The Pigeonhole Principle

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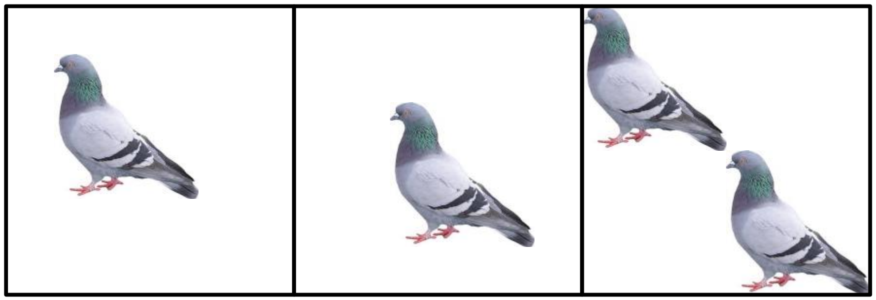

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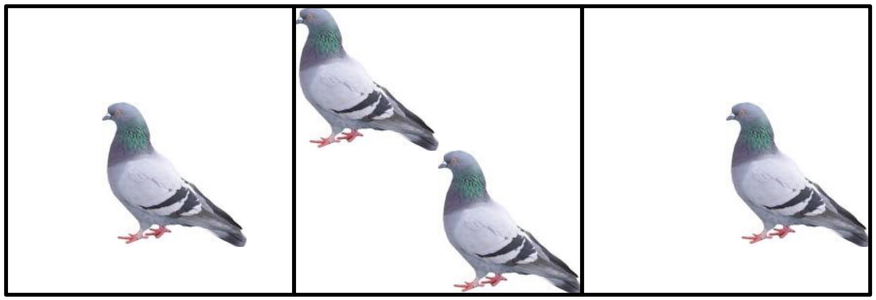

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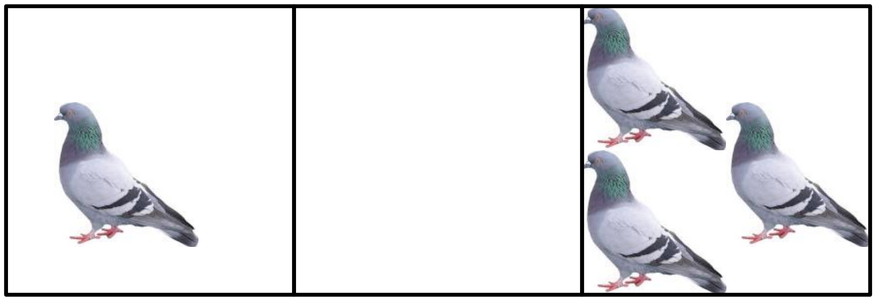

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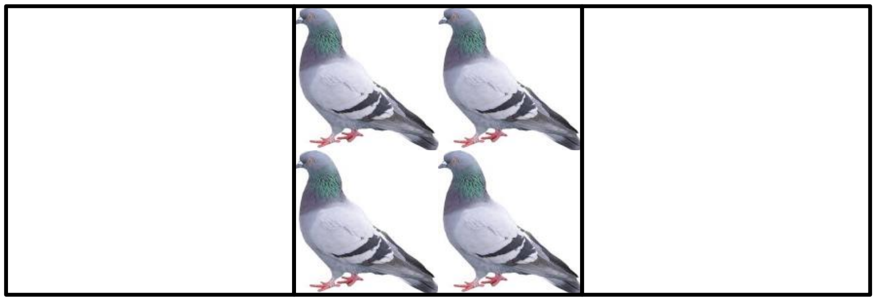
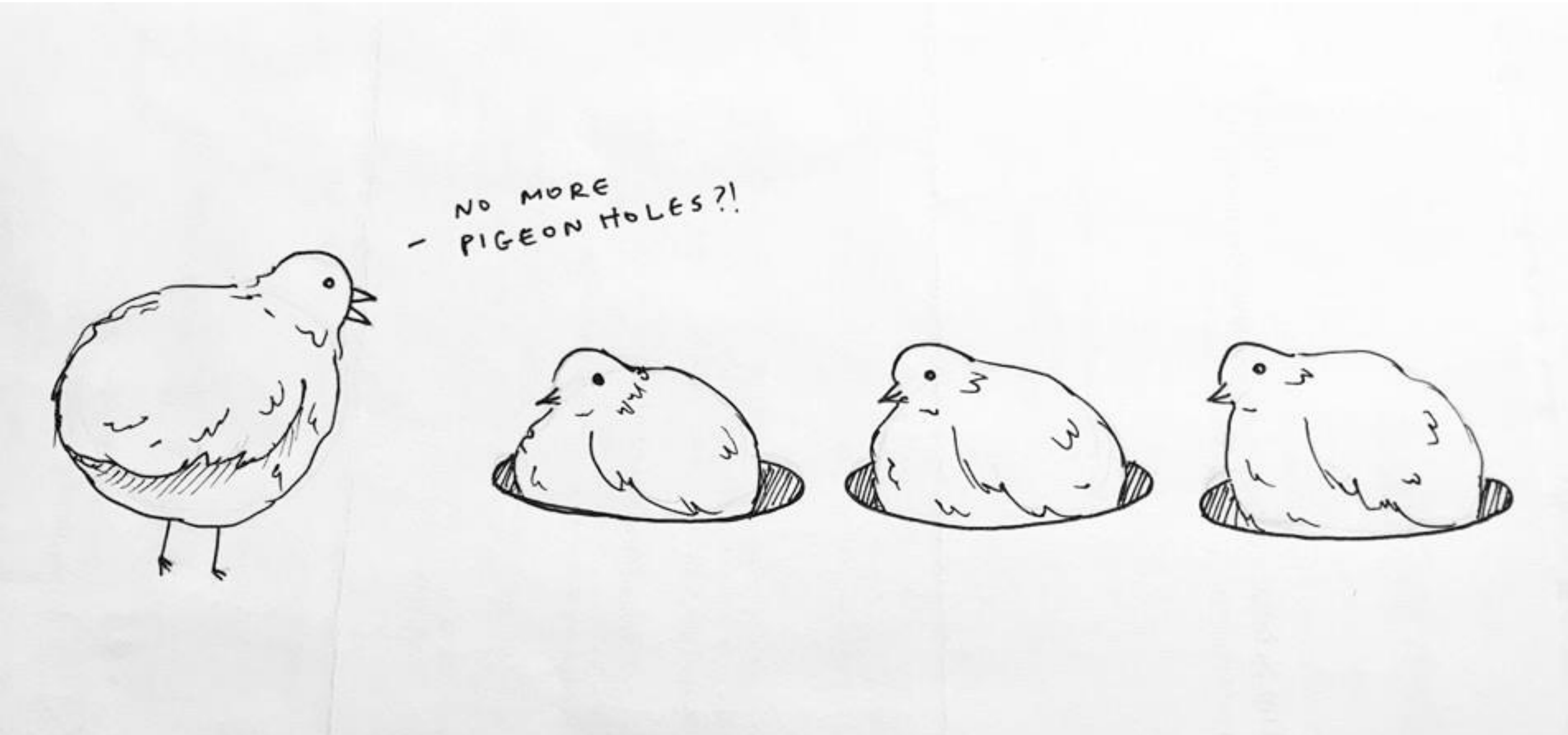




$m = 4, n = 3$

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
- 366 possible birthdays (pigeonholes)
- 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
- Maximum number of hairs ever found on a human head is no greater than 500,000.
- There are over 800,000 people in San Francisco.

Proving the Pigeonhole Principle

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. There are m objects in total, so we know that

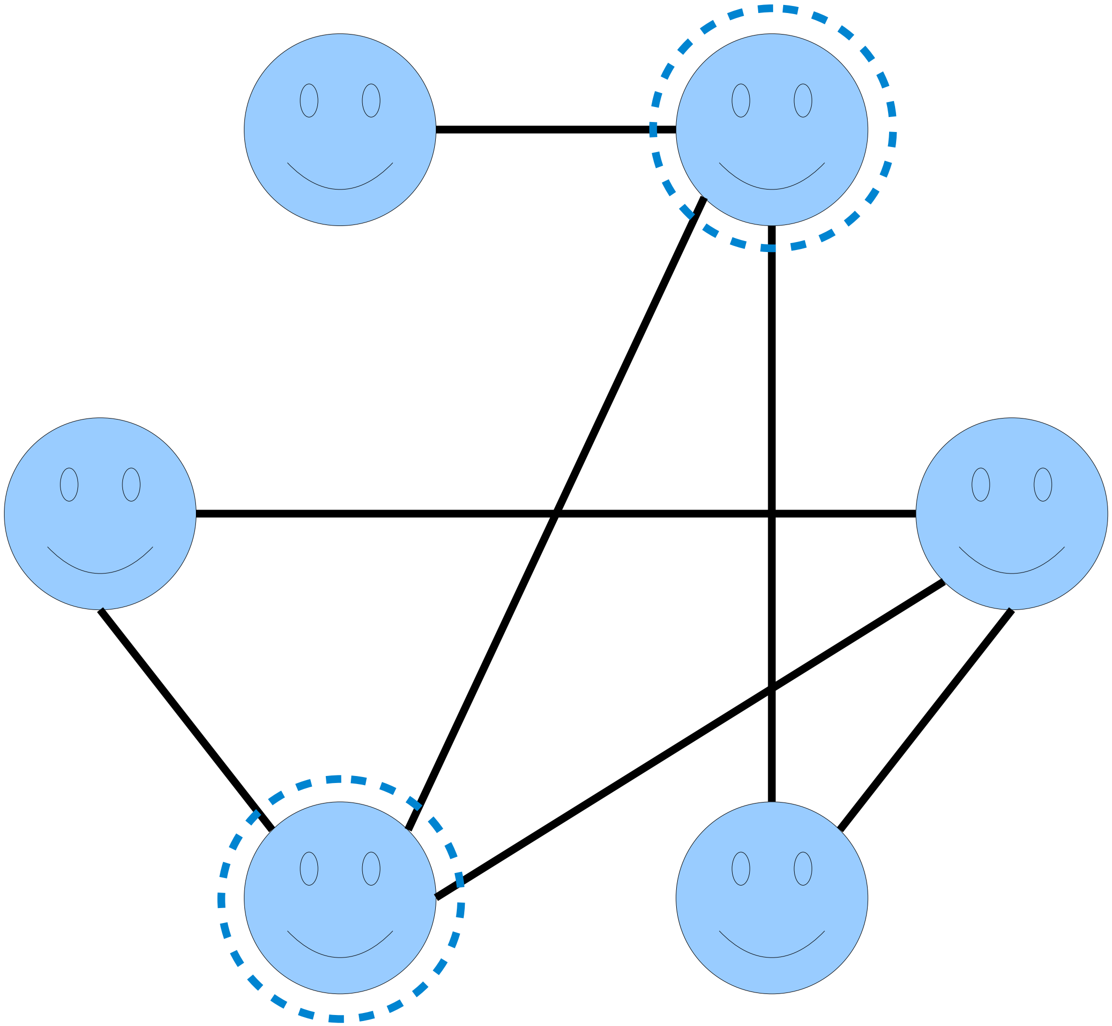
$$m = x_1 + x_2 + \ldots + x_n.$$

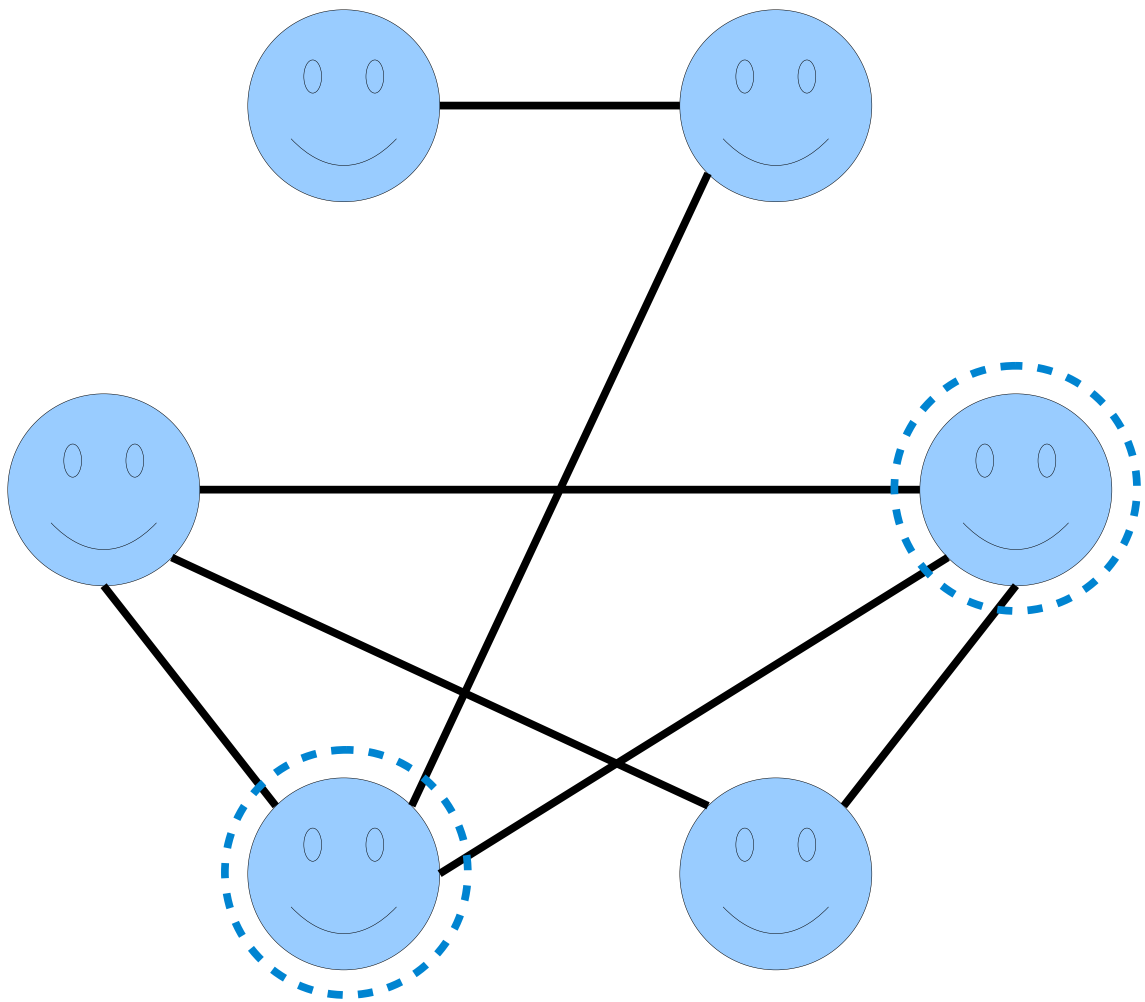
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i. This means that

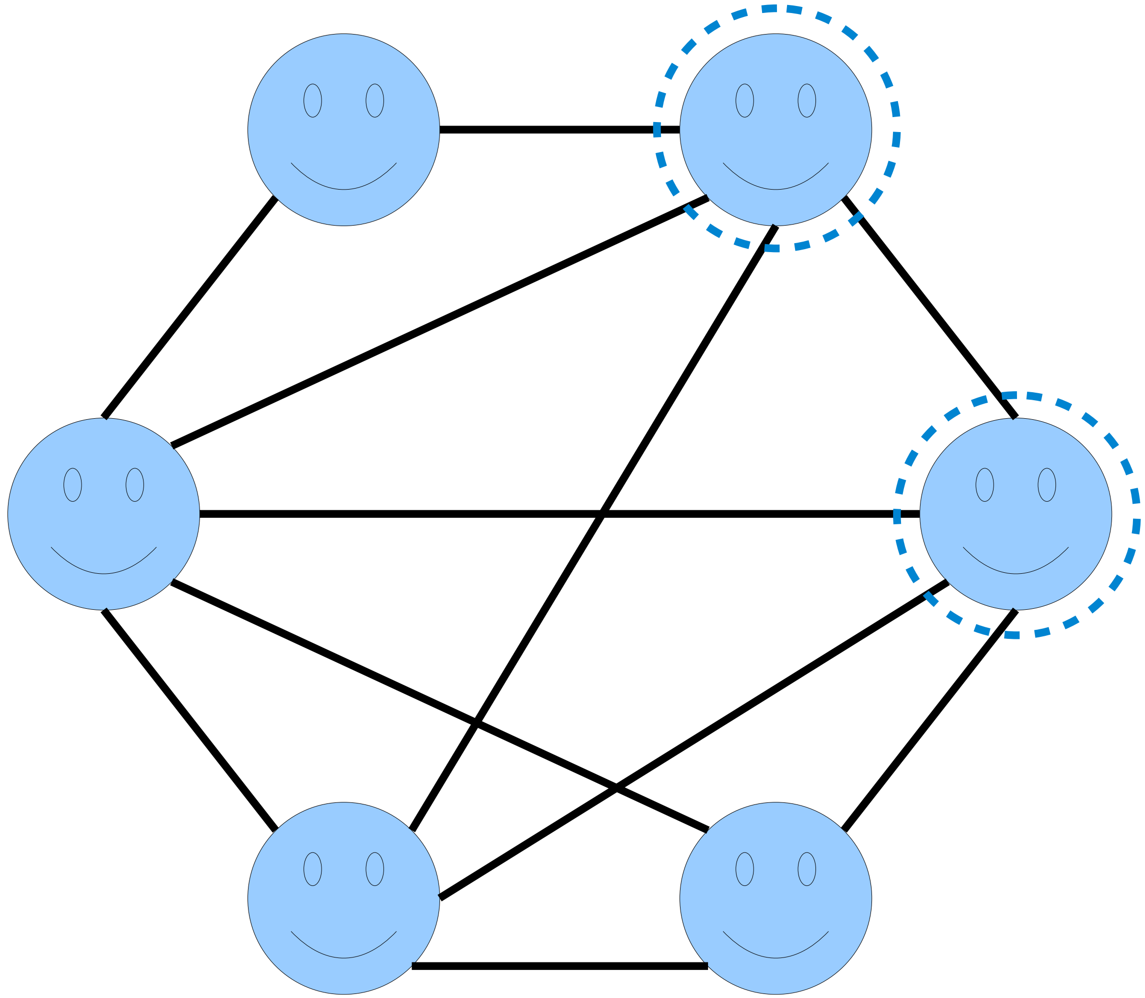
$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \quad (n \text{ times}) \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks

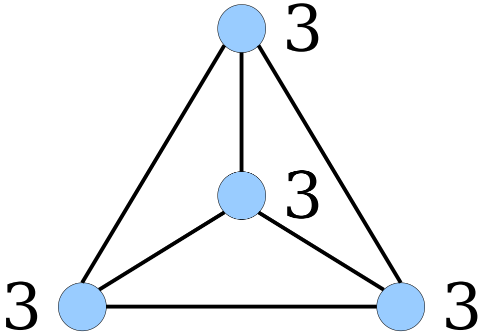






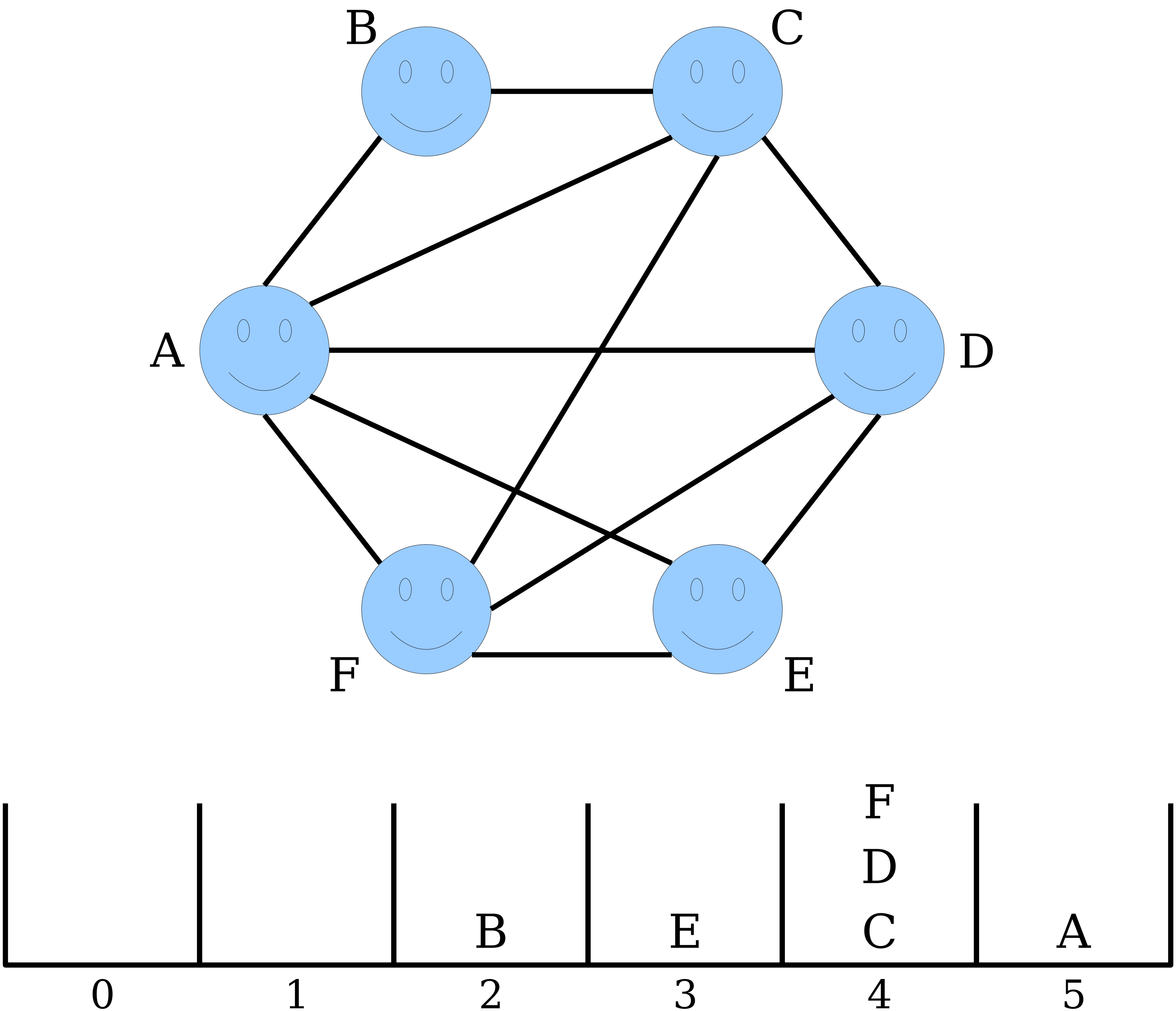
Degrees

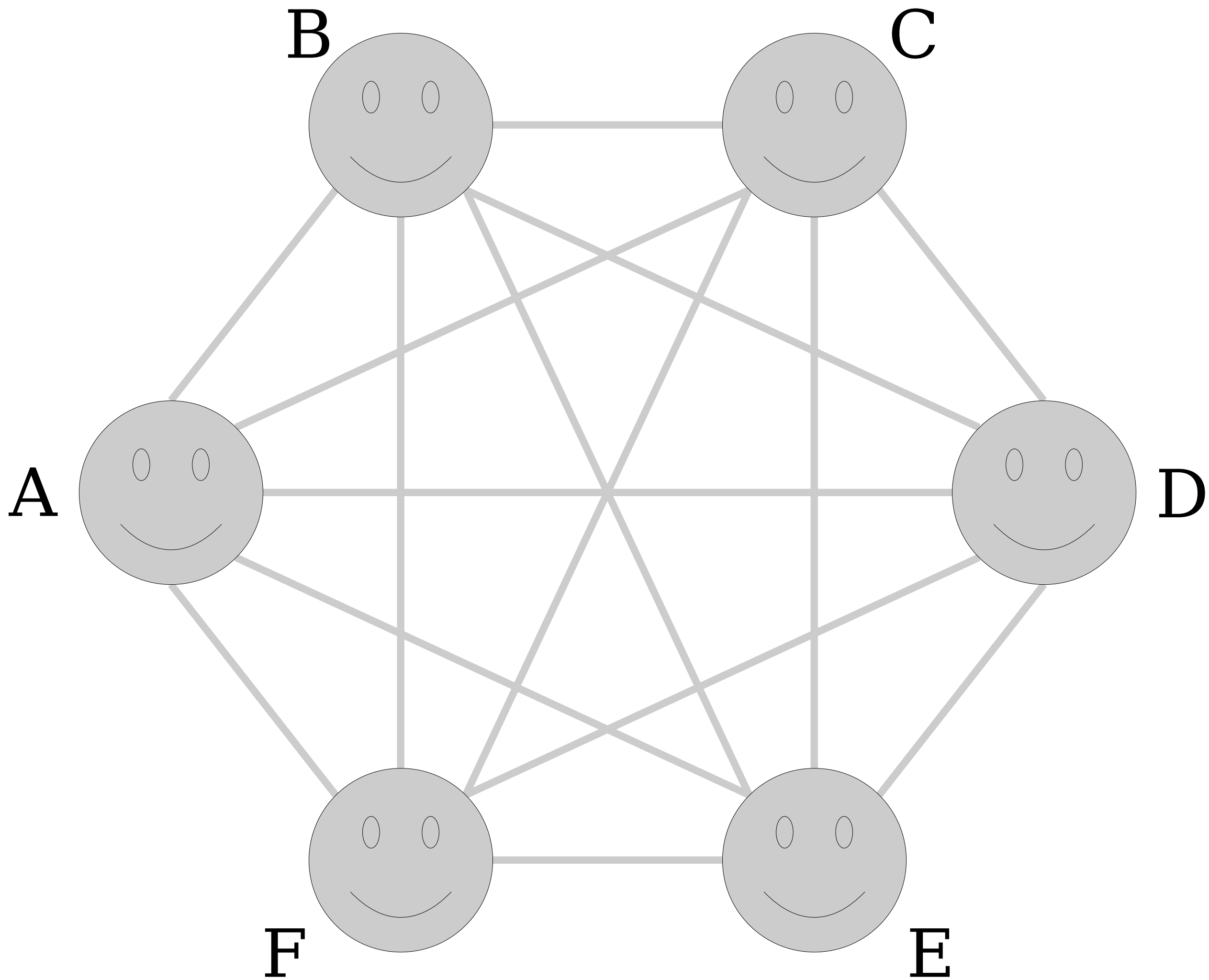
The ***degree*** of a node *v* in a graph is the number of nodes that *v* is adjacent to.



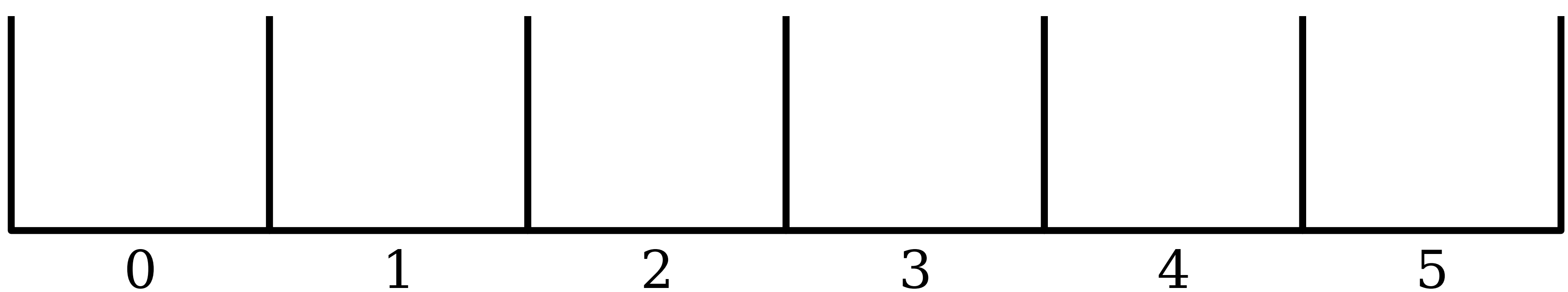
Theorem: Every graph with at least two nodes has at least two nodes with the same degree.

Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.

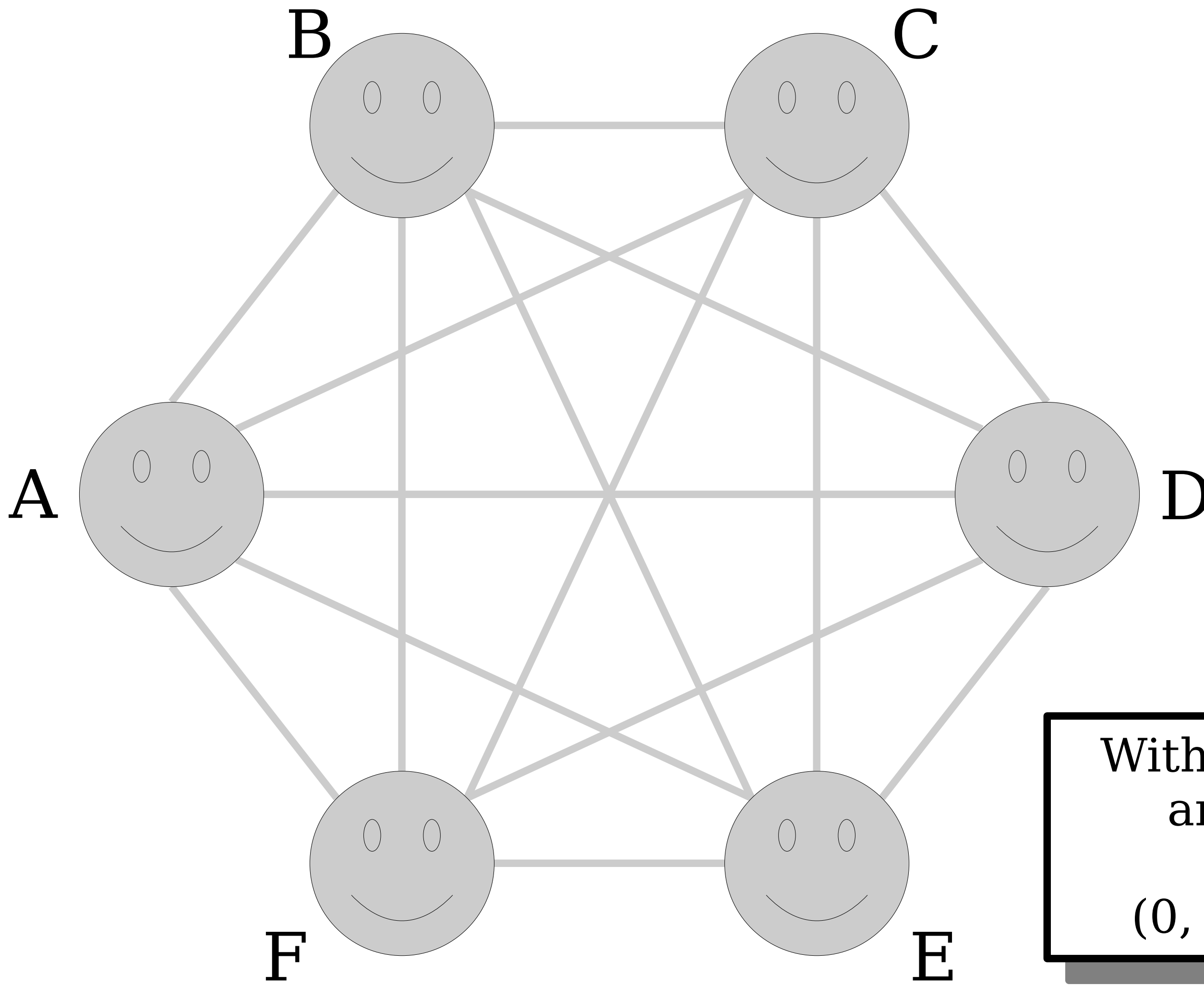

B
C
A
D
F
E
F
D
B
E
C
A
0
1
2
3
4
5



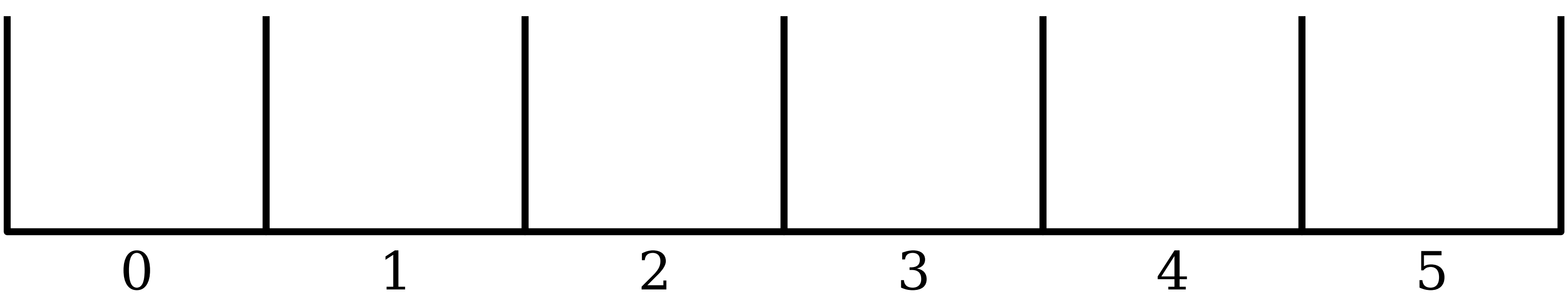
B
C
A
D
F
E

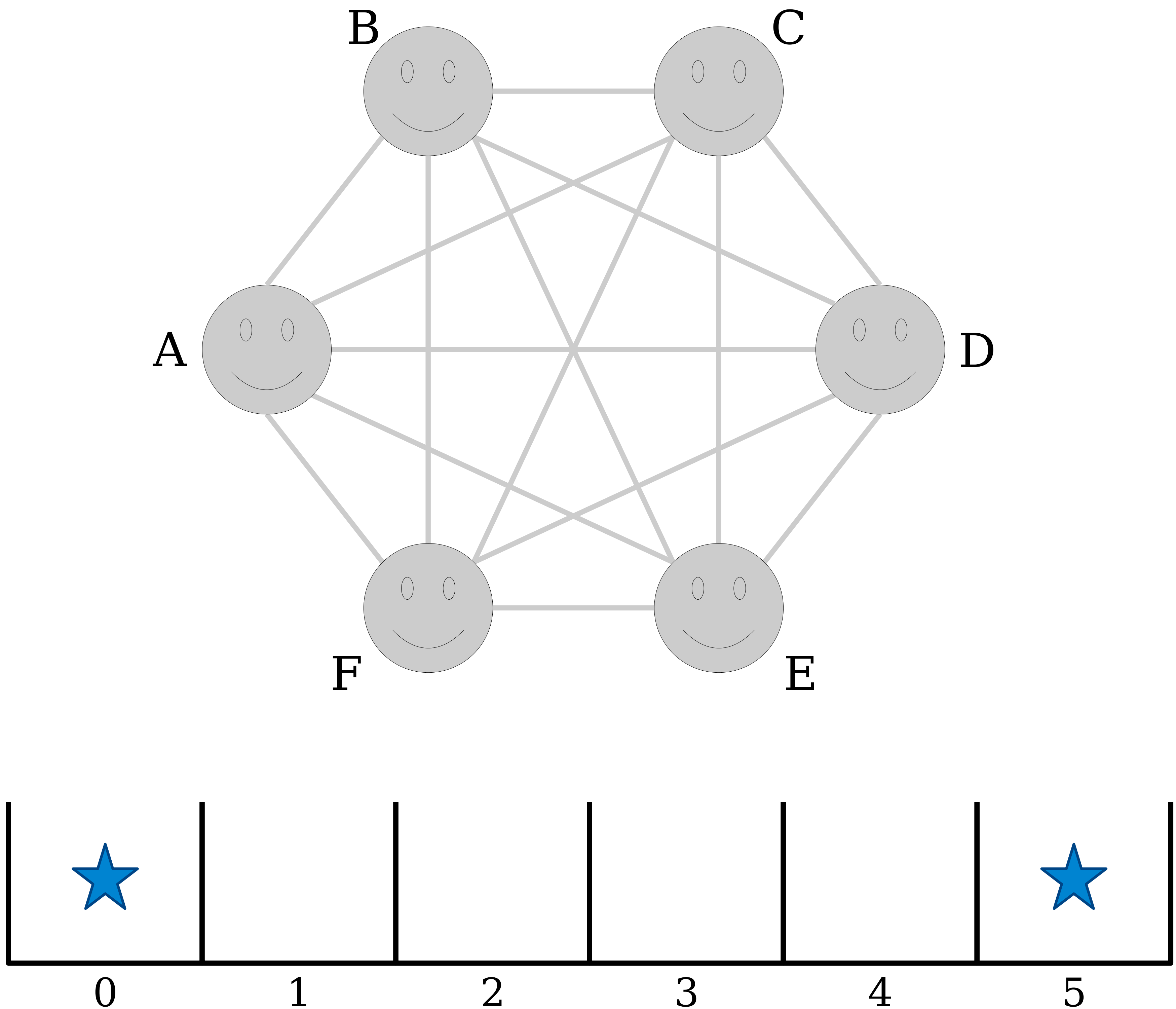


0
1
2
3
4
5



With *n* nodes, there are *n* possible degrees (0, 1, 2, …, *n* – 1)




B
C
A
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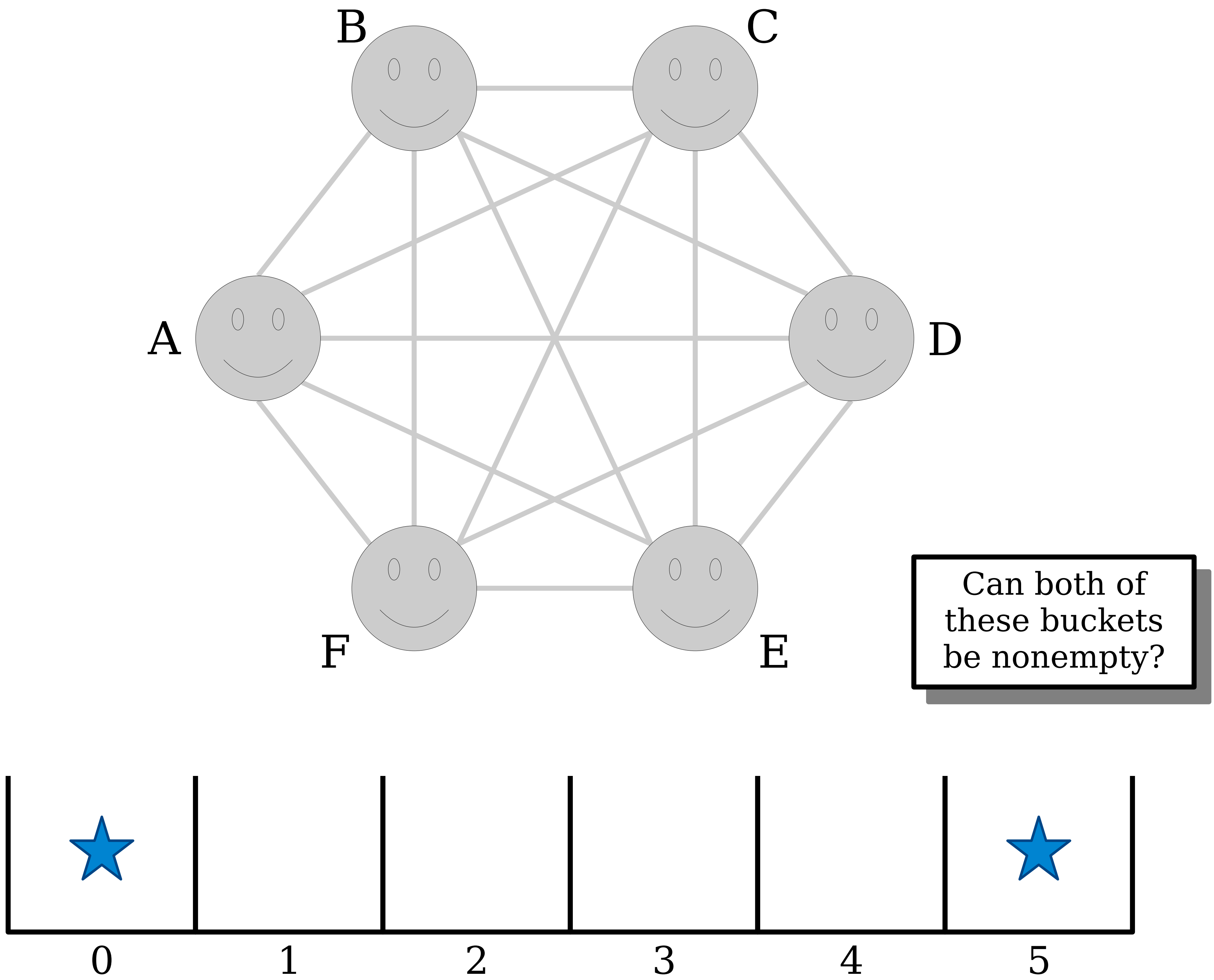
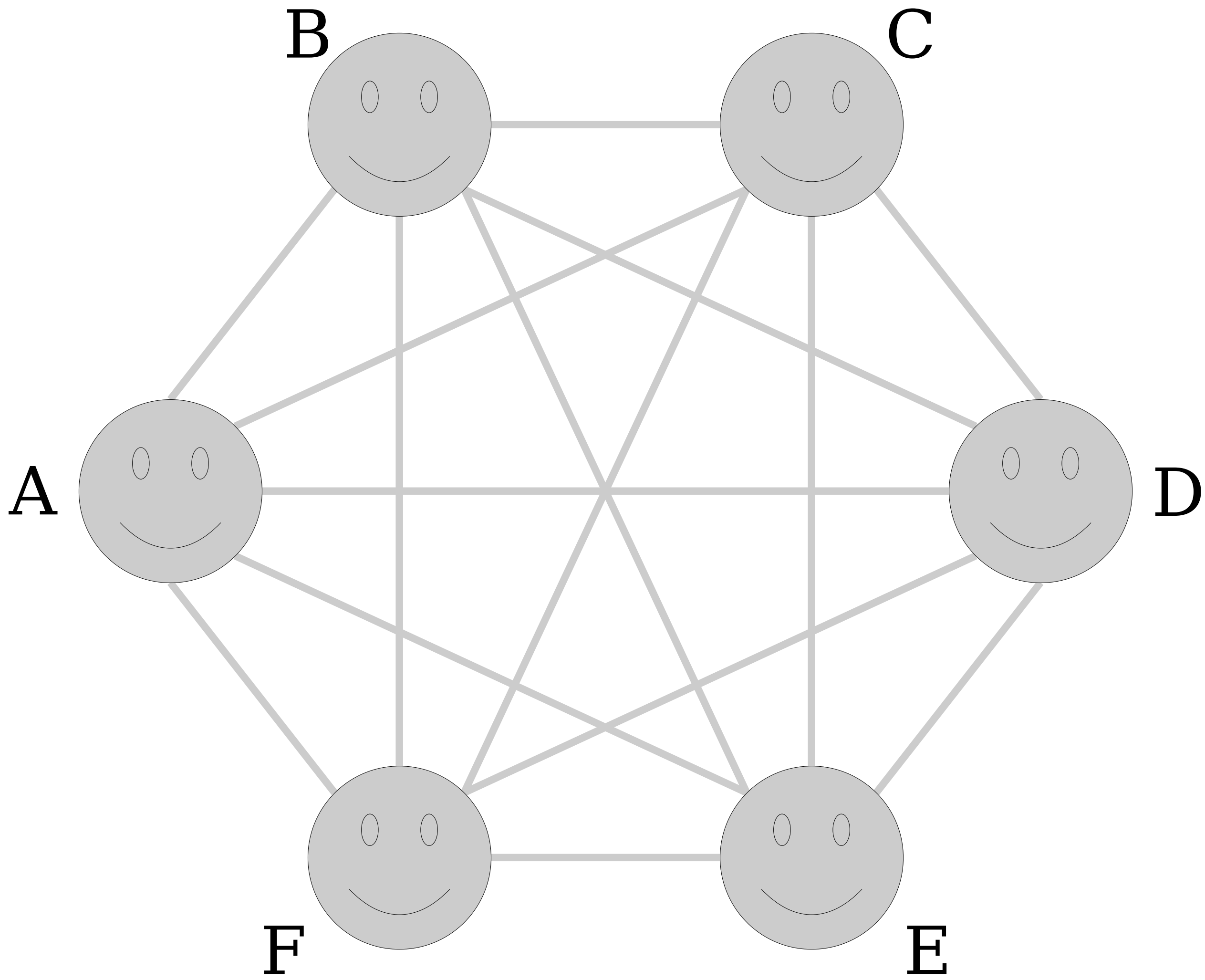
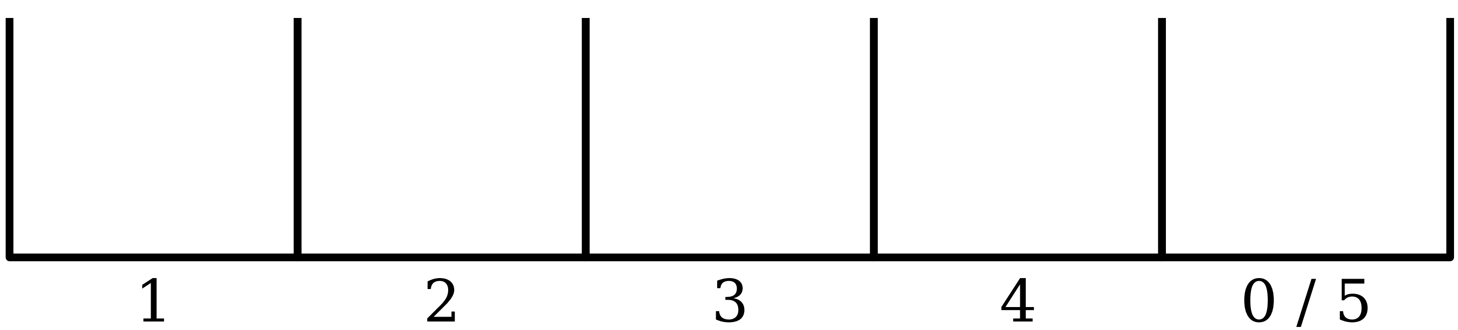
F
E
0
1
2
3
4
5



B
C
A
D
F
E
Can both of these buckets be nonempty?
0
1
2
3
4
5


B
C
A
D
F
E


1
2
3
4
0 / 5

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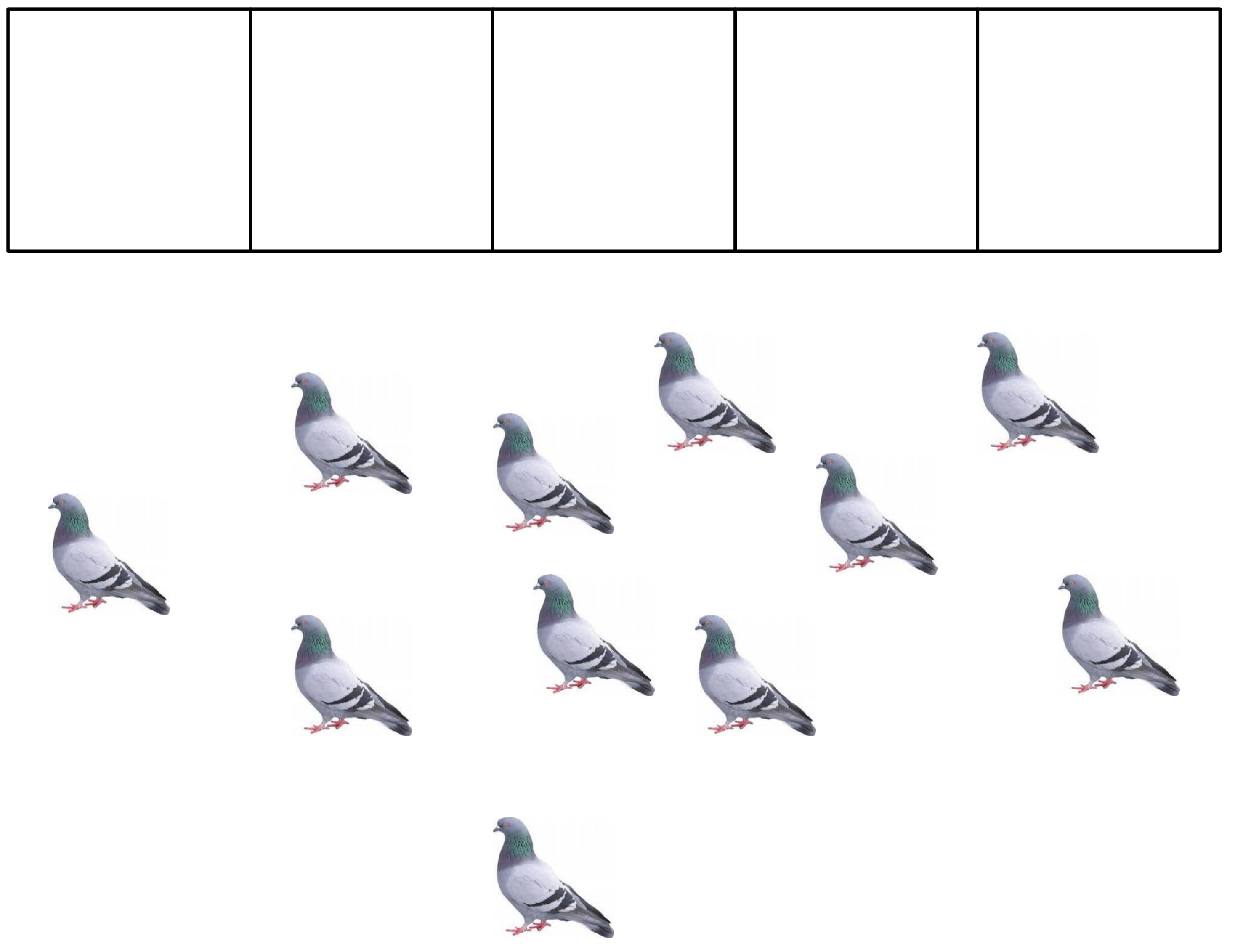
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G, namely 0, 1, 2, …, $n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

The Pigeonhole Principle



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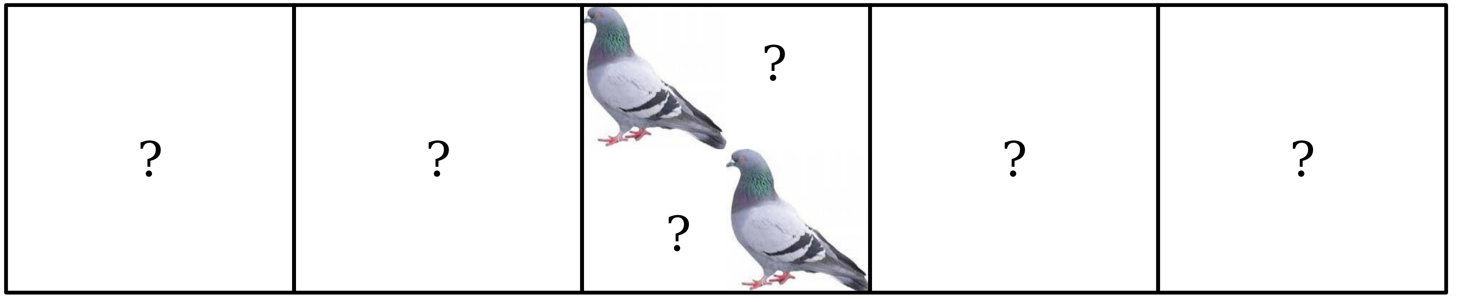

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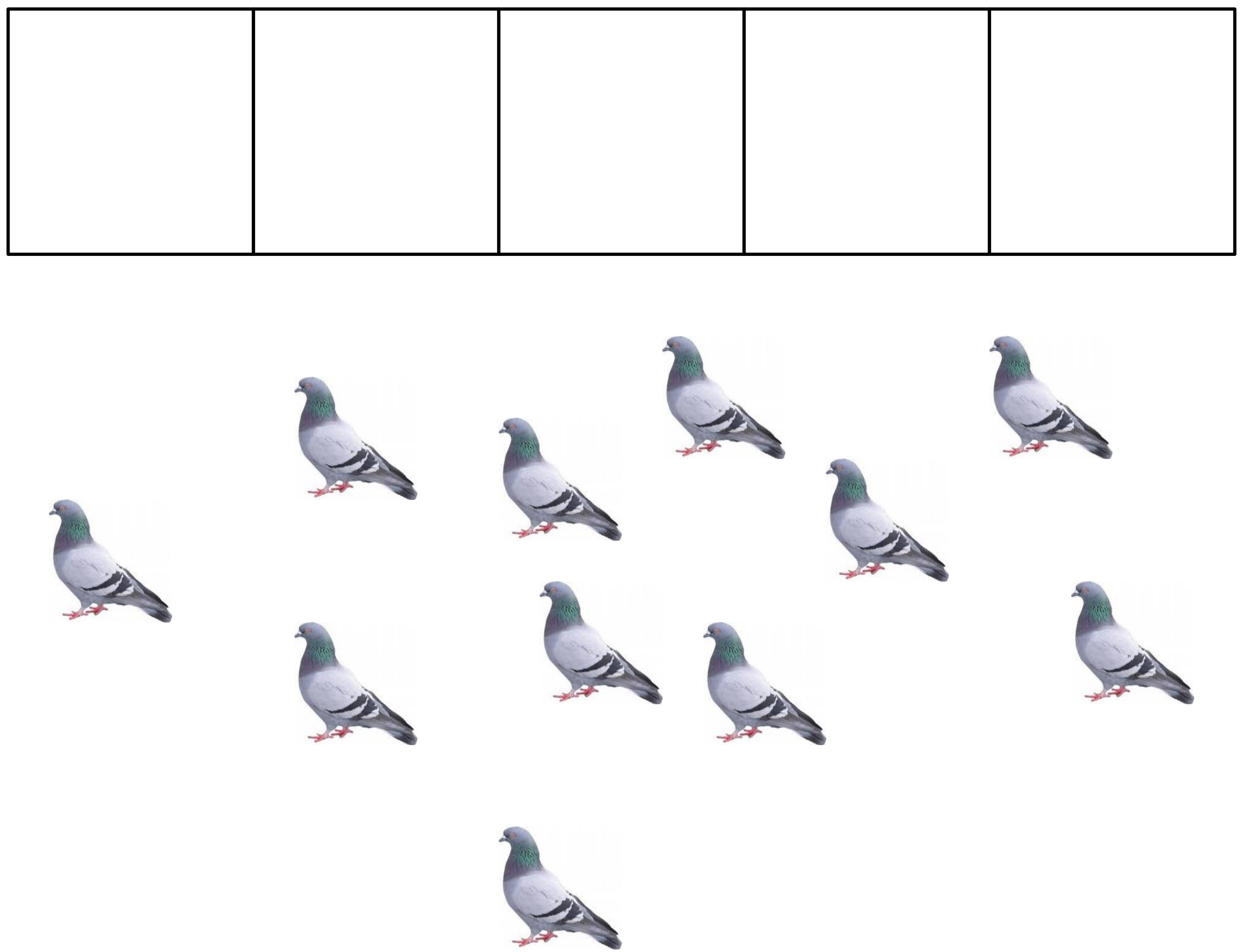

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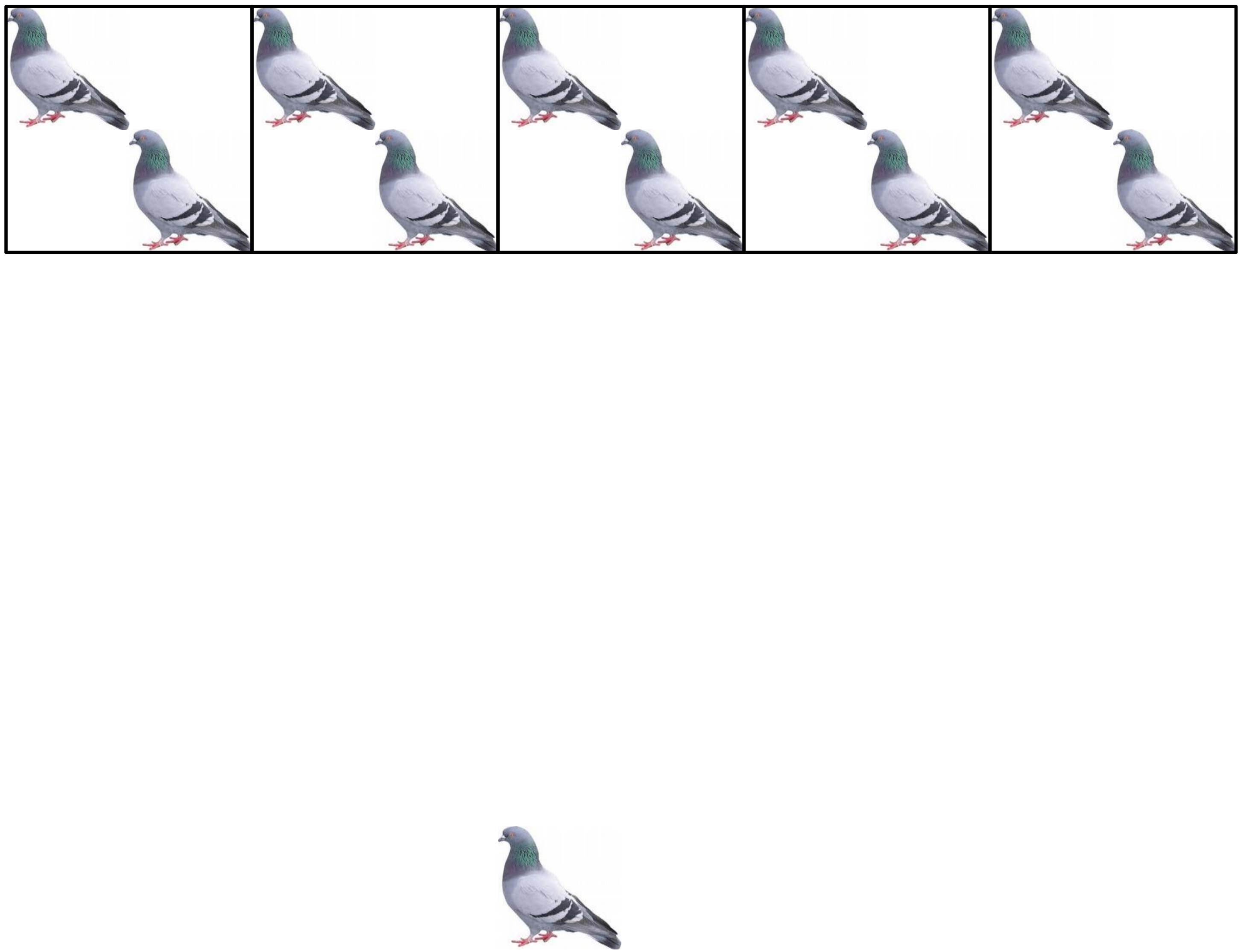

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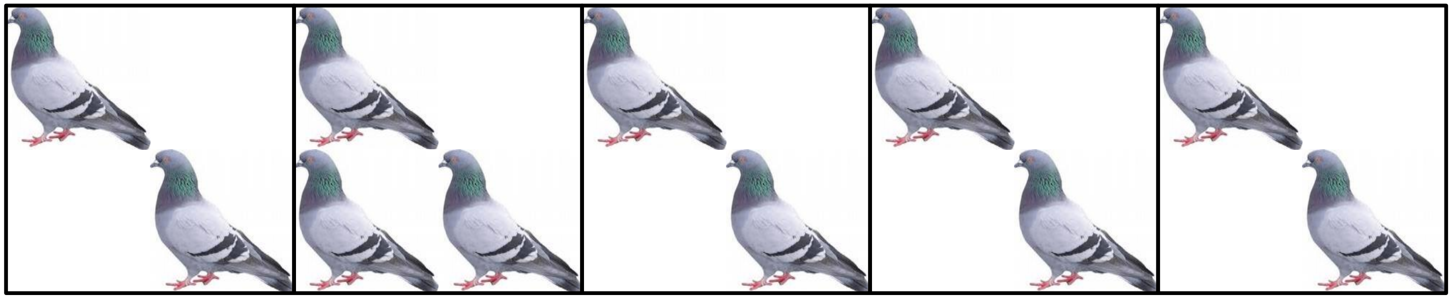

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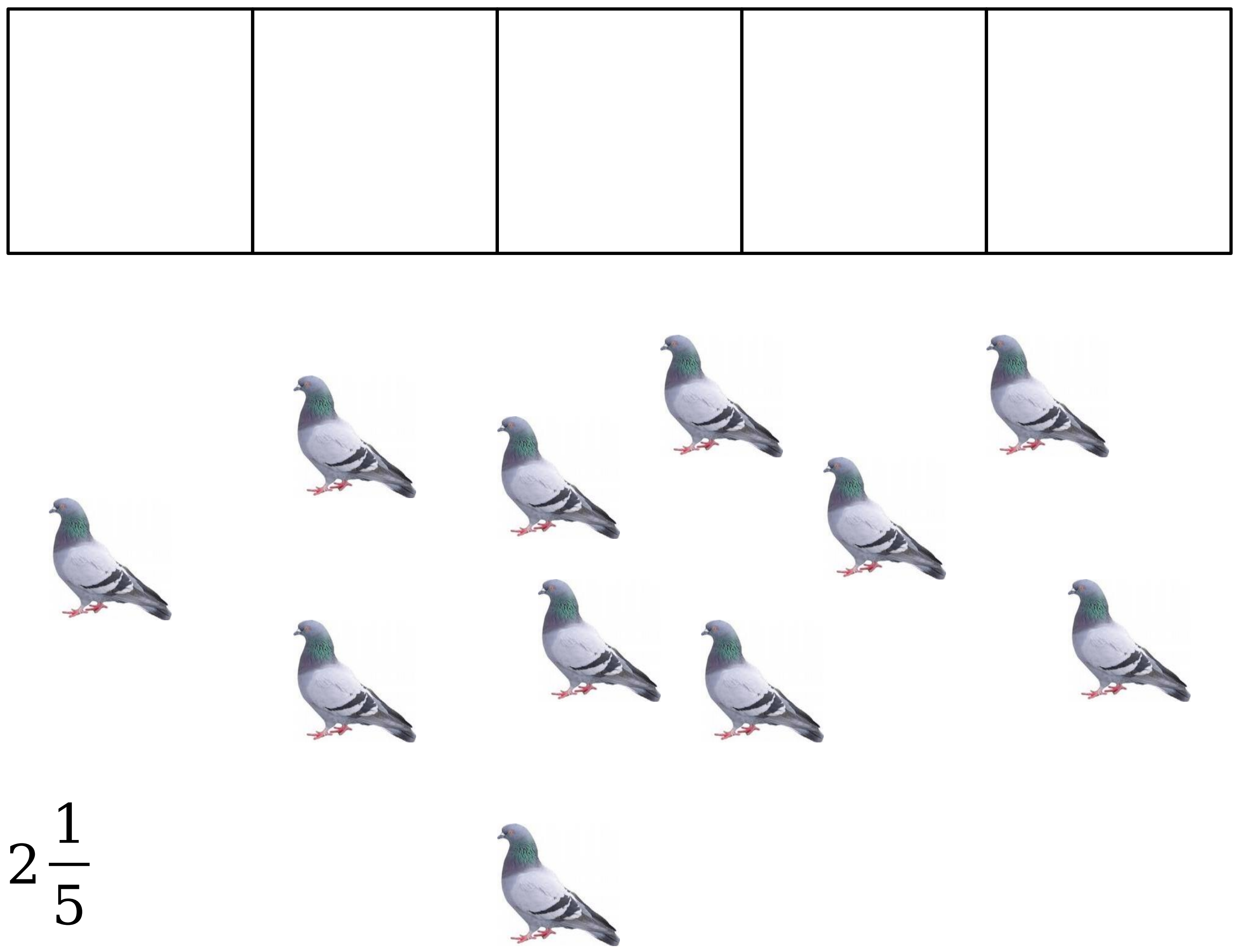

$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

The ***generalized pigeonhole principle*** says that if you distribute m objects into n bins, then

some bin will have at least $\lceil {}^{m}/_{n} \rceil$ objects in it, and

some bin will have at most $\lfloor {}^{m}/_{n} \rfloor$ objects in it.

$\lceil {}^{m}/_{n} \rceil$ means "${}^{m}/_{n}$, rounded up."
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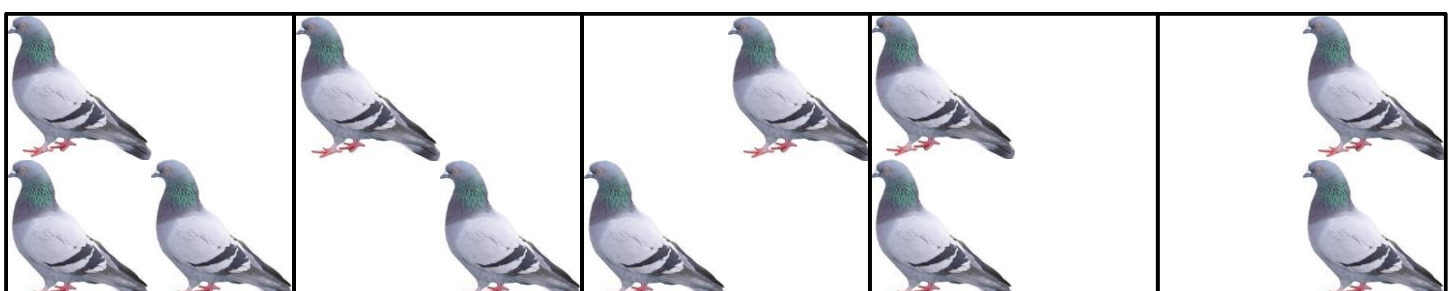

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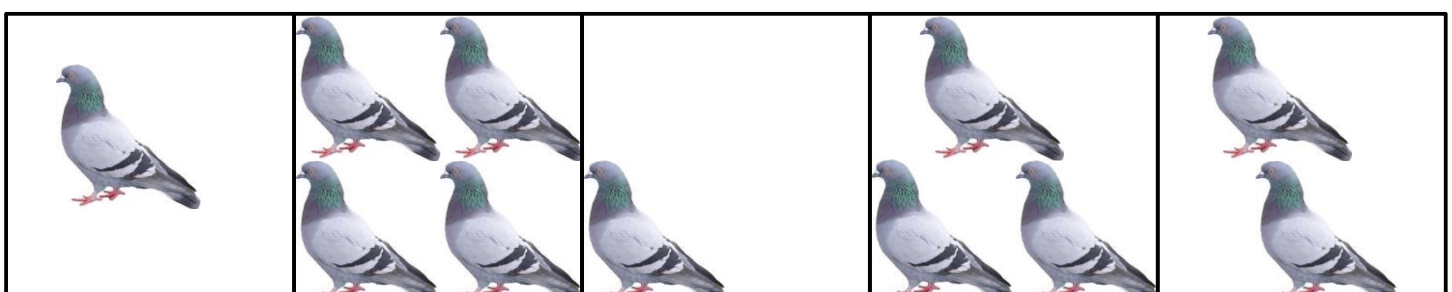

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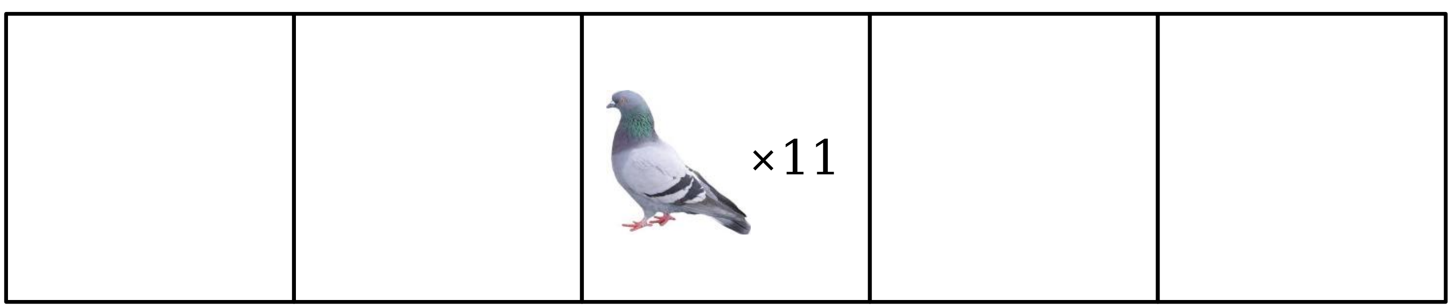

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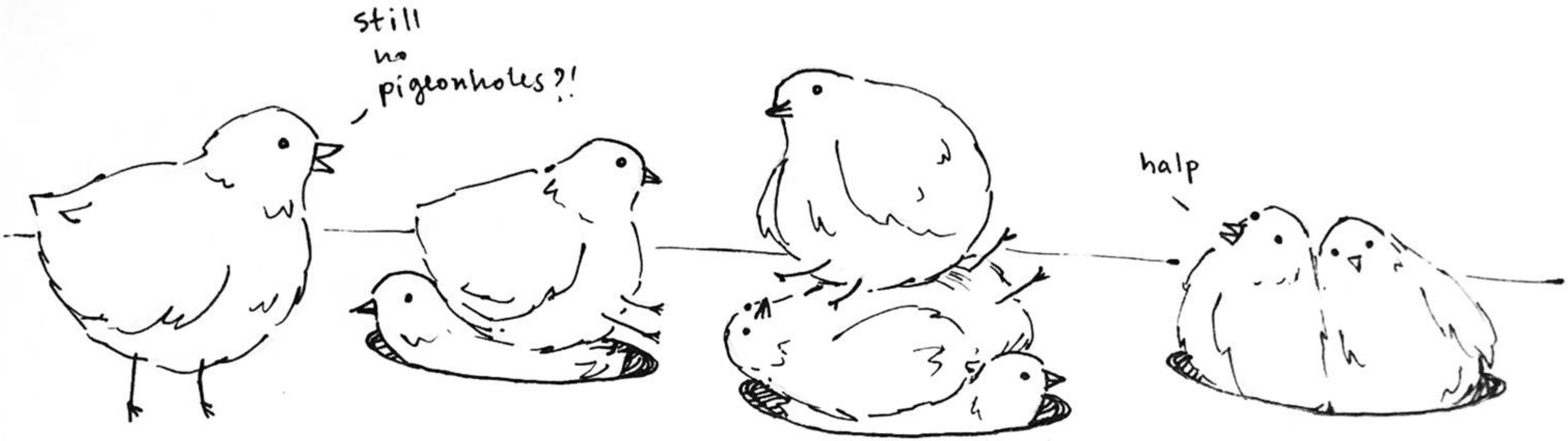


$m = 8, n = 3$

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least m/n objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n, there is a way to distribute m objects into n bins such that each bin contains fewer than m/n objects.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. Since there are m objects in total, we know that

$$m = x_1 + x_2 + \ldots + x_n.$$

Since each bin contains fewer than m/n objects, we see that $x_i < m/n$ for each i. Therefore, we have that

$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &< m/n + m/n + \ldots + m/n \quad (n \text{ times}) \\ &= m. \end{aligned}$$

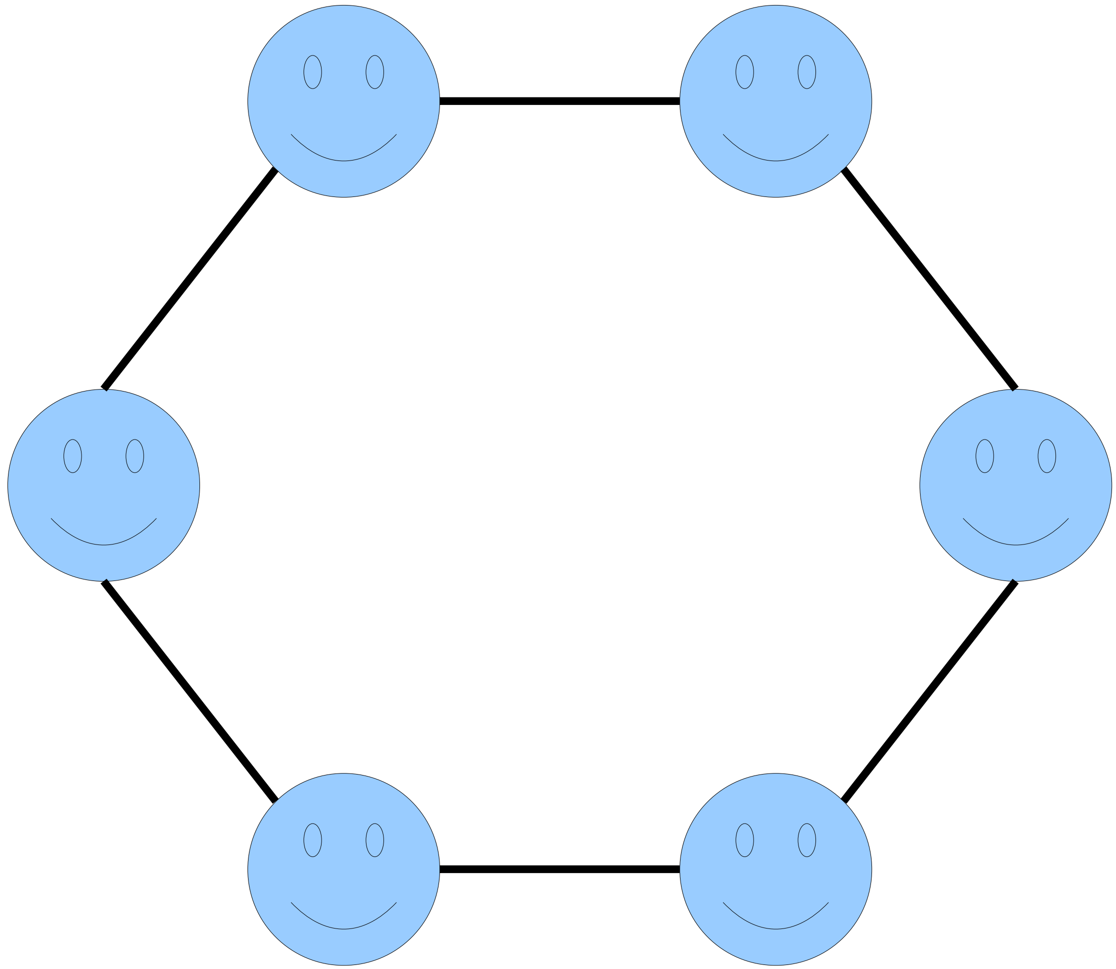
But this means that $m < m$, which is impossible. We have reached a contradiction, so our initial assumption must have been wrong. Therefore, if m objects are distributed into n bins, some bin must contain at least $\lceil m/n \rceil$ objects. ■

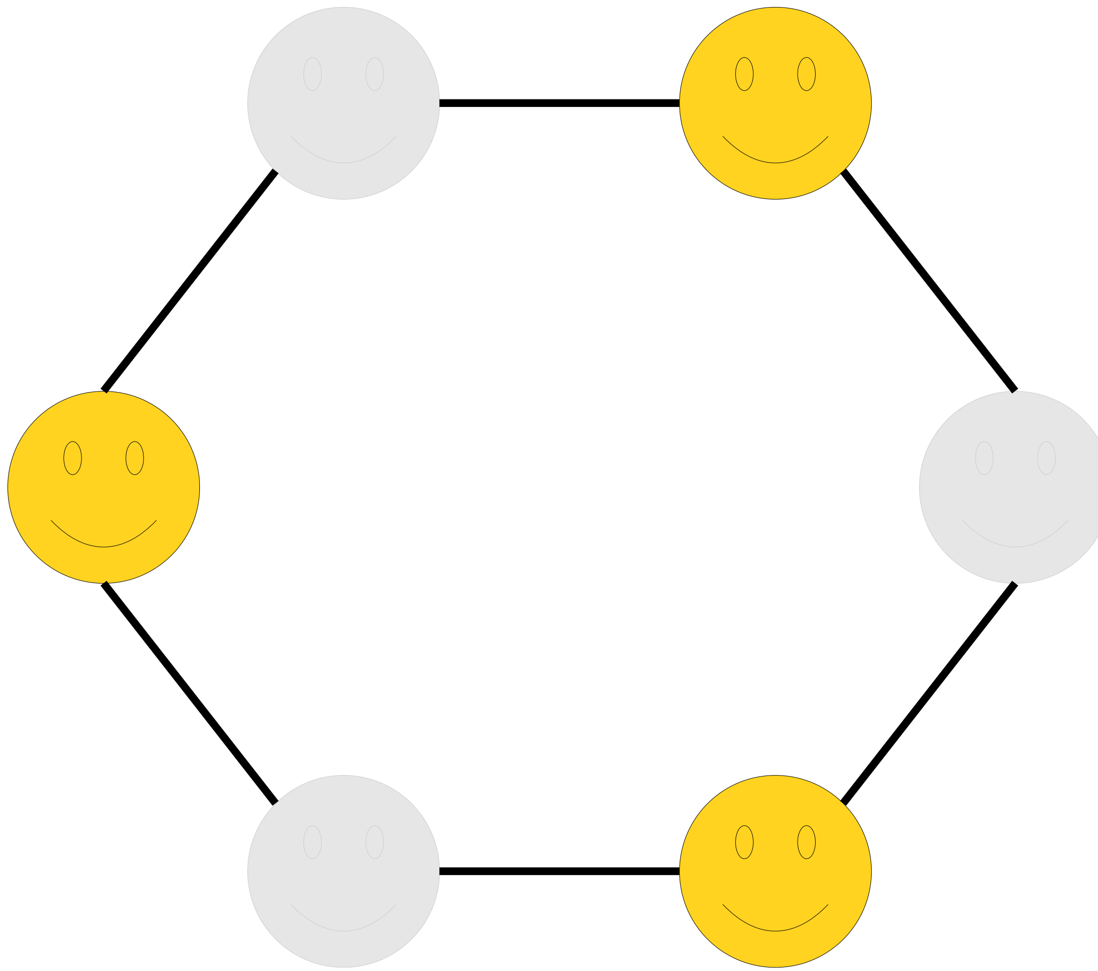
An Application: Friends and Strangers

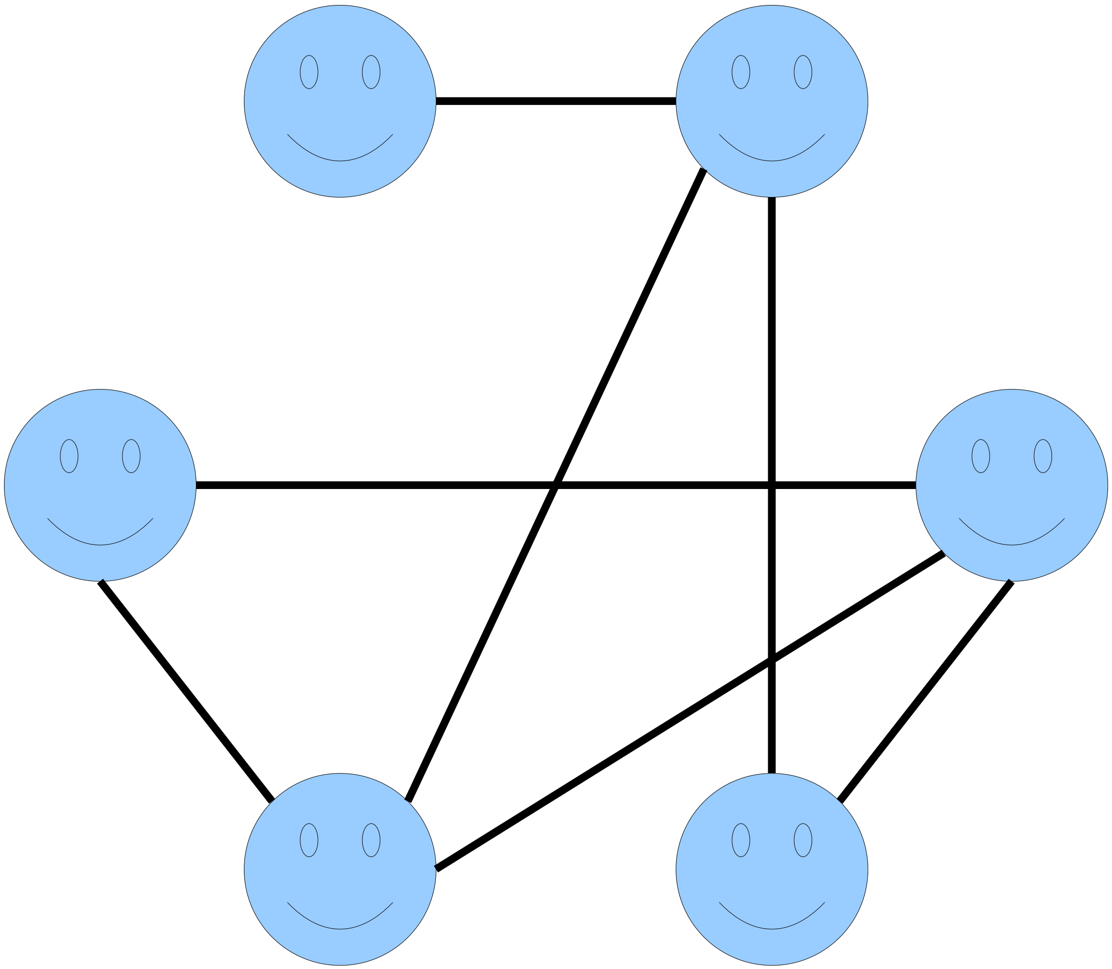
Friends and Strangers

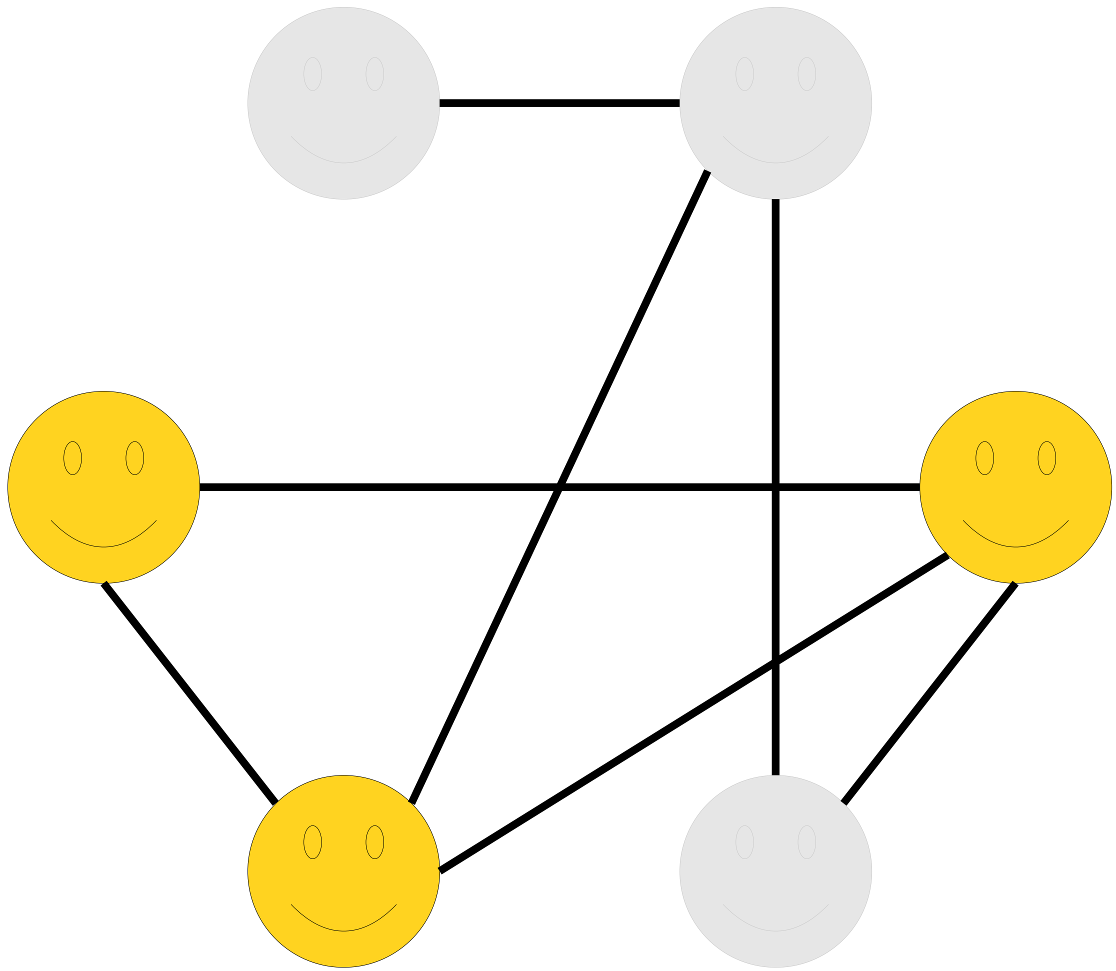
Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).

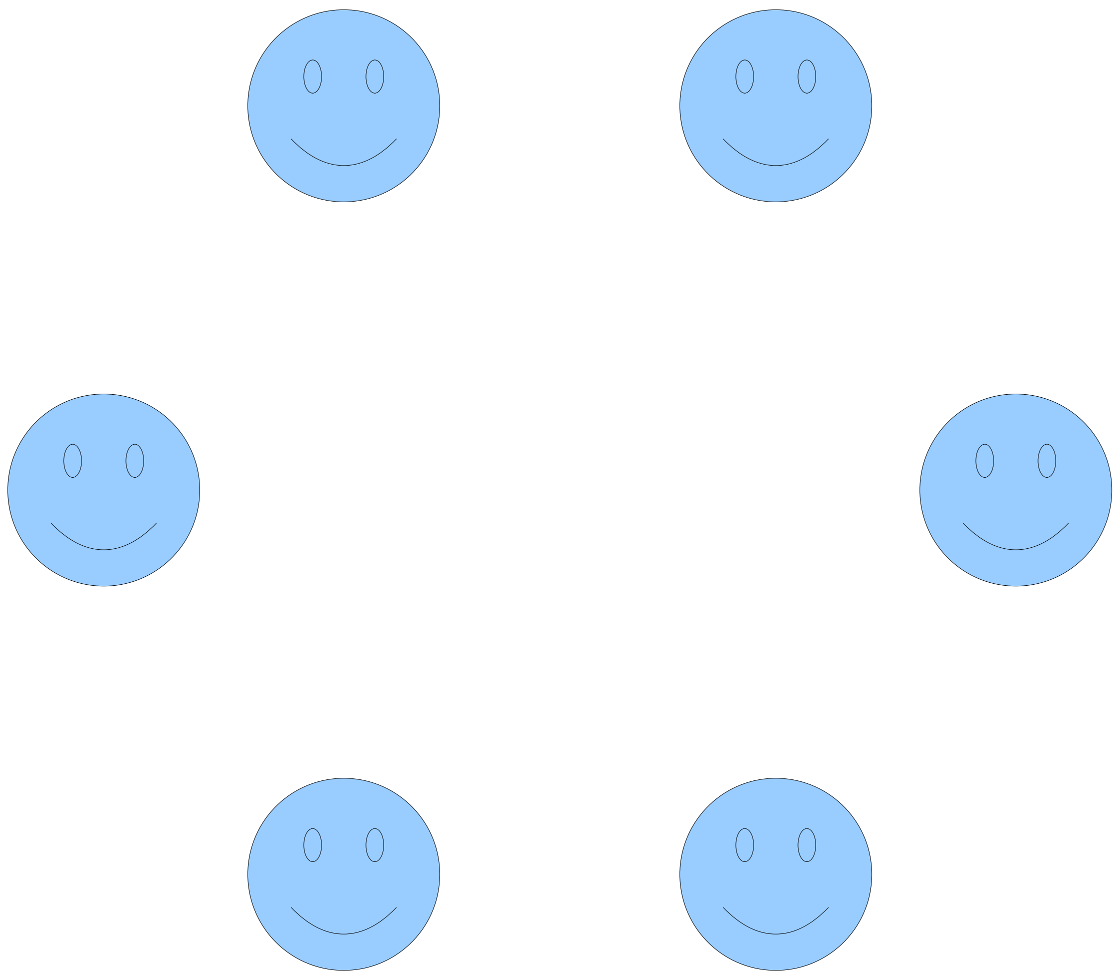
Theorem: Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

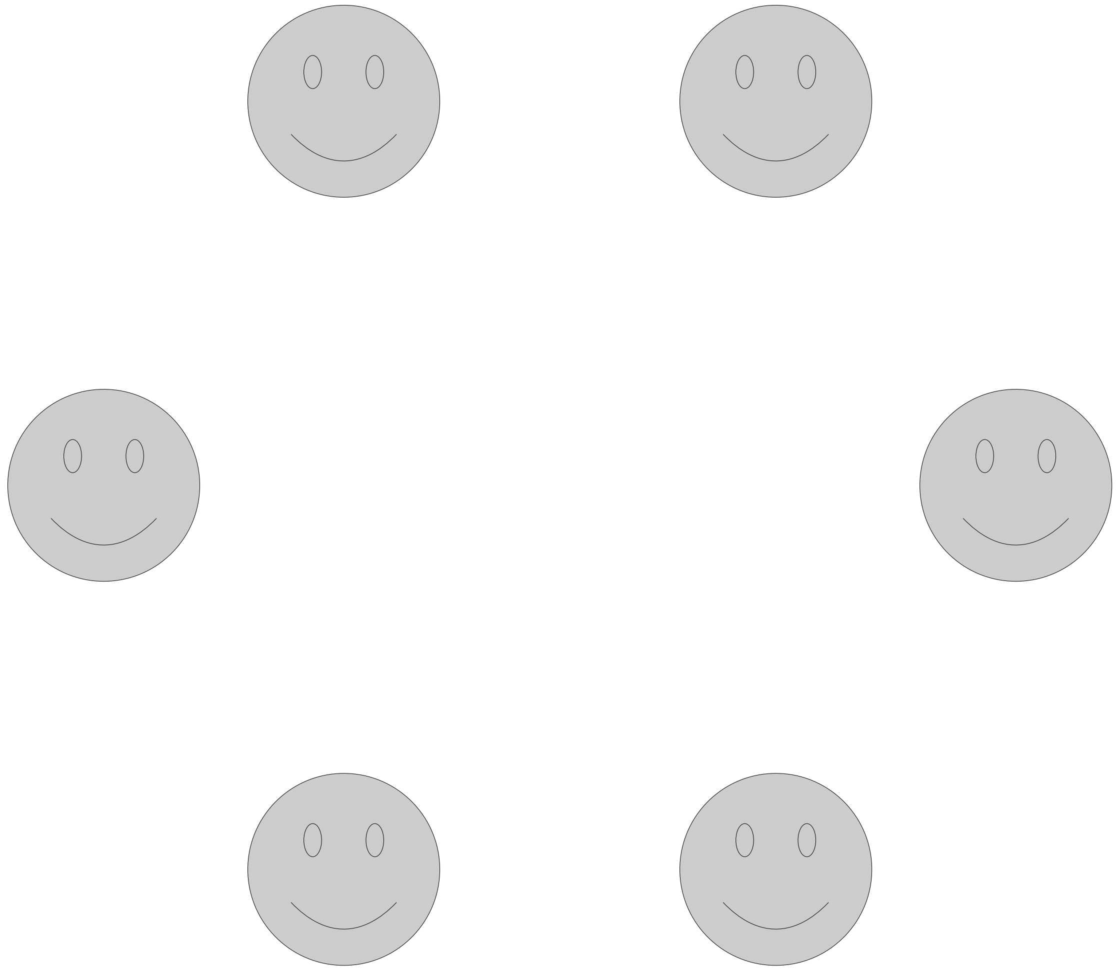


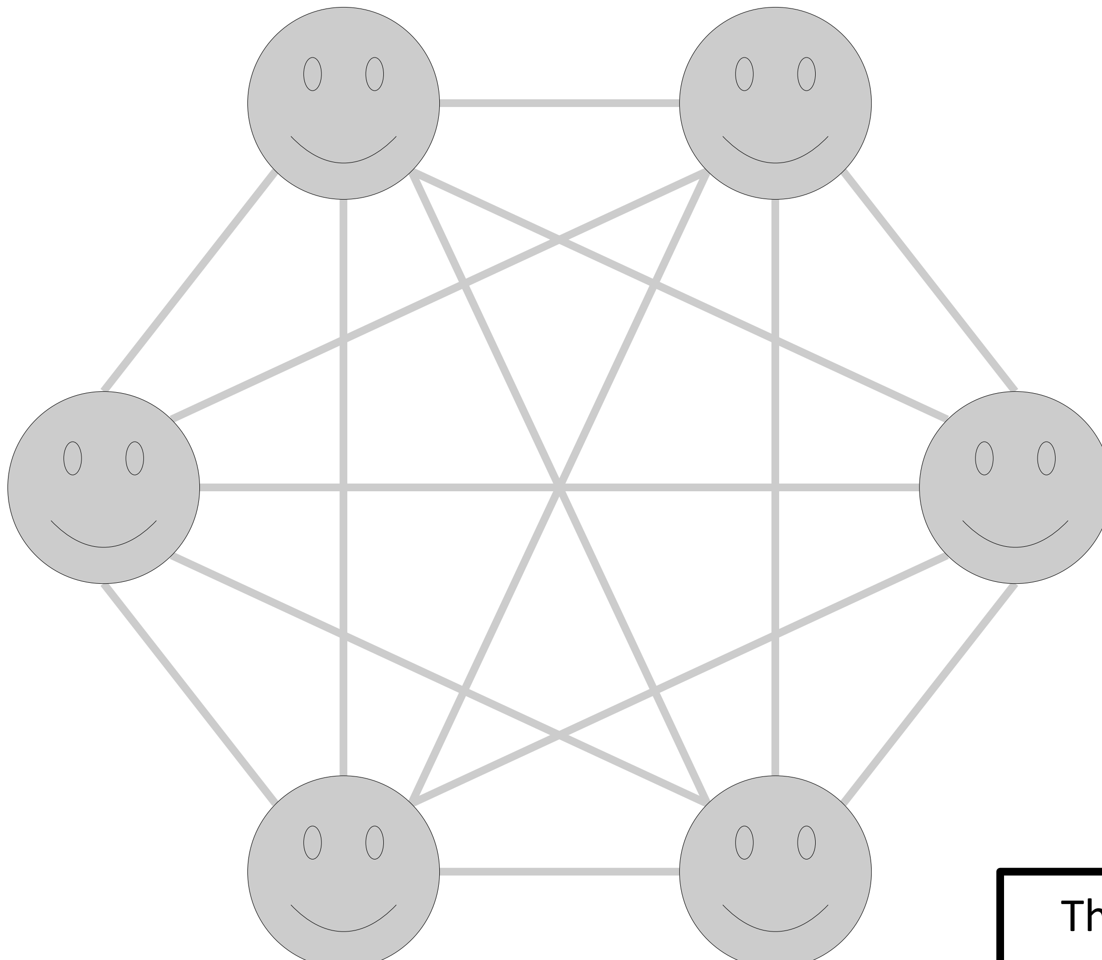




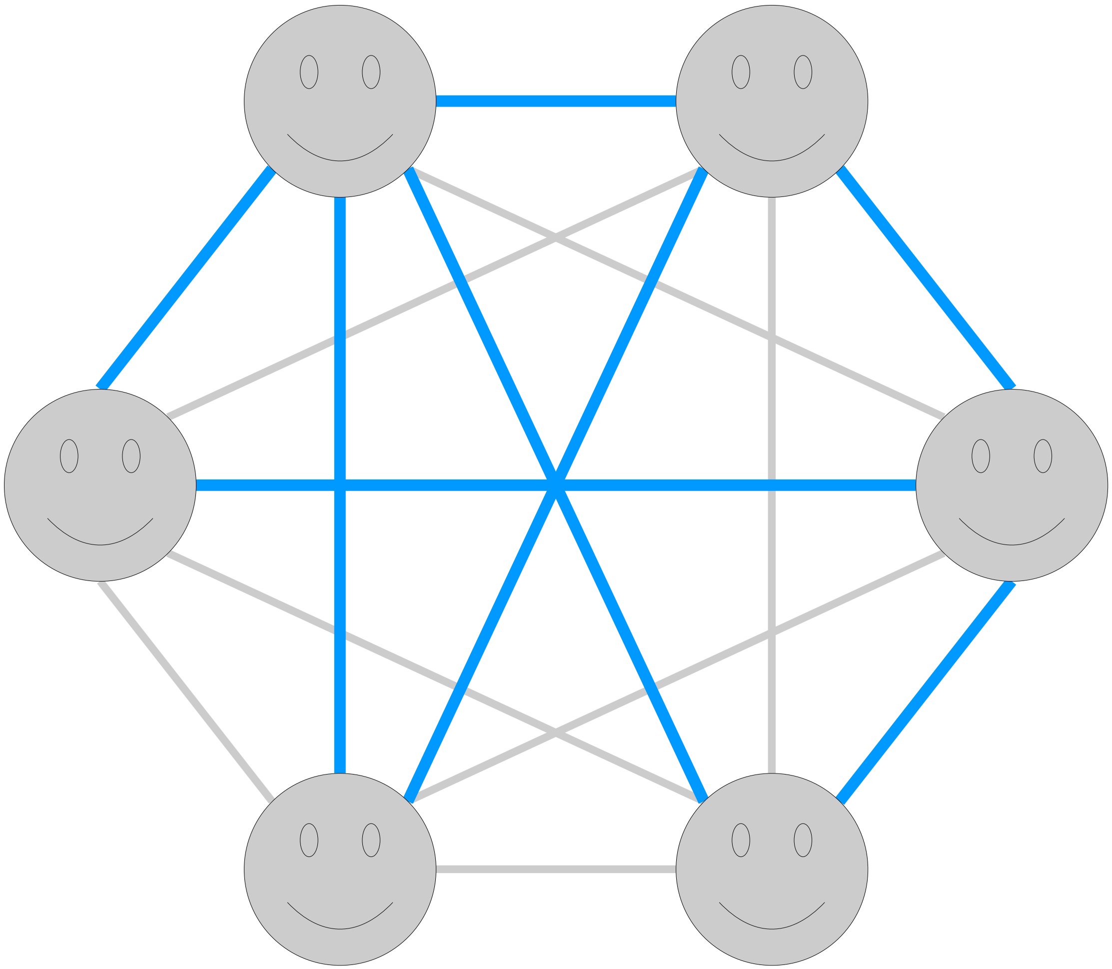


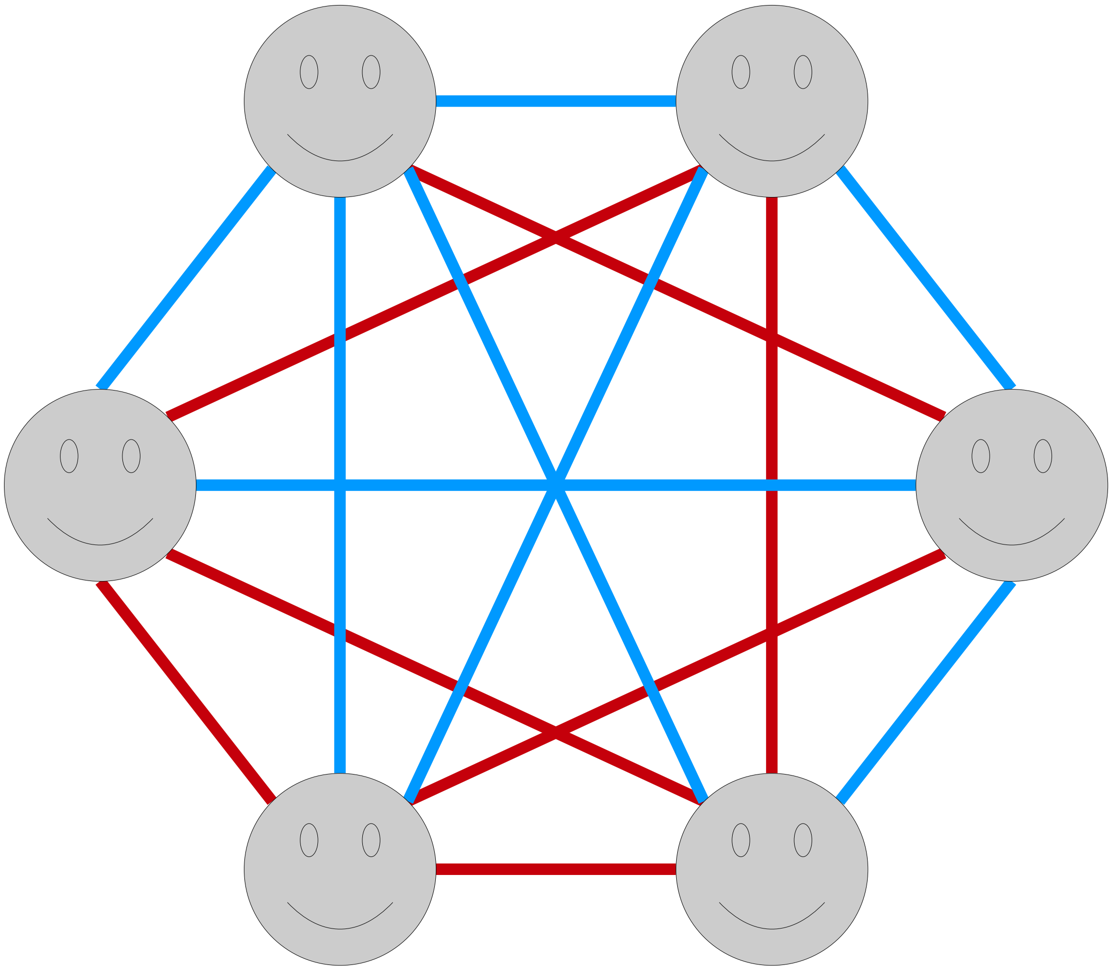


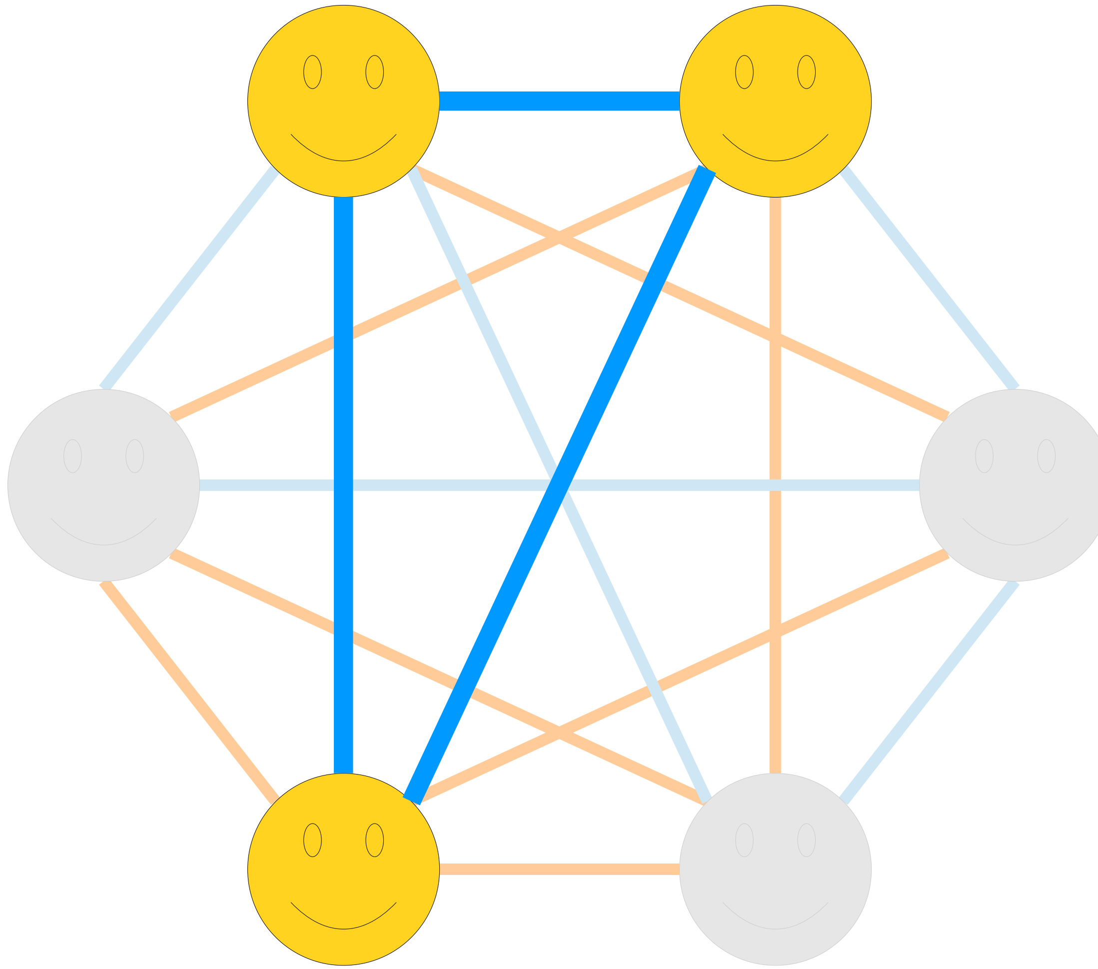


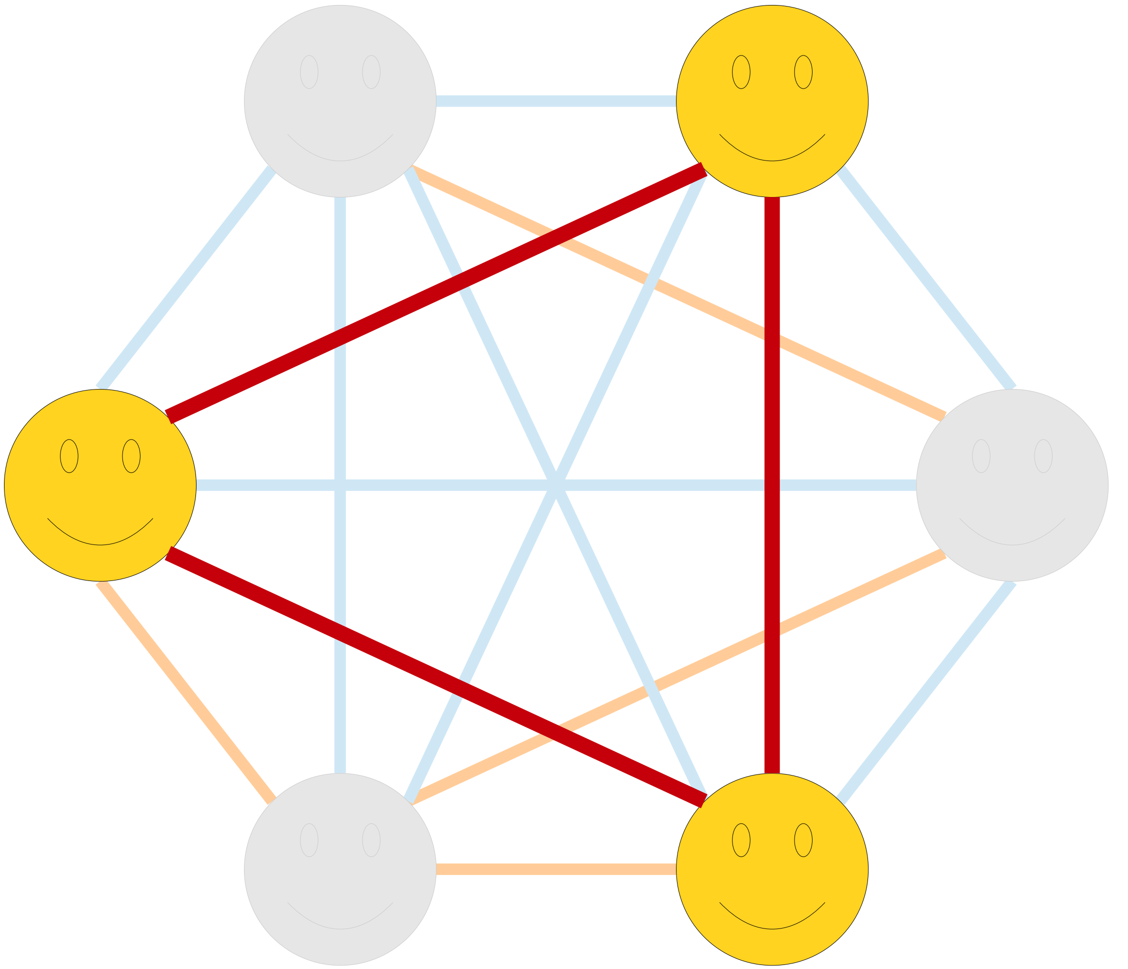


This graph is called a ***6-clique***, by the way.







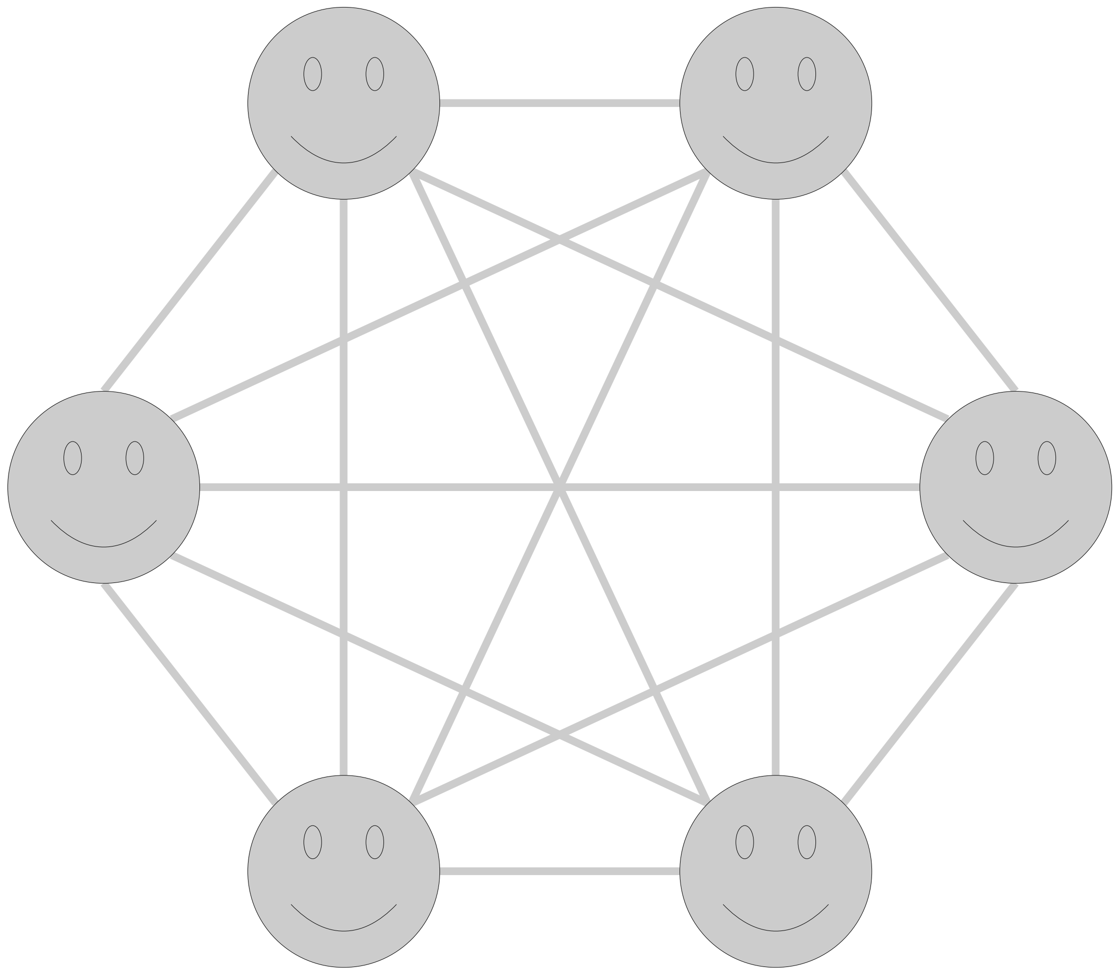


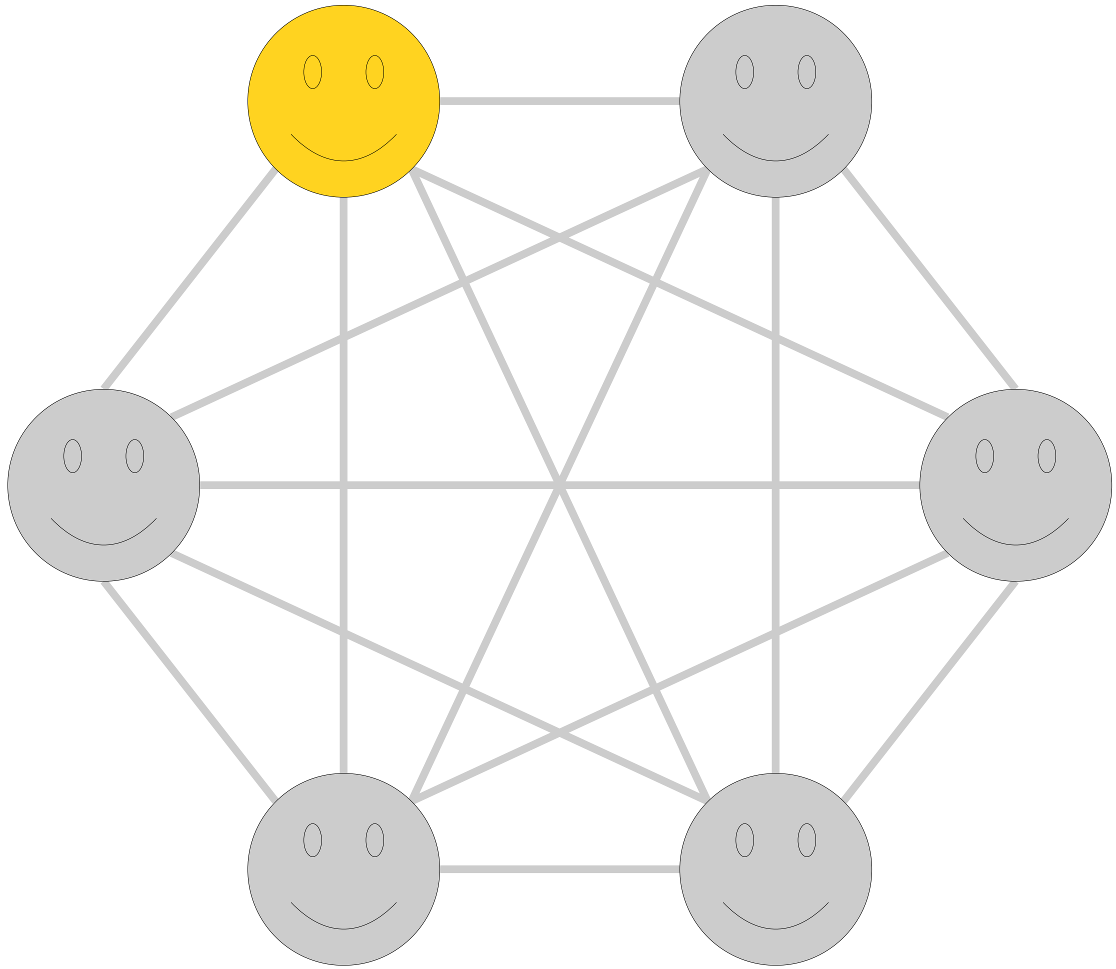
Friends and Strangers Restated

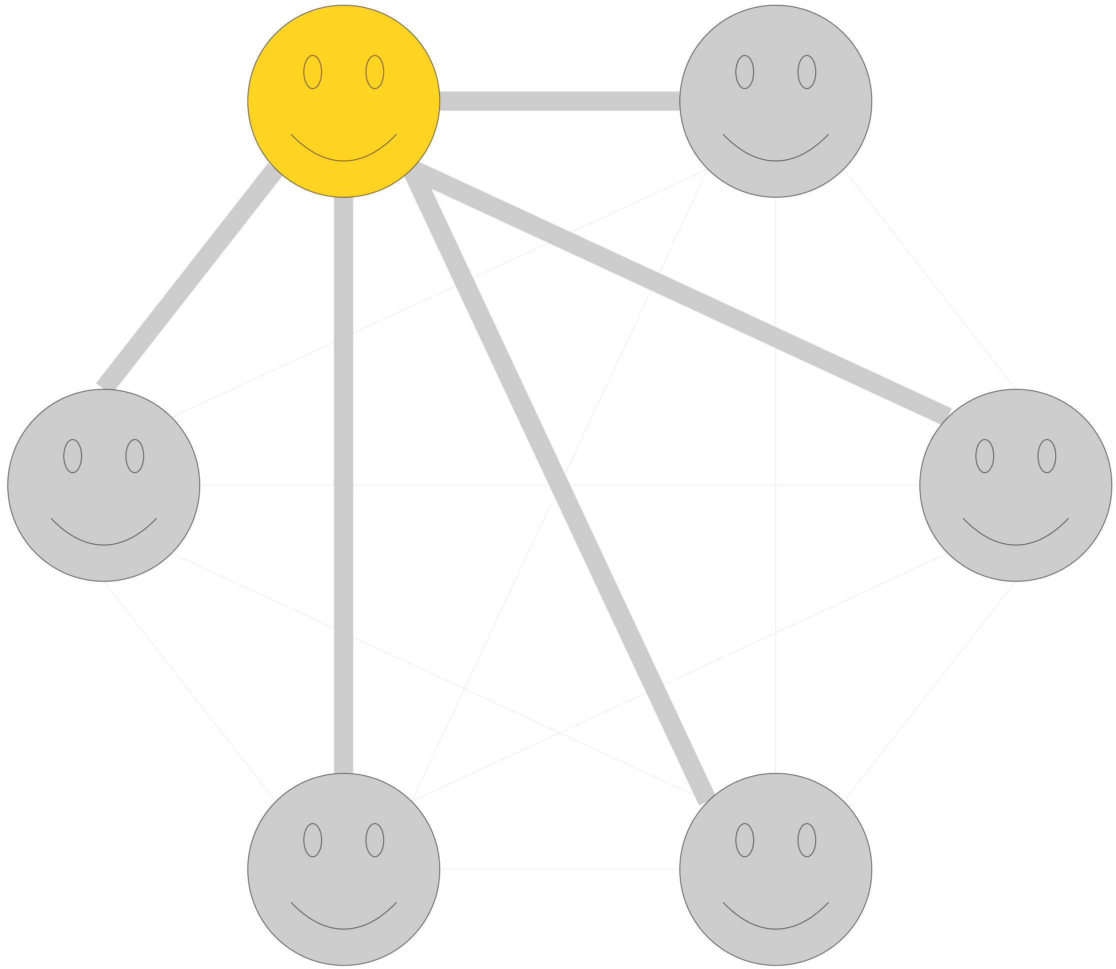
From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

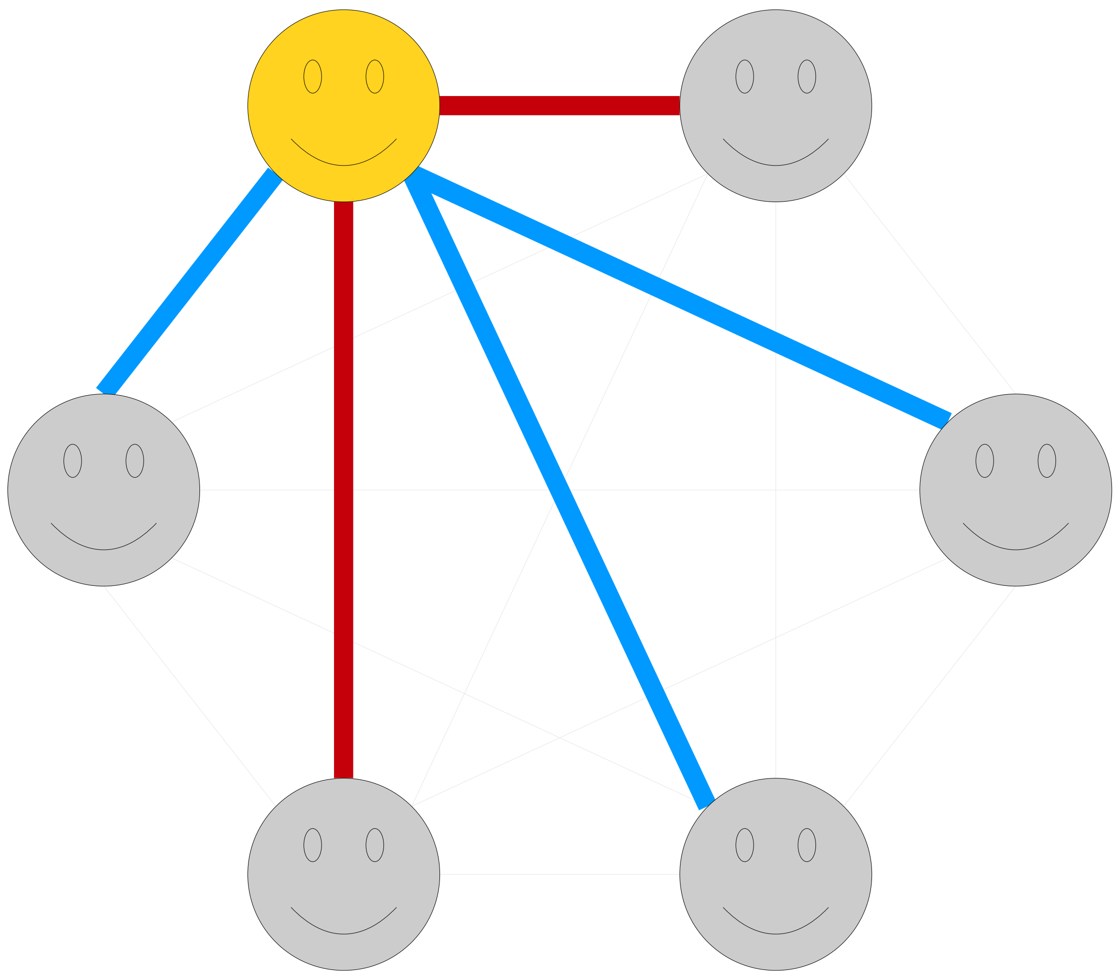
Theorem: Any 6-clique whose edges are colored red and blue contains a red triangle or a blue triangle (or both).

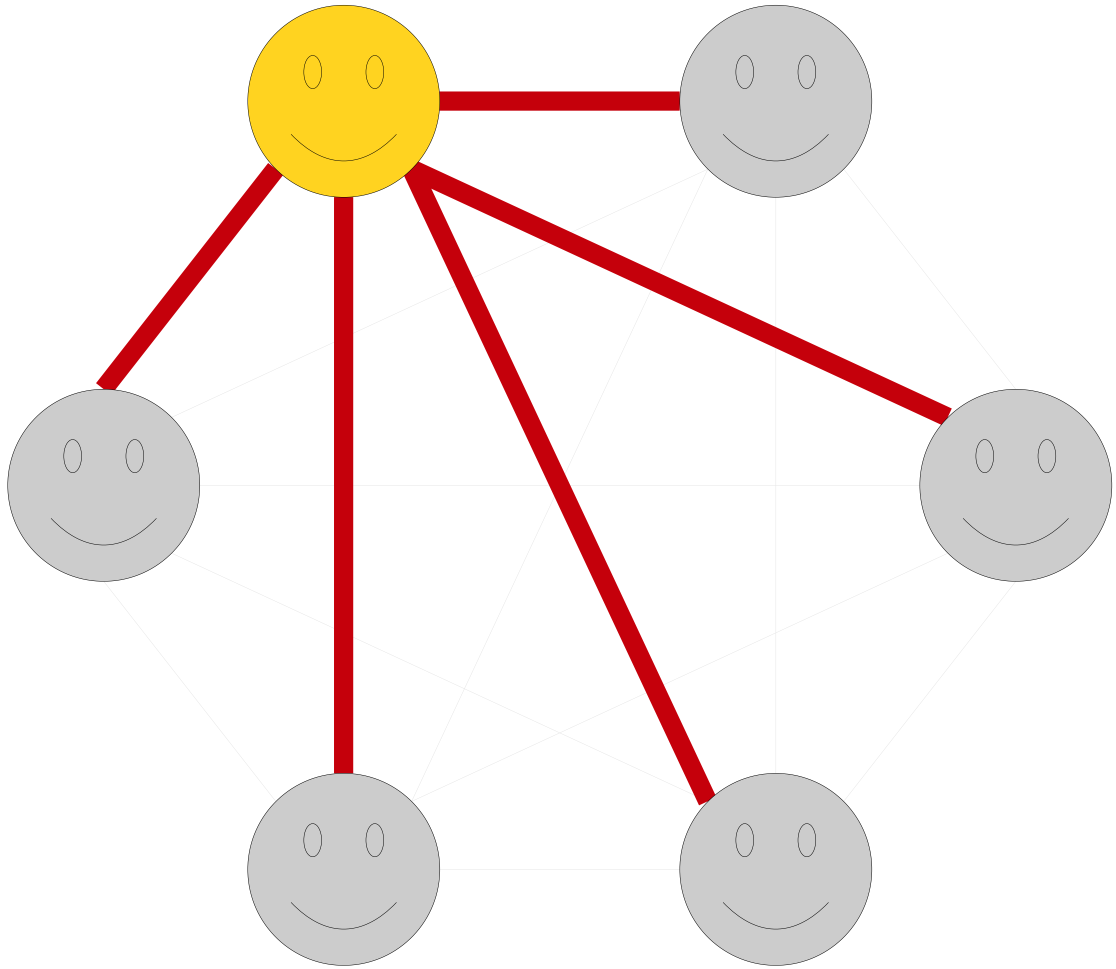
How can we prove this?

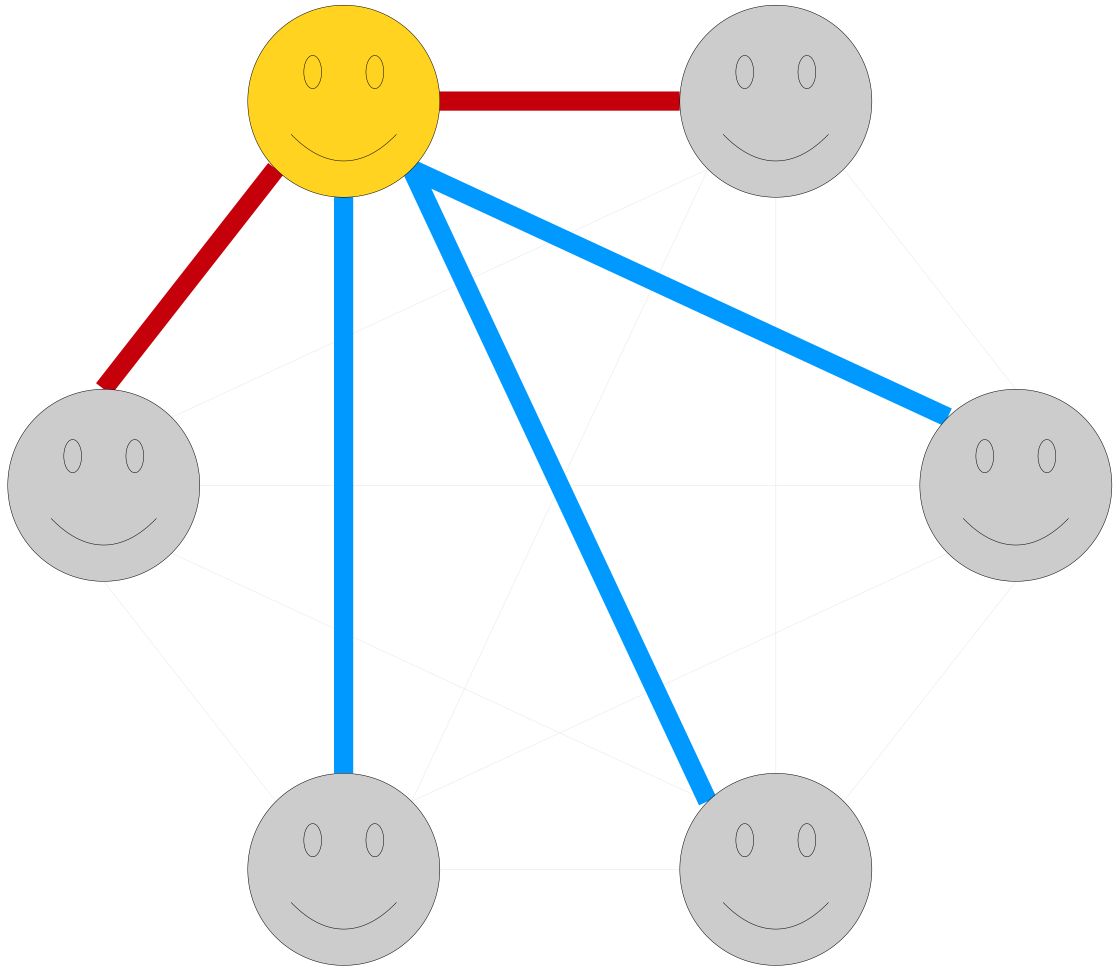


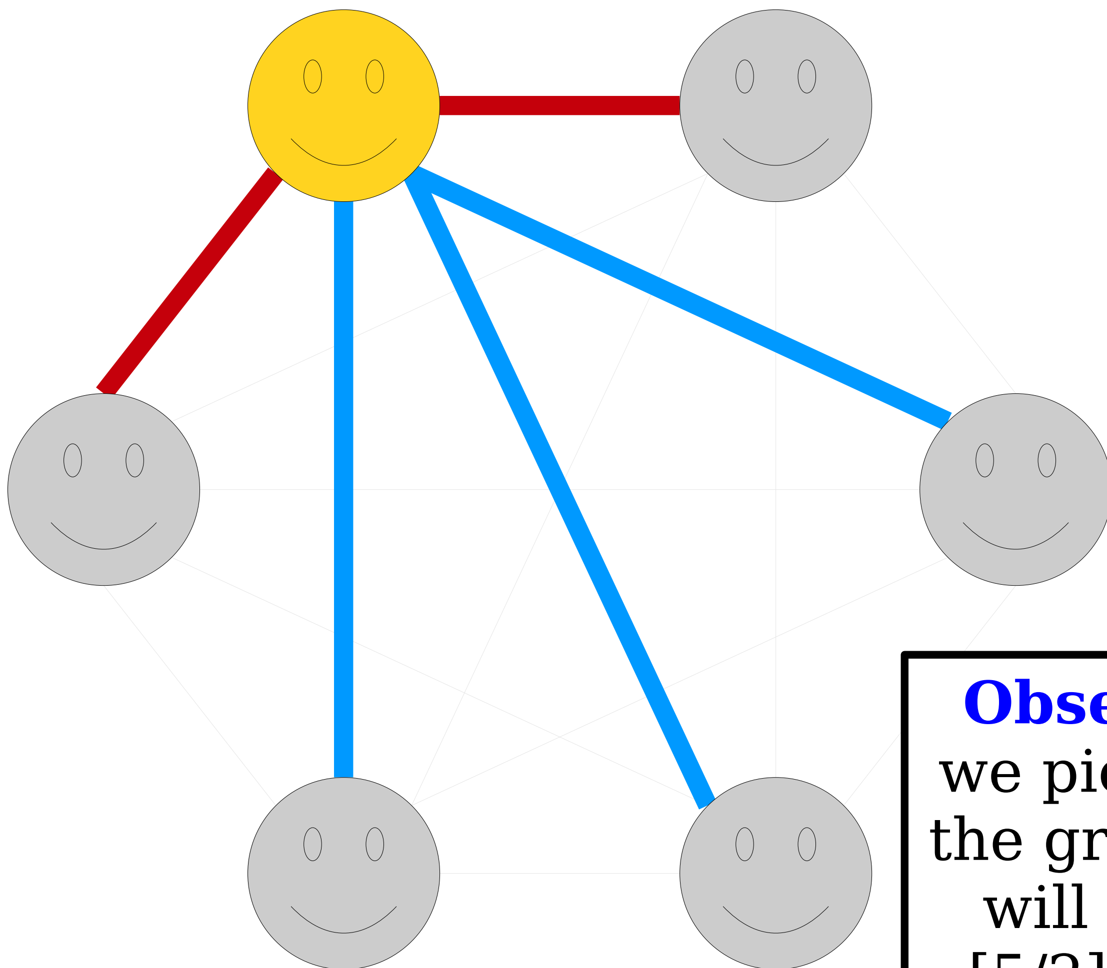




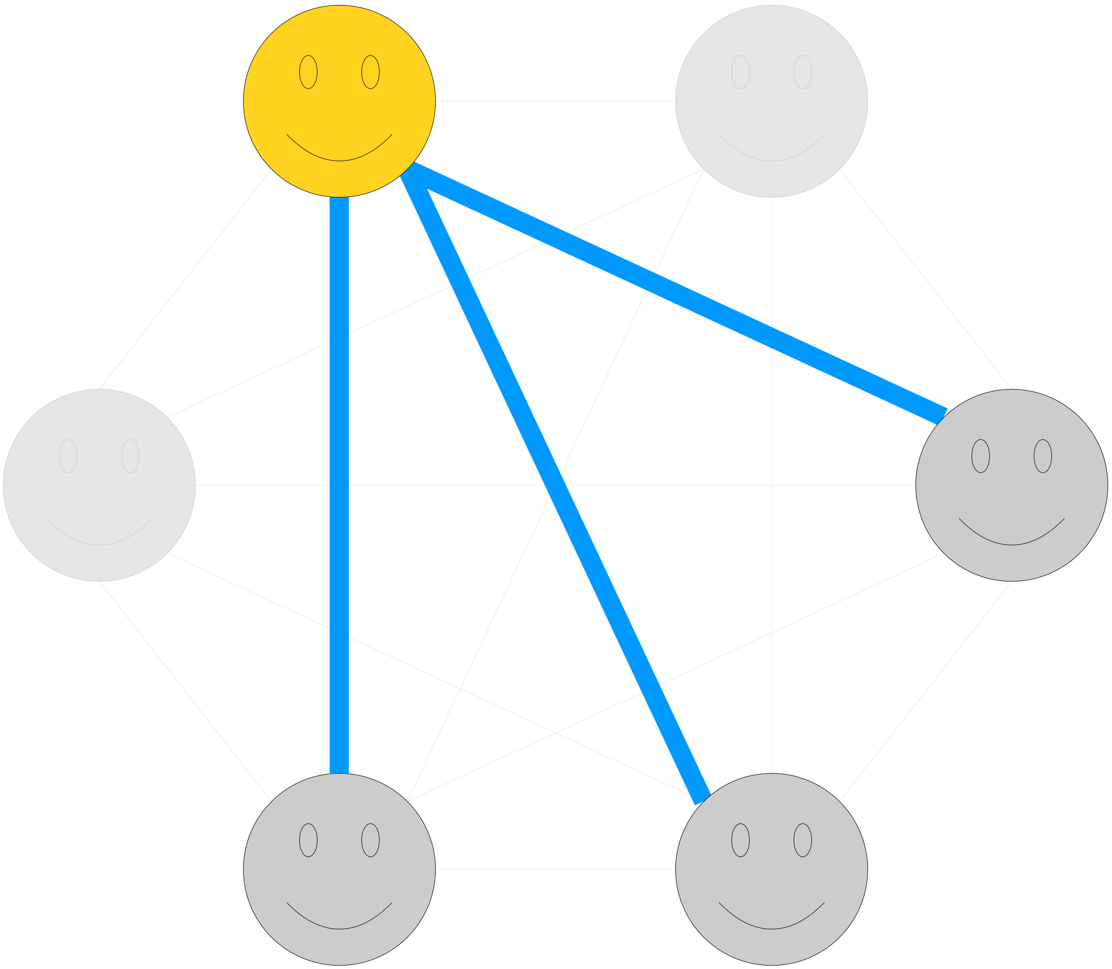


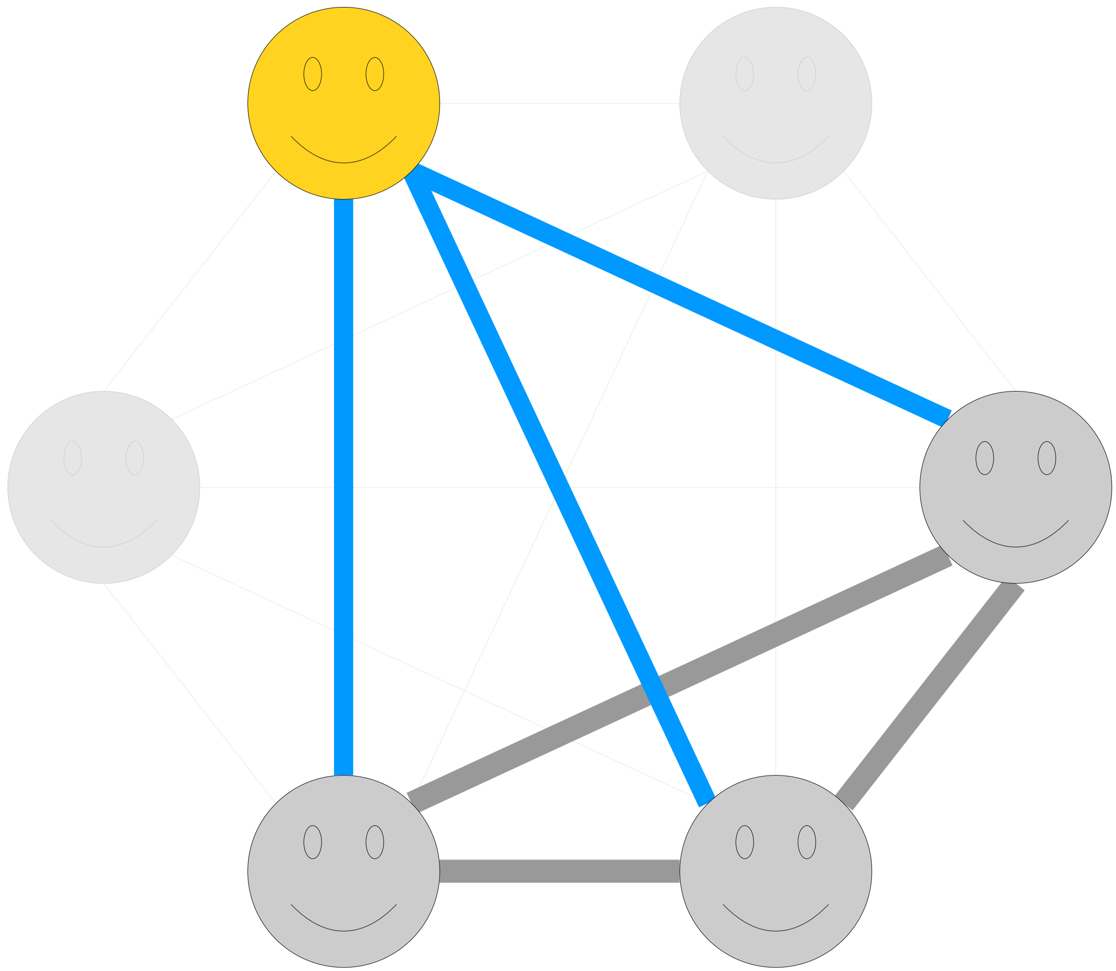


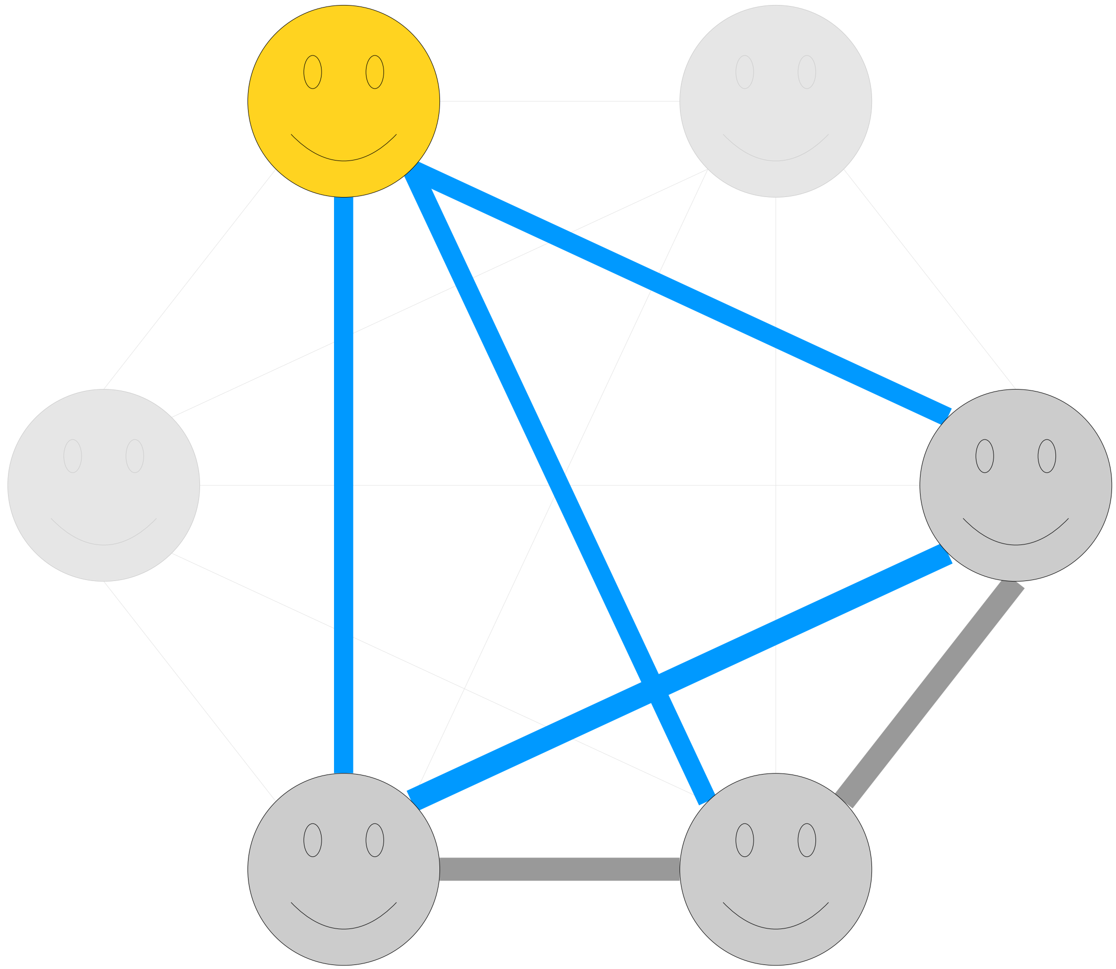


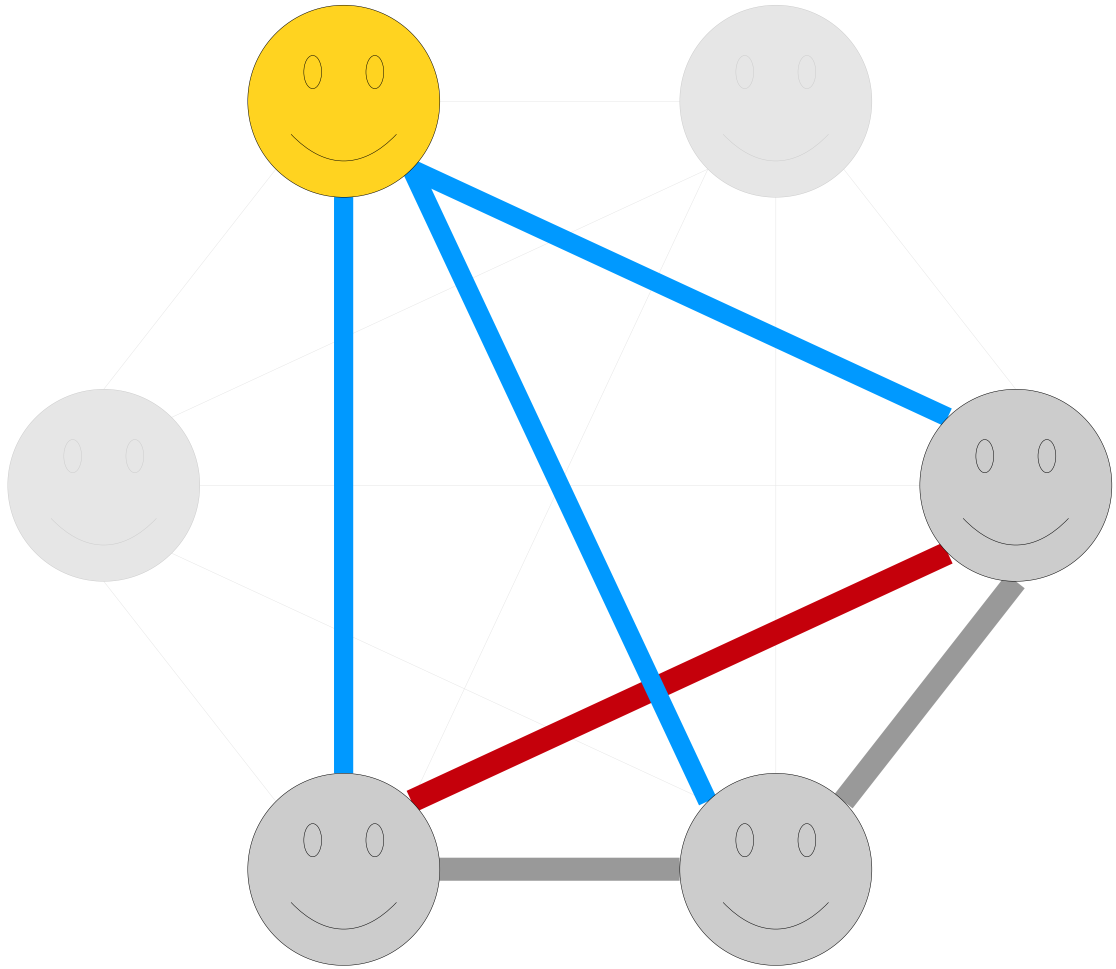


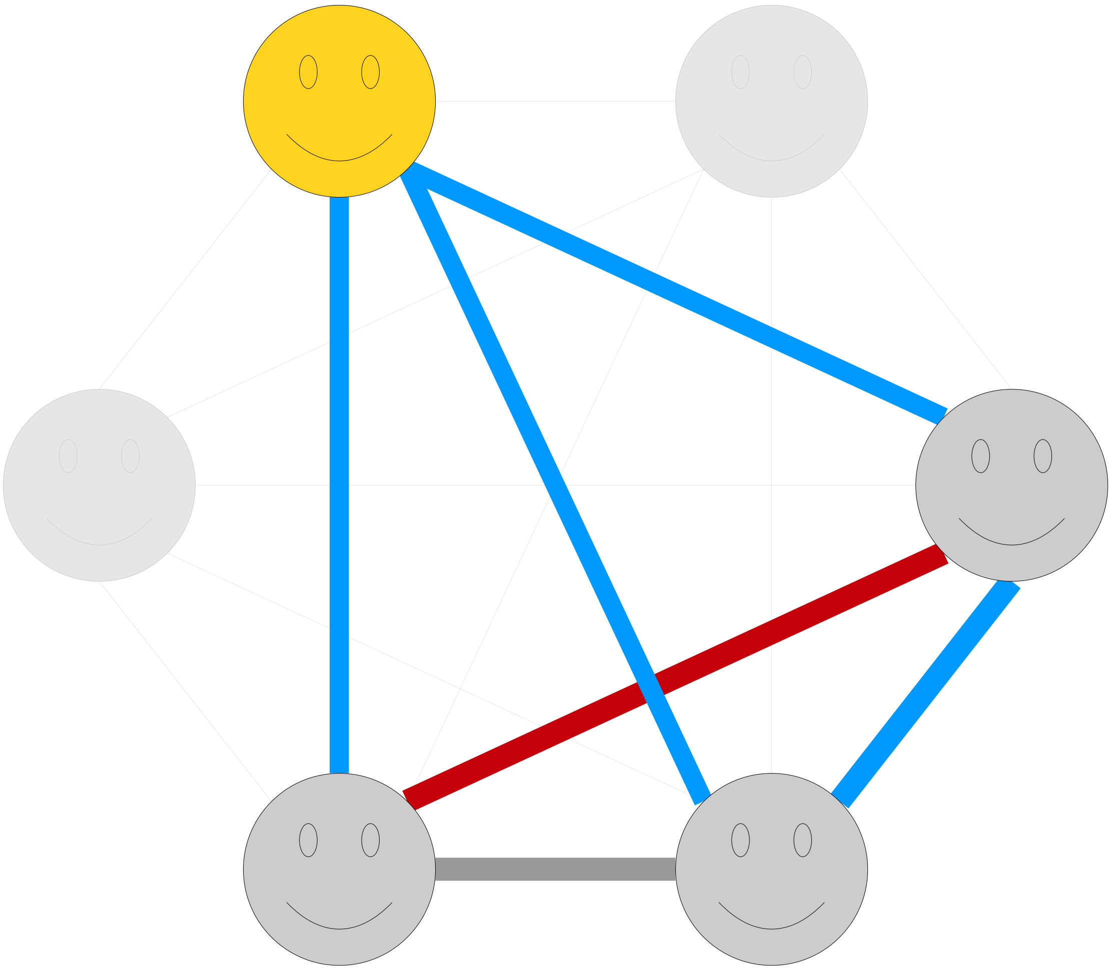
Observation 1: If we pick any node in the graph, that node will have at least ⌈5/2⌉ = 3 edges of the same color incident to it.

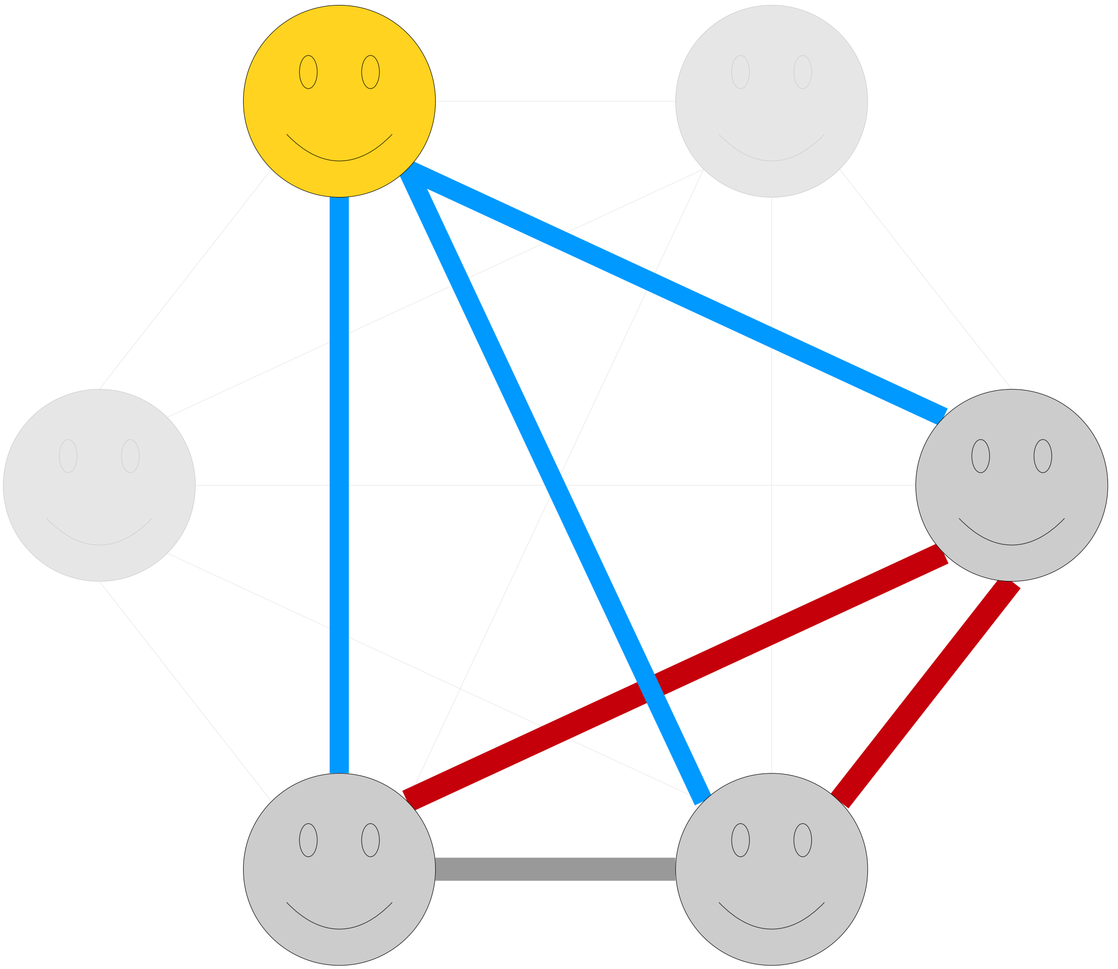


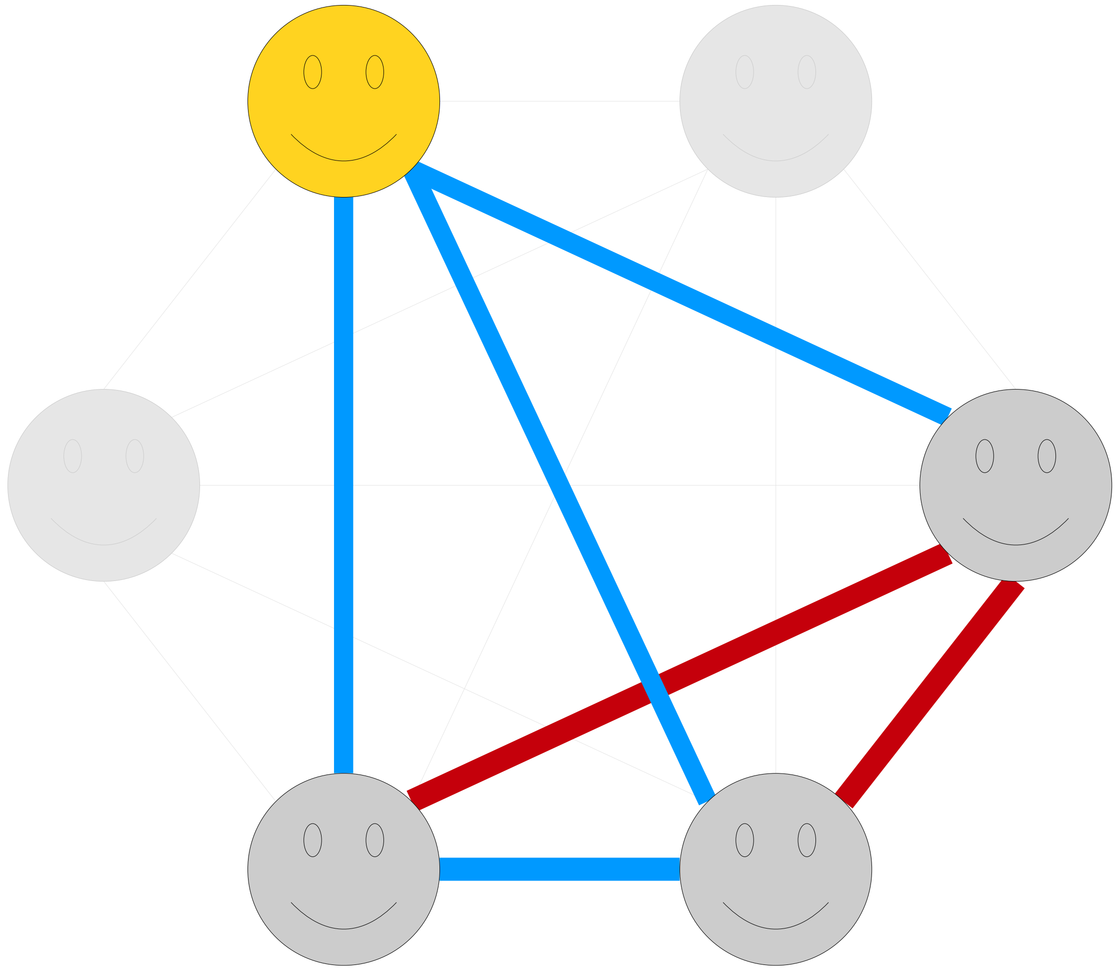


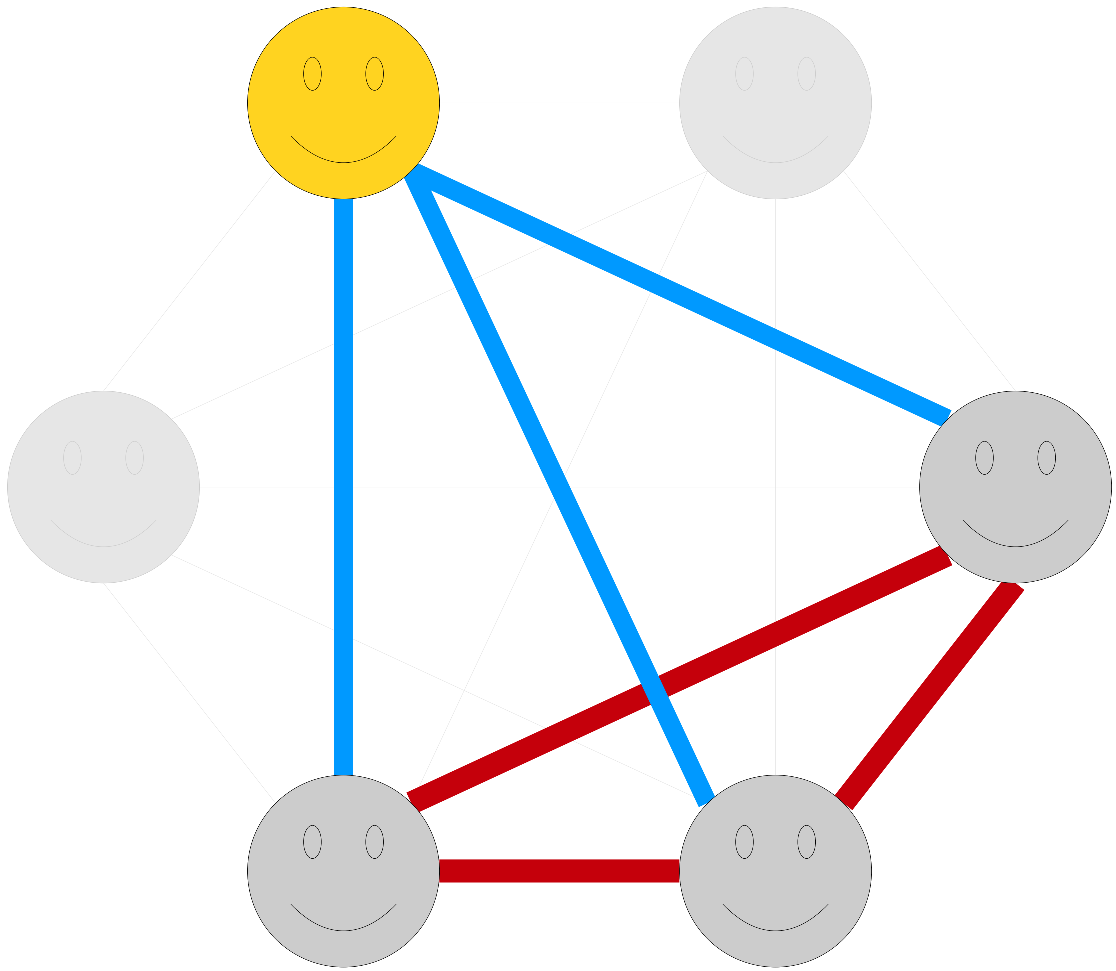












Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: Color the edges of the 6-clique either red or blue arbitrarily. Let x be any node in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Call that color c_1 and let the other color be c_2.

Let r, s, and t be three of the nodes adjacent to node x along an edge of color c_1. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are of color c_1, then one of those edges plus the two edges connecting back to node x form a triangle of color c_1. Otherwise, all three of those edges are of color c_2, and they form a triangle of color c_2. Overall, this gives a red triangle or a blue triangle, as required. ■

Ramsey Theory

The proof we did is a special case of a broader result.

Theorem (Ramsey's Theorem): For any natural number n, there is a smallest natural number $R(n)$ such that if the edges of an $R(n)$-clique are colored red or blue, the resulting graph will contain either a red n-clique or a blue n-clique.

Our proof was that $R(3) \leq 6$.

A more philosophical take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.

A Little Math Puzzle

"In a group of $n > 0$ people …

- 90% of those people enjoyed *Get Out*,
- 80% of those people enjoyed *Lady Bird*,
- 70% of those people enjoyed *Arrival*, and
- 60% of those people enjoyed *Zootopia*.

No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?"

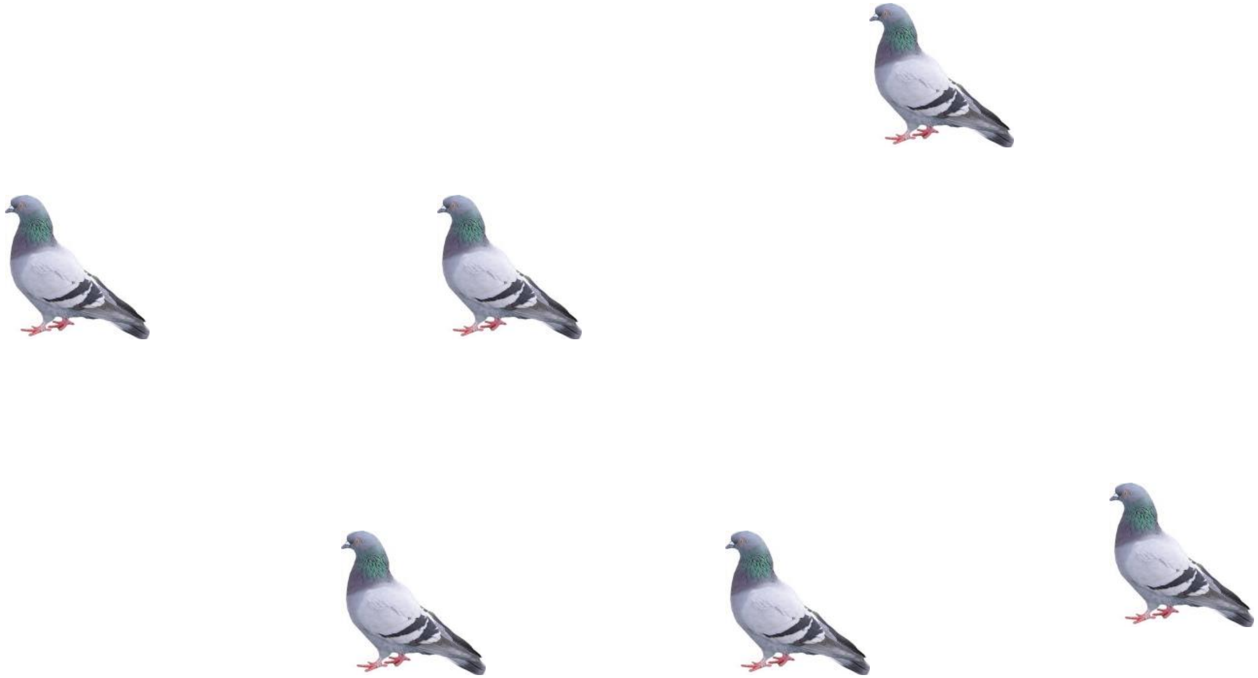
(Adapted from here.)

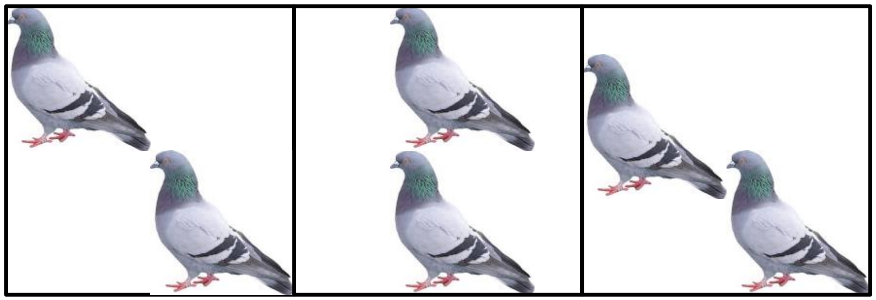
Other Pigeonhole-Type Results

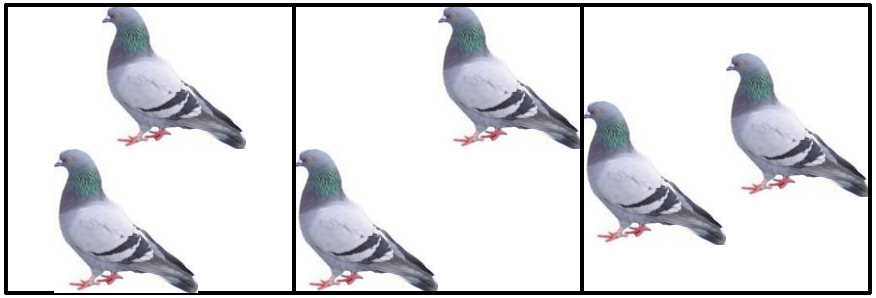
If m objects are distributed into n boxes, then [condition] holds.

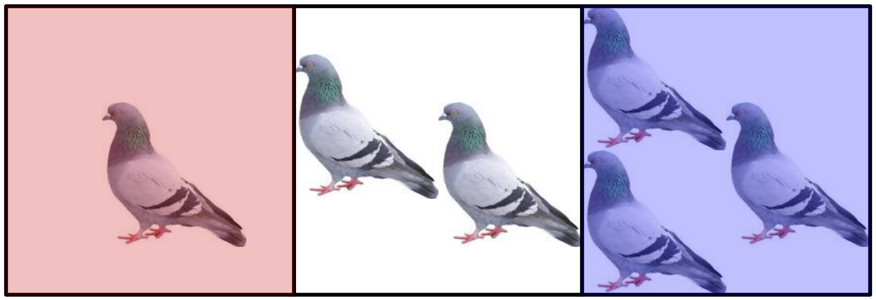
If m objects are distributed into n boxes, then some box is loaded to at least the average $^m/_n$, and some box is loaded to at most the average $^m/_n$.

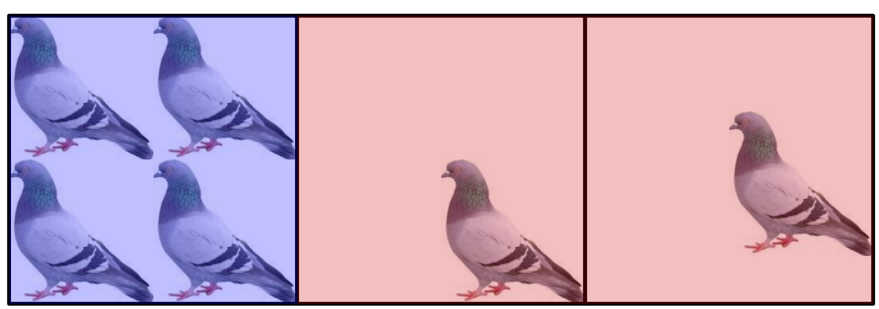
If m objects are distributed into n boxes, then [condition] holds.











Theorem: If m objects are distributed into n bins, then there is a bin containing more than $^m/_n$ objects if and only if there is a bin containing fewer than $^m/_n$ objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

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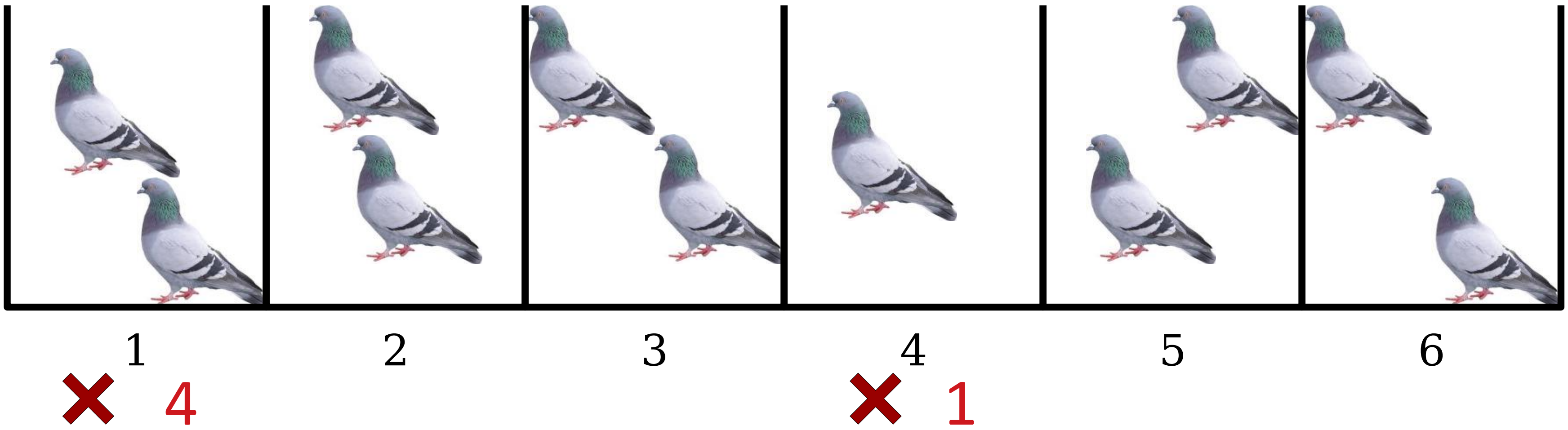
This magic phrase means "we get to pick how we're labeling things anyway, so if it doesn't work out, just relabel things."



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$$m = x_1 + x_2 + x_3 + \ldots + x_n$$

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This third step follows because each remaining bin has at most m/n objects. Grouping the n copies of the m/n term here tells us that

$$\begin{aligned} m &< m/n + m/n + m/n + \ldots + m/n \\ &= m. \end{aligned}$$

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"In a group of $n > 0$ people …

- 90% of those people enjoyed ***Get Out***,
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- 70% of those people enjoyed ***Arrival***, and
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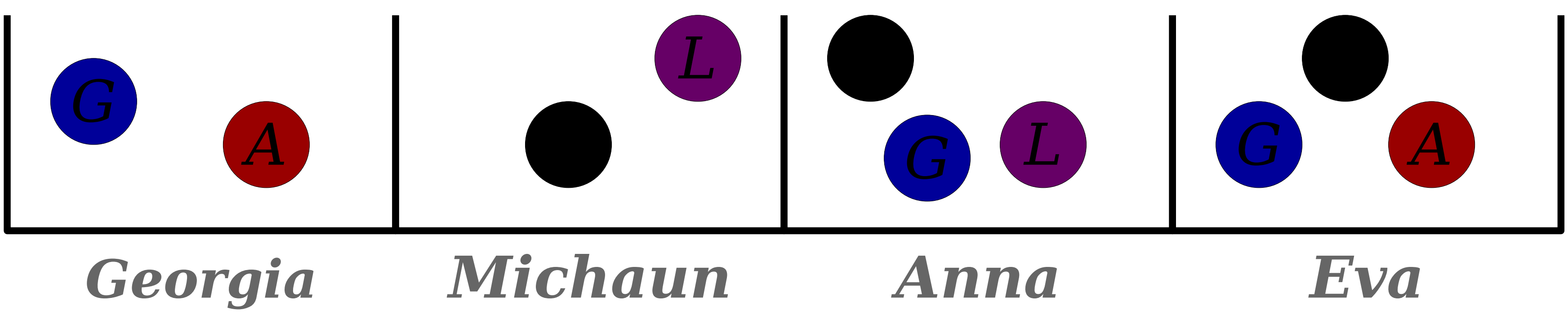
No one enjoyed all four movies. How many people enjoyed at least one of *Get Out* and *Arrival*?"

Insight 1: Model movie preferences as balls (movies) in bins (people).

Insight 2: There are n total bins, one for each person.



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$$.9n + .8n + .7n + .6n$$

$$= 3n$$

Insight 3: There are $3n$ balls being distributed into n bins.

Insight 4: The average number of balls in each bin is 3.

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Insight 5: No one enjoyed more than three movies…

Insight 6: … so no one enjoyed fewer than three movies …

Insight 7: … so everyone enjoyed exactly three movies.

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Insight 8: You have to enjoy at least one of these movies to enjoy three of the four movies.

Conclusion: **Everyone** liked at least one of these two movies!

Theorem: In the scenario described here, all n people enjoyed at least one of *Get Out* and *Arrival*.

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Proof: Suppose there is a group of n people meeting these criteria.

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and since there are n people, there are n bins.

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Now suppose for the sake of contradiction that someone didn't enjoy *Get Out* and didn't enjoy *Arrival*.

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Going Further

- The pigeonhole principle can be used to prove a *ton* of amazing theorems. Here's a sampler:
- There is always a way to fairly split rent among multiple people, even if different people want different rooms. *(Sperner's lemma)*
- You and a friend can climb any mountain from two different starting points so that the two of you maintain the same altitude at each point in time. *(Mountain-climbing theorem)*
- If you model coffee in a cup as a collection of infinitely many points and then stir the coffee, some point is always where it initially started. *(Brower's fixed-point theorem)*
- A complex process that doesn't parallelize well must contain a large serial subprocess. *(Mirksy's theorem)*
- Any positive integer *n* has a nonzero multiple that can be written purely using the digits 1 and 0. *(Doesn't have a name, but still cool!)*

Next Time

Mathematical Induction

- Reasoning about stepwise processes

Applications of Induction

- To numbers!
- To anticounterfeiting!
- To puzzles!