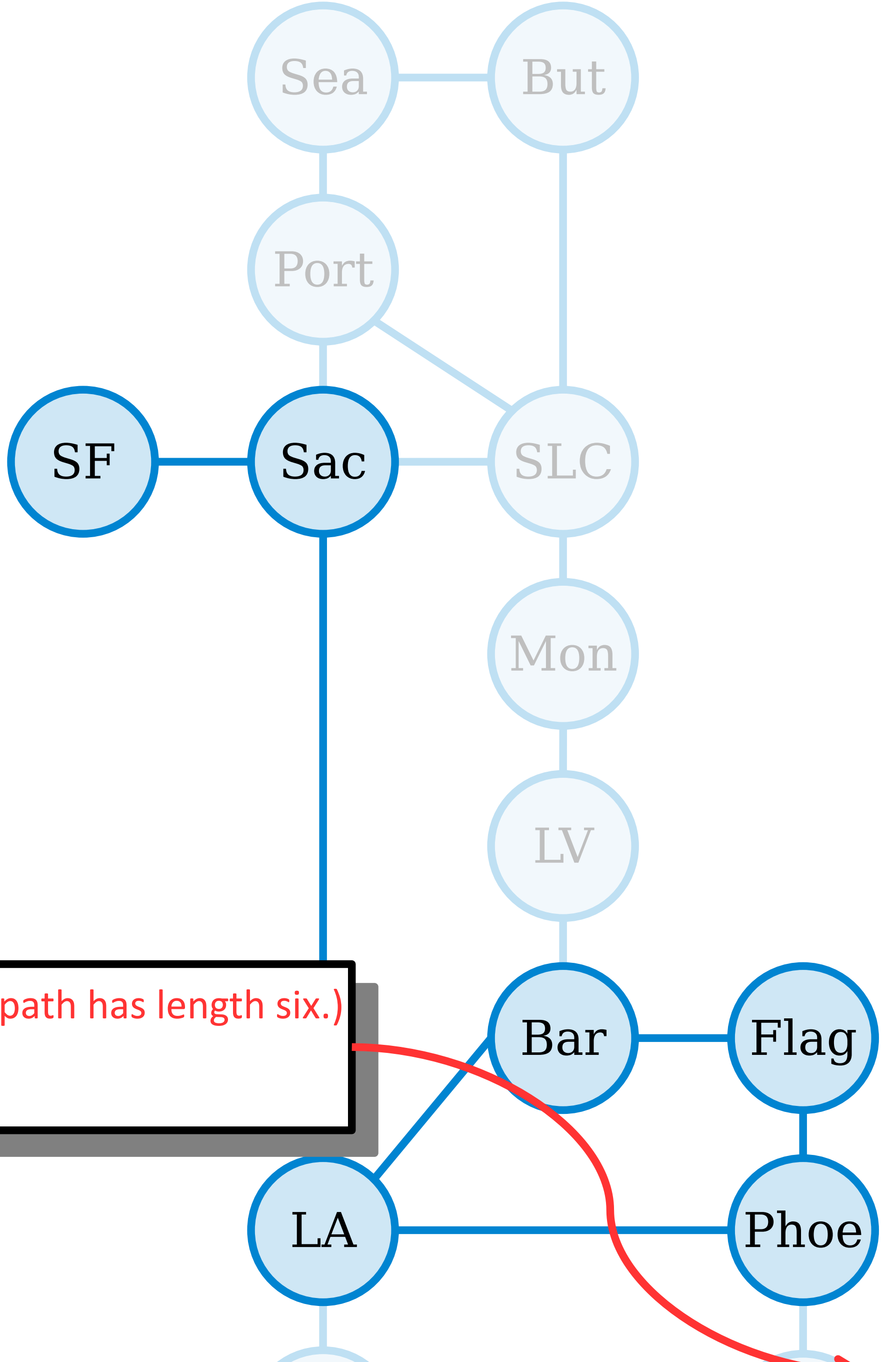


(A path, not a simple path.)

SF, Sac, LA, Phoe, Flag, Bar, LA



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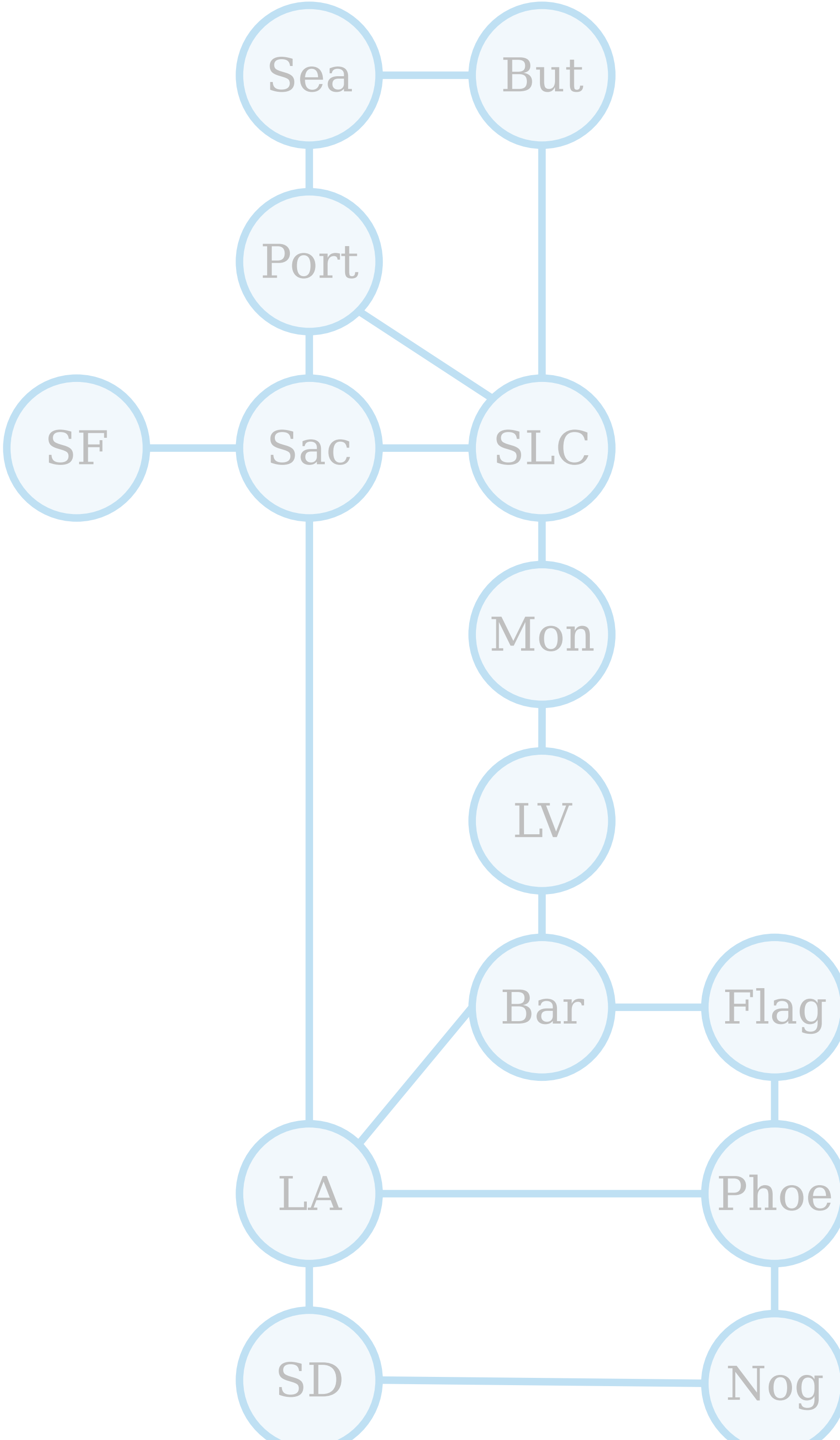
A ***simple path*** in a graph is path that does not repeat any nodes or edges.

(This path has length six.)

SF, Sac, LA, Phoe, Flag, Bar, LA



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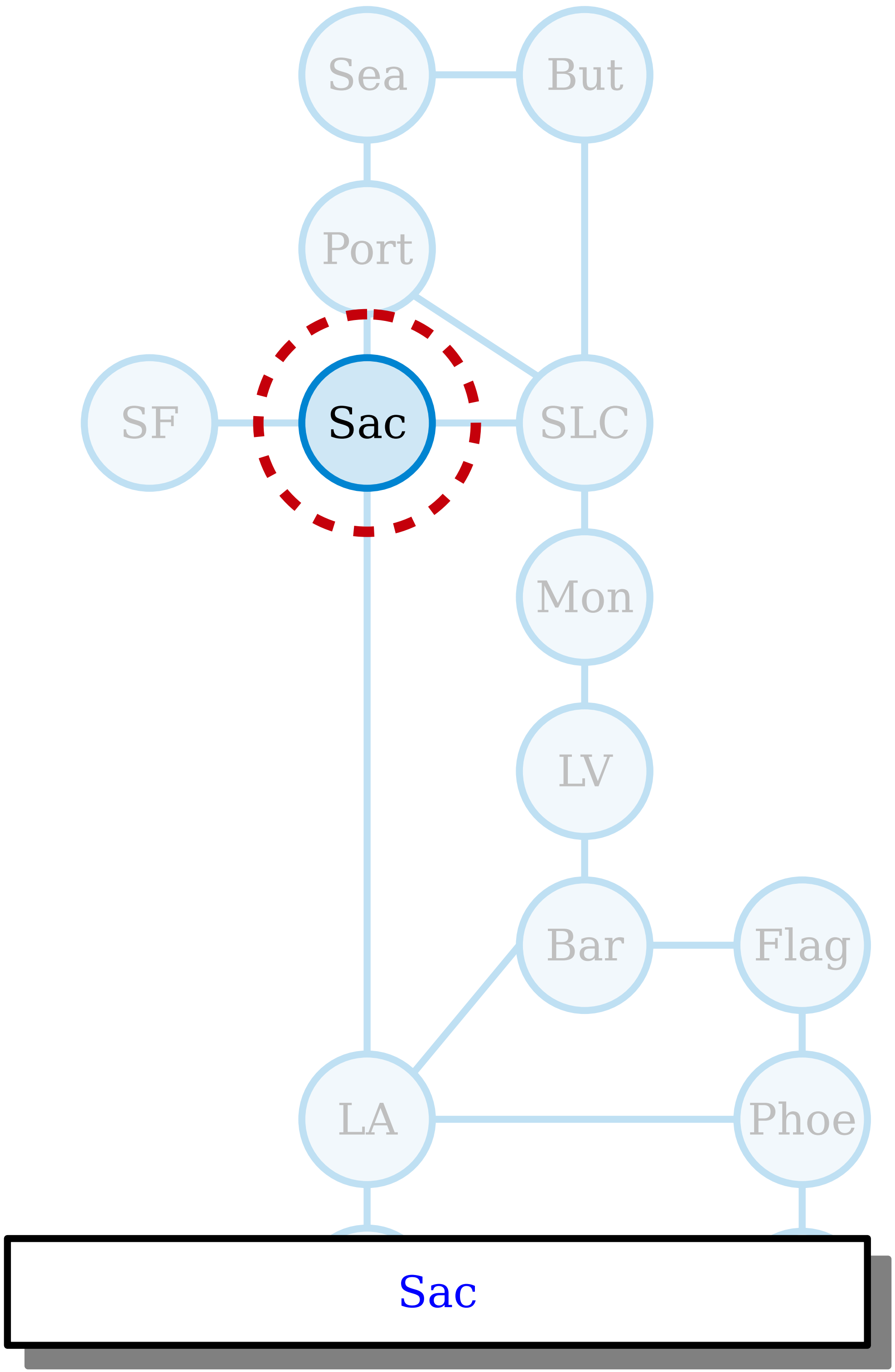

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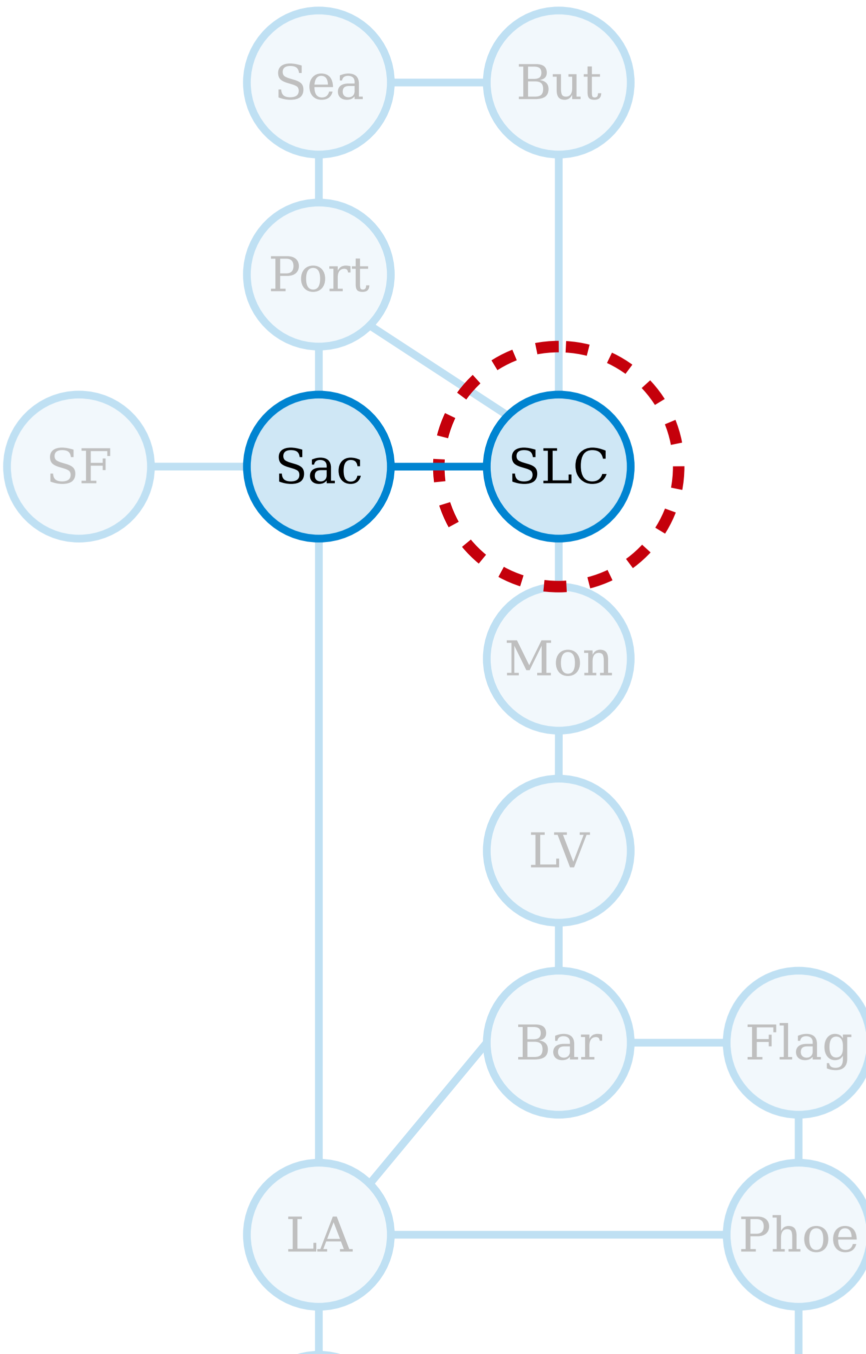

Sac, SLC

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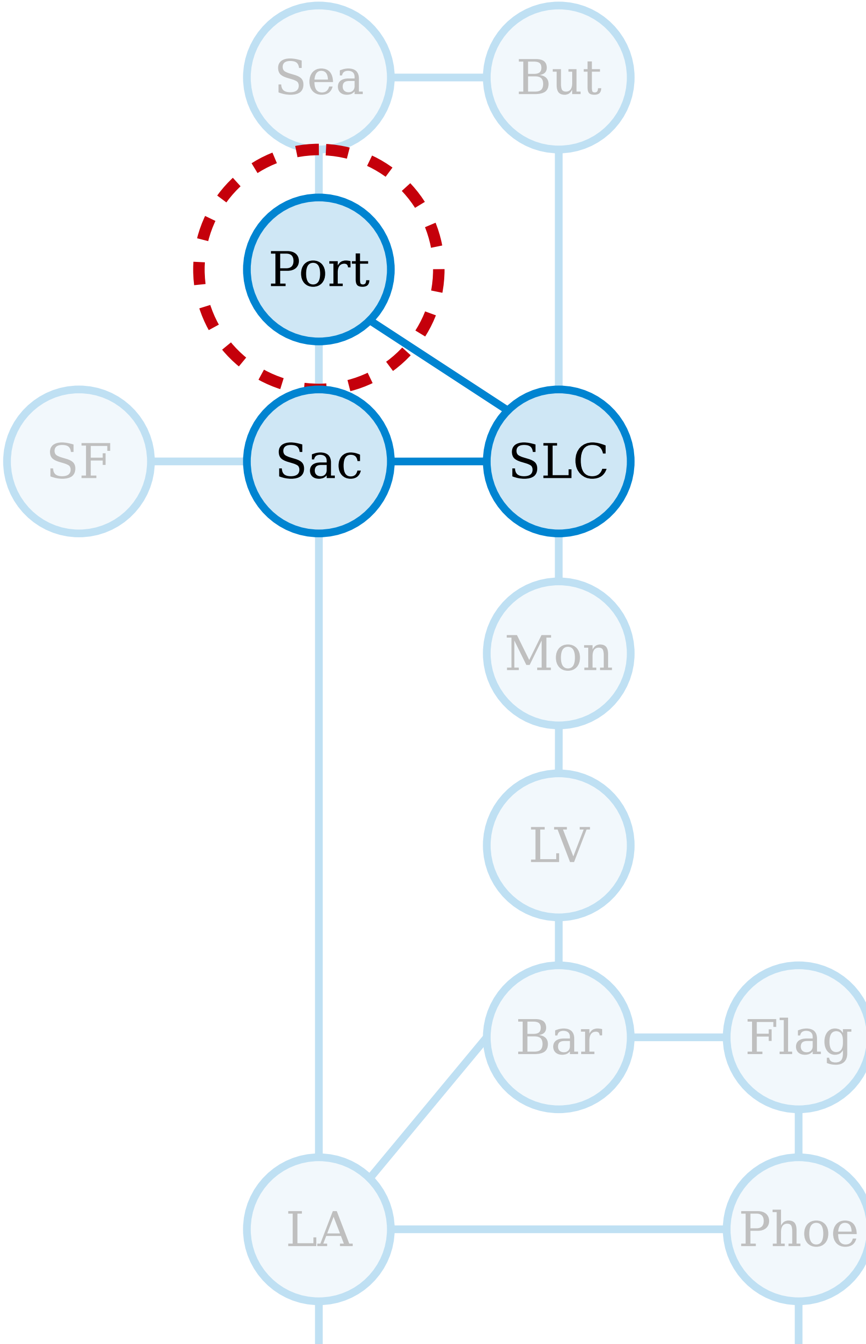

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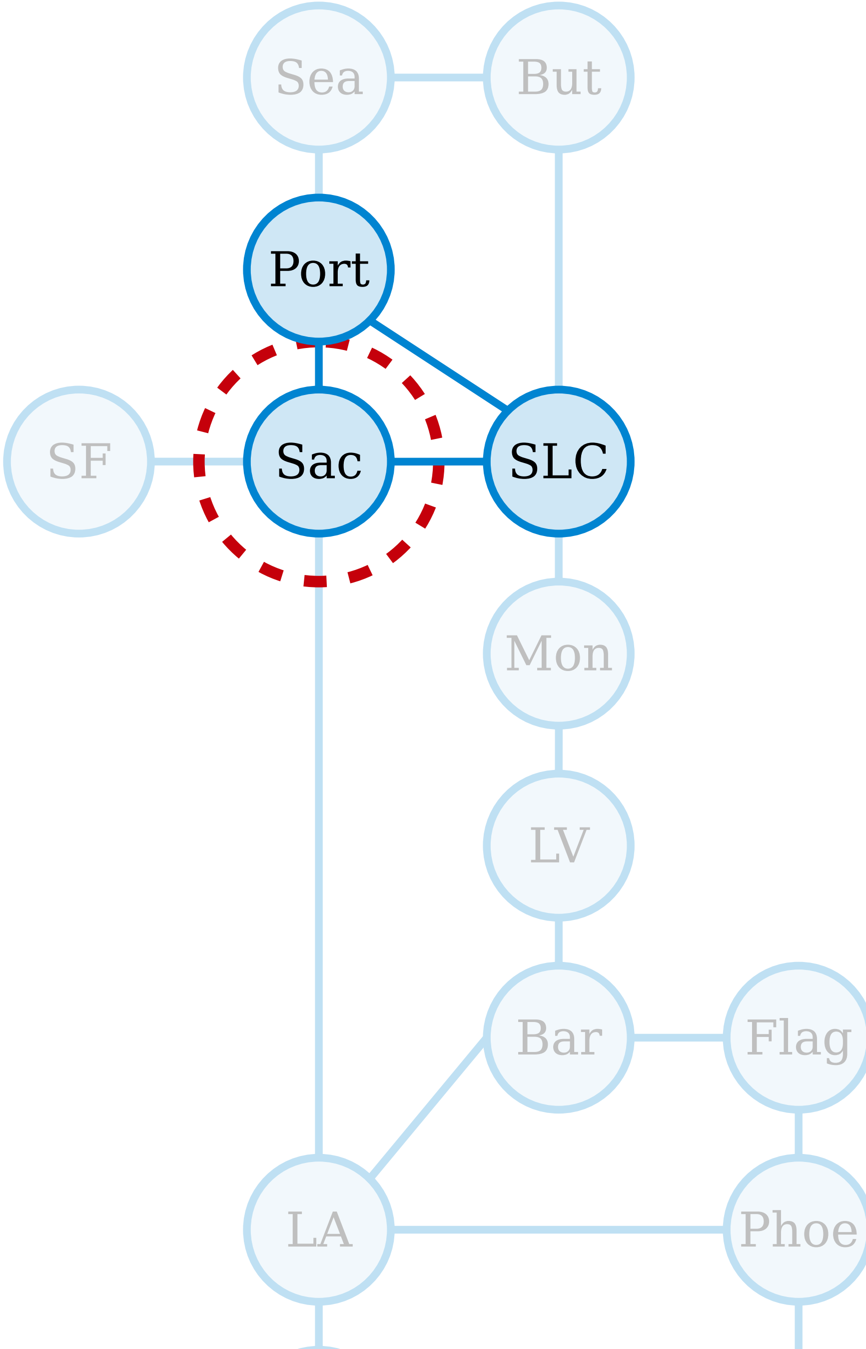

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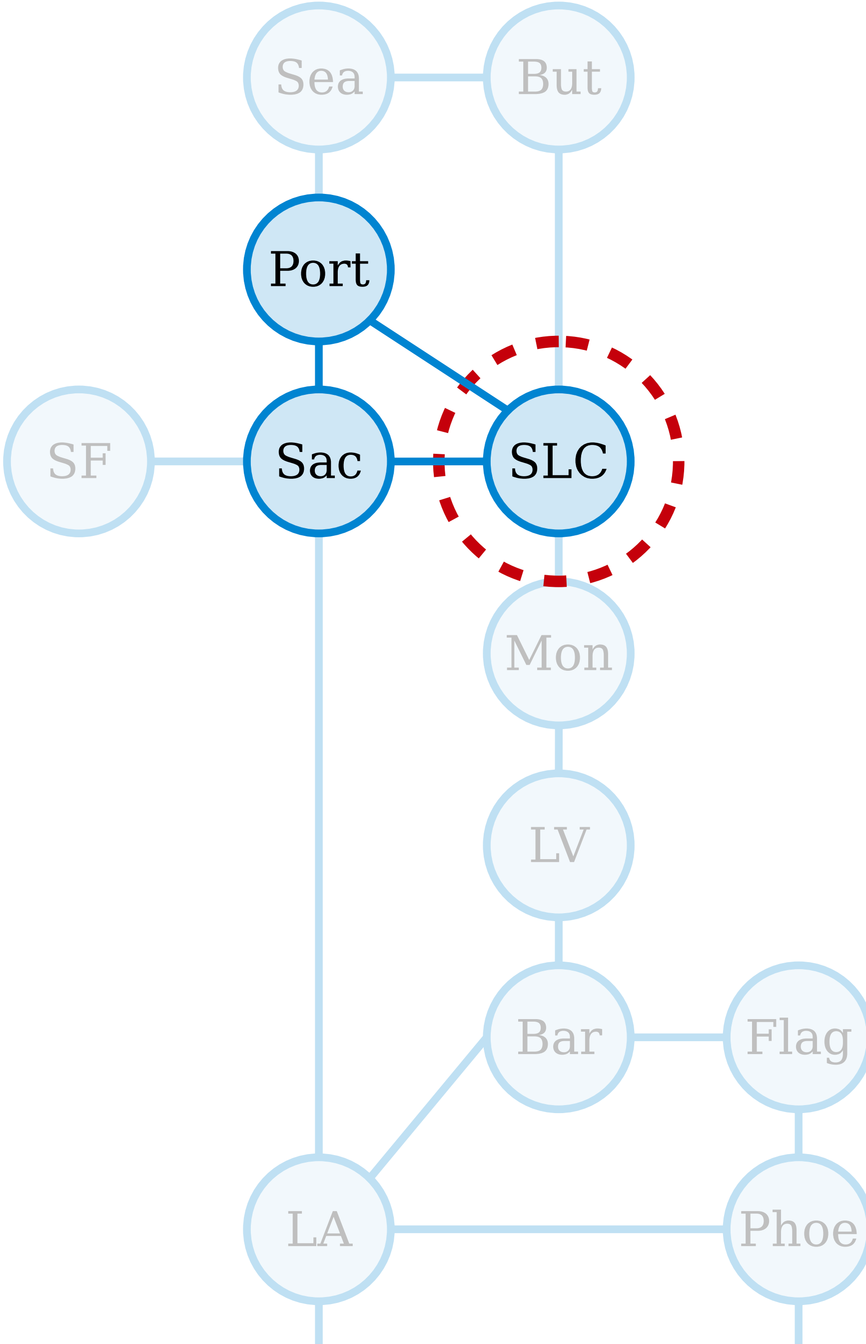

Sac, SLC, Port, Sac, SLC

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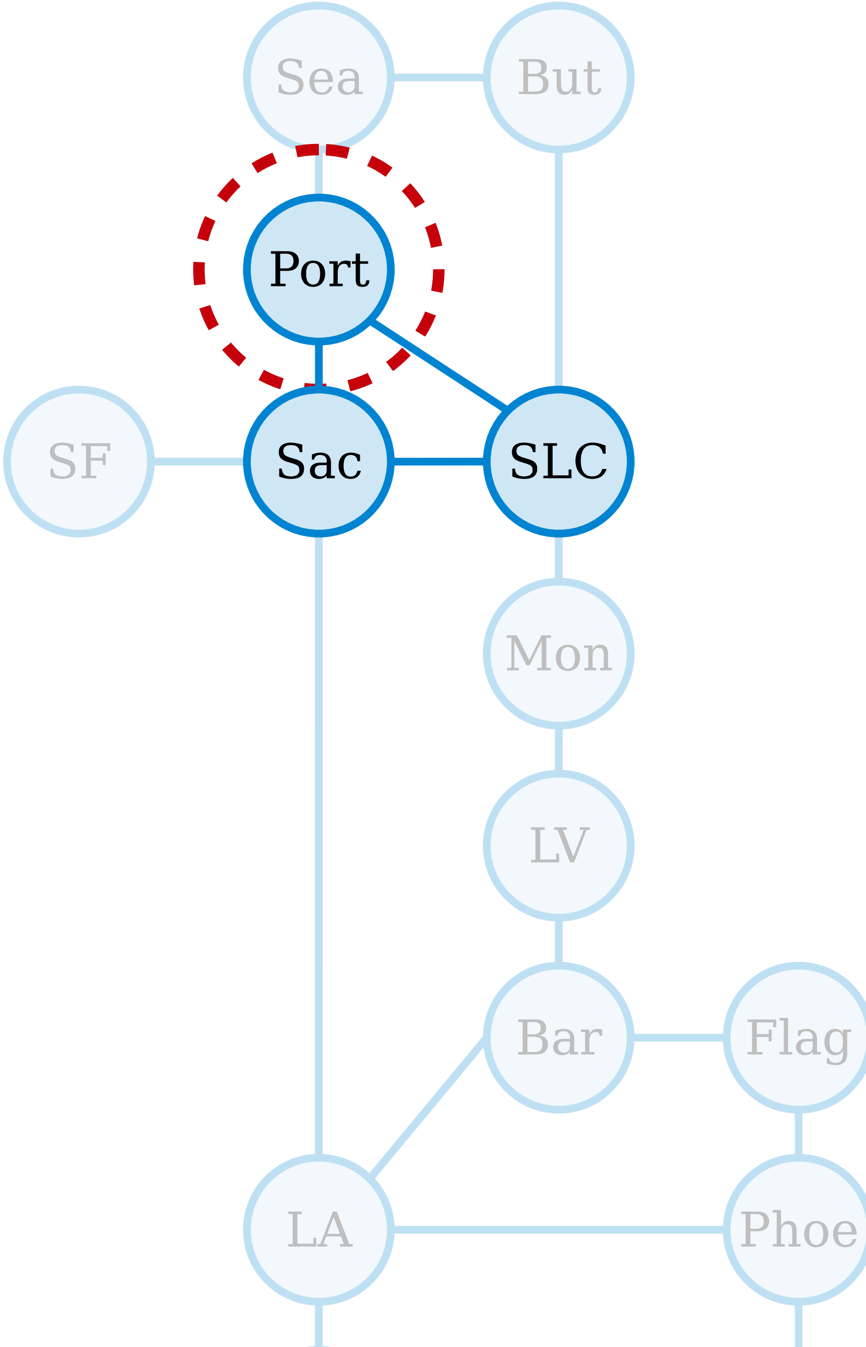

Sac, SLC, Port, Sac, SLC, Port

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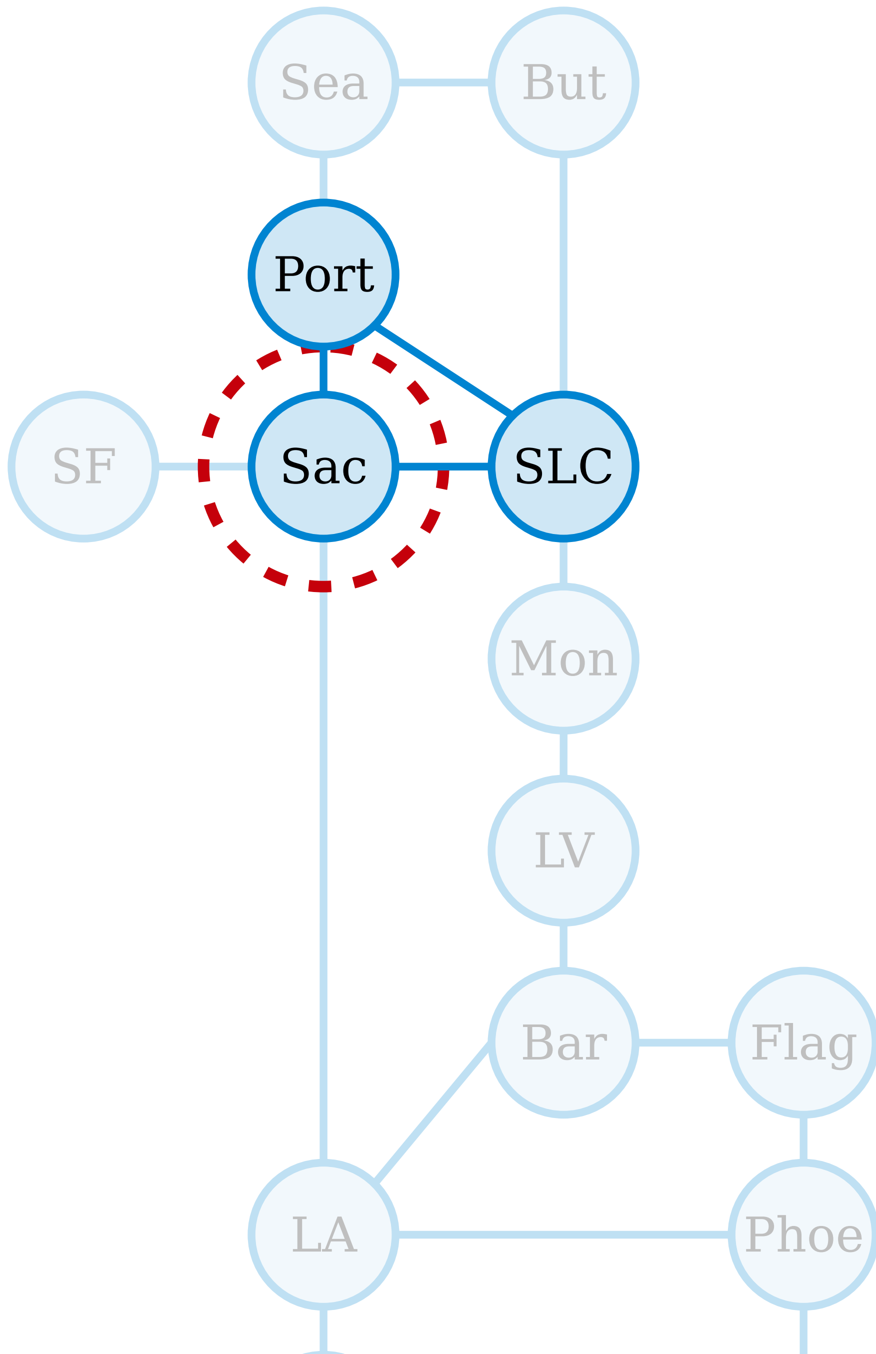

Sac, SLC, Port, Sac, SLC, Port, Sac

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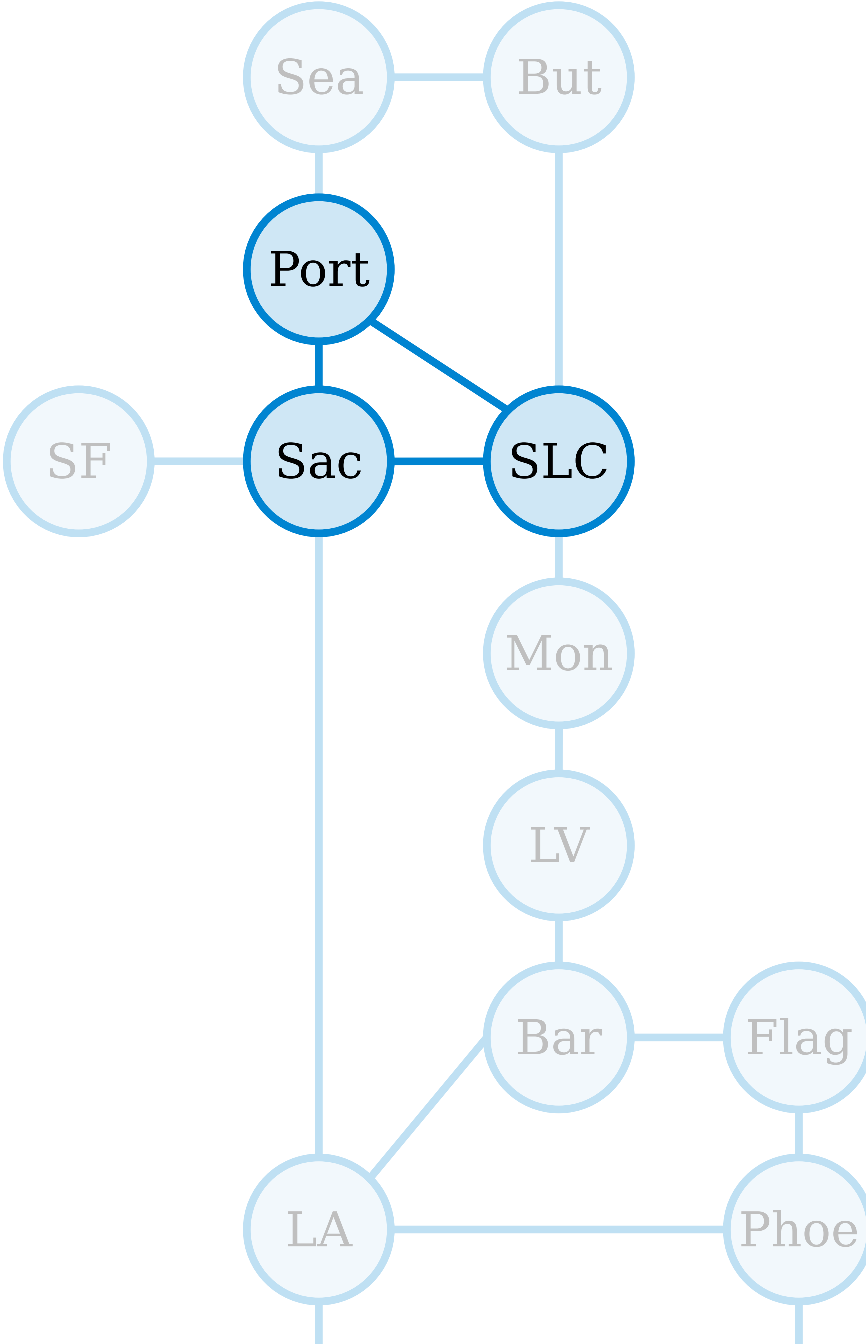

Sac, SLC, Port, Sac, SLC, Port, Sac

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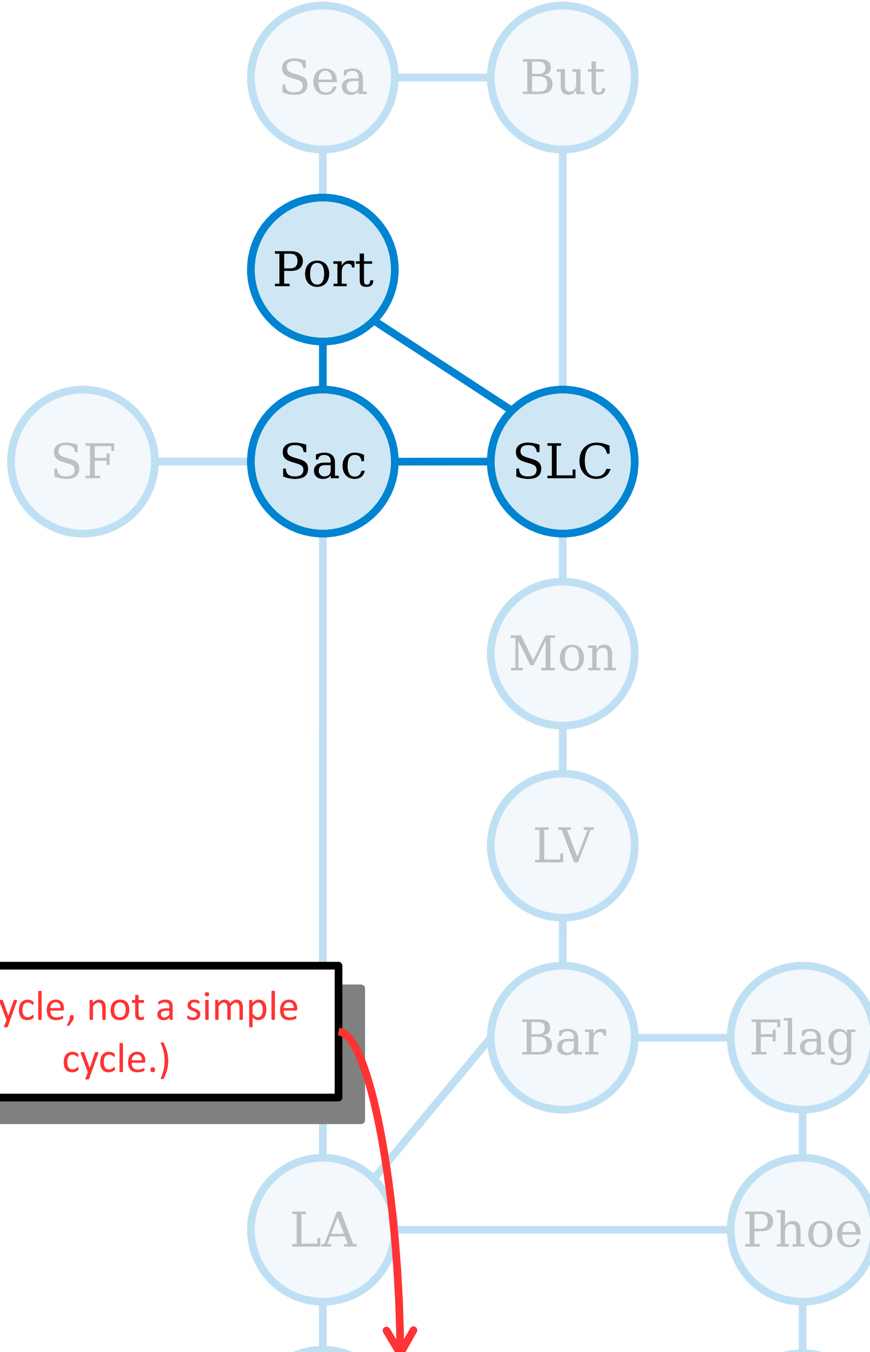

(A cycle, not a simple cycle.)

Sac, SLC, Port, Sac, SLC, Port, Sac

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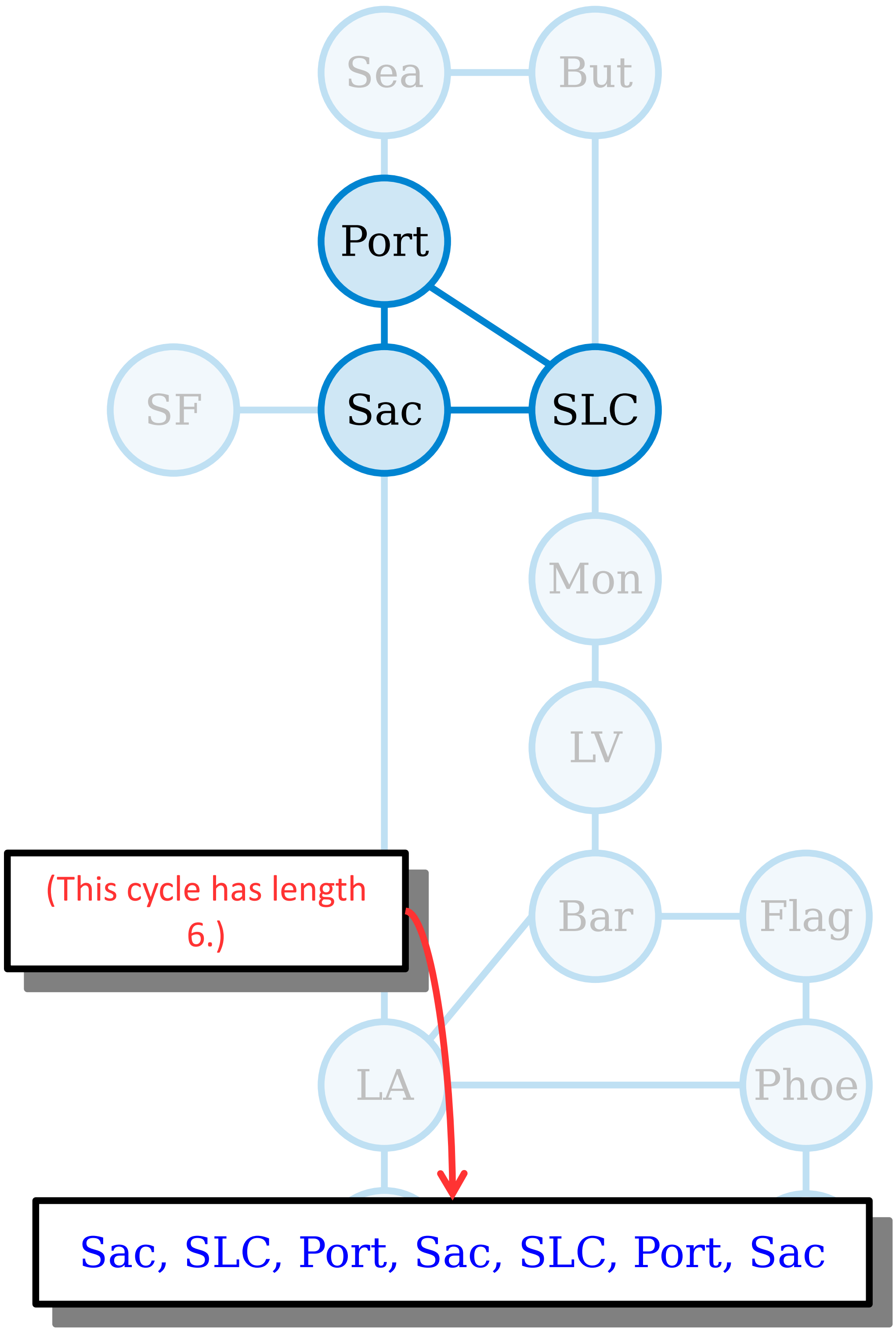

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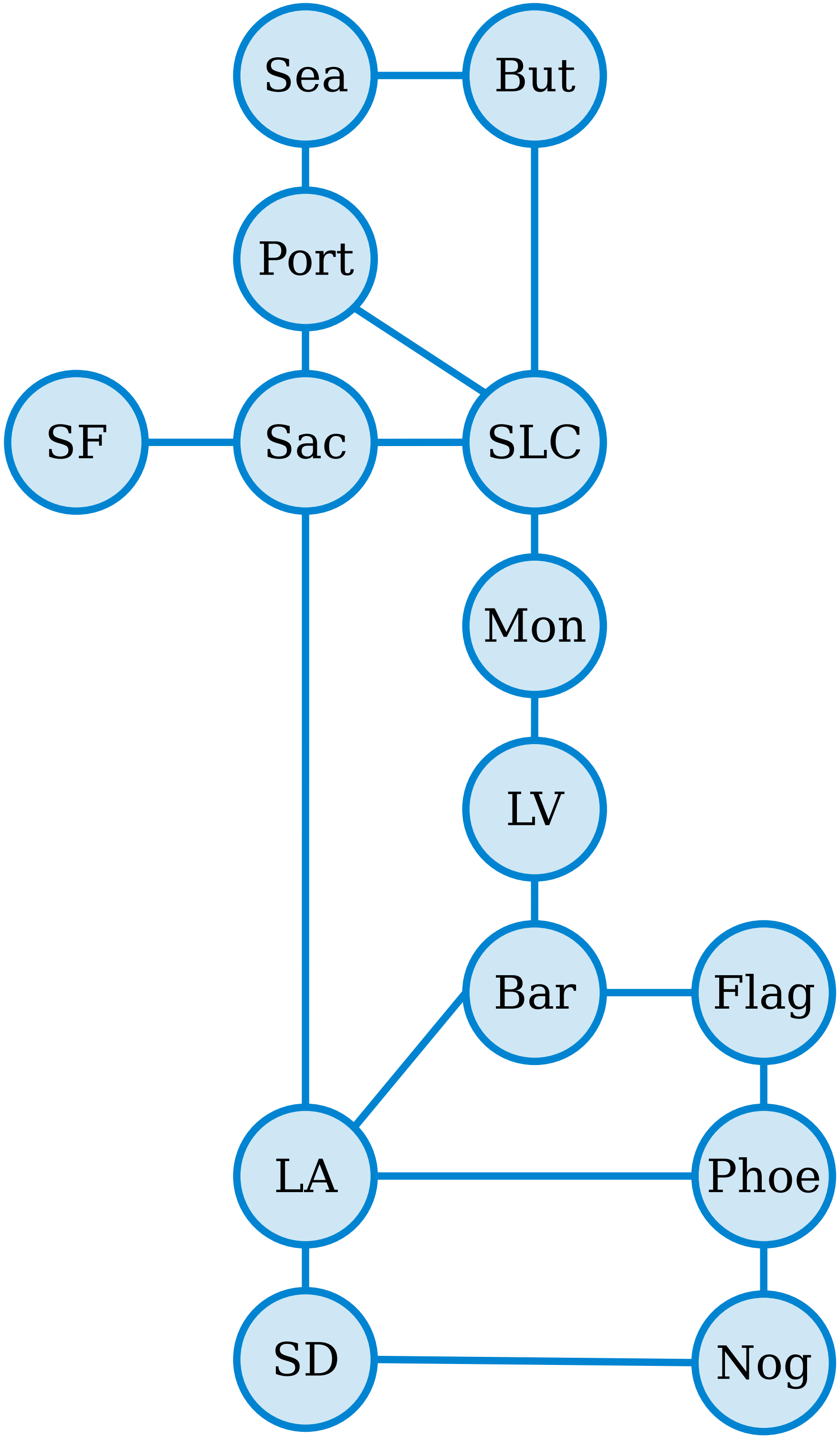

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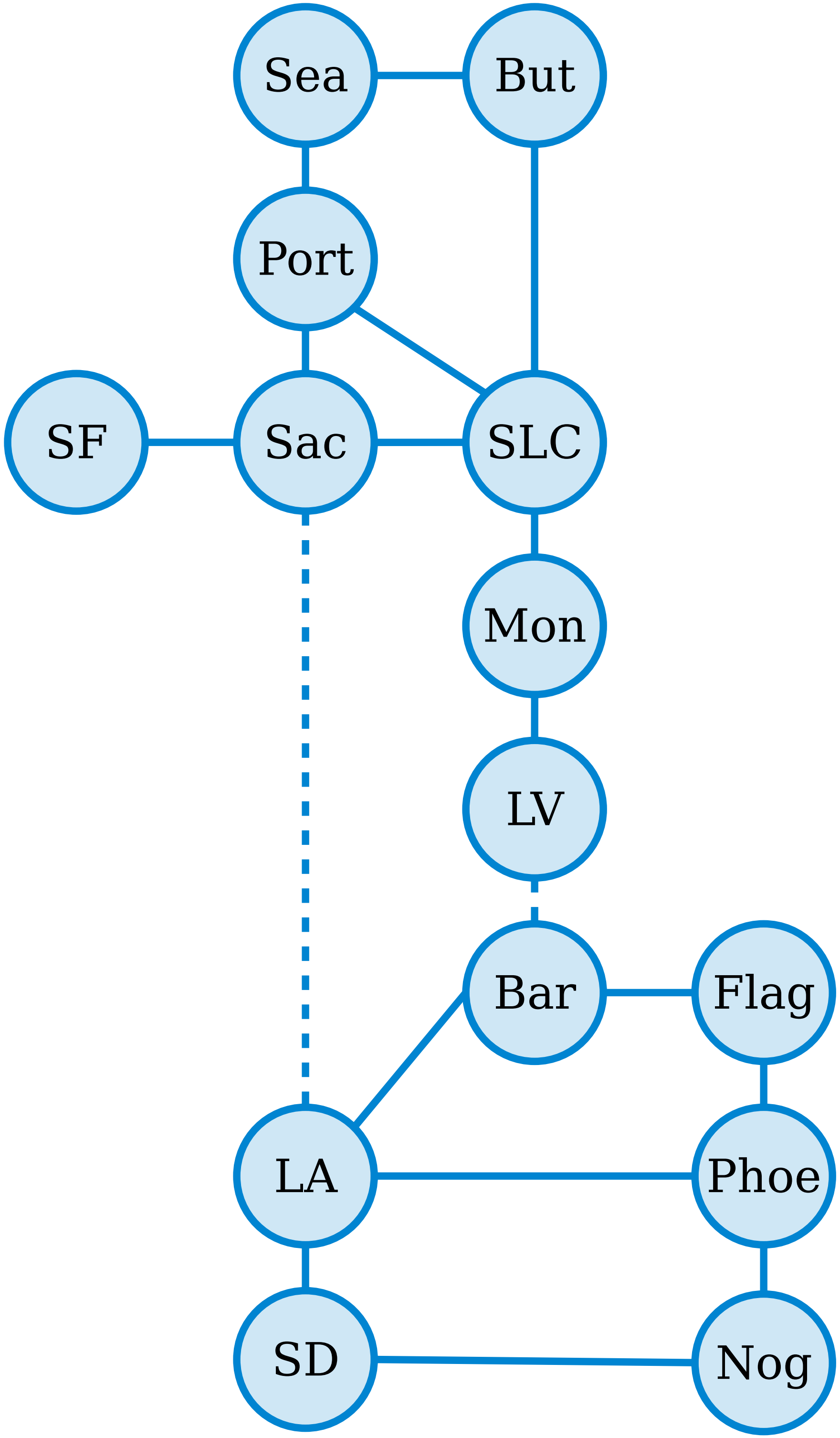

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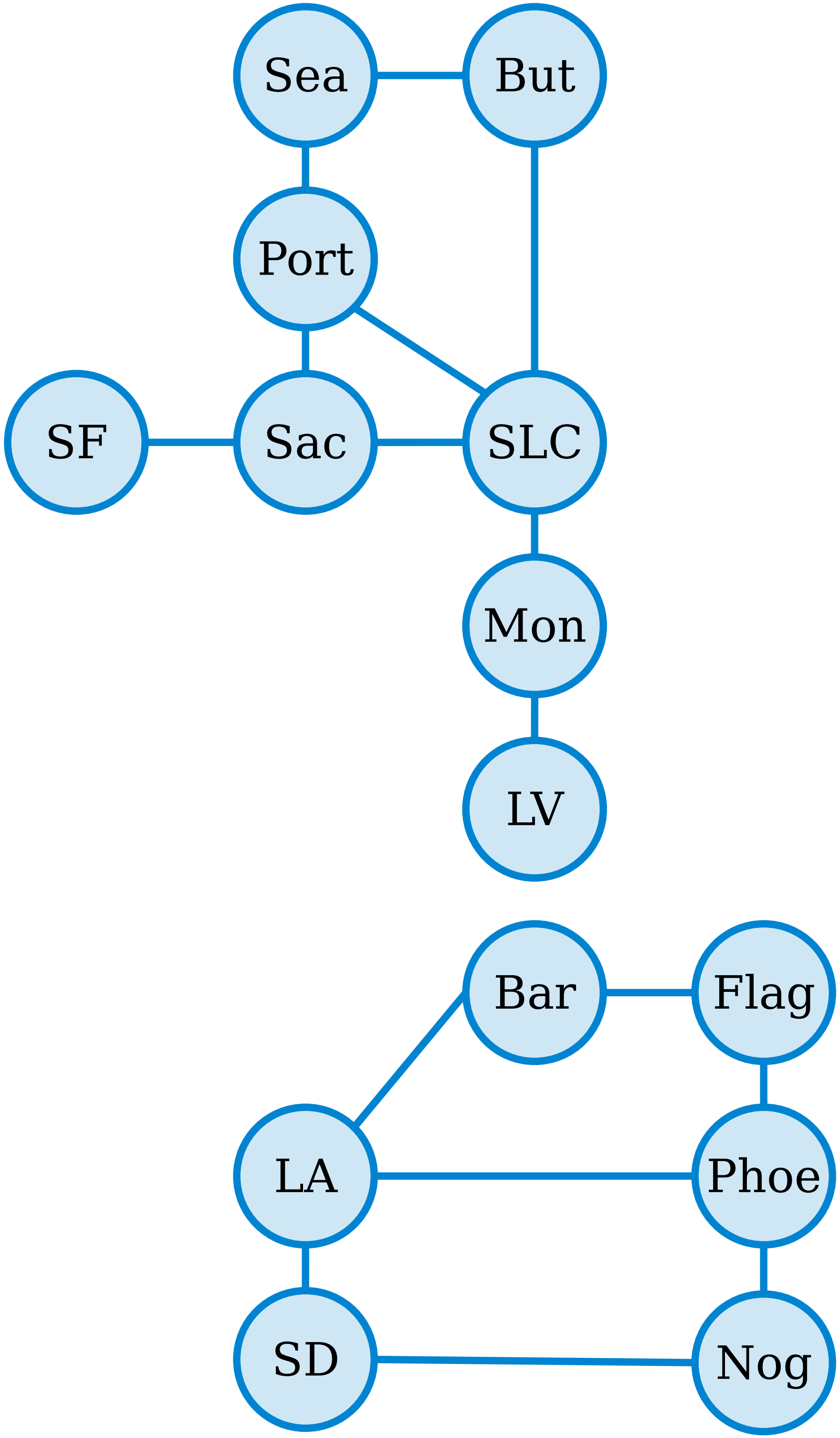

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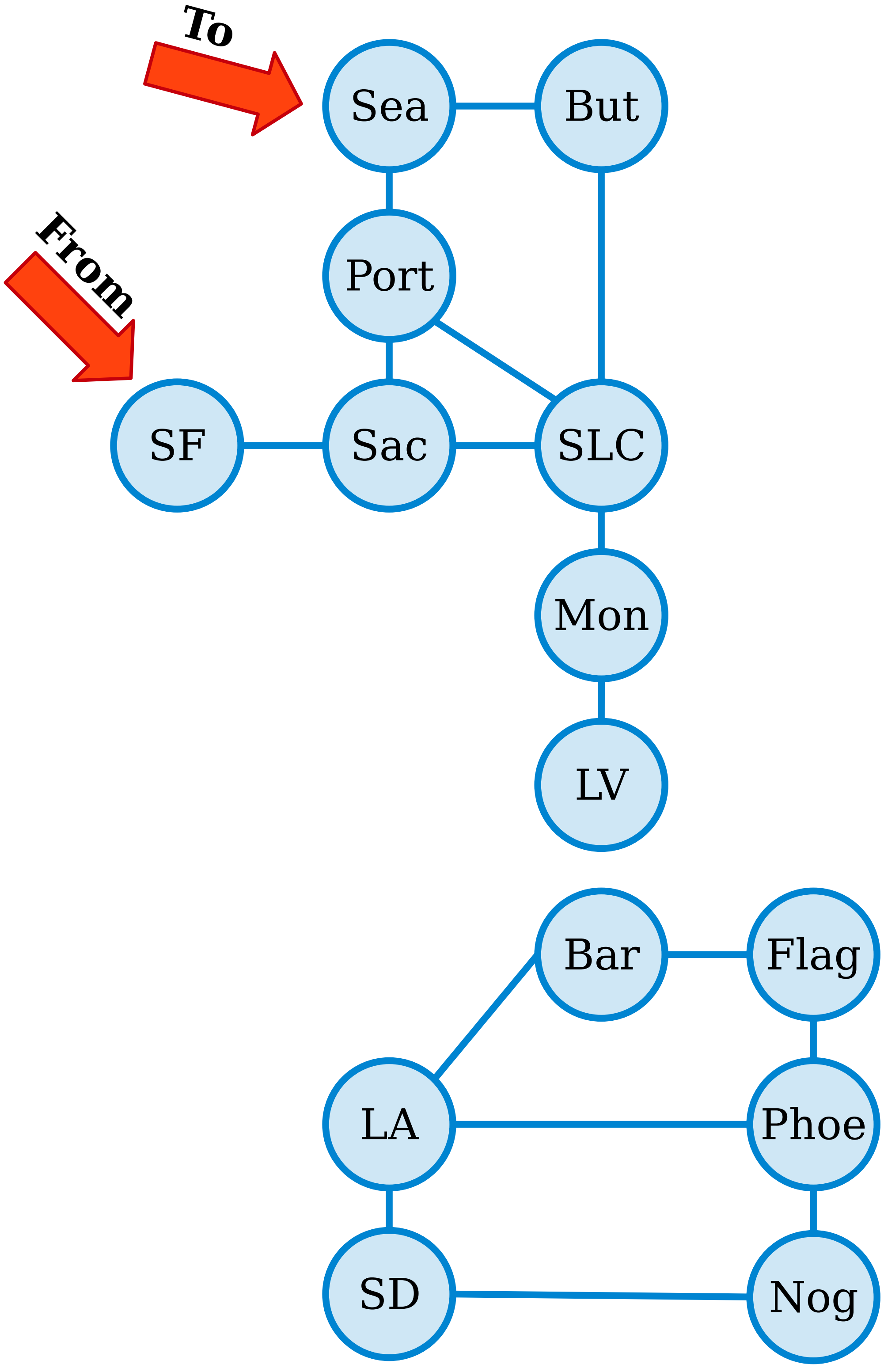

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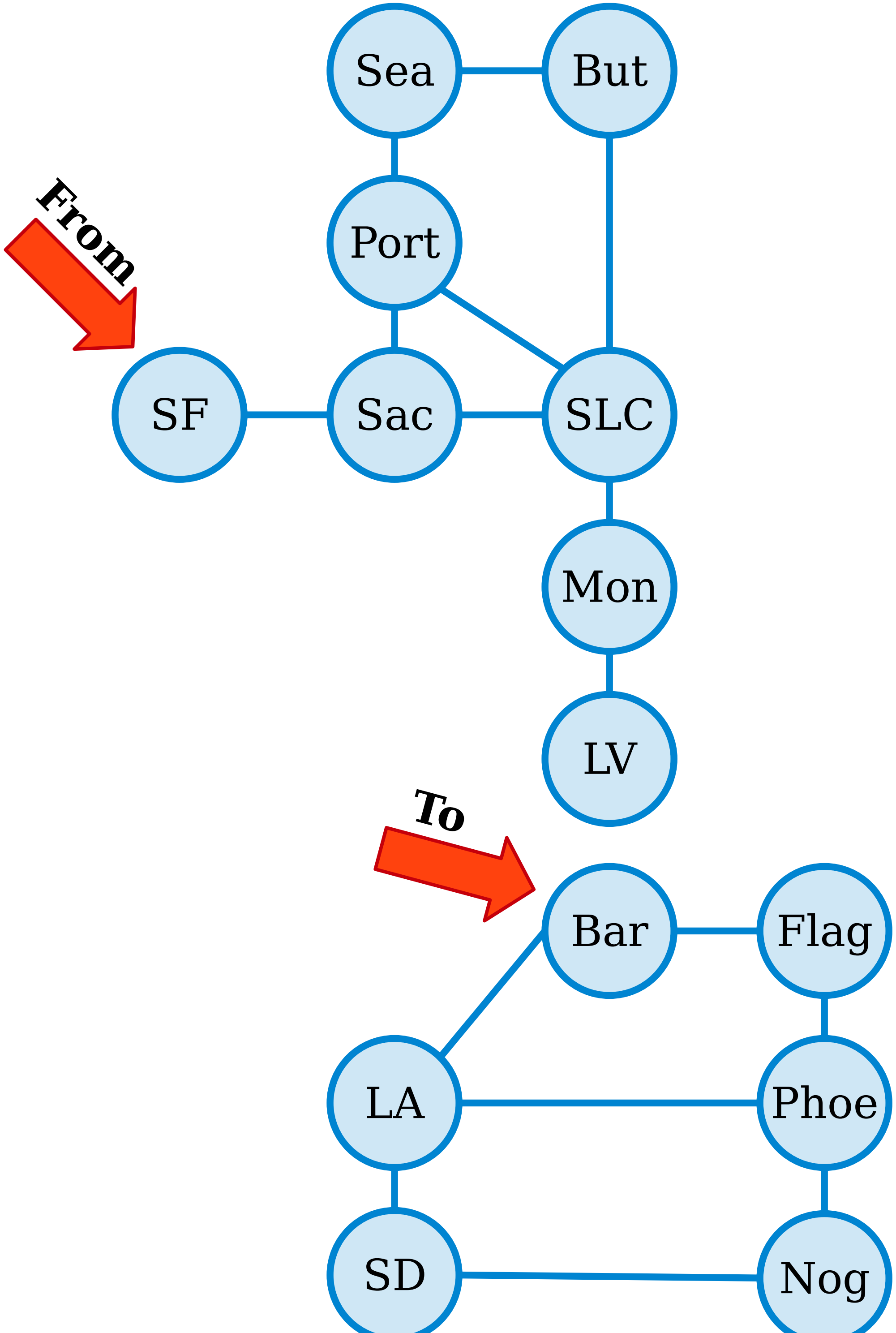

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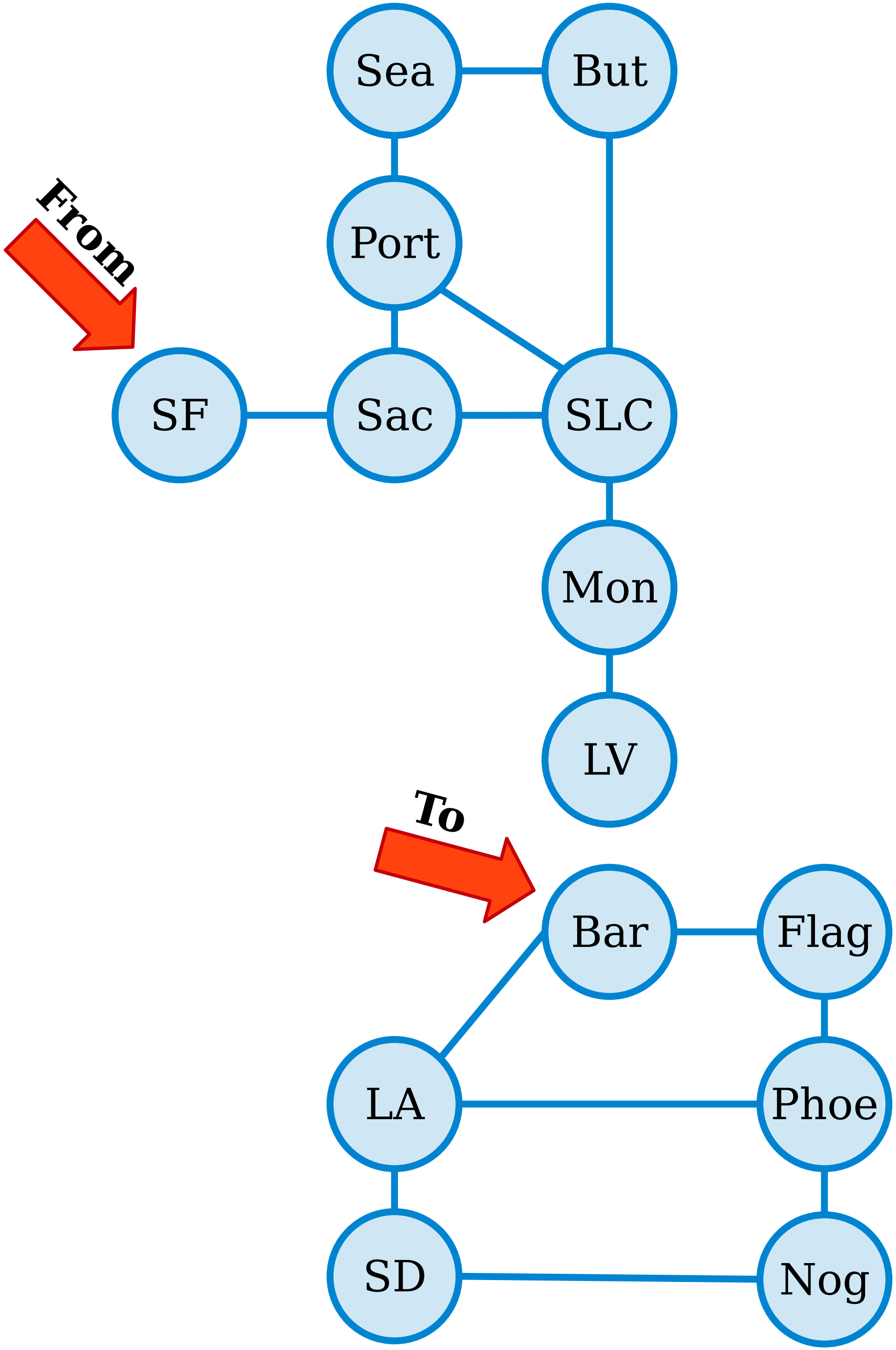

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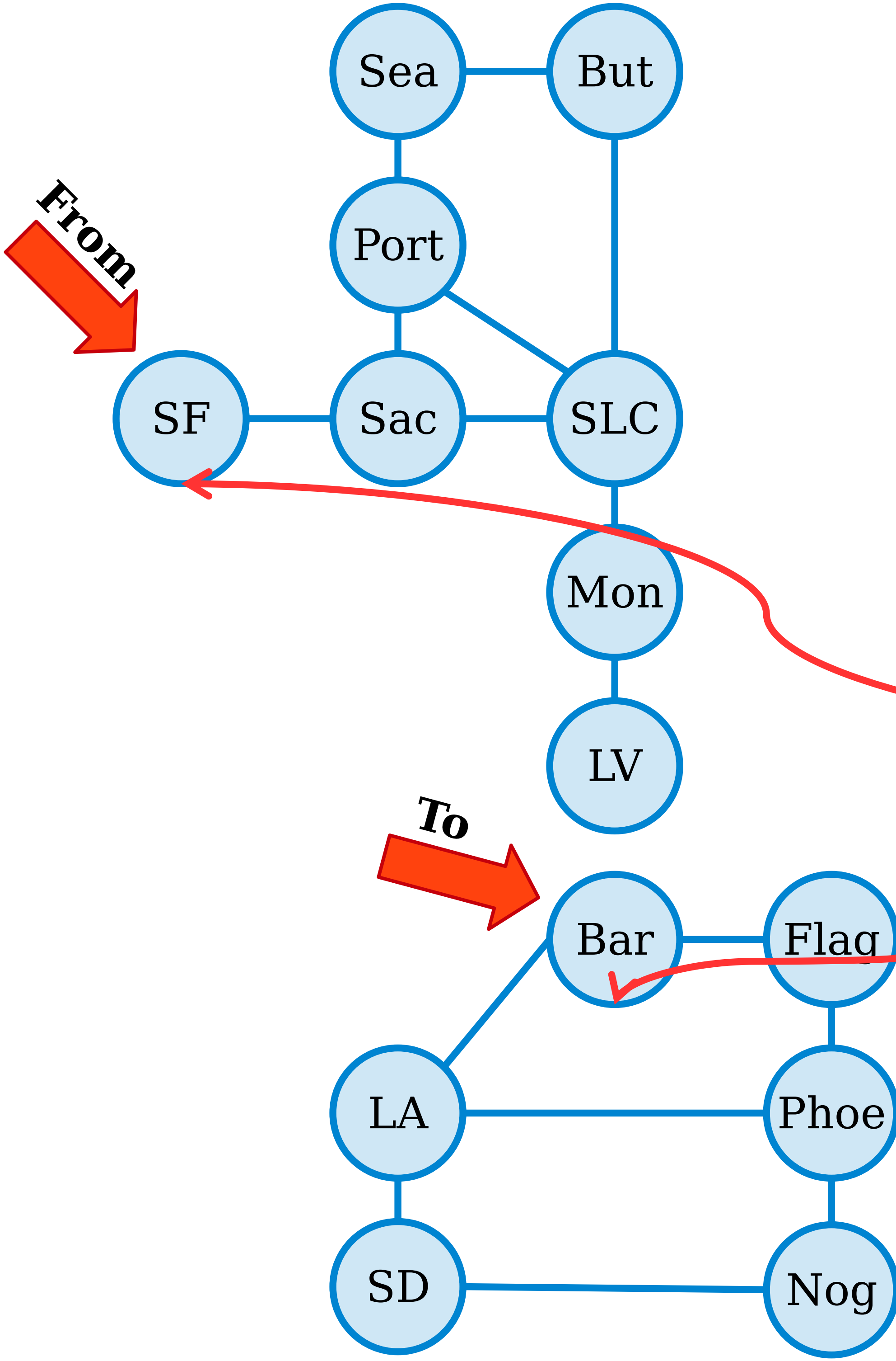
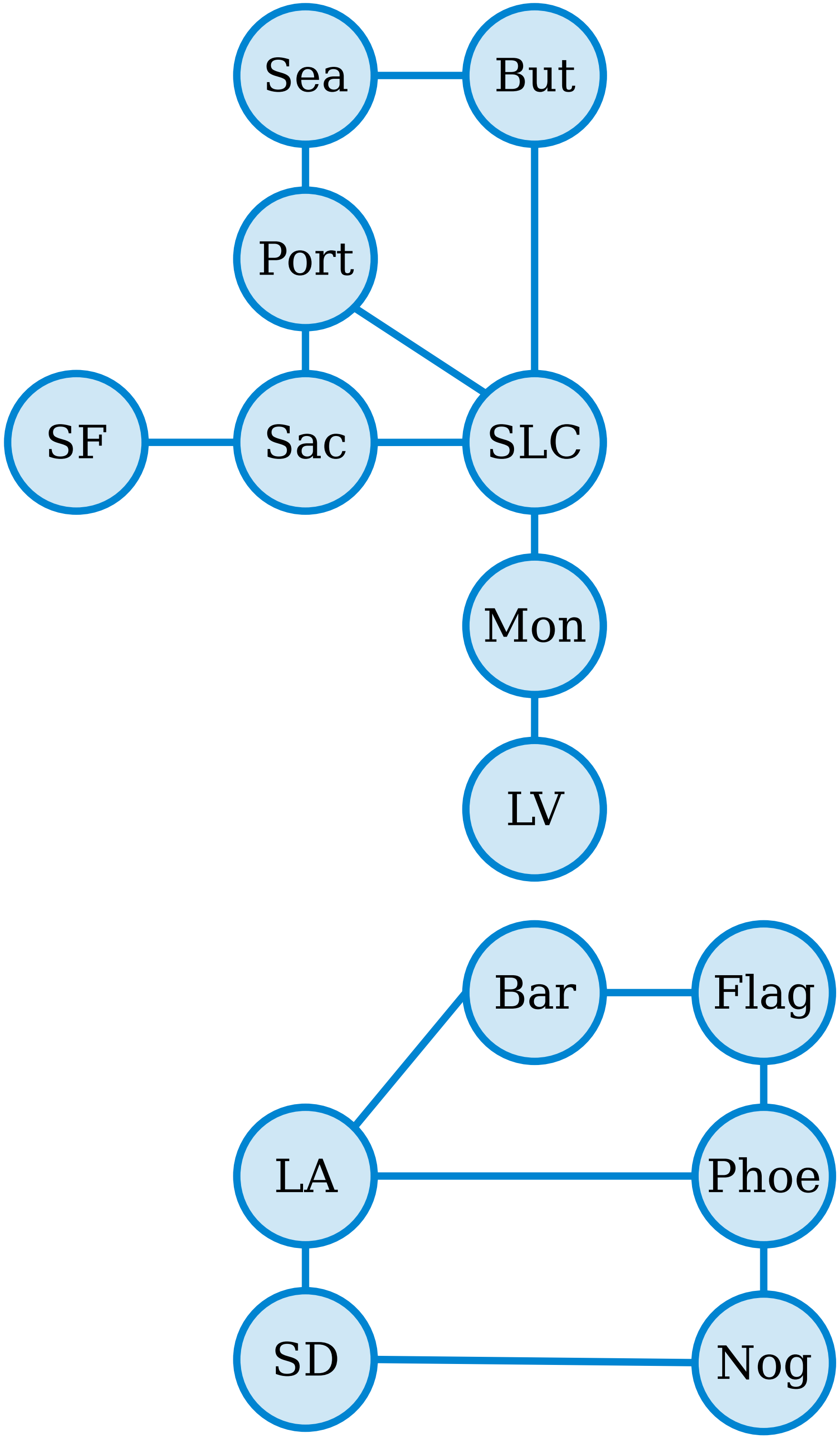


(These nodes are not connected. No Grand Canyon for you.)



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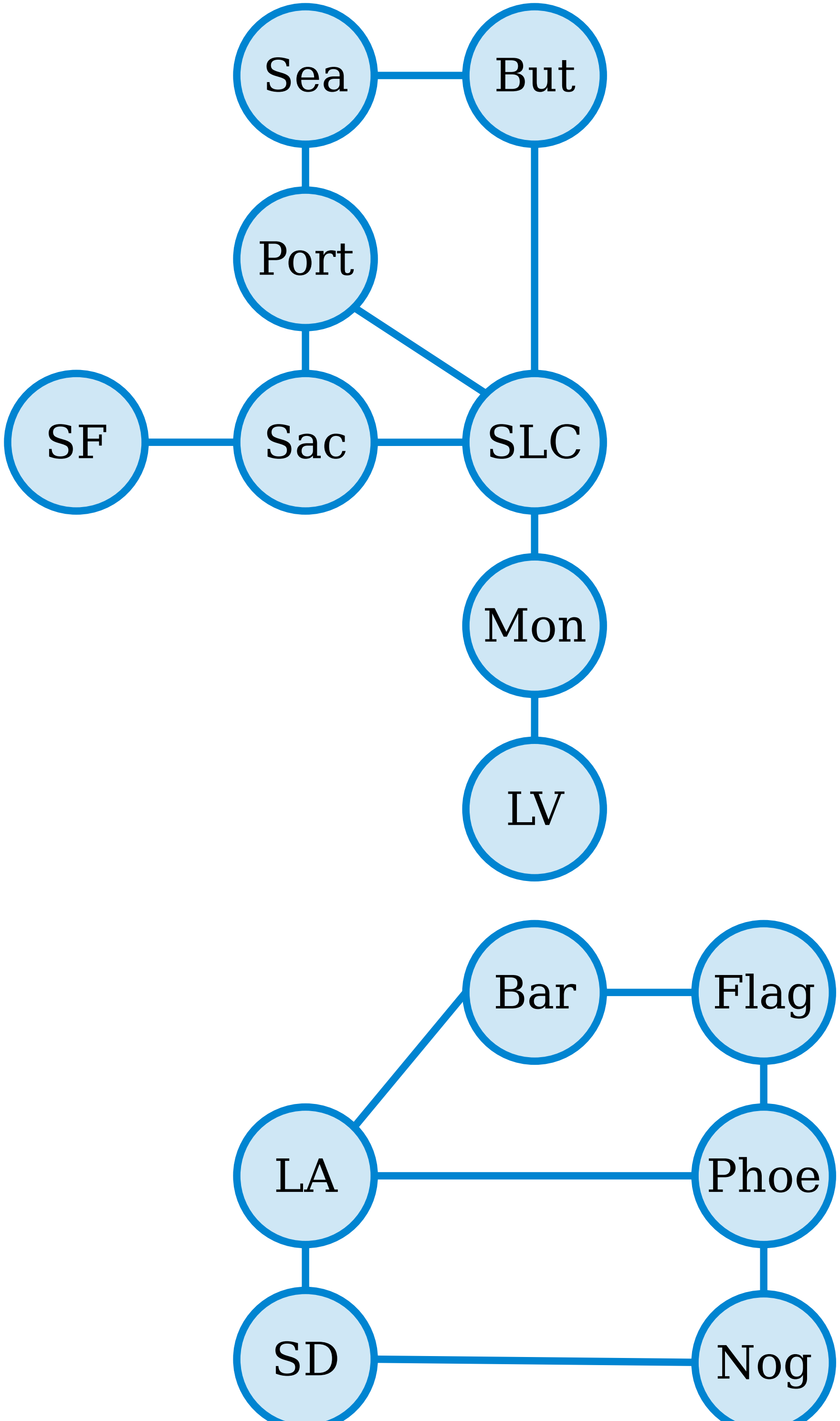

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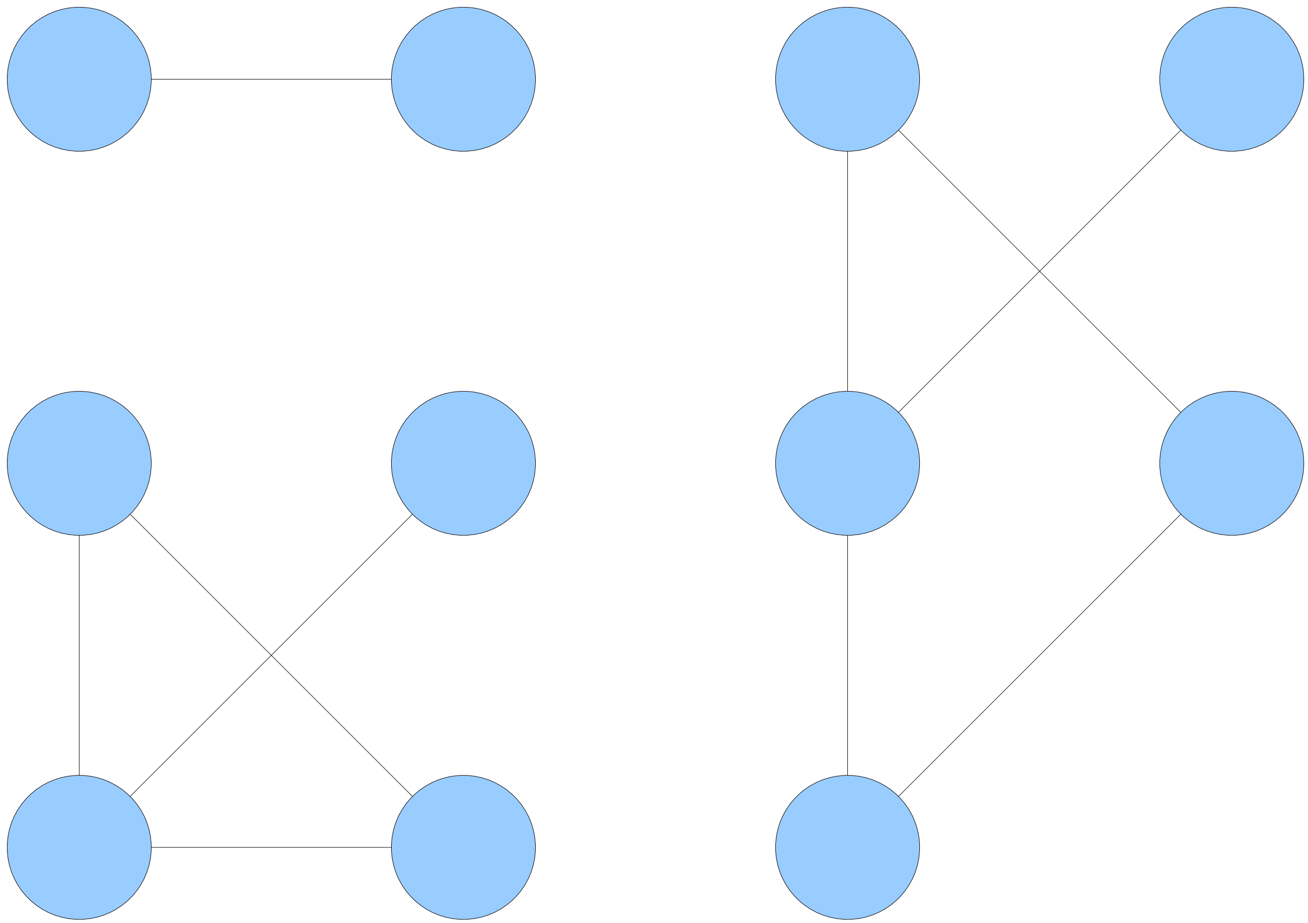
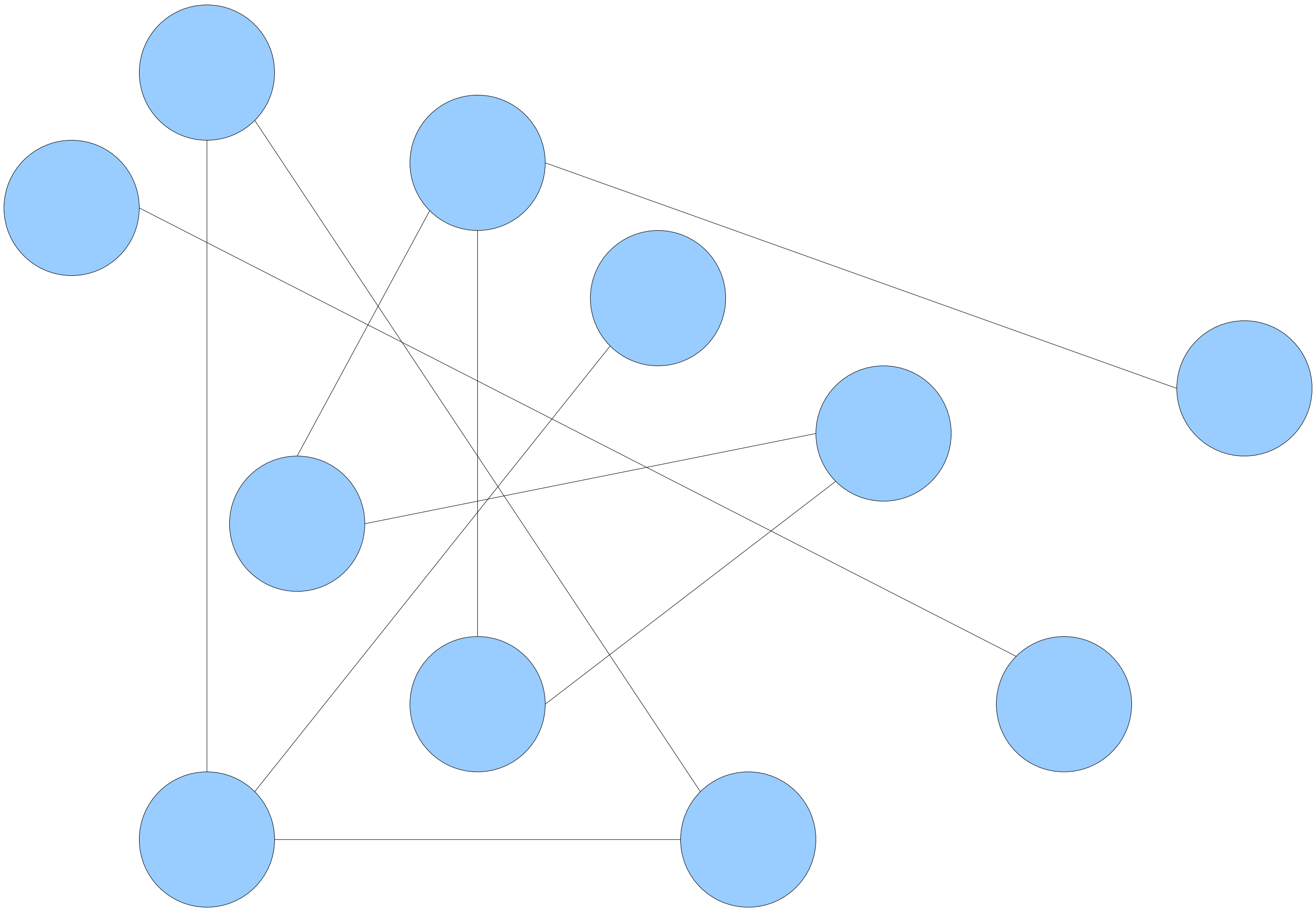
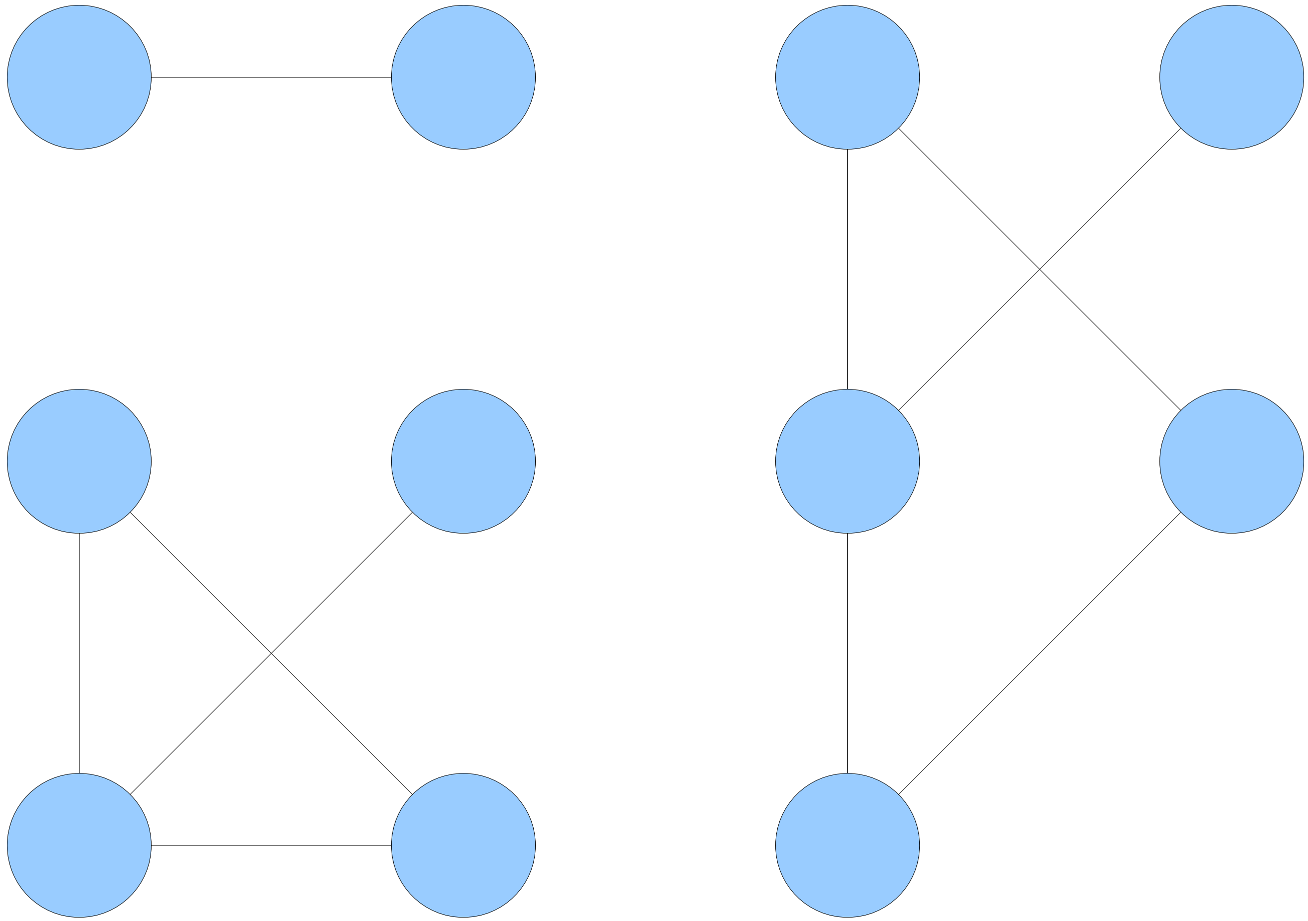
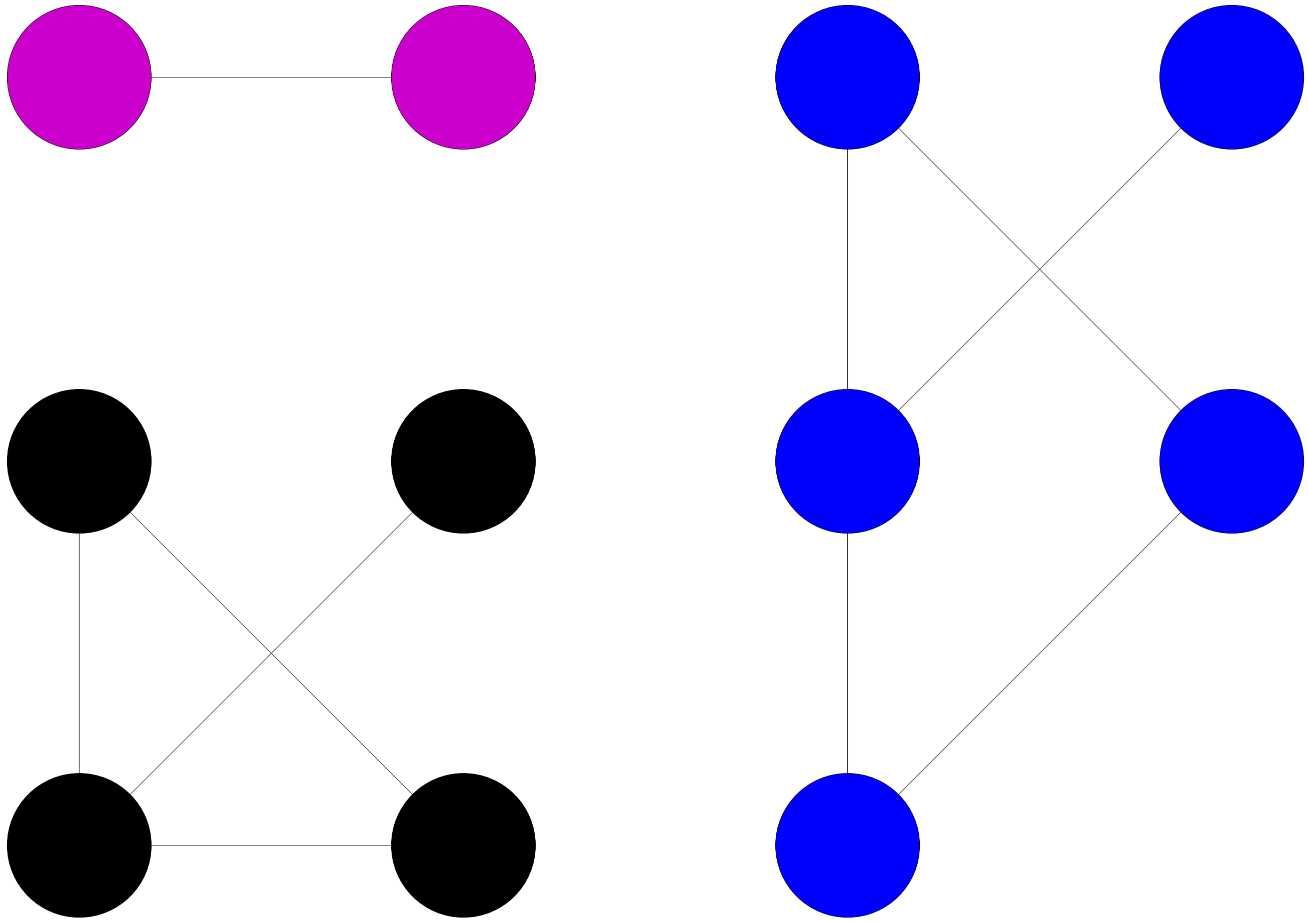
(This graph is not connected.)

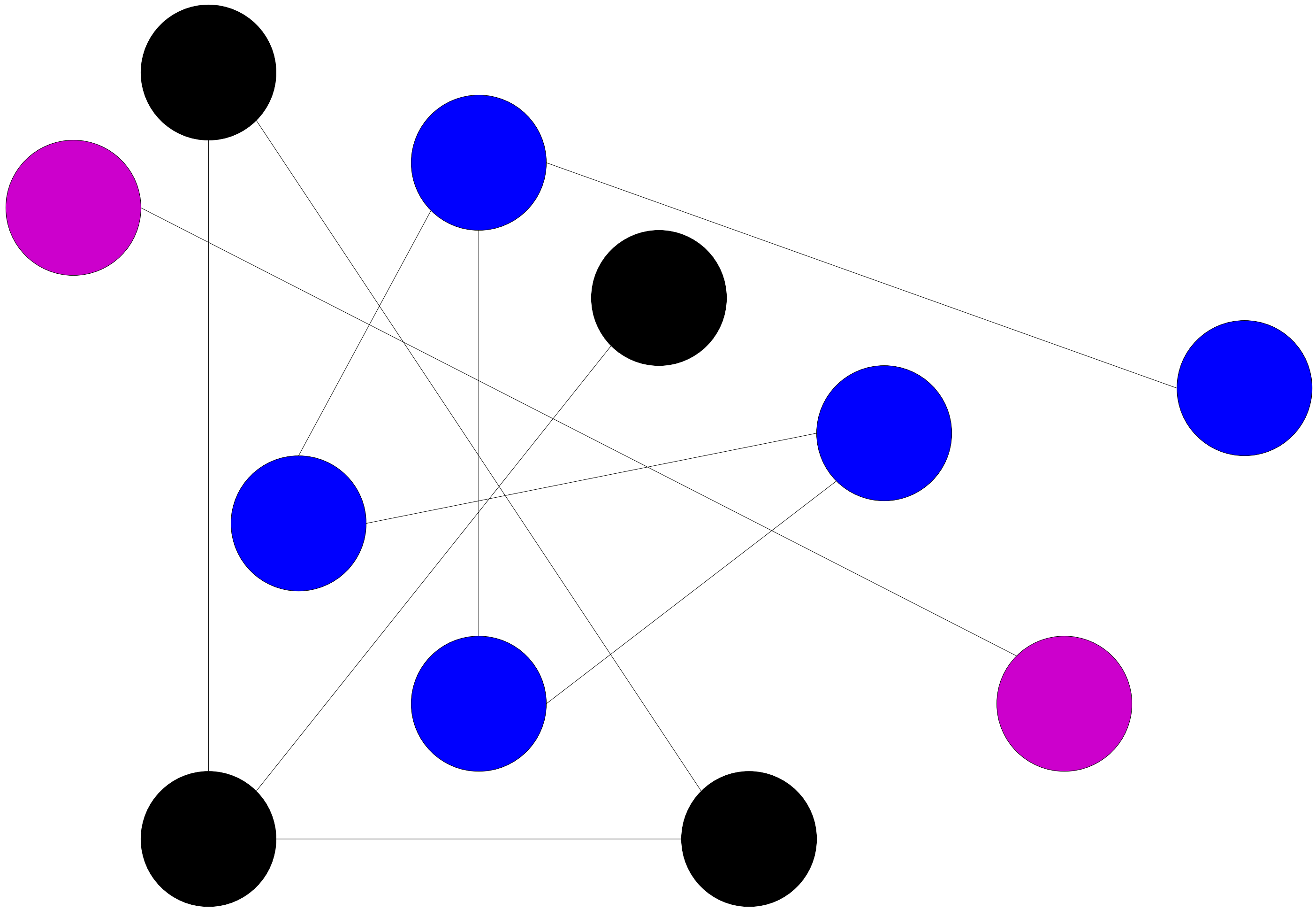
Connected Components

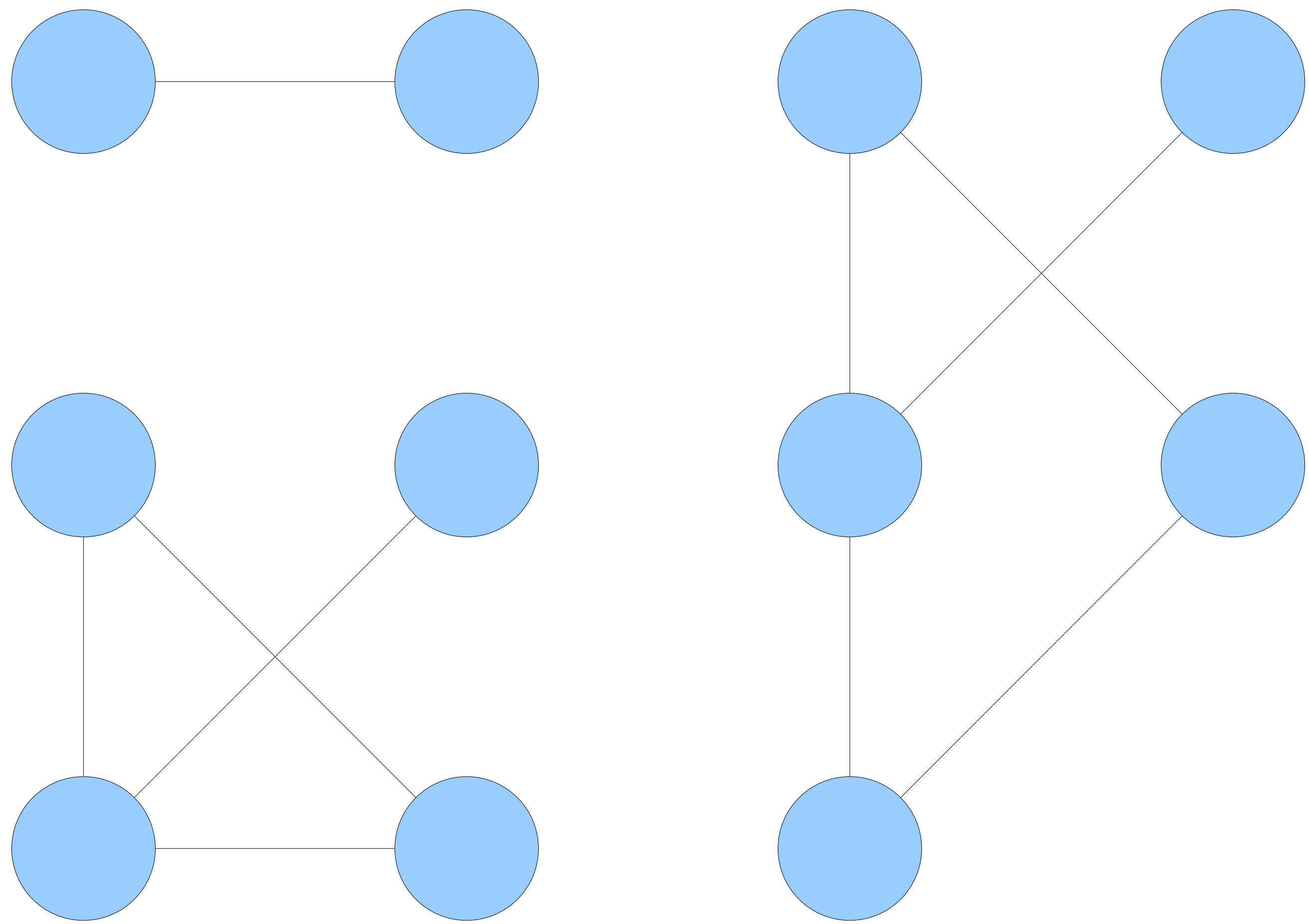


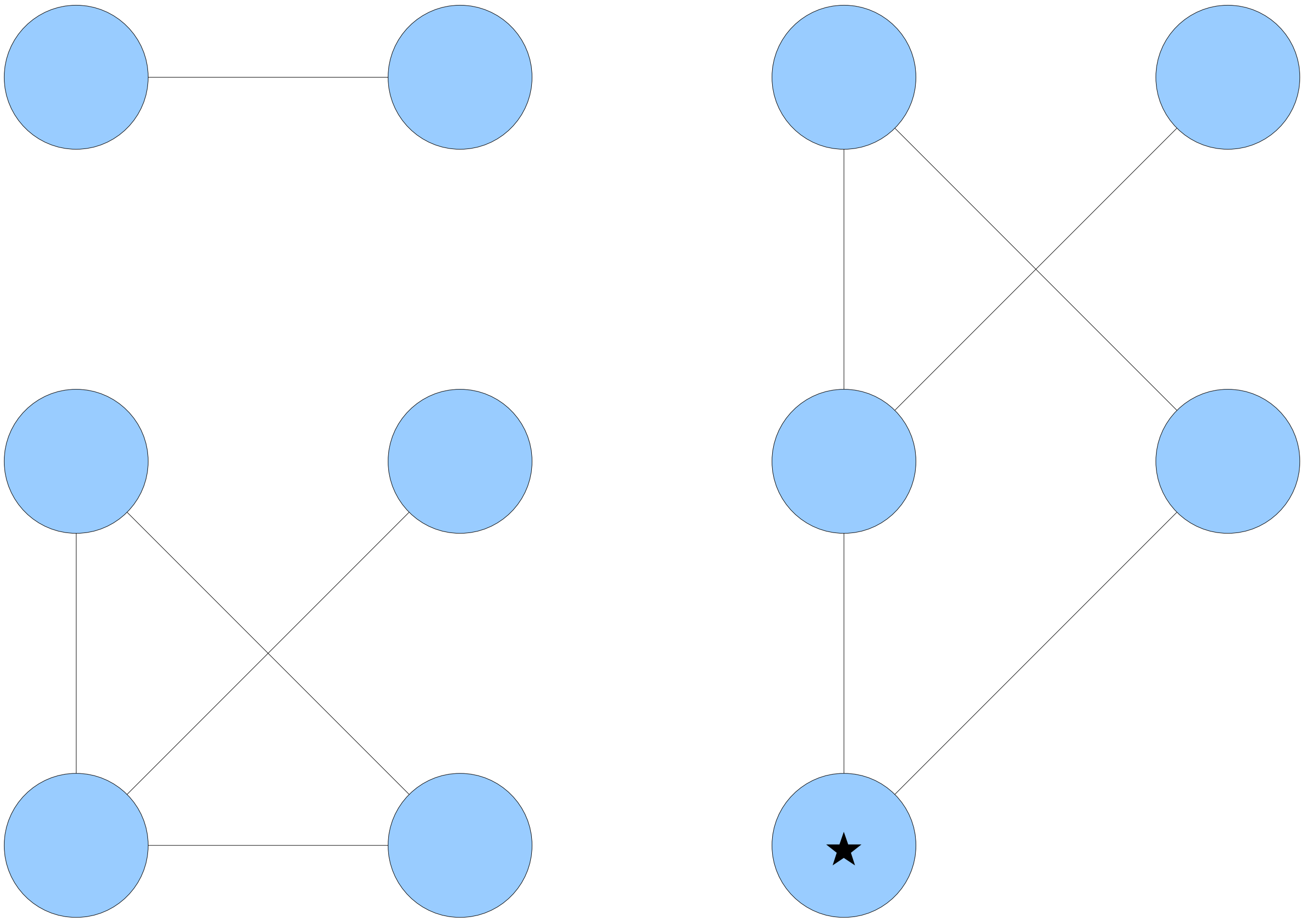


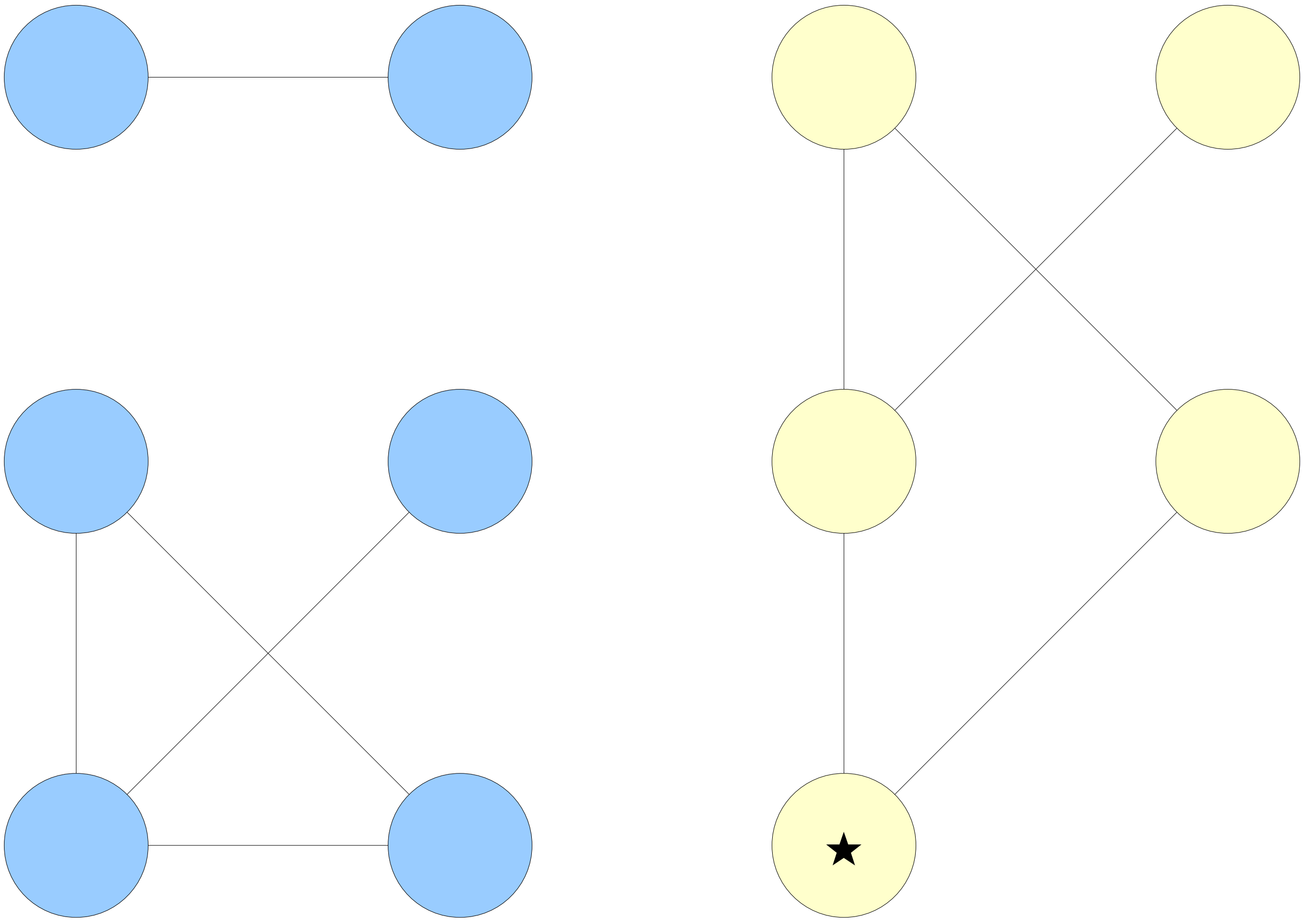












Connected Components

Let $G = (V, E)$ be a graph. For each $v \in V$, the ***connected component*** containing v is the set

$$[v] = \{ x \in V \mid v \text{ is connected to } x \}$$

Intuitively, a connected component is a "piece" of a graph in the sense we just talked about.

Question: How do we know that this particular definition of a "piece" of a graph is a good one?

Goal: Prove that any graph can be broken apart into different connected components.

We’re trying to reason about some way of partitioning the nodes in a graph into different groups.

What structure have we studied that captures the idea of a partition?

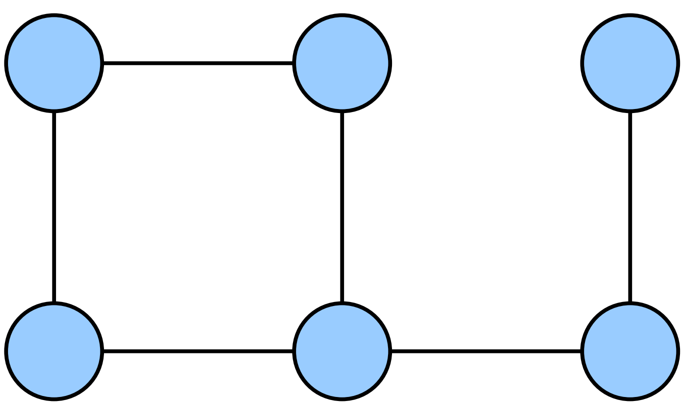
Connectivity

Claim: For any graph G, the “is connected to” relation is an equivalence relation.

Is it reflexive?

Is it symmetric?

Is it transitive?



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Claim: For any graph G, the "is connected to" relation is an equivalence relation.

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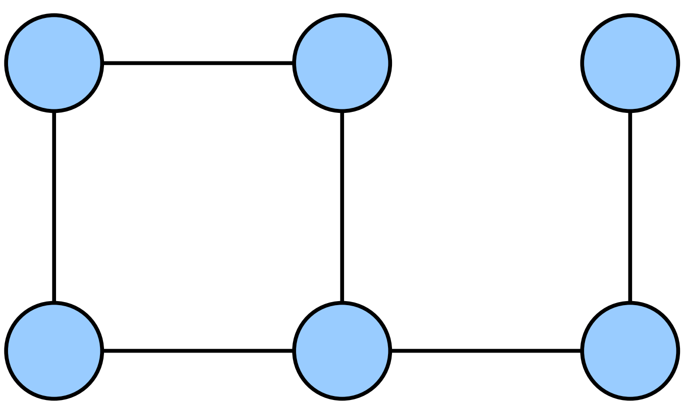
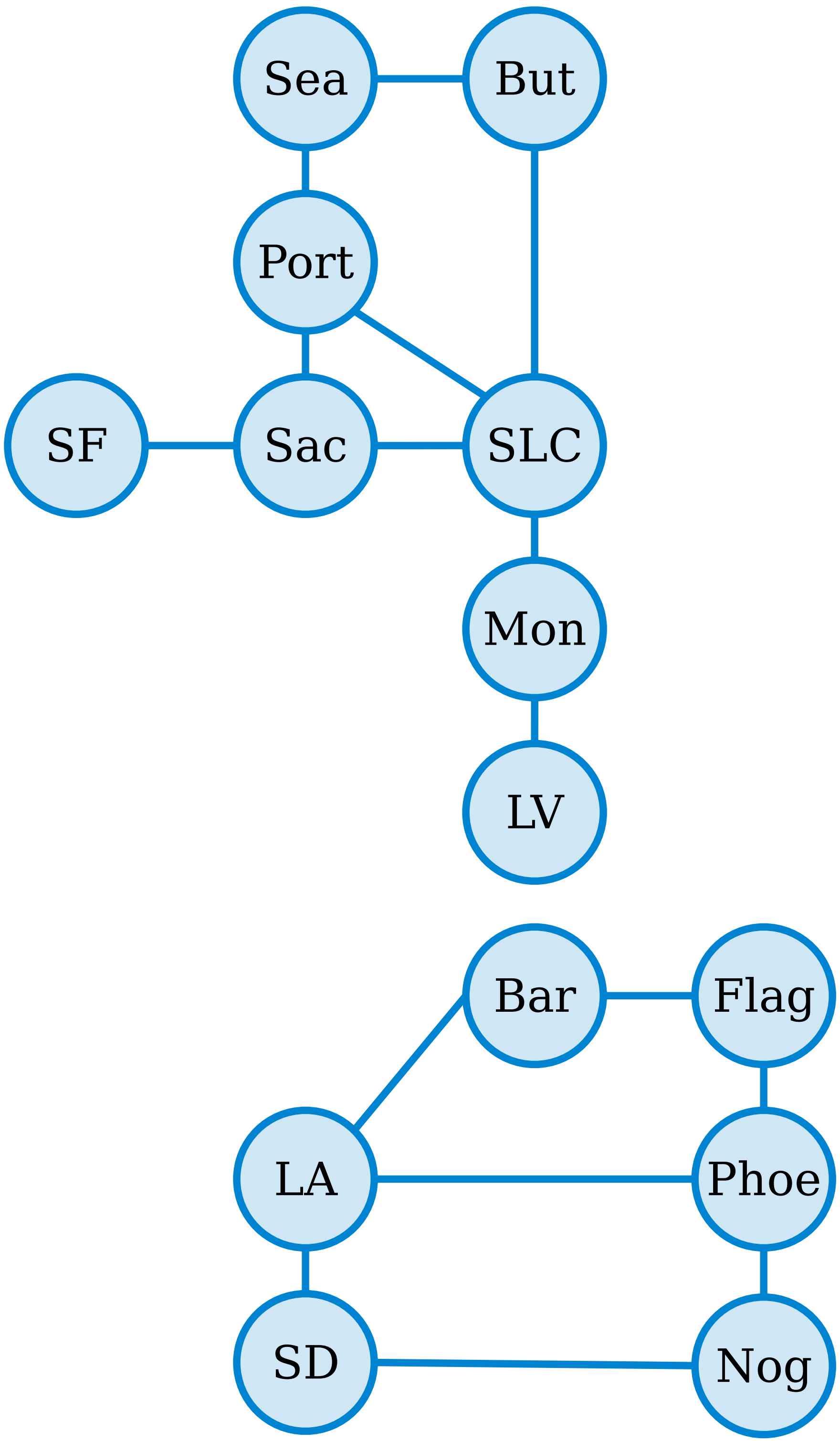
$$\forall v \in V.\ Conn(v, v)$$





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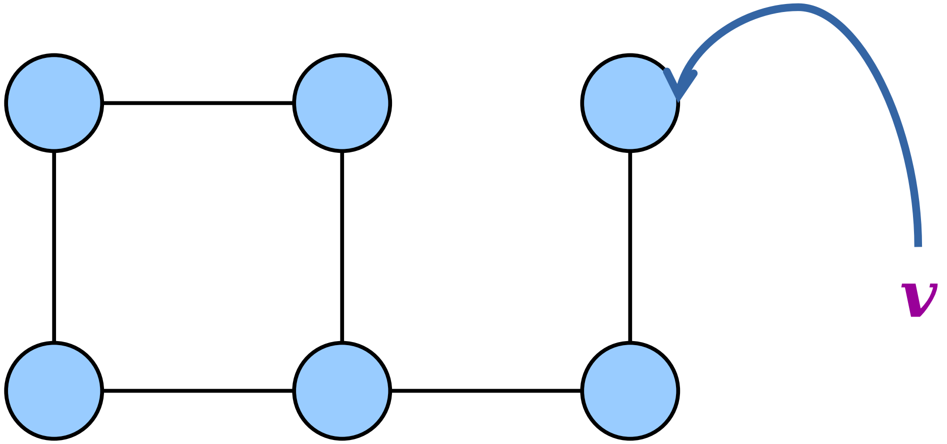

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Connectivity

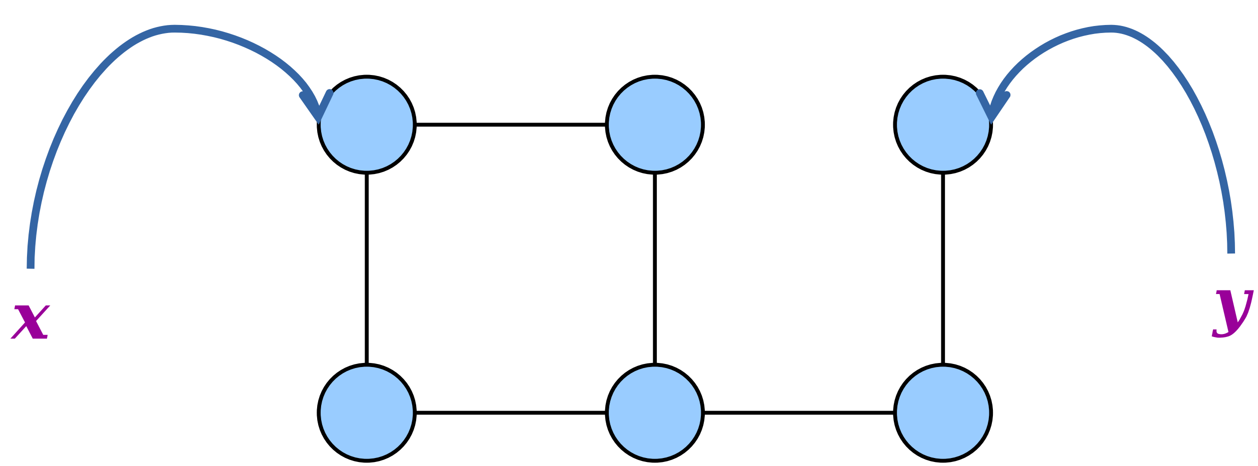
Claim: For any graph G, the “is connected to” relation is an equivalence relation.

$$\forall x \in V.\ \forall y \in V.\ (Conn(x, y) \rightarrow Conn(y, x))$$

Is it reflexive?

Is it symmetric?

Is it transitive?



Connectivity

Claim: For any graph G, the "is connected to" relation is an equivalence relation.

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Is it reflexive?

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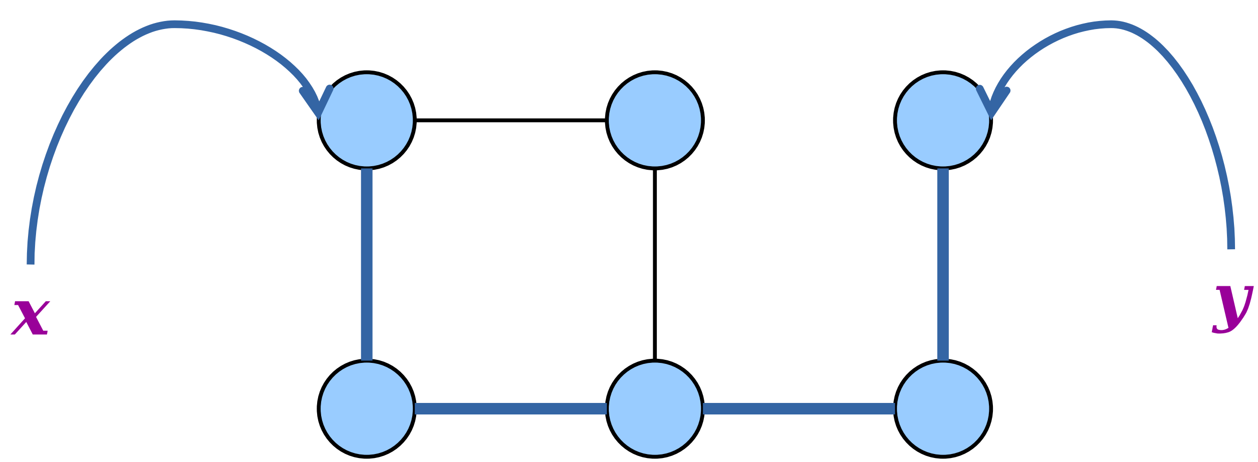

Connectivity

Claim: For any graph *G*, the "is connected

$$\forall x \in V.\ \forall y \in V.\ \forall z \in V.\ (Conn(x, y) \land Conn(y, z) \rightarrow Conn(x, z))$$

Is it reflexive?

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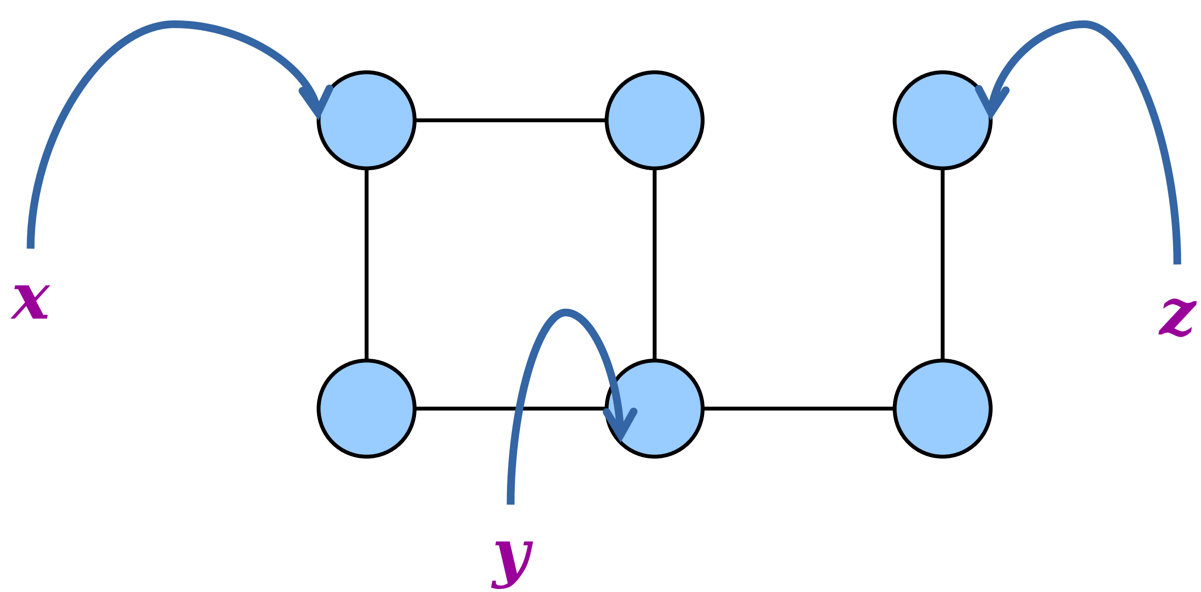

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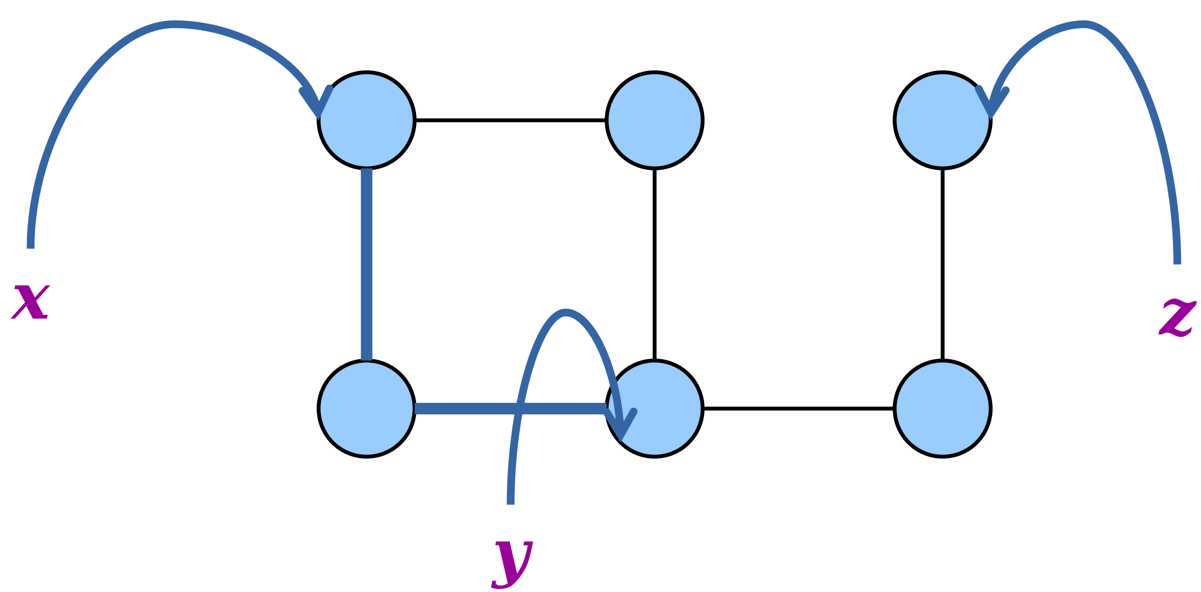

Theorem: Let $G = (V, E)$ be a graph. Then the connectivity relation over V is an equivalence relation.

Proof: Consider an arbitrary graph $G = (V, E)$. We will prove that the connectivity relation over V is reflexive, symmetric, and transitive.

To show that connectivity is reflexive, consider any $v \in V$. Then the singleton path v is a path from v to itself. Therefore, v is connected to itself, as required.

To show that connectivity is symmetric, consider any $x, y \in V$ where x is connected to y. We need to show that y is connected to x. Since x is connected to y, there is some path $x, v_1, \ldots, v_n, y$ from x to y. Then $y, v_n, \ldots, v_1, x$ is a path from y back to x, so y is connected to x.

Finally, to show that connectivity is transitive, let $x, y, z \in V$ be arbitrary nodes where x is connected to y and y is connected to z. We will prove that x is connected to z. Since x is connected to y, there is a path $x, u_1, \ldots, u_n, y$ from x to y. Since y is connected to z, there is a path $y, v_1, \ldots, v_k, z$ from y to z. Then the path $x, u_1, \ldots, u_n, y, v_1, \ldots, v_k, z$ goes from x to z. Thus x is connected to z, as required. ■

Putting Things Together

Earlier, we defined the connected component of a node v to be

$$[v] = \{\ x \in V \mid v \text{ is connected to } x\ \}$$

Connectivity is an equivalence relation! So what's the equivalence class of a node v with respect to connectivity?

$$[v]_{conn} = \{\ x \in V \mid v \text{ is connected to } x\ \}$$

Connected components are equivalence classes of the connectivity relation!

Theorem: If $G = (V, E)$ is a graph, then every node in G belongs to exactly one connected component of G.

Proof: Let $G = (V, E)$ be an arbitrary graph and let $v \in V$ be any node in G. The connected components of G are just the equivalence classes of the connectivity relation in G. The Fundamental Theorem of Equivalence Relations guarantees that v belongs to exactly one equivalence class of the connectivity relation. Therefore, v belongs to exactly one connected component in G. ■

Time out for announcements!

Problem Set 3

- Due tomorrow (Thursday) at 11:59pm PDT.
- Use a late period to extend this to Saturday at 11:59pm PDT.
- Any last questions? Come to office hours or ask on Campuswire.

Midterm

- Thursday July 23rd

- Will cover material up to Monday's lecture (psets 1, 2, 3).

- 24-hour window to start the exam. Begins at 9:30AM PDT on Thursday July 23rd.

- Once you click start on Gradscope, Gradescope will give you access to the exam. You'll have 3 hours to complete the exam, plus 15 minutes to upload your exam to Gradescope.

- Please make sure any OAE letters get sent to the staff mailing list as soon as possible.