Mathematical Induction

Part One

Everybody – do the wave!

The Wave

If done properly, everyone will eventually end up joining in.

Why is that?

Someone (me!) started everyone off.

Once the person before you did the wave, you did the wave.

Let P be some predicate. The ***principle of mathematical induction*** states that if

$P(0)$ is true

If it starts true…

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Induction, Intuitively

$$\boldsymbol{P(0)}$$

$$\boldsymbol{\forall k \in \mathbb{N}. \ (P(k) \rightarrow P(k+1))}$$

It's true for 0.

Since it's true for 0, it's true for 1.

Since it's true for 1, it's true for 2.

Since it's true for 2, it's true for 3.

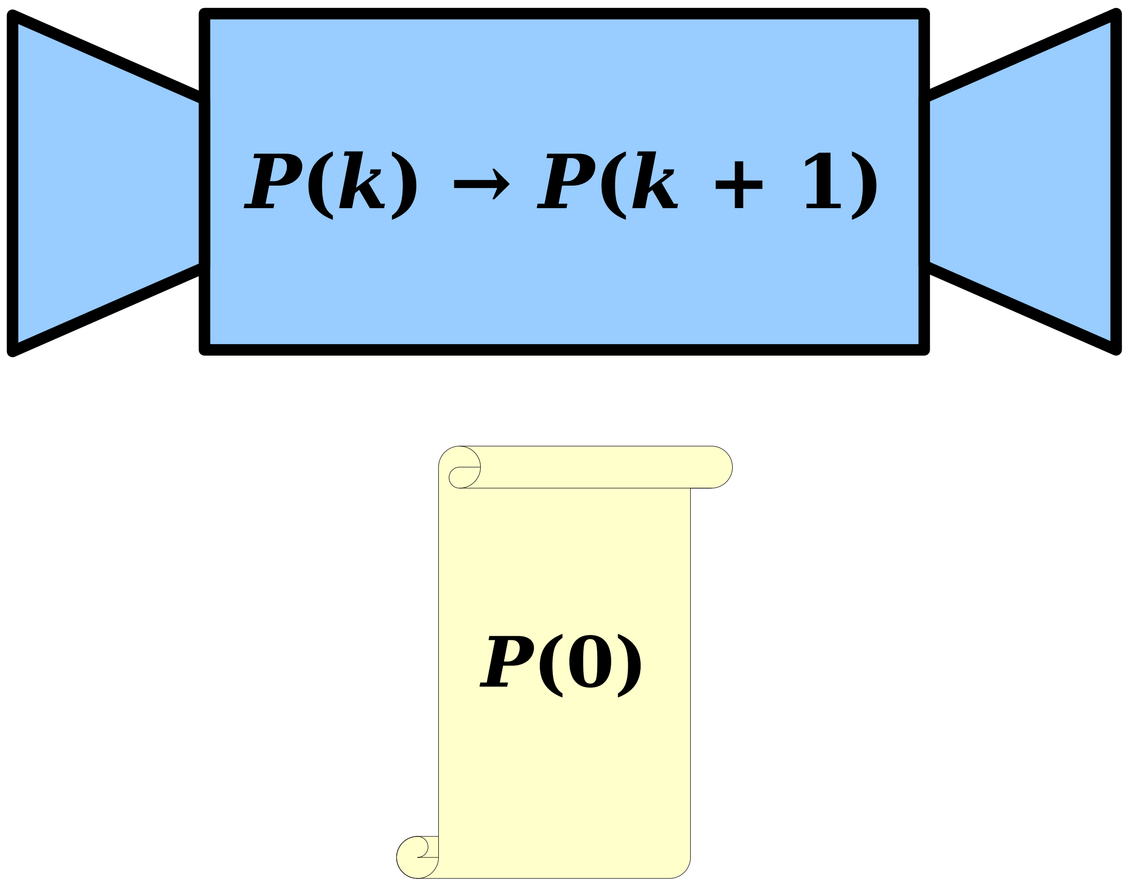
Since it's true for 3, it's true for 4.

Since it's true for 4, it's true for 5.

Since it's true for 5, it's true for 6.

…

Why Induction Works



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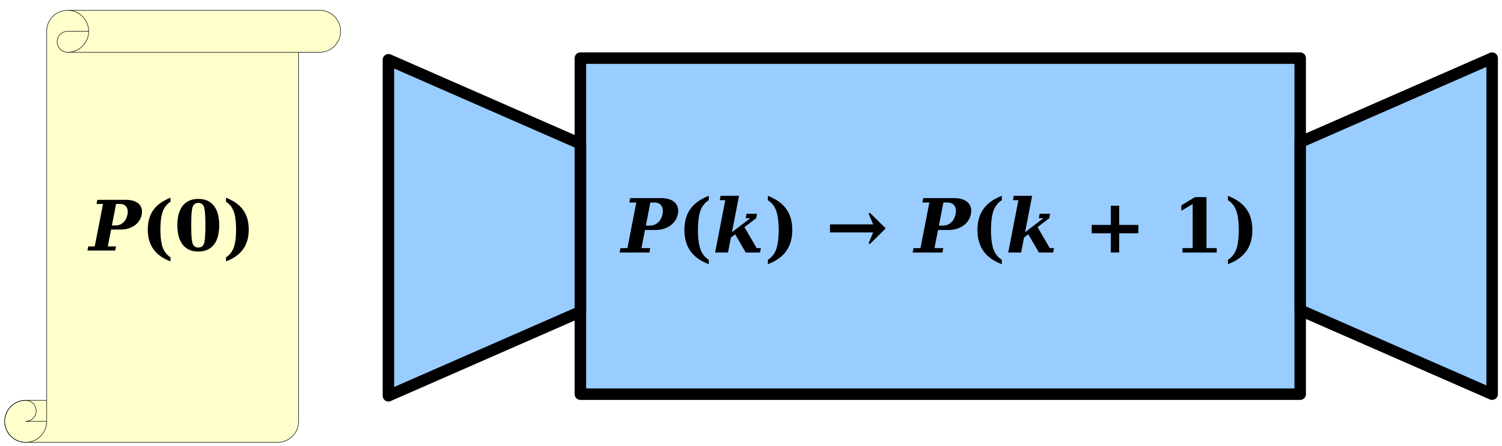

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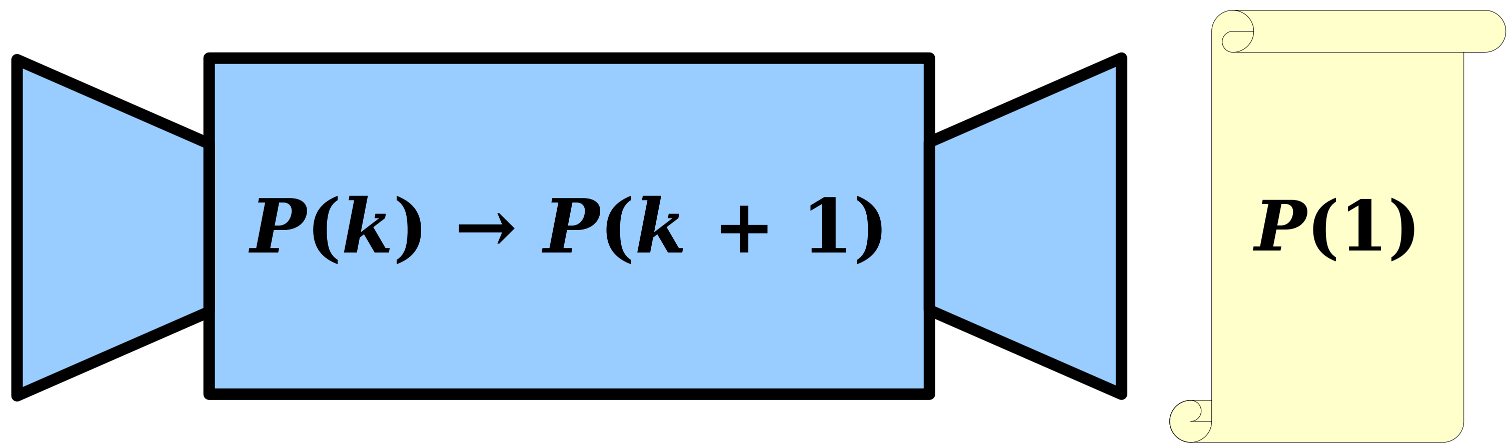

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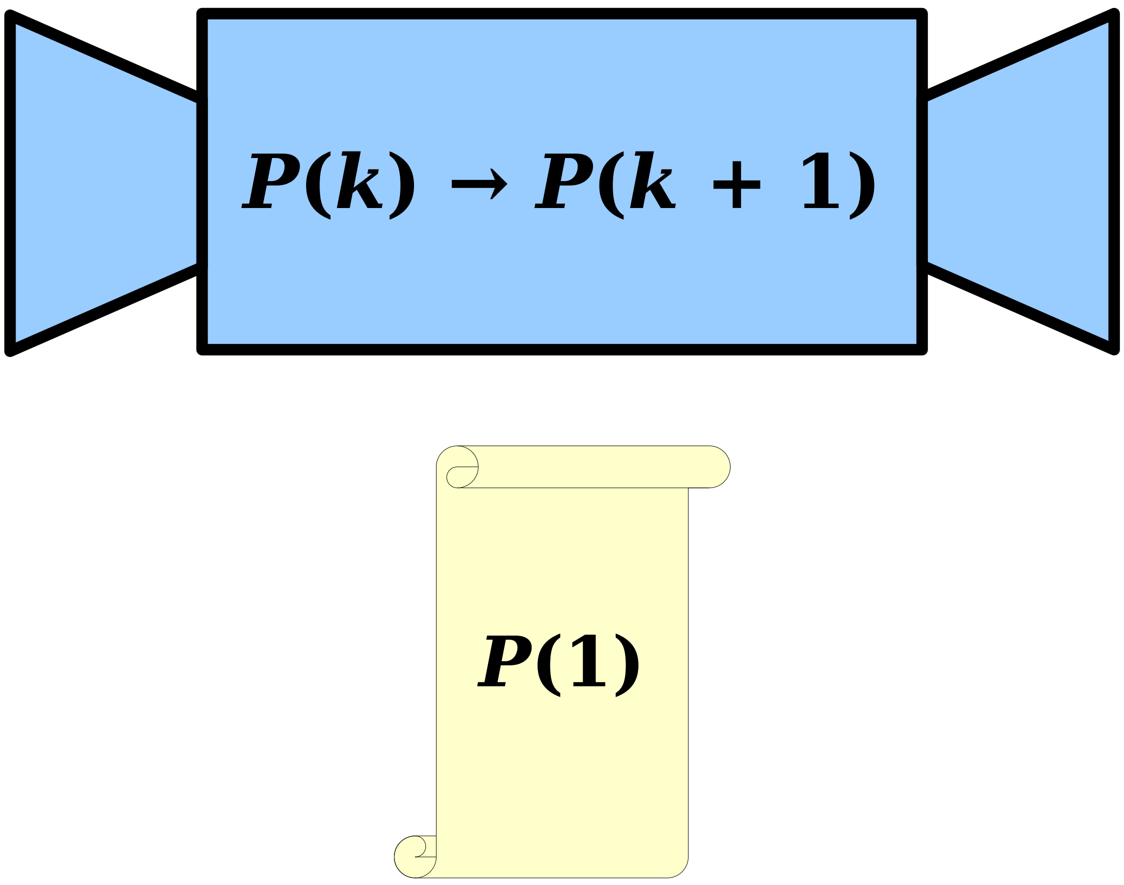

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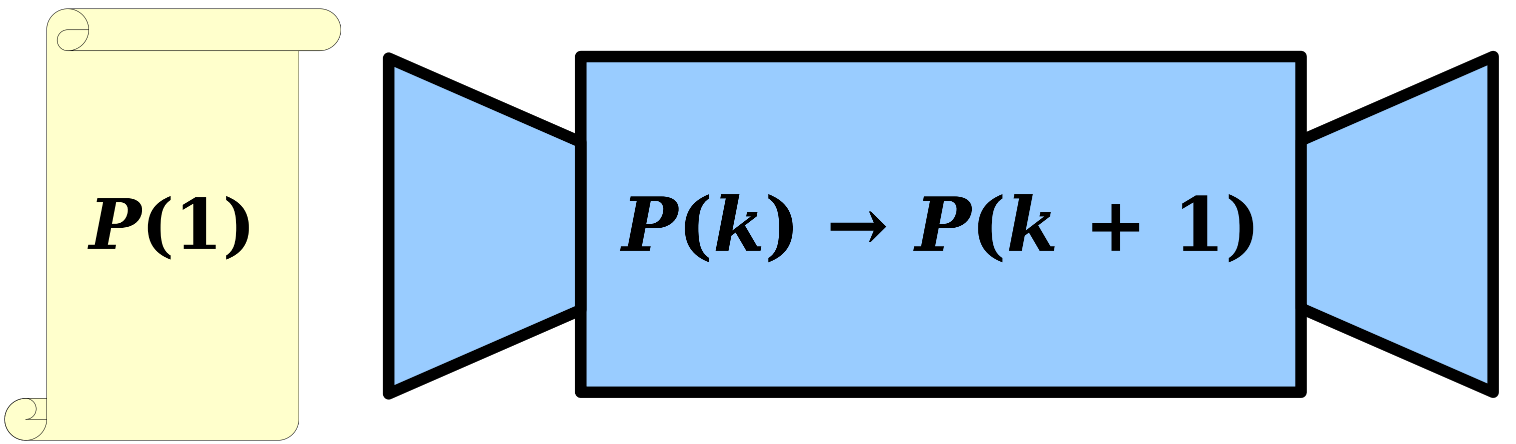

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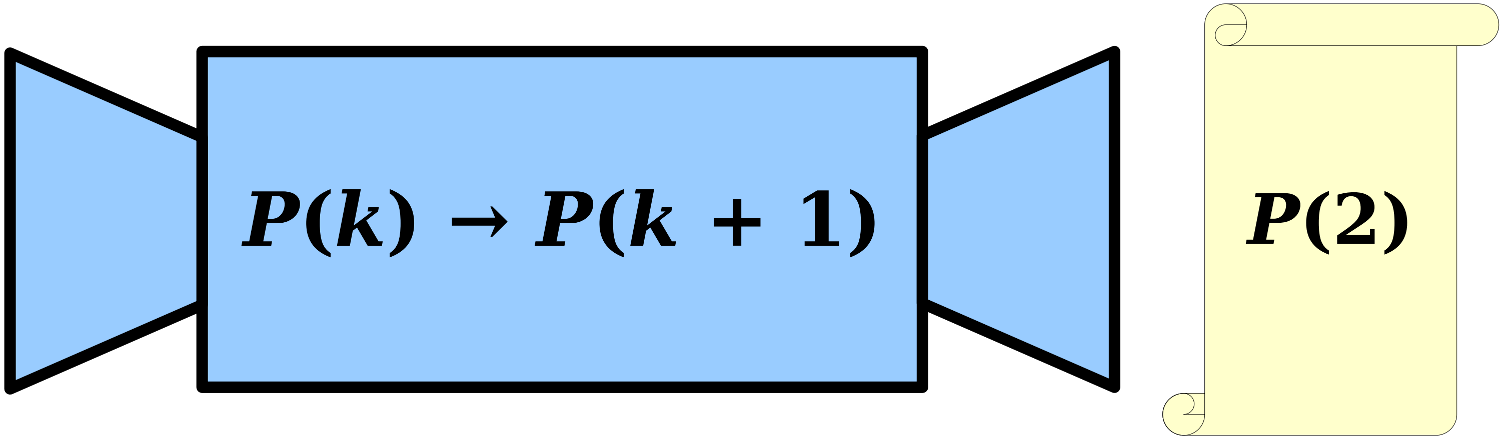

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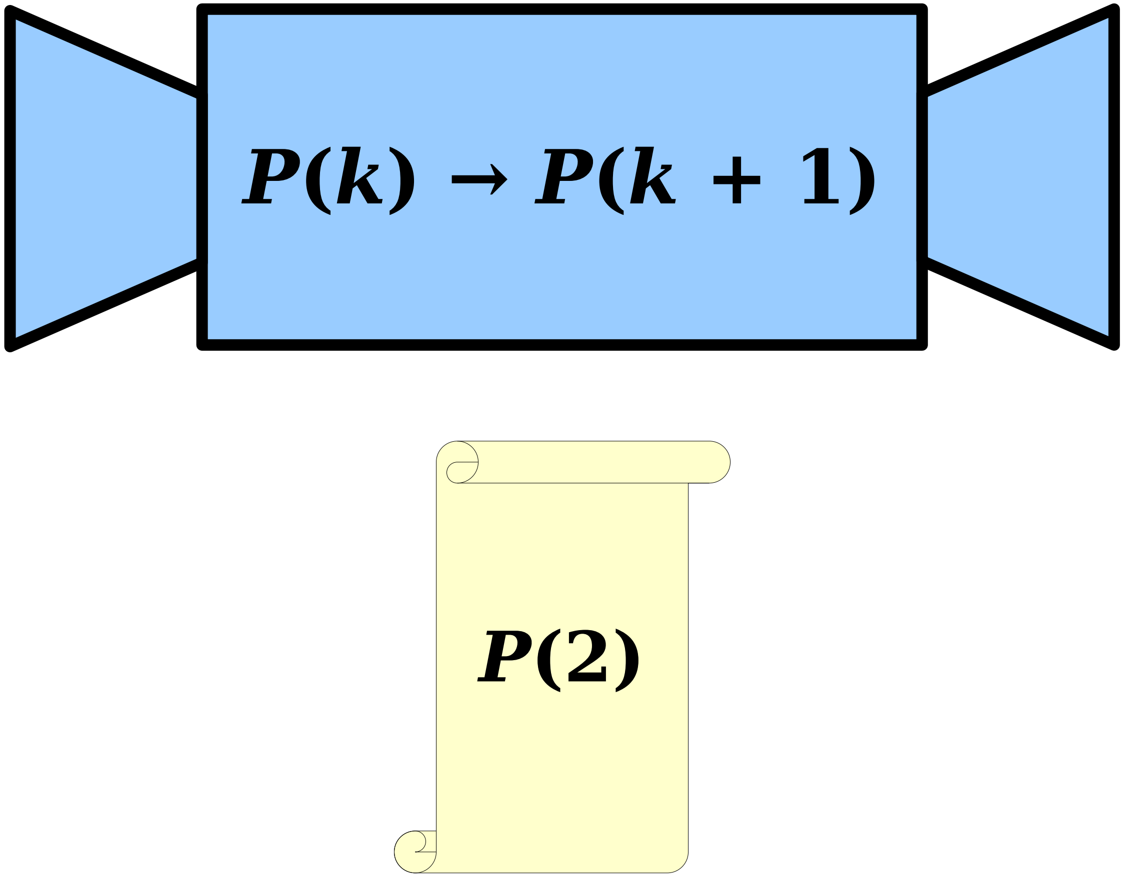

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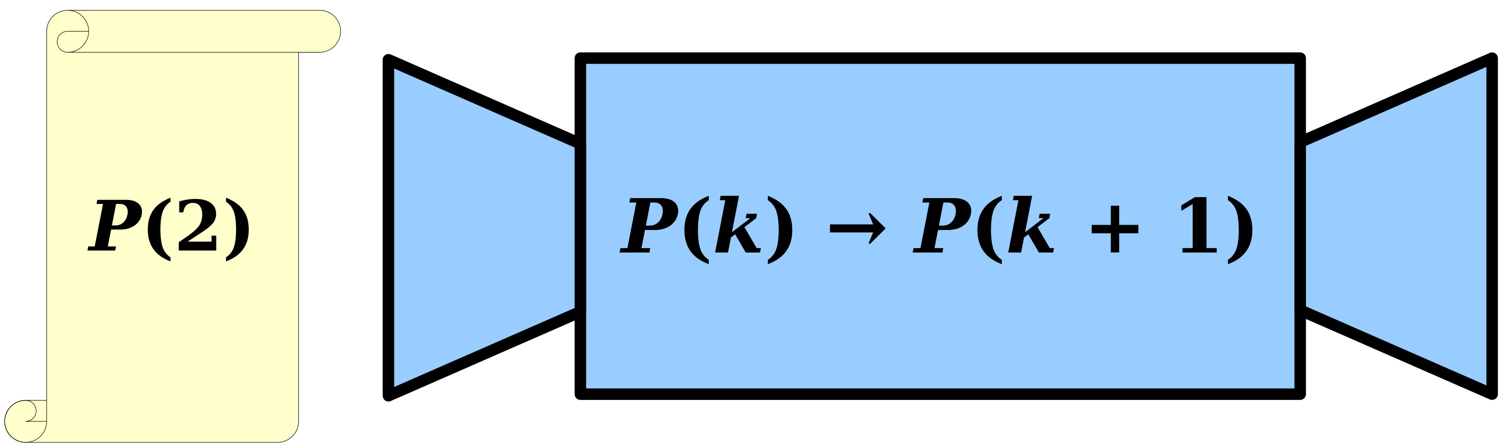

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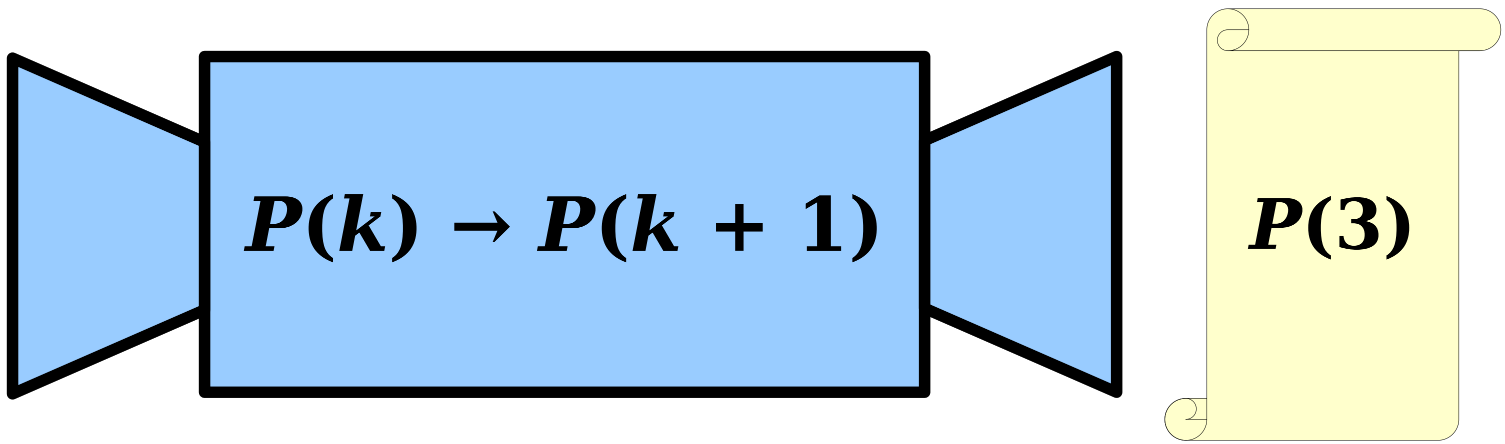

Proof by Induction

A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers n.

In a proof by induction, there are three steps:

- Prove that $P(0)$ is true.
  - This is called the ***basis*** or the ***base case***.
- Prove that if $P(k)$ is true, then $P(k+1)$ is true.
  - This is called the ***inductive step***.
  - The assumption that $P(k)$ is true is called the ***inductive hypothesis***.
- Conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Some Sums

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

Theorem: The sum of the first n powers of two is $2^n - 1$.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers *n*, then tell them we're going to prove it by induction.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to prove it.

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$$2^0 + 2^1 + \ldots + 2^{k-1} + 2^k \qquad = (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k$$

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Here, we'll use our **inductive hypothesis** (the assumption that *P*(*k*) is true) to simplify a complex expression. This is a common theme in inductive proofs.

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A Quick Aside

- This result helps explain the range of numbers that can be stored in an **int**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. If we have time, we'll see one today.

Structuring a Proof by Induction

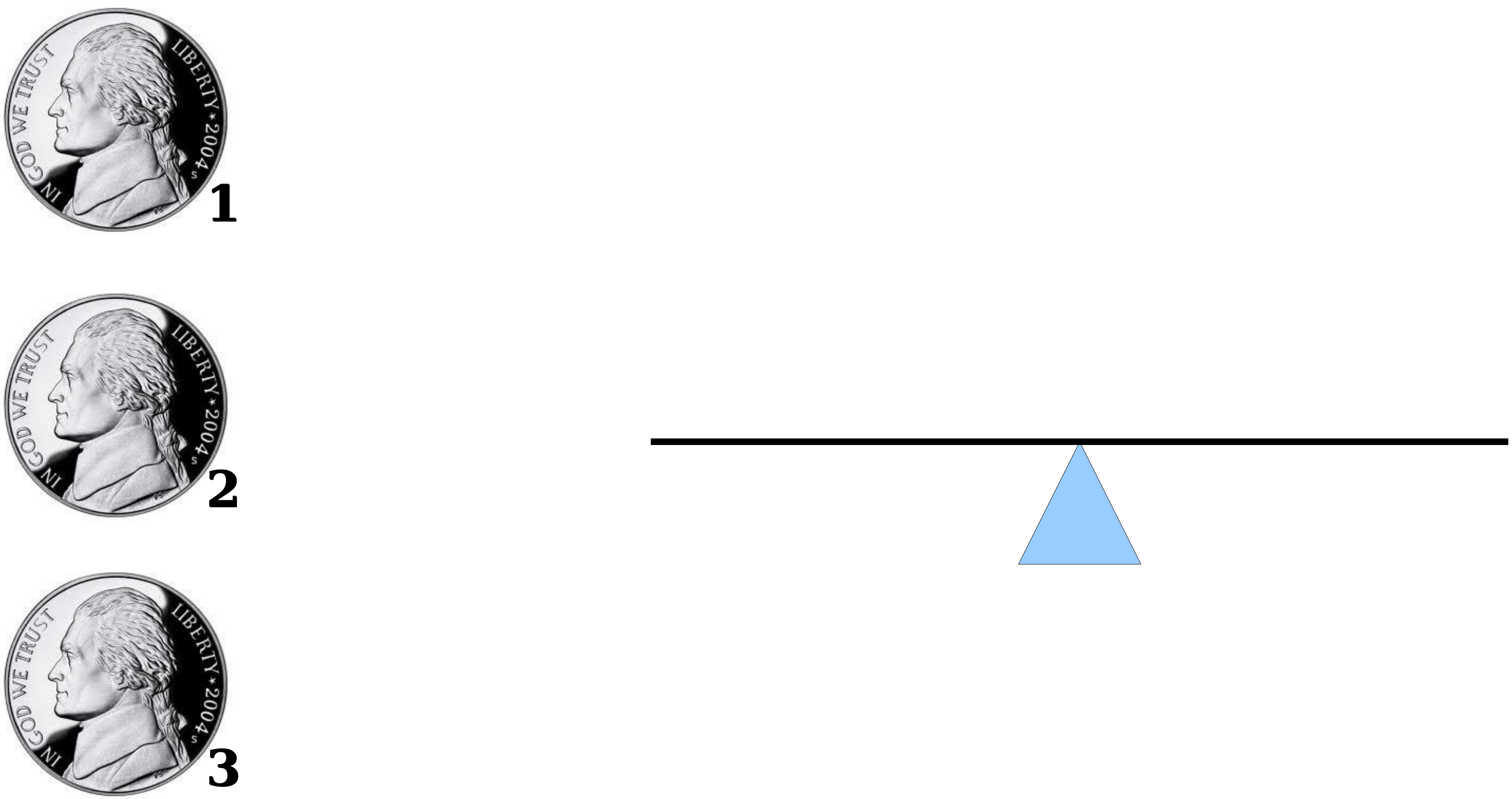
- Define some predicate P that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that $P(0)$ is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming $P(k)$ for some arbitrary natural number k, then write out exactly what that means.
  - Say that you're going to prove $P(k+1)$, then write out exactly what that means.
  - Prove that $P(k+1)$ using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

The Counterfeit Coin Problem

Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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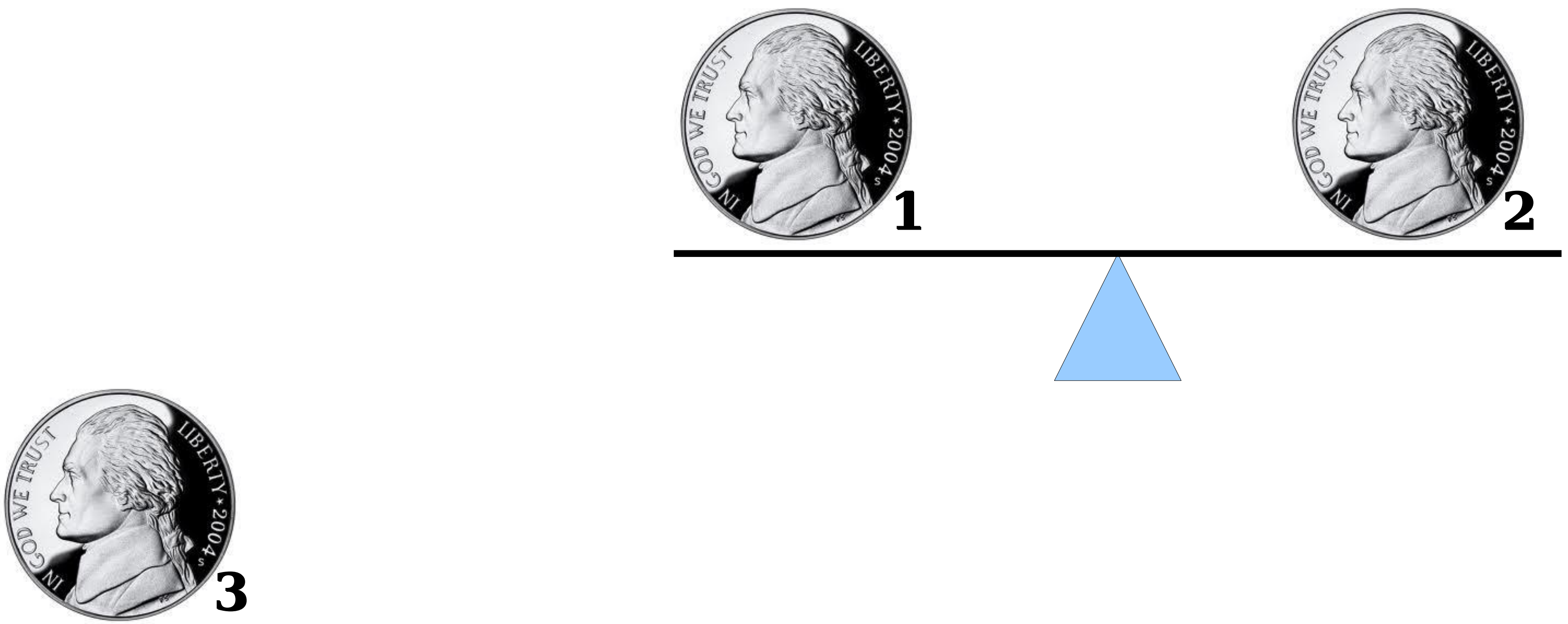

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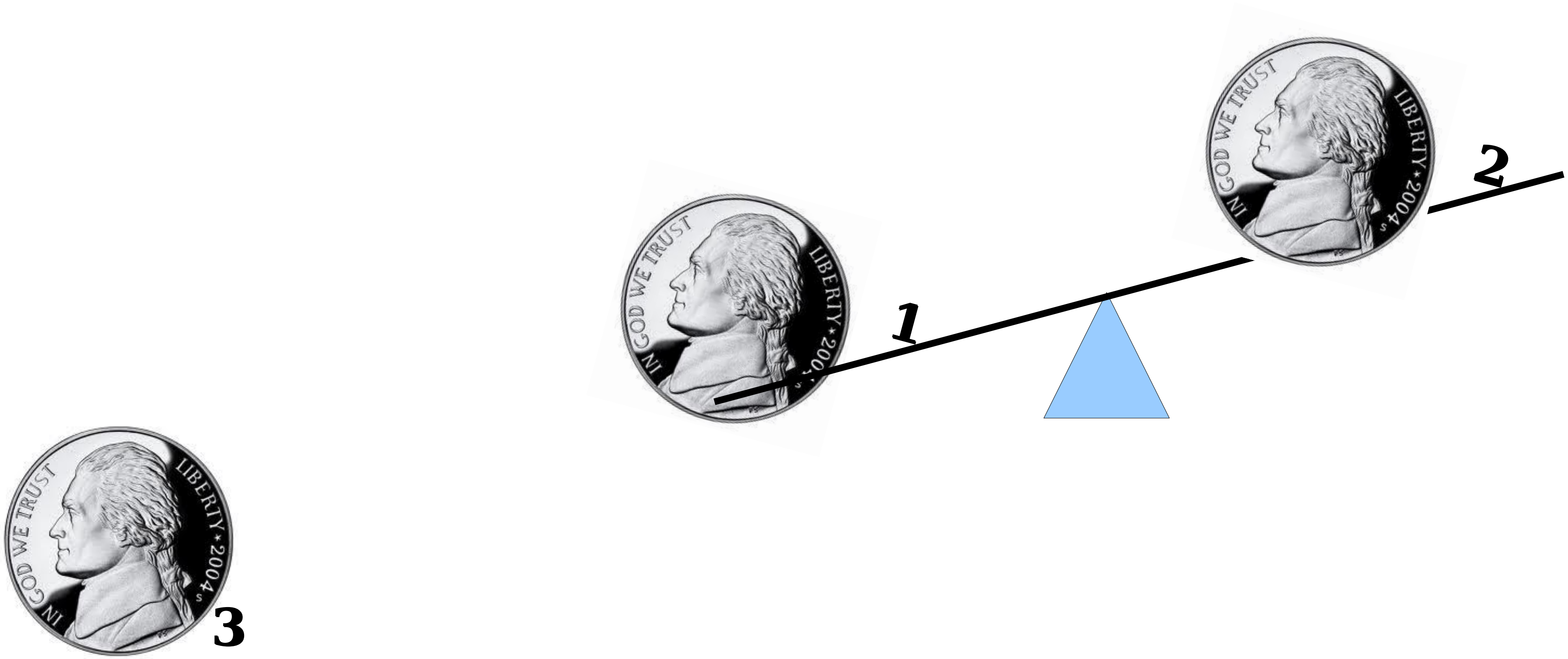

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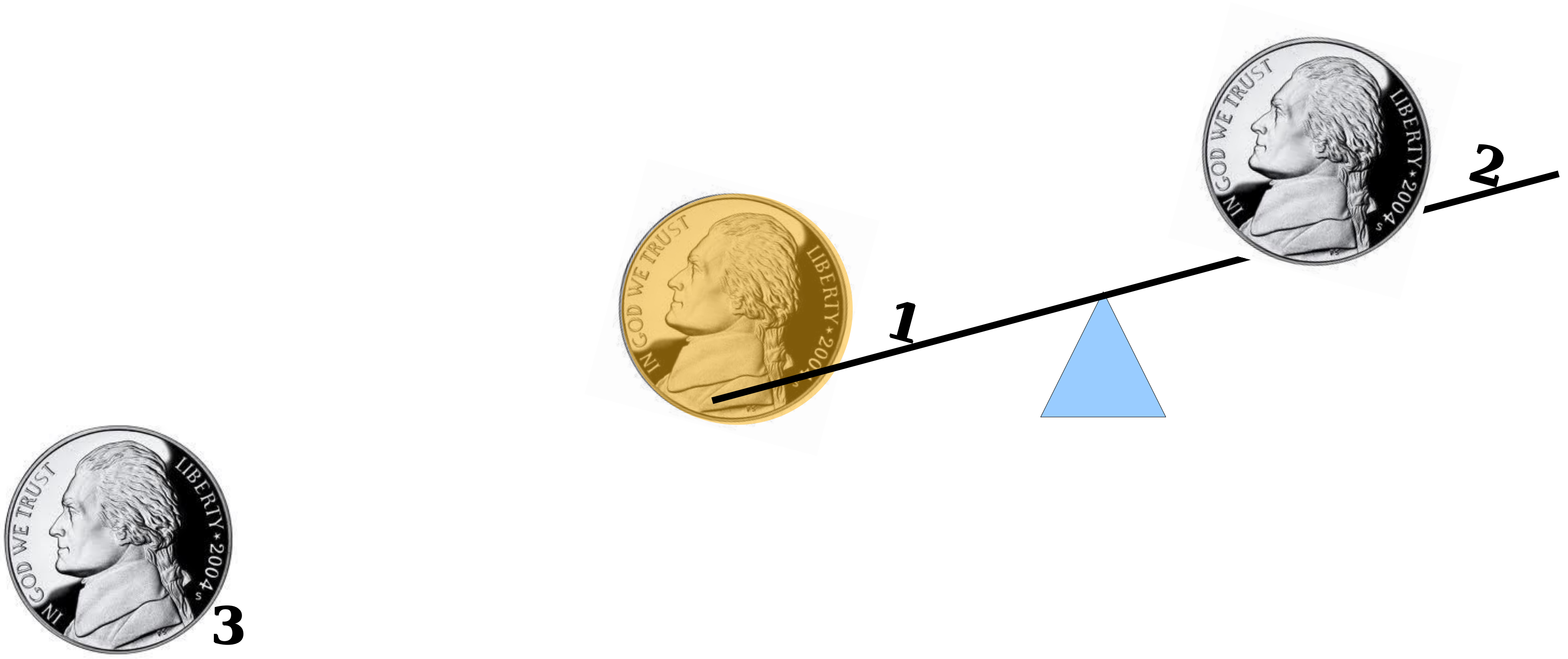

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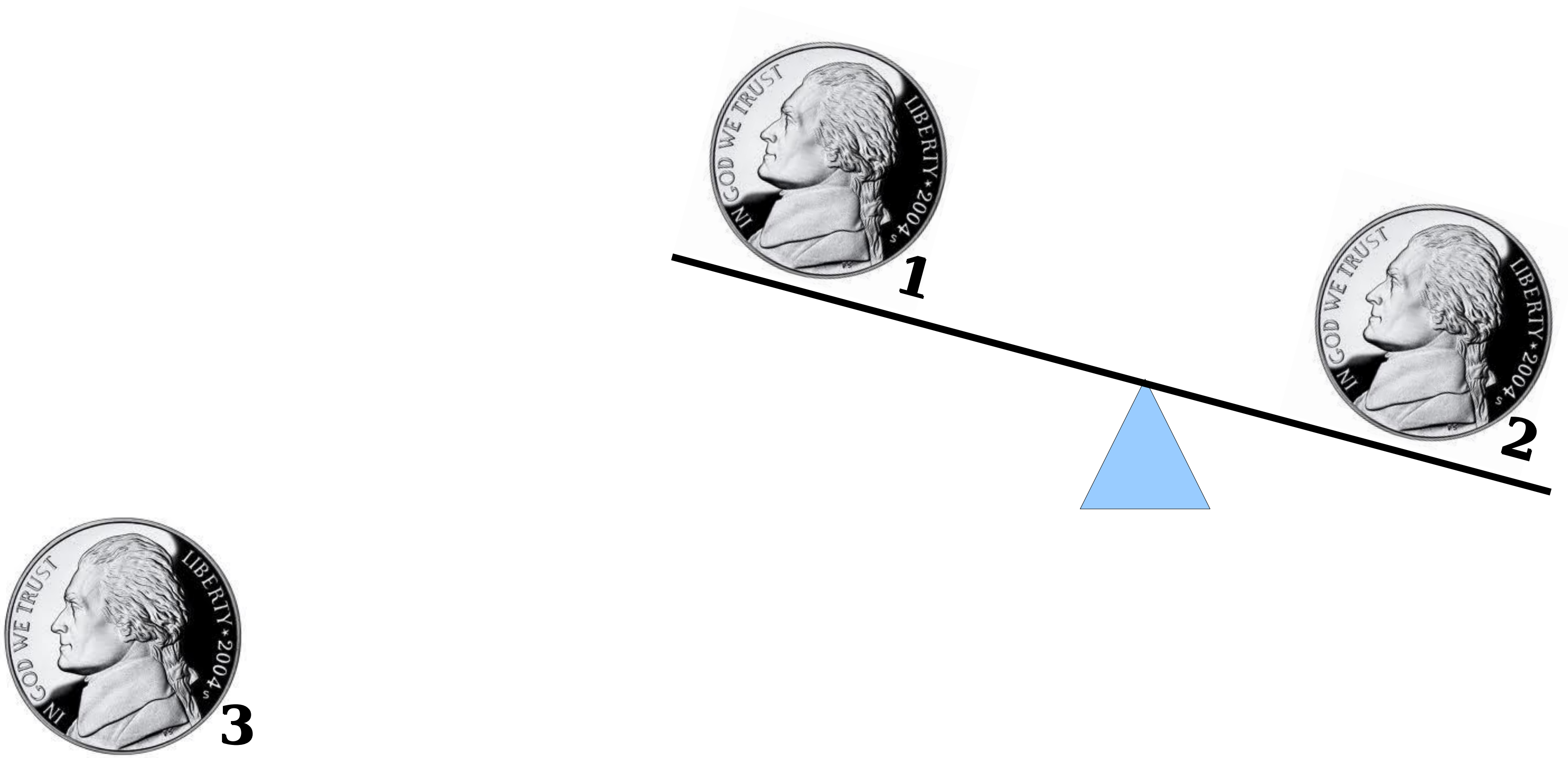

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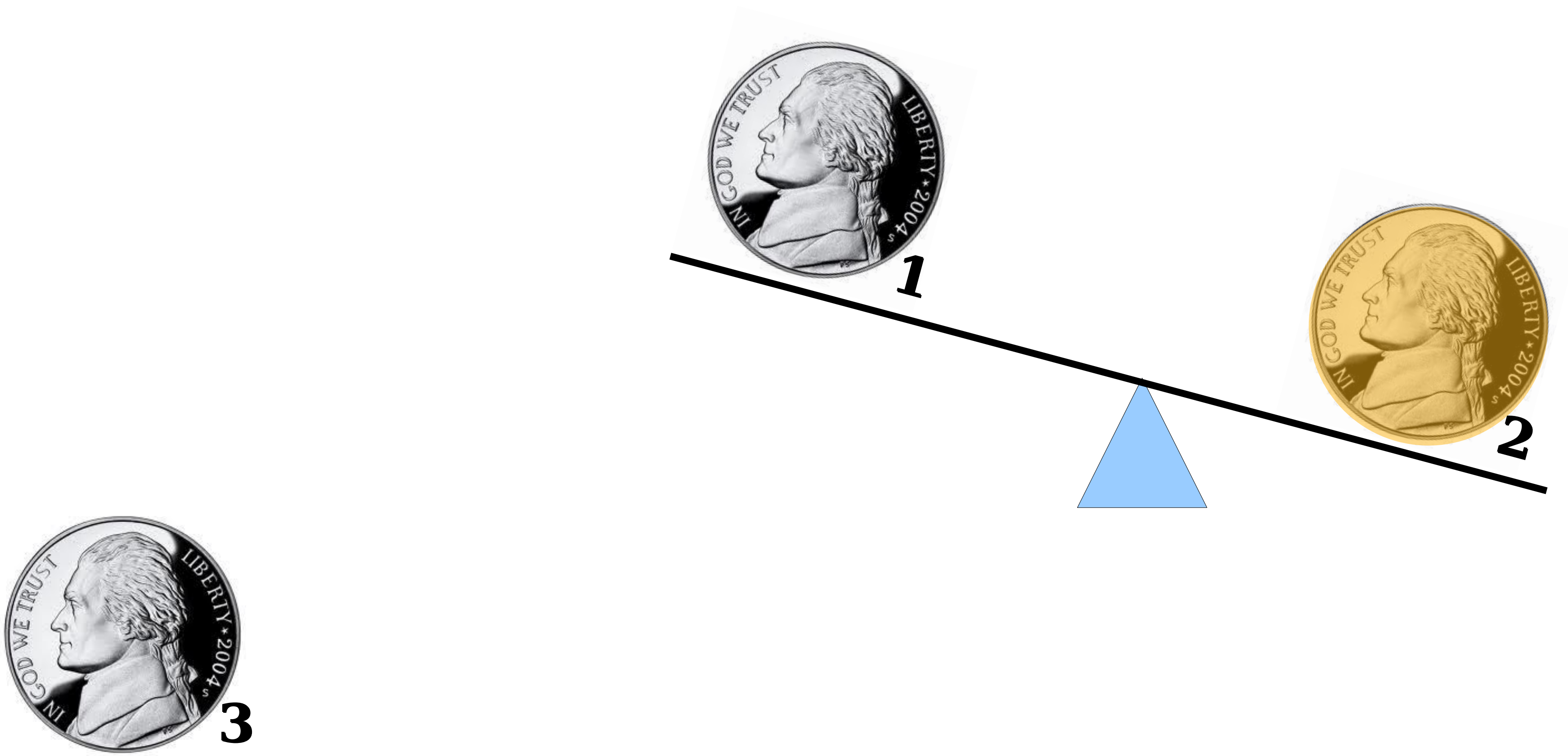

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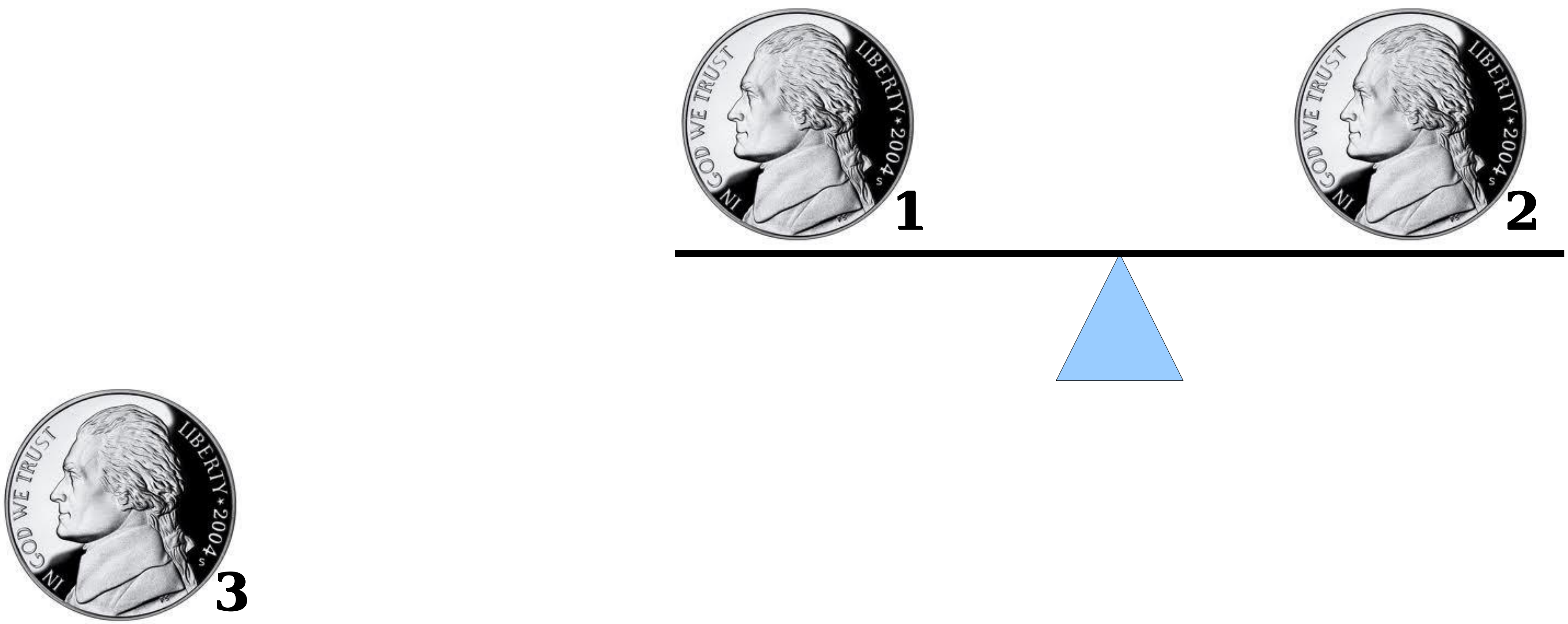

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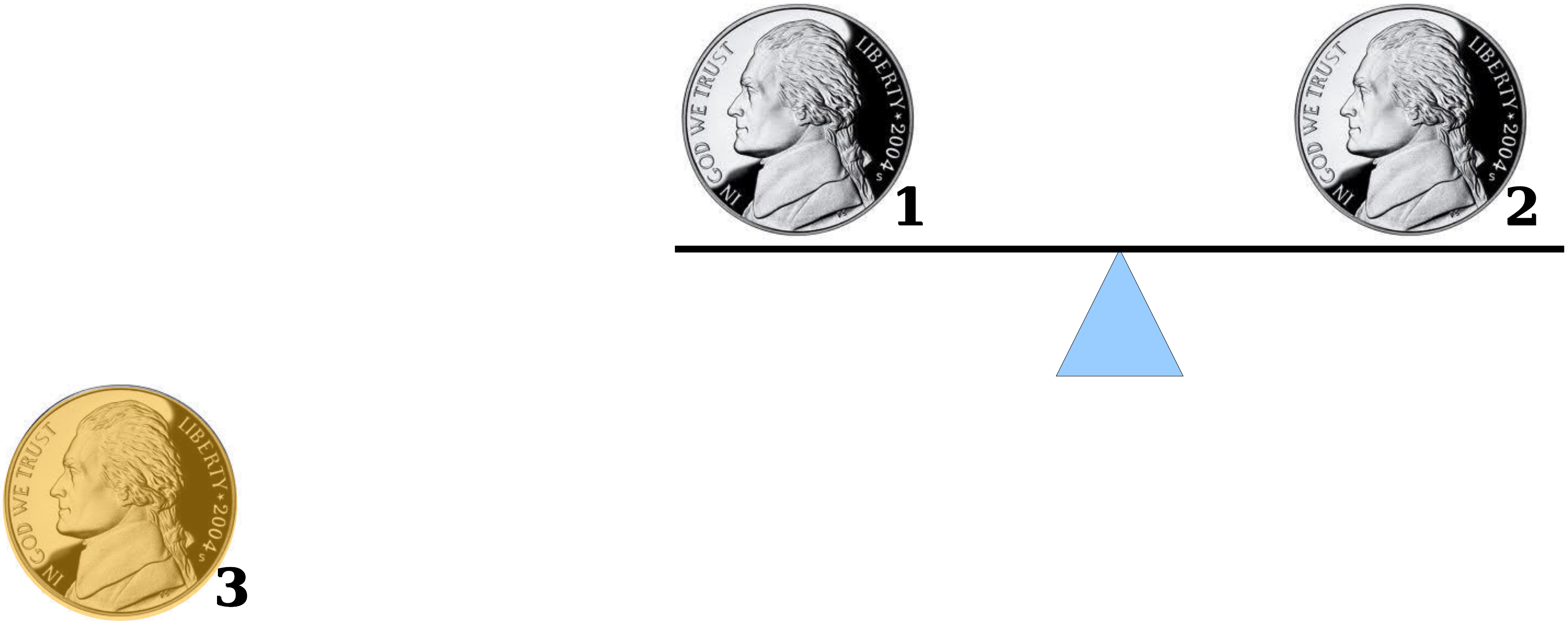


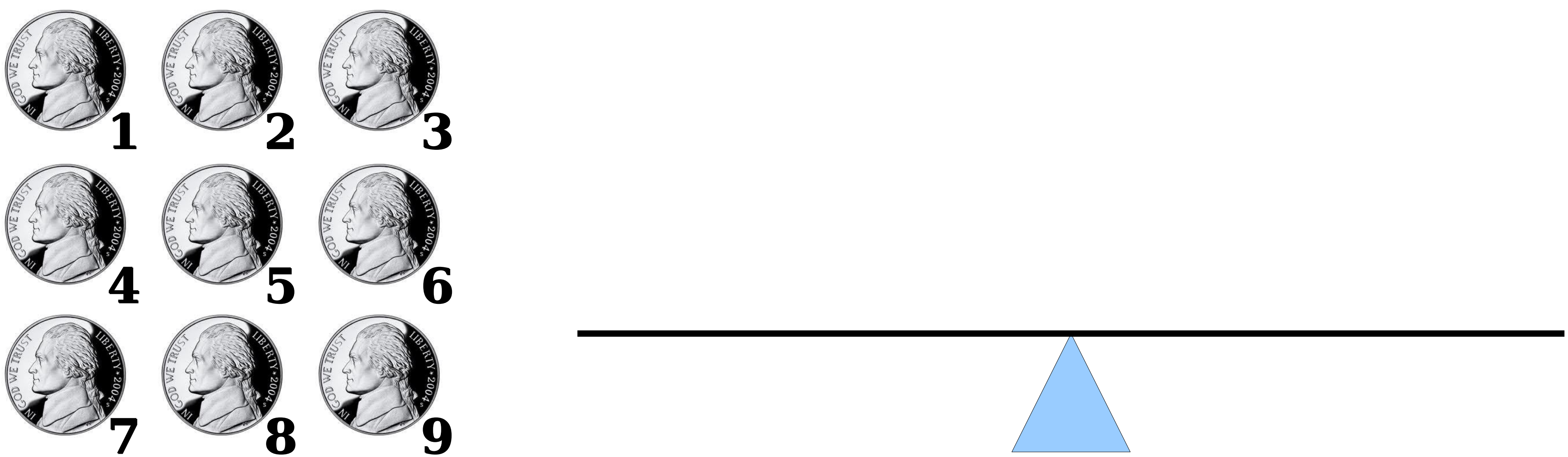
A Harder Problem

You are given a set of ***nine*** seemingly identical coins, eight of which are real and one of which is counterfeit.

The counterfeit coin weighs more than the rest of the coins.

You are given a balance. Using only ***two*** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



Finding the Counterfeit Coin

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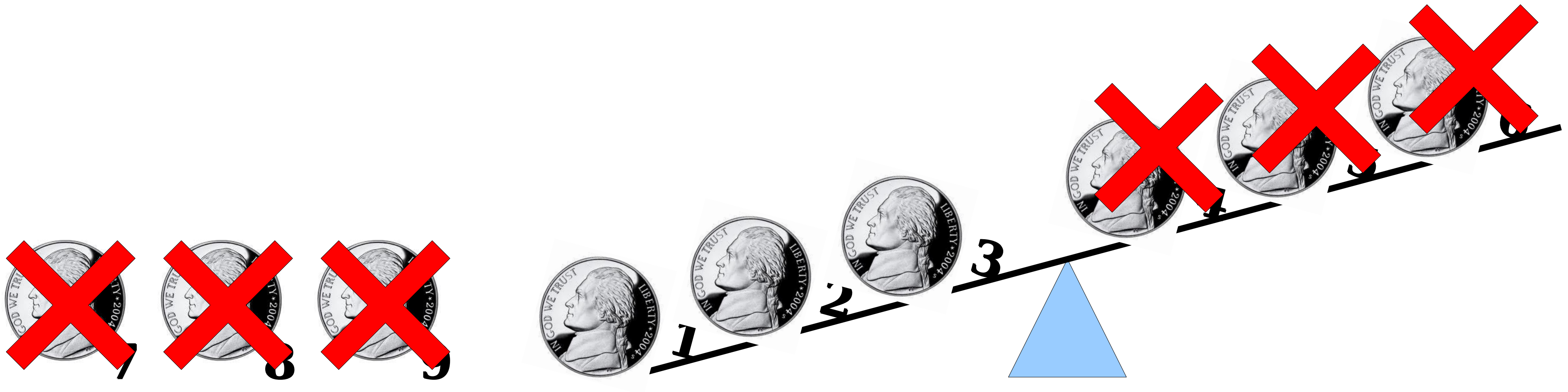
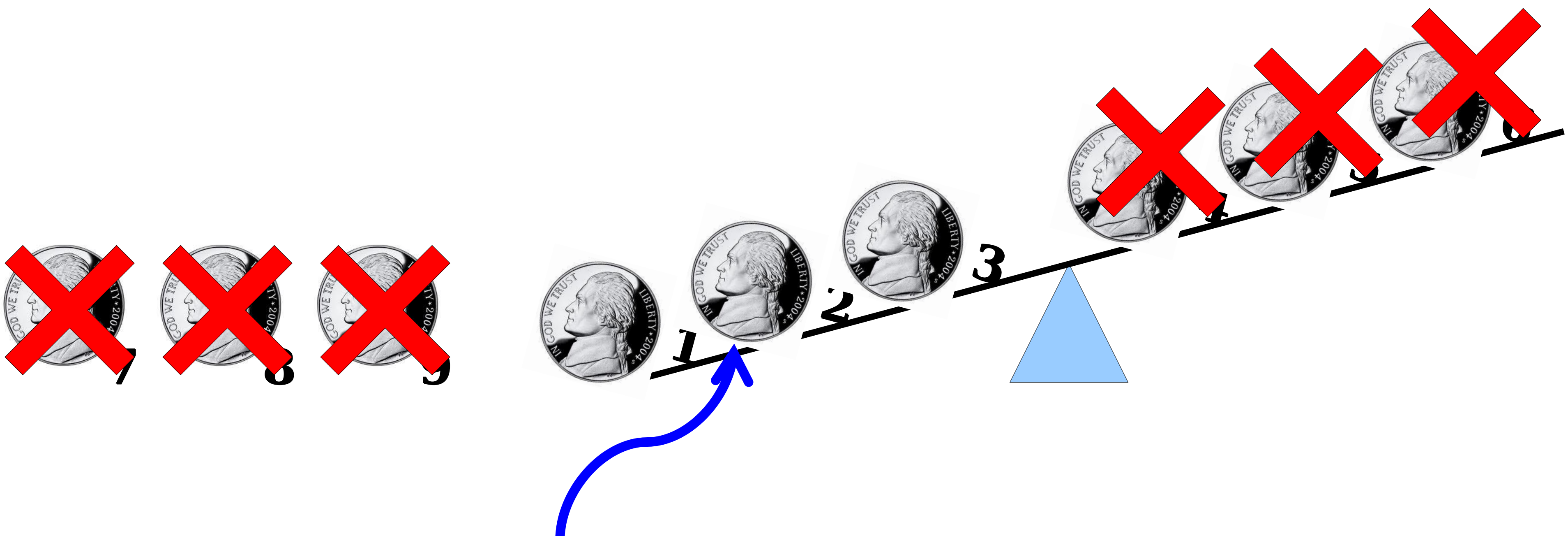
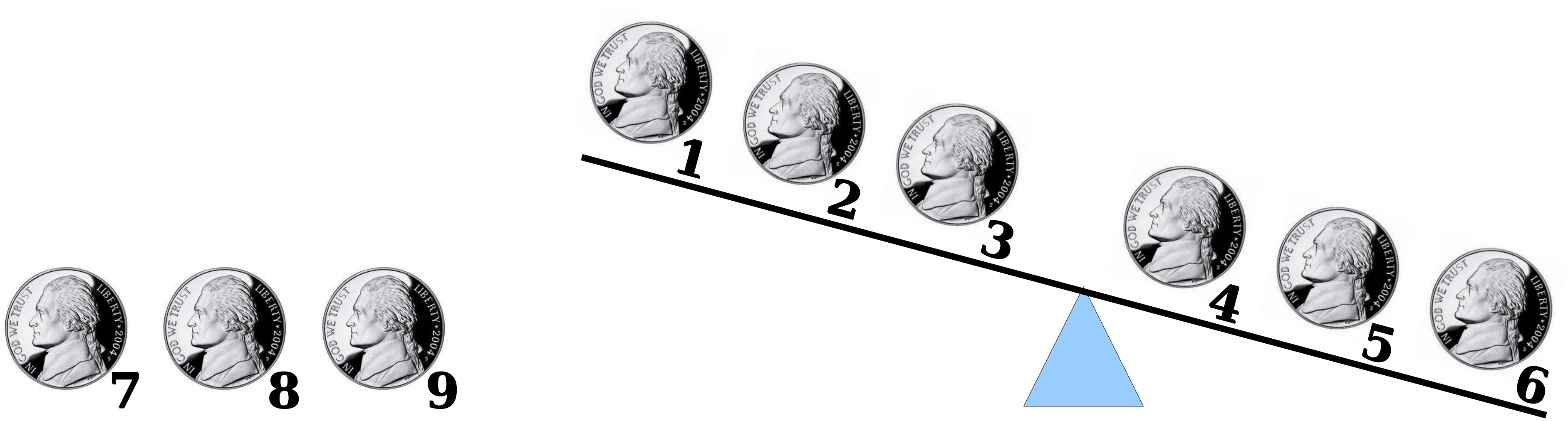


Finding the Counterfeit Coin



Now we have one weighing to find the counterfeit out of these three coins.

Finding the Counterfeit Coin



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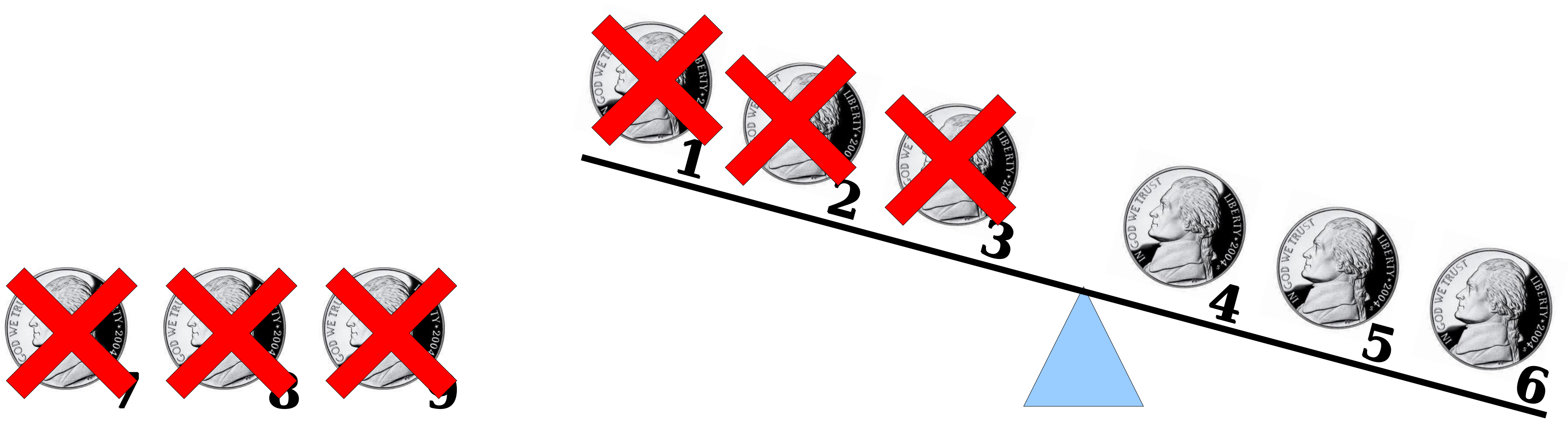

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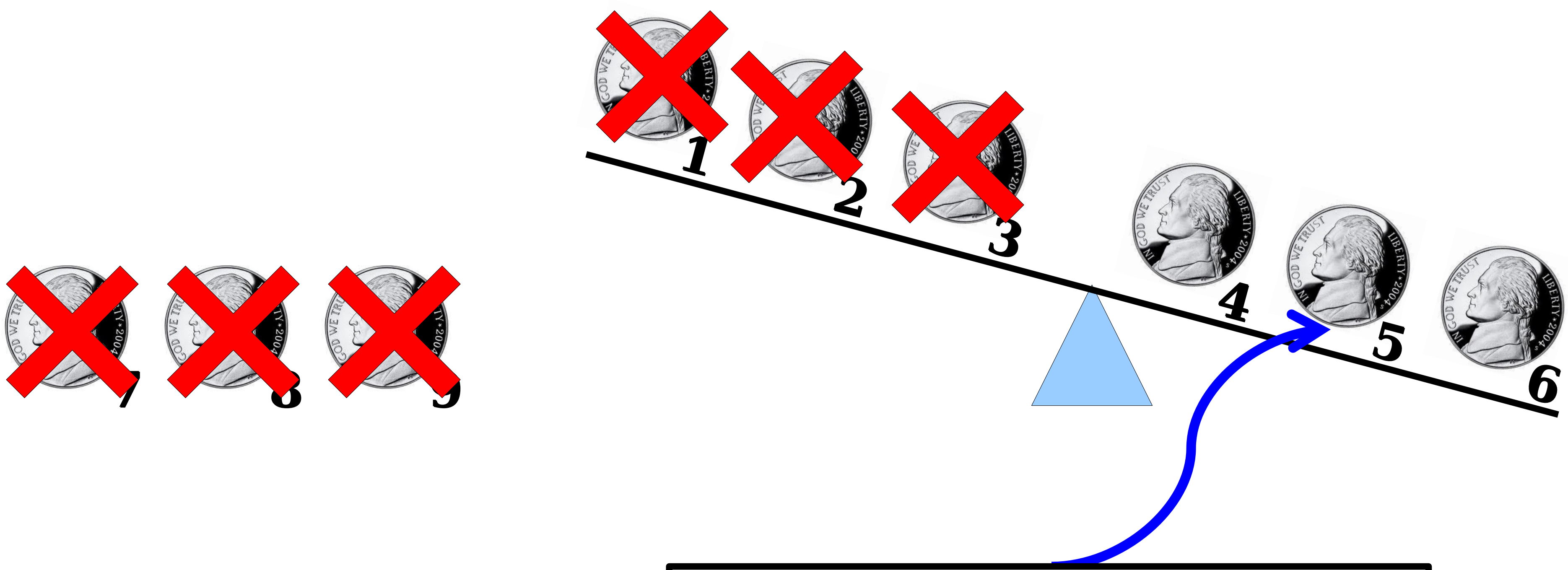

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Finding the Counterfeit Coin

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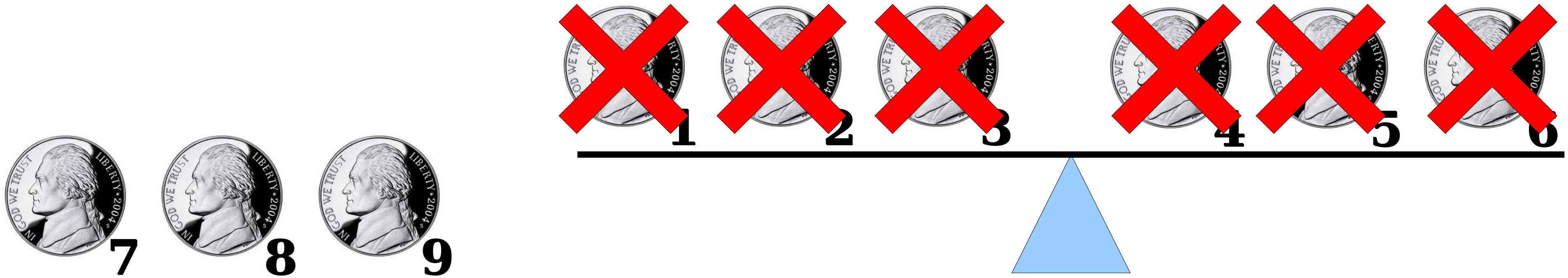

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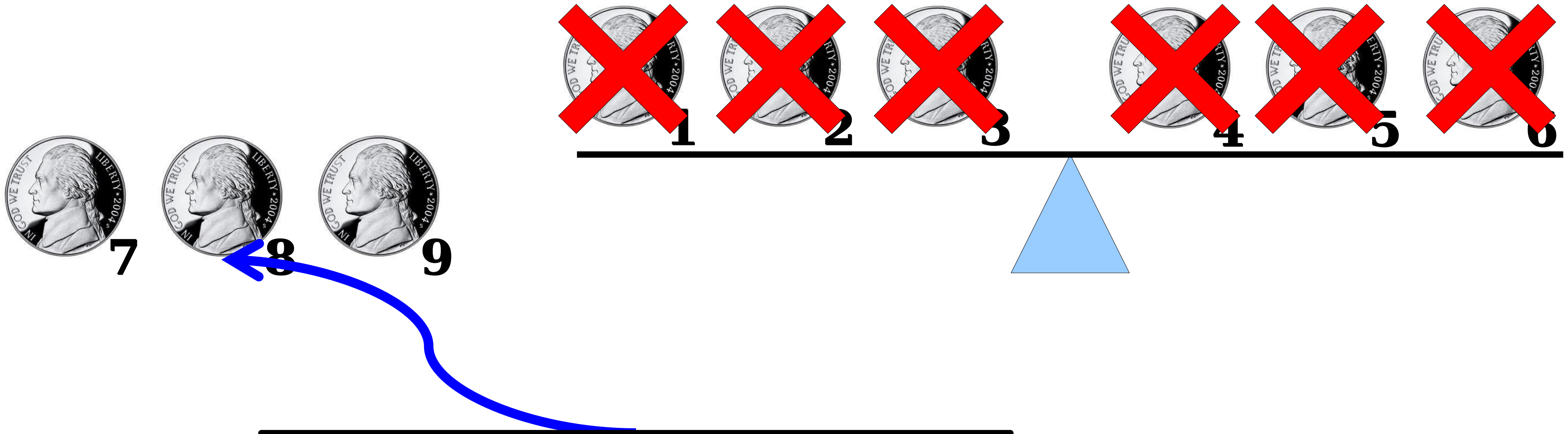

Now we have one weighing to find the counterfeit out of these three coins.

Can we generalize this?

A Pattern

Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.

If we have no weighings, how many coins can we have while still being able to find the counterfeit?

One coin, since that coin has to be the counterfeit!

If we have one weighing, we can find the counterfeit out of ***three*** coins.

If we have two weighings, we can find the counterfeit out of ***nine*** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers *n*, then tell them we're going to prove it by induction.

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In a proof by induction, we need to prove that

- ☐ $P(0)$ is true
- ☐ If $P(k)$ is true, then $P(k+1)$ is true.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Proof: Let $P(n)$ be the following statement:

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We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0 = 1$ coins with one coin heavier than the rest, we can find that coin with zero weighings.

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As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0=1$ coins with one coin heavier than the rest, we can find that coin with zero weighings.

Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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For the inductive step, suppose $P(k)$ is true for some arbitrary $k \in \mathbb{N}$, so we can find the heavier of 3^k coins in k weighings.

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For the inductive step, suppose $P(k)$ is true for some arbitrary $k \in \mathbb{N}$, so we can find the heavier of 3^k coins in k weighings. We'll prove $P(k+1)$: that we can find the heavier of 3^{k+1} coins in $k+1$ weighings.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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For the inductive step, suppose $P(k)$ is true for some arbitrary $k \in \mathbb{N}$, so we can find the heavier of 3^k coins in k weighings. We'll prove $P(k+1)$: that we can find the heavier of 3^{k+1} coins in $k+1$ weighings.

The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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For the inductive step, suppose $P(k)$ is true for some arbitrary $k \in \mathbb{N}$, so we can find the heavier of 3^k coins in k weighings. We'll prove $P(k+1)$: that we can find the heavier of 3^{k+1} coins in $k+1$ weighings.

Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Suppose we have 3^{k+1} coins with one heavier than the others.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Proof: Let $P(n)$ be the following statement:

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Proof: Let $P(n)$ be the following statement:

If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin. Otherwise, the heavier coin must be in the group we didn't put on the scale.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Proof: Let $P(n)$ be the following statement:

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin. Otherwise, the heavier coin must be in the group we didn't put on the scale. Therefore, with one weighing, we can find a group of 3^k coins containing the heavy coin.

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If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin. Otherwise, the heavier coin must be in the group we didn't put on the scale. Therefore, with one weighing, we can find a group of 3^k coins containing the heavy coin. We can then use k more weighings to find the heavy coin in that group.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Proof: Let $P(n)$ be the following statement:

If

Here, we use our **inductive hypothesis** (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

We'll us ... from which the theorem follows.

As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0=1$ coins with one coin heavier than the rest, we can find that coin with zero weighings. This is true because if we have just one coin, it's vacuously heavier than all the others, and no weighings are needed.

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We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0=1$ coins with one coin heavier than the rest, we can find that coin with zero weighings. This is true because if we have just one coin, it's vacuously heavier than all the others, and no weighings are needed.

For the inductive step, suppose $P(k)$ is true for some arbitrary $k \in \mathbb{N}$, so we can find the heavier of 3^k coins in k weighings. We'll prove $P(k+1)$: that we can find the heavier of 3^{k+1} coins in $k+1$ weighings.

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We've given a way to use $k+1$ weighings and find the heavy coin out of a group of 3^{k+1} coins.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

Proof: Let $P(n)$ be the following statement:

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Some Fun Problems

Here's some nifty variants of this problem that you can work through:

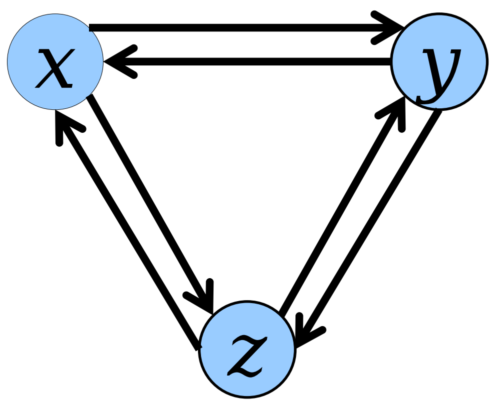
- Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get k weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
- What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have k weighings?
- Can you find the counterfeit out of a group of more than 3^k coins with k weighings?
- Can you find the counterfeit out of any group of at most 3^k coins with k weighings?

Time-Out for Announcements!

Problem Sets

PS3 due last night. Use a late period to extend this to Saturday at 11:59pm.

Great question to ponder: Why *isn't* this relation transitive?



Problem Set 4 and Midterm

- Problem Set Four is due two weeks from now on Thursday, July 30th at 11:59pm.
- This is because of the midterm next week.
- Full details going on Campuswire after lecture.
- Practice midterms will be posted to the website: look for them after the lecture.

Back to CS103!

How Not To Induct

Something's Wrong...

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Theorem: The sum of the first n powers of two is 2^n.

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(1)

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… + …

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When writing a proof by induction,

make sure to prove the base case!

Otherwise, your argument is invalid!

Why did this work?

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You can prove ***anything*** from a faulty assumption. This is called the **principle of explosion**. To see why, read "Animal, Vegetable, or Minister" for a silly example.

$$= 2^k + 2^k \qquad \textit{(via (1))}$$

$$= 2(2^k)$$

$$= 2^{k+1}.$$

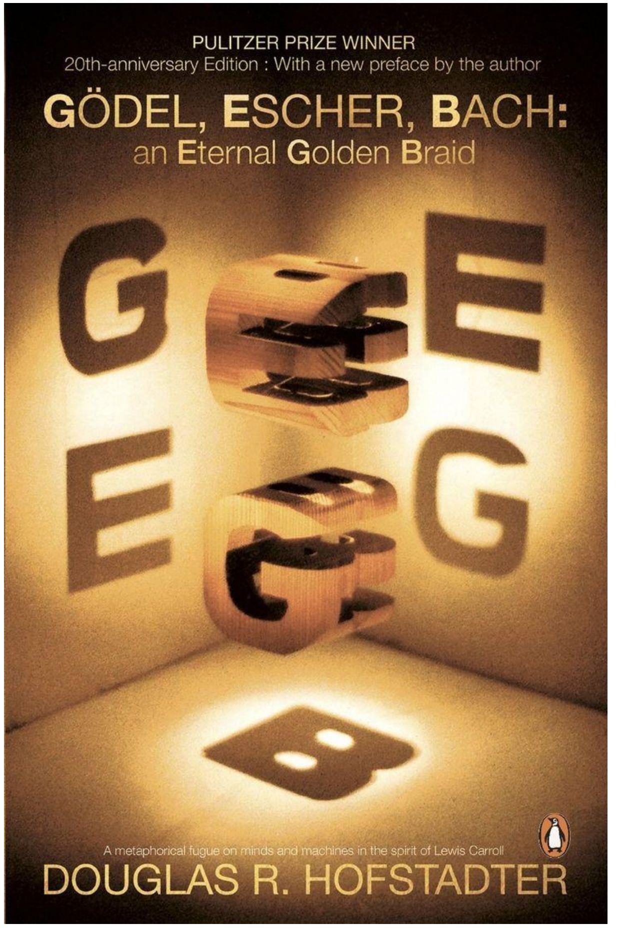
Therefore, $P(k + 1)$ is true, completing the induction. ■

The MU Puzzle

Gödel, Escher Bach: An Eternal Golden Braid

Douglas Hofstadter, cognitive scientist at the University of Indiana, wrote this Pulitzer-Prize-winning mind trip of a book.

It's a great read after you've finished CS103 – you'll see so many of the ideas we'll cover presented in a totally different way!



The **MU** Puzzle

Begin with the string **MI**.

Repeatedly apply one of the following operations:

Double the contents of the string after the **M**: for example, **MIIU** becomes **MIIUIIU**, or **MI** becomes **MII**.

Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**.

Append **U** to the string if it ends in **I**: **MI** becomes **MIU**.

Remove any **UU**: **MUUU** becomes **MU**.

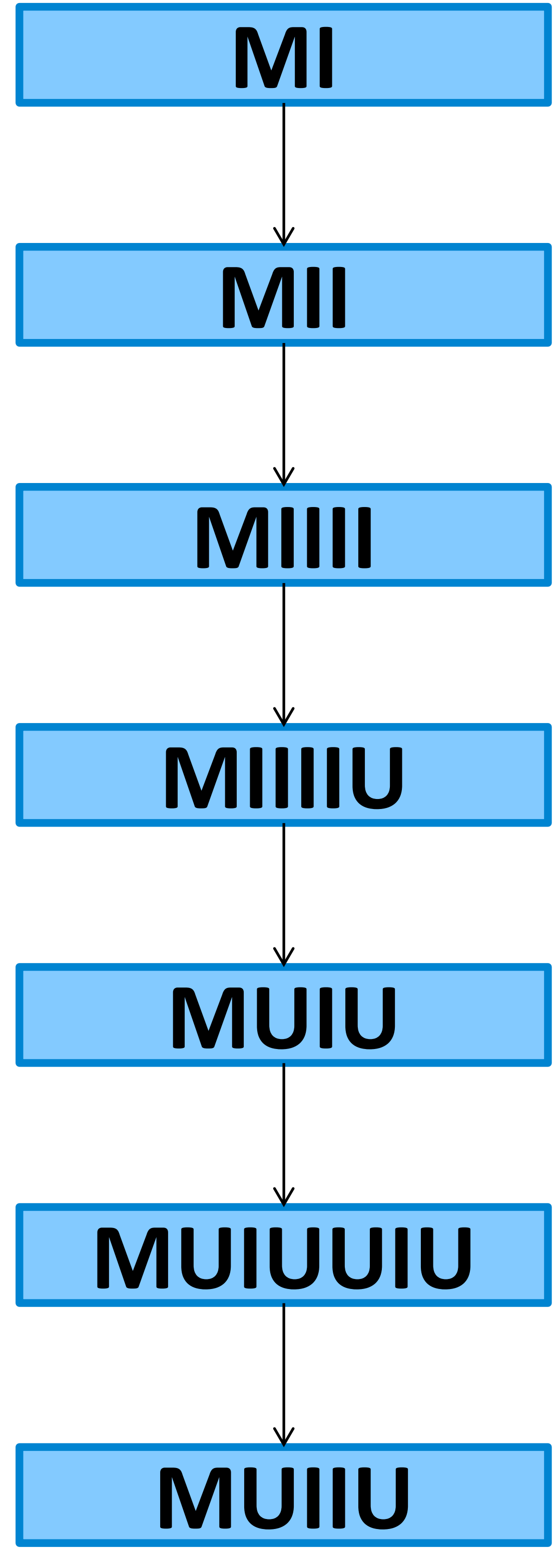
Question: How do you transform **MI** to **MU**?

(a) Double the string after an **M**.

(b) Replace **III** with **U**.

(c) Append **U**, if the string ends in **I**.

(d) Delete **UU** from the string.



Try It!

Starting with **MI**, apply these operations to make **MU**:

(a) Double the string after an **M**.

(b) Replace **III** with **U**.

(c) Append **U**, if the string ends in **I**.

(d) Delete **UU** from the string.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting **I**'s


MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

The Key Insight

- Initially, the number of **I**'s is *not* a multiple of three.
- To make **MU**, the number of **I**'s must end up as a multiple of three.
- Can we *ever* make the number of **I**'s a multiple of three?

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

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Proof: By contrapositive; we'll prove that if $n - 3$ is a multiple of three, then n is also a multiple of three. Because $n - 3$ is a multiple of three, we can write $n - 3 = 3k$ for some integer k. Then $n = 3(k+1)$, so n is also a multiple of three, as required. ■

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Proof: Let n be a number that isn't a multiple of three. If n is congruent to one modulo three, then $n = 3k + 1$ for some integer k. This means $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$, so $2n$ is not a multiple of three. Otherwise, n must be congruent to two modulo three, so $n = 3k + 2$ for some integer k. Then $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$, and so $2n$ is not a multiple of three. ■

Lemma: No matter which moves are made, the number of **I**'s in the string never becomes multiple of three.

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As a base case, we'll prove $P(0)$, that the number of **I**'s after 0 moves is not a multiple of three. After no moves, the string is **MI**, which has one **I** in it. Since one isn't a multiple of three, $P(0)$ is true.

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Case 1: Double the string after the **M**.

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Case 1: Double the string after the **M**. After this, we will have $2r$ **I**'s in the string, and from our lemma $2r$ isn't a multiple of three.

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Case 1: Double the string after the **M**. After this, we will have $2r$ **I**'s in the string, and from our lemma $2r$ isn't a multiple of three.

Case 2: Replace **III** with **U**. After this, we will have $r - 3$ **I**'s in the string, and by our lemma $r - 3$ is not a multiple of three.

Case 3: Either append **U** or delete **UU**.

Lemma: No matter which moves are made, the number of **I**'s in the string never becomes multiple of three.

Proof: Let $P(n)$ be the statement "after any n moves, the number of **I**'s in the string will not be multiple of three." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

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Theorem: The **MU** puzzle has no solution.

Proof: Assume for the sake of contradiction that the **MU** puzzle has a solution and that we can convert **MI** to **MU**. This would mean that at the very end, the number of **I**'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of **I**'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the **MU** puzzle has no solution. ■

Algorithms and Loop Invariants

The proof we just made had the form

"If P is true before we perform an action, it is true after we perform an action."

We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.

In algorithmic analysis, this is called a ***loop invariant***.

Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.

Take CS161 for more details!

Let’s take a five minute break!

Variations on Induction: ***Starting Later***

Induction Starting at 0

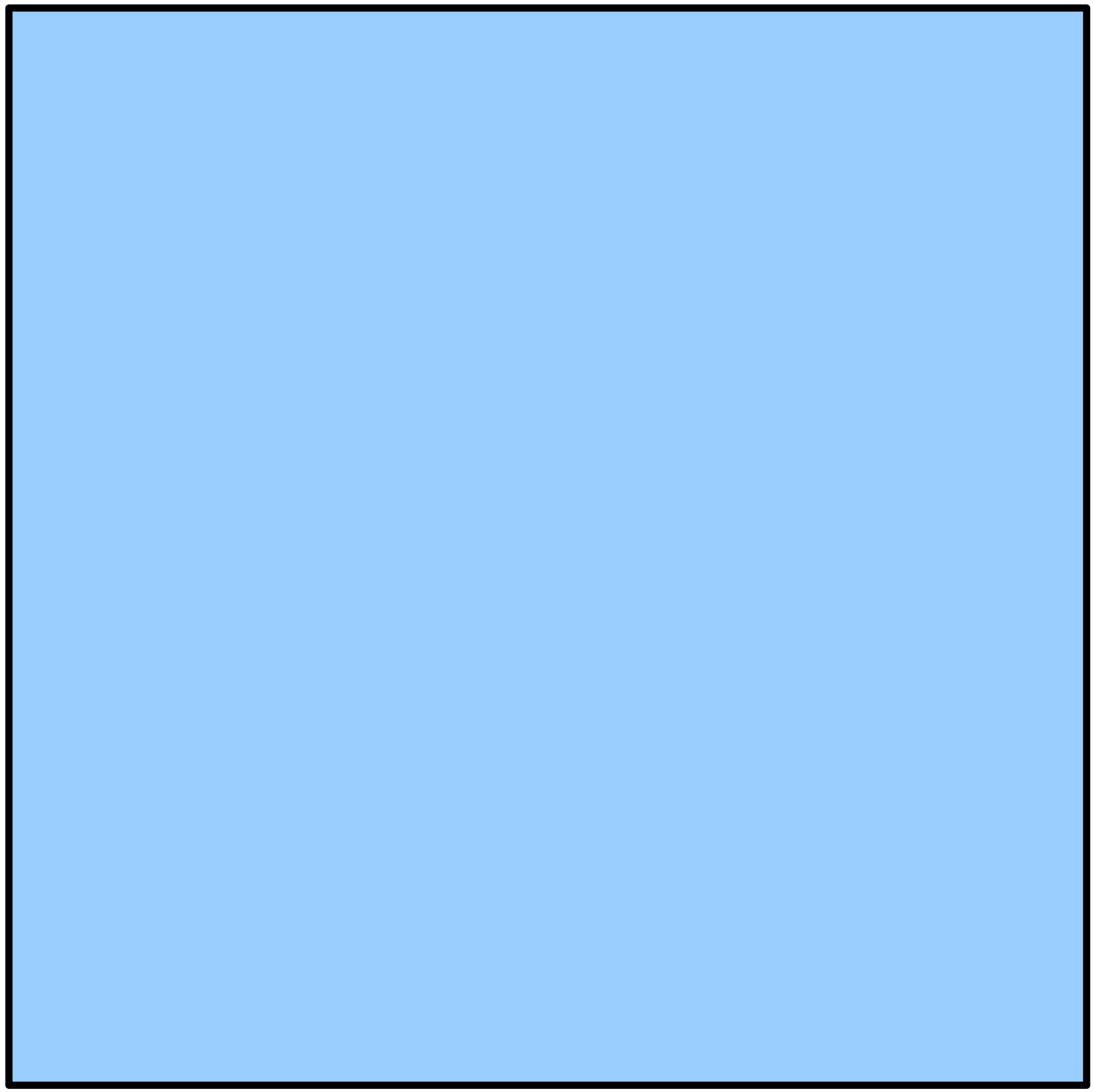
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
- Show that $P(0)$ is true.
- Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
- Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

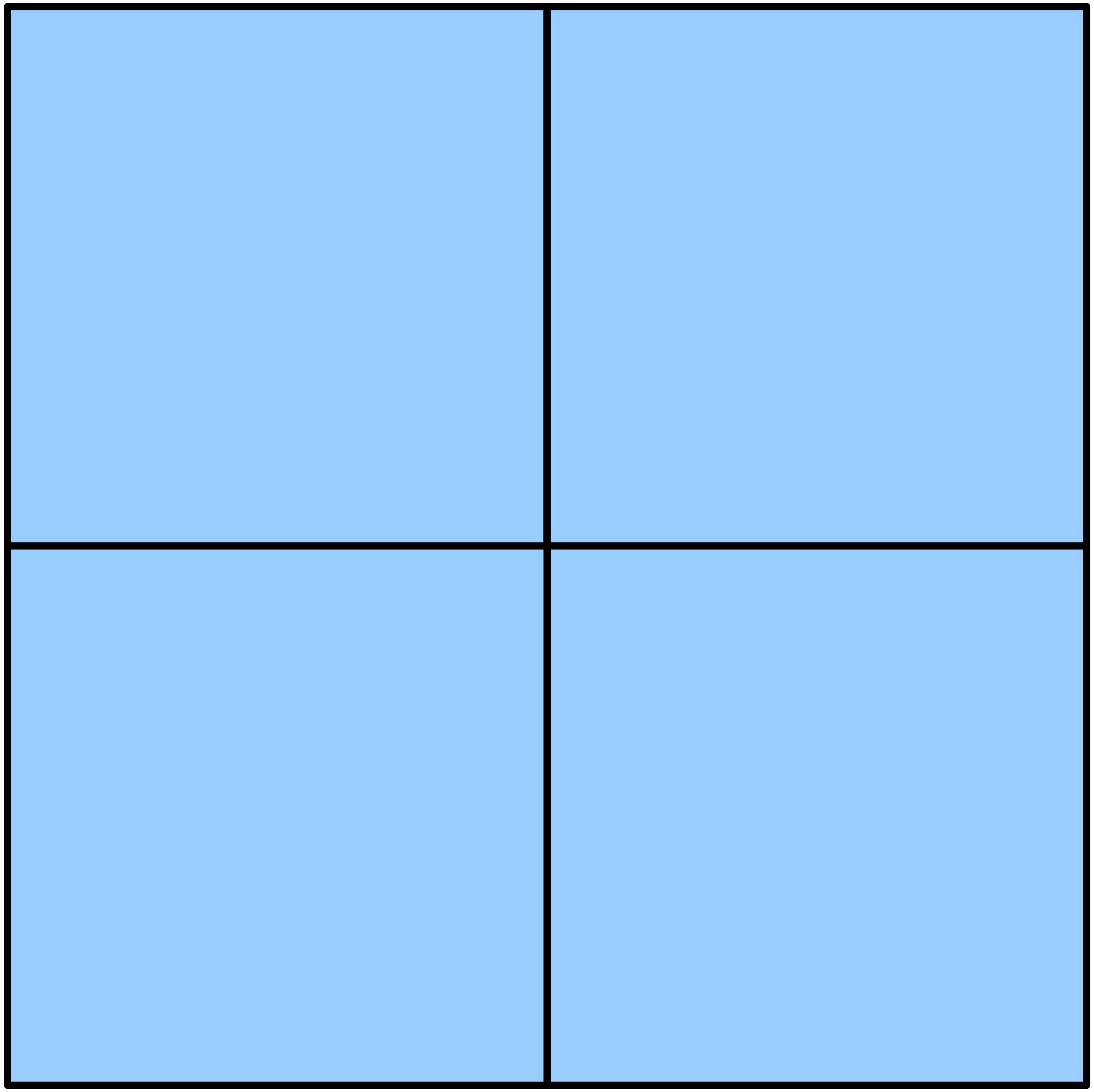
- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
- Show that $P(\boldsymbol{m})$ is true.
- Show that for any $k \geq \boldsymbol{m}$, that
  if $P(k)$ is true, then $P(k+1)$ is true.
- Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: ***Bigger Steps***

Subdividing a Square



Subdividing a Square



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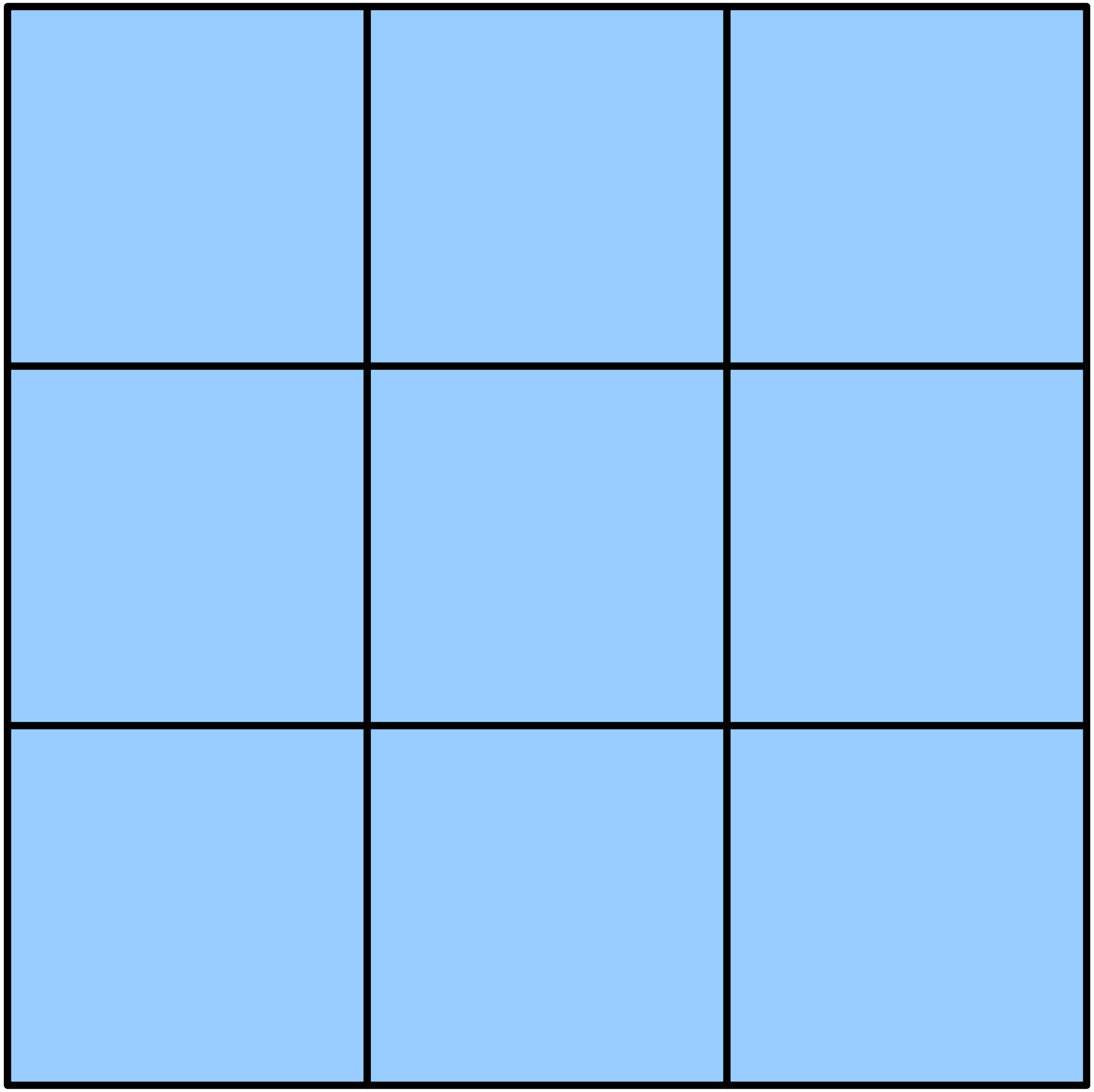

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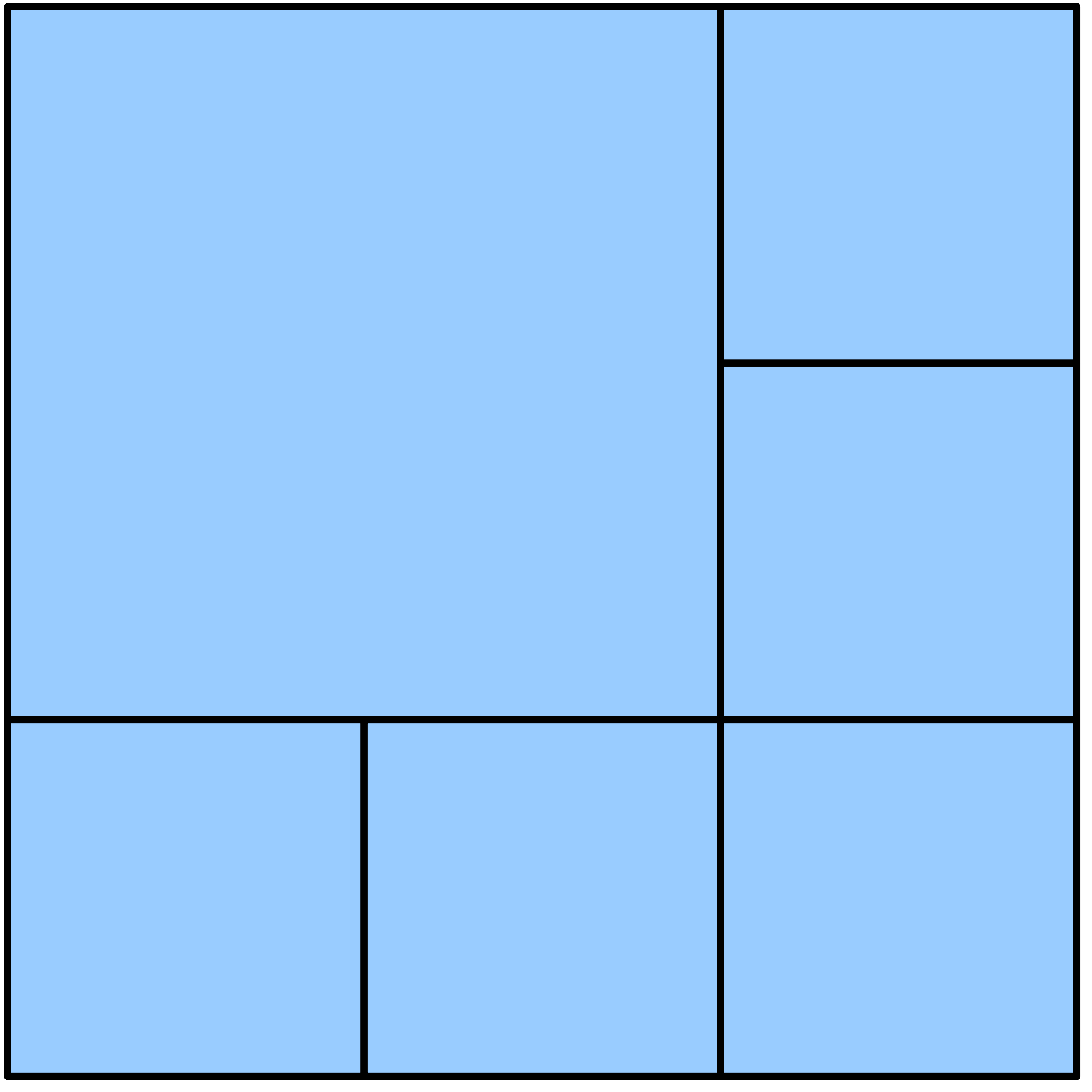
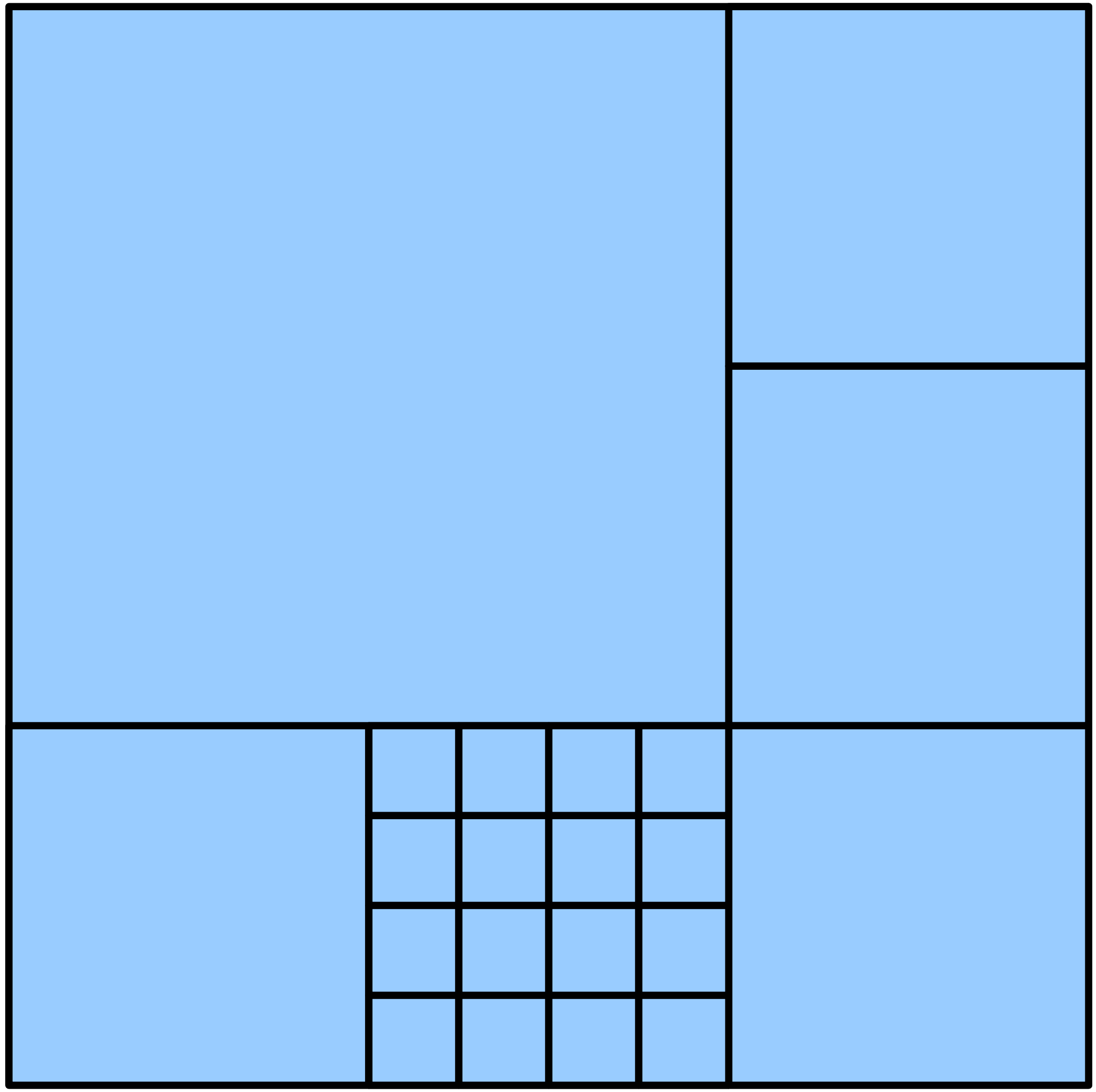


Subdividing a Square



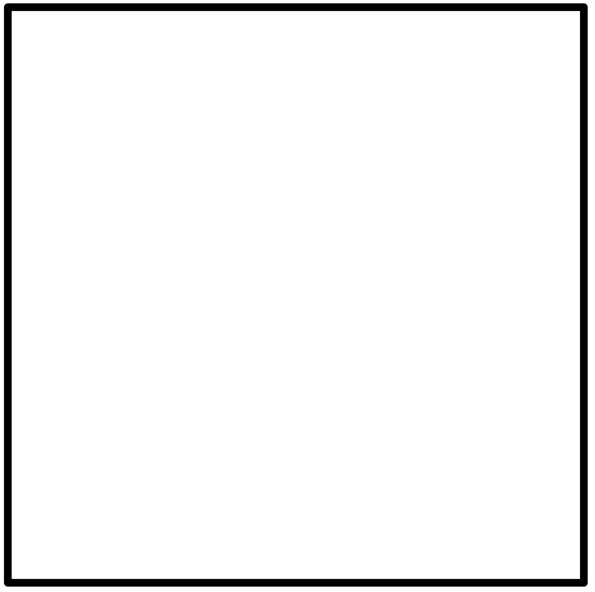
For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

1 2 3 4 5 6 7 8 9 10 11 12

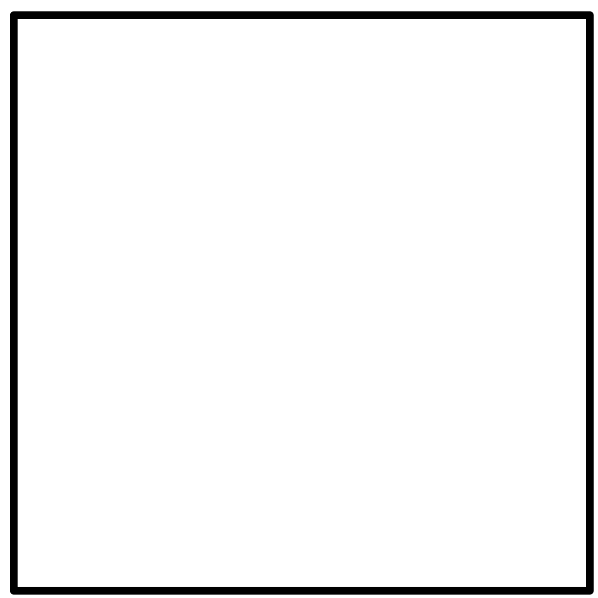
1

1 2 3 4 5 6 7 8 9 10 11 12



1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



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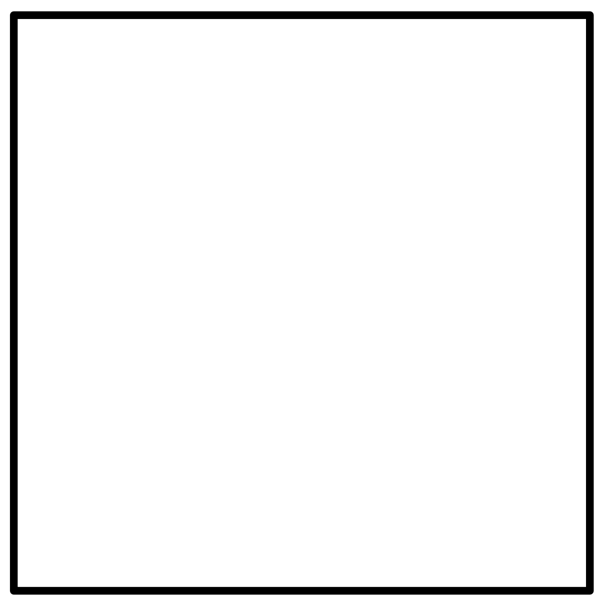
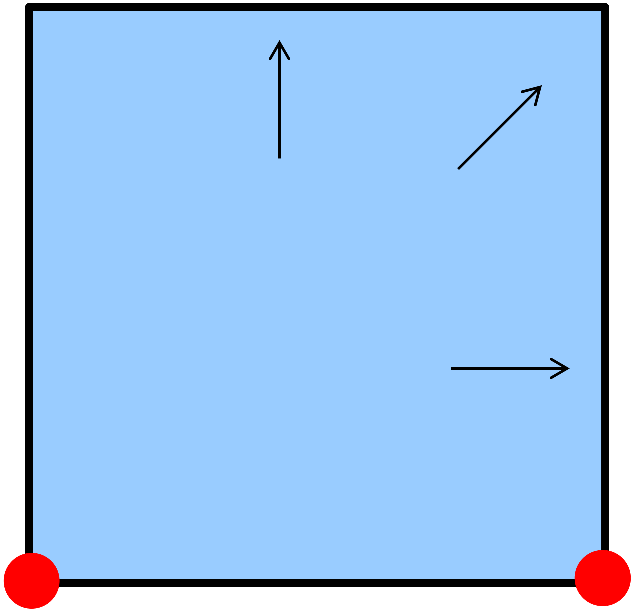


Number of corners = 4

Number of squares < 4

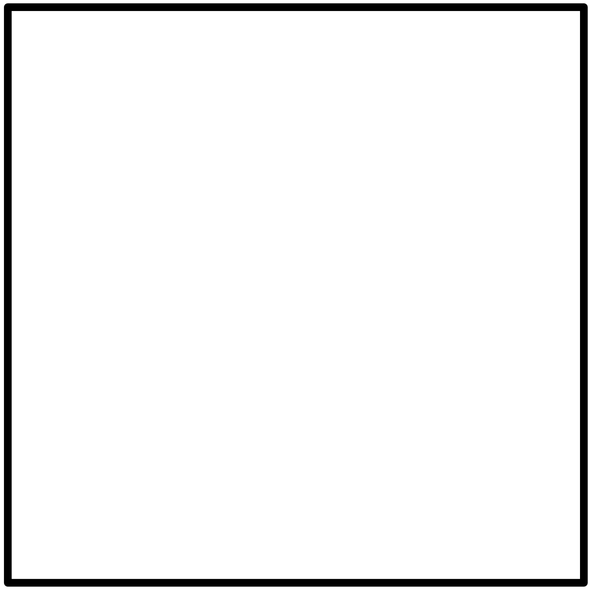
1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



By the pigeonhole principle, at least one smaller square needs to cover at least *two* of the original square's corners.

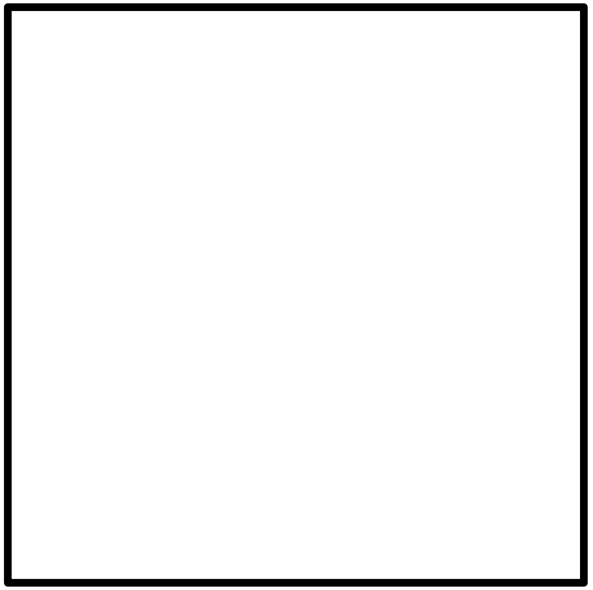
1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

1	2
4	3

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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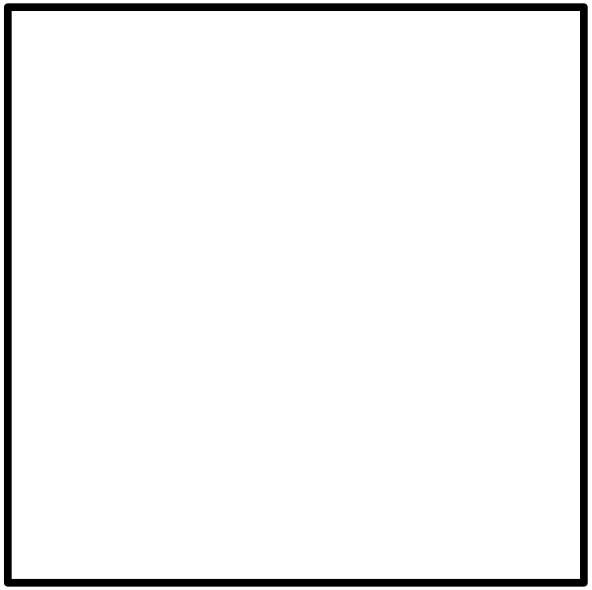
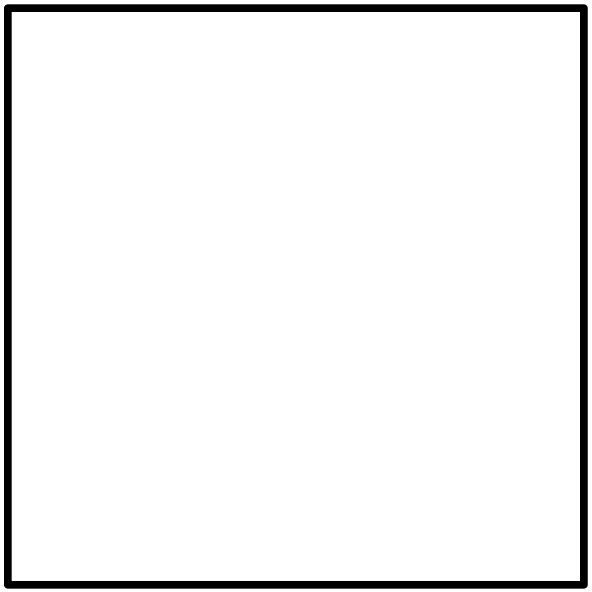


Number of corners = 4

Number of squares = 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

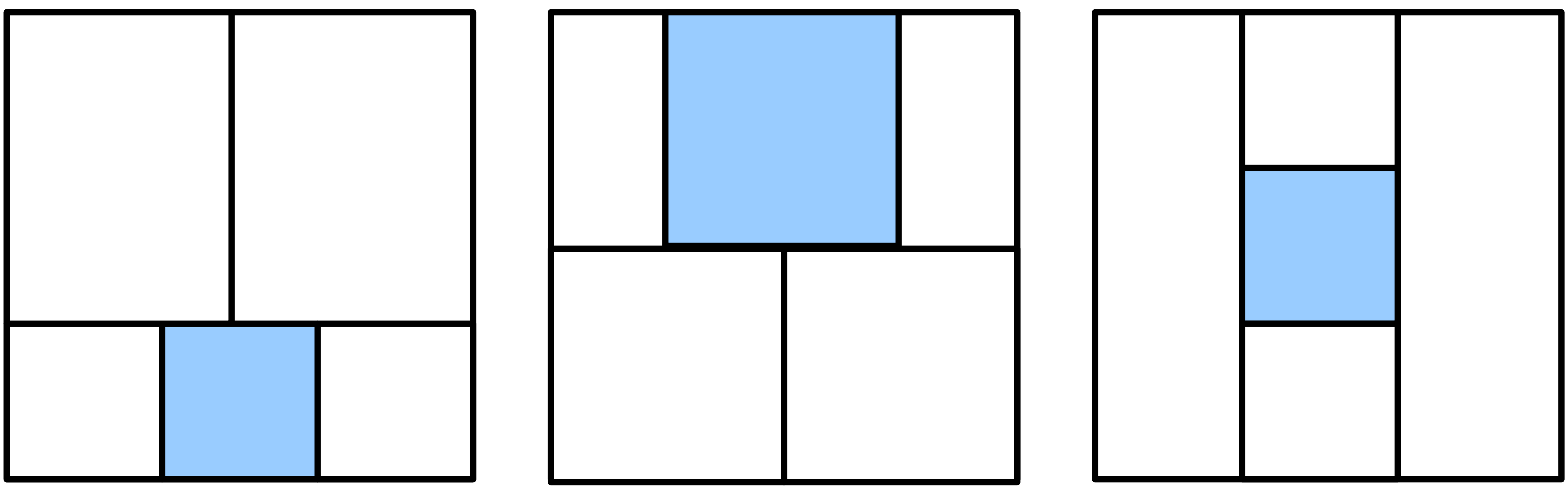
At least one square cannot be covering *any* of the original corners



Number of corners = 4

Number of squares = 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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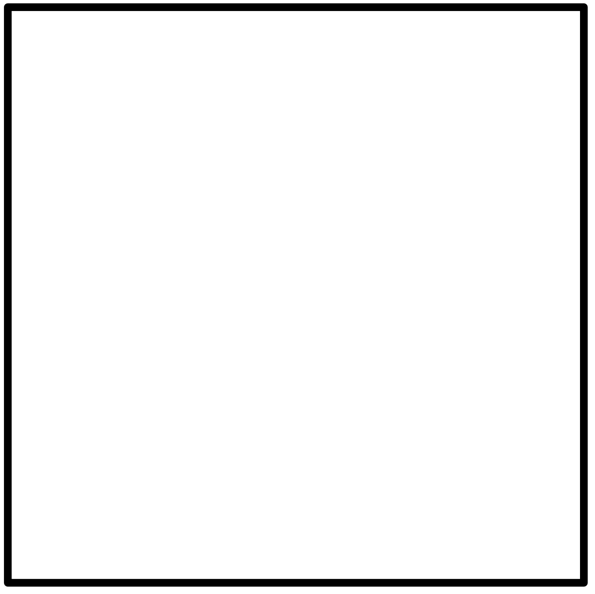

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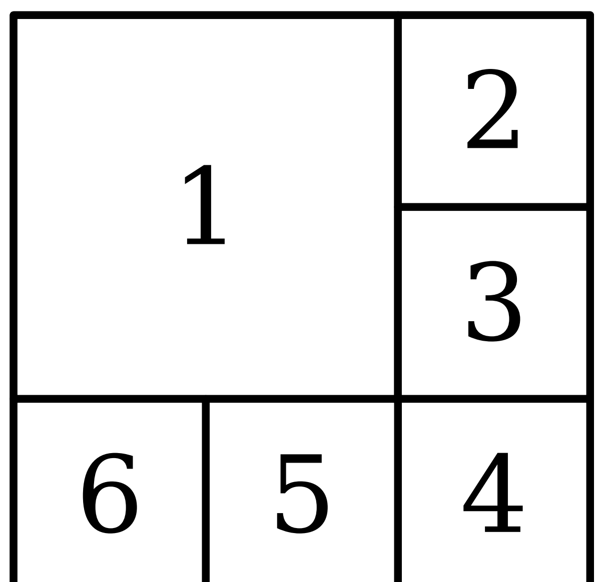

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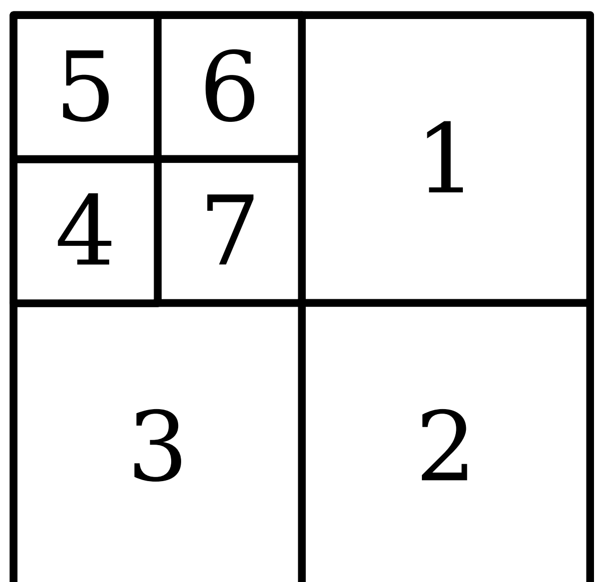

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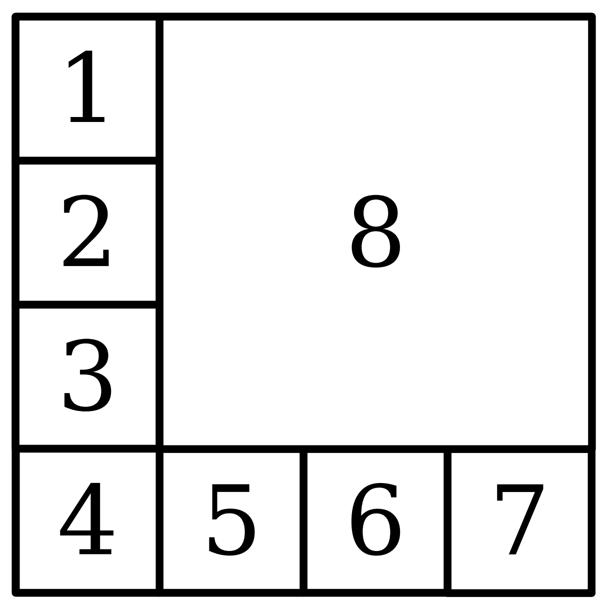
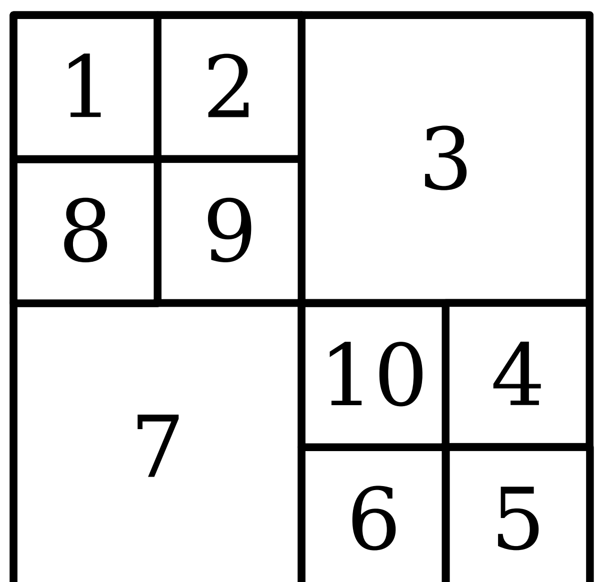


1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12



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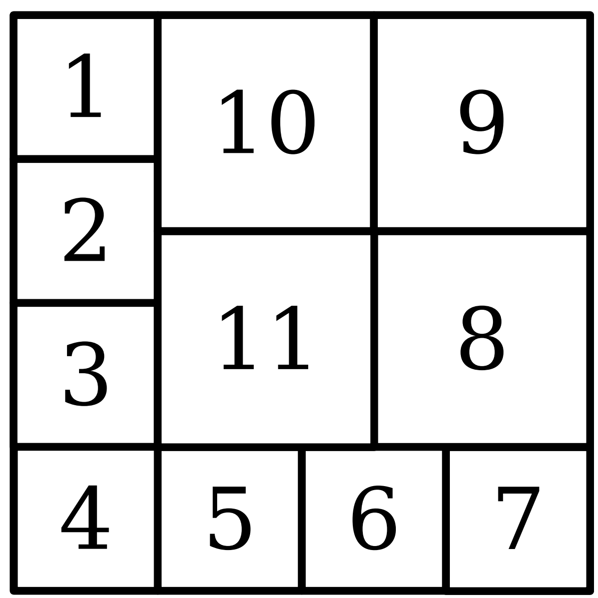

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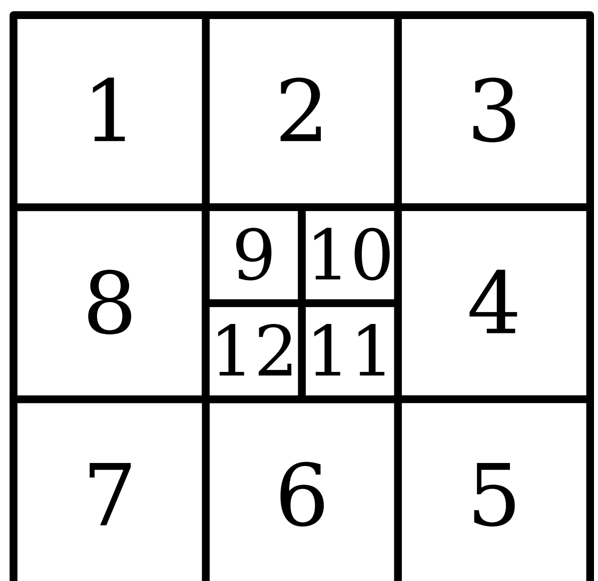

The Key Insight

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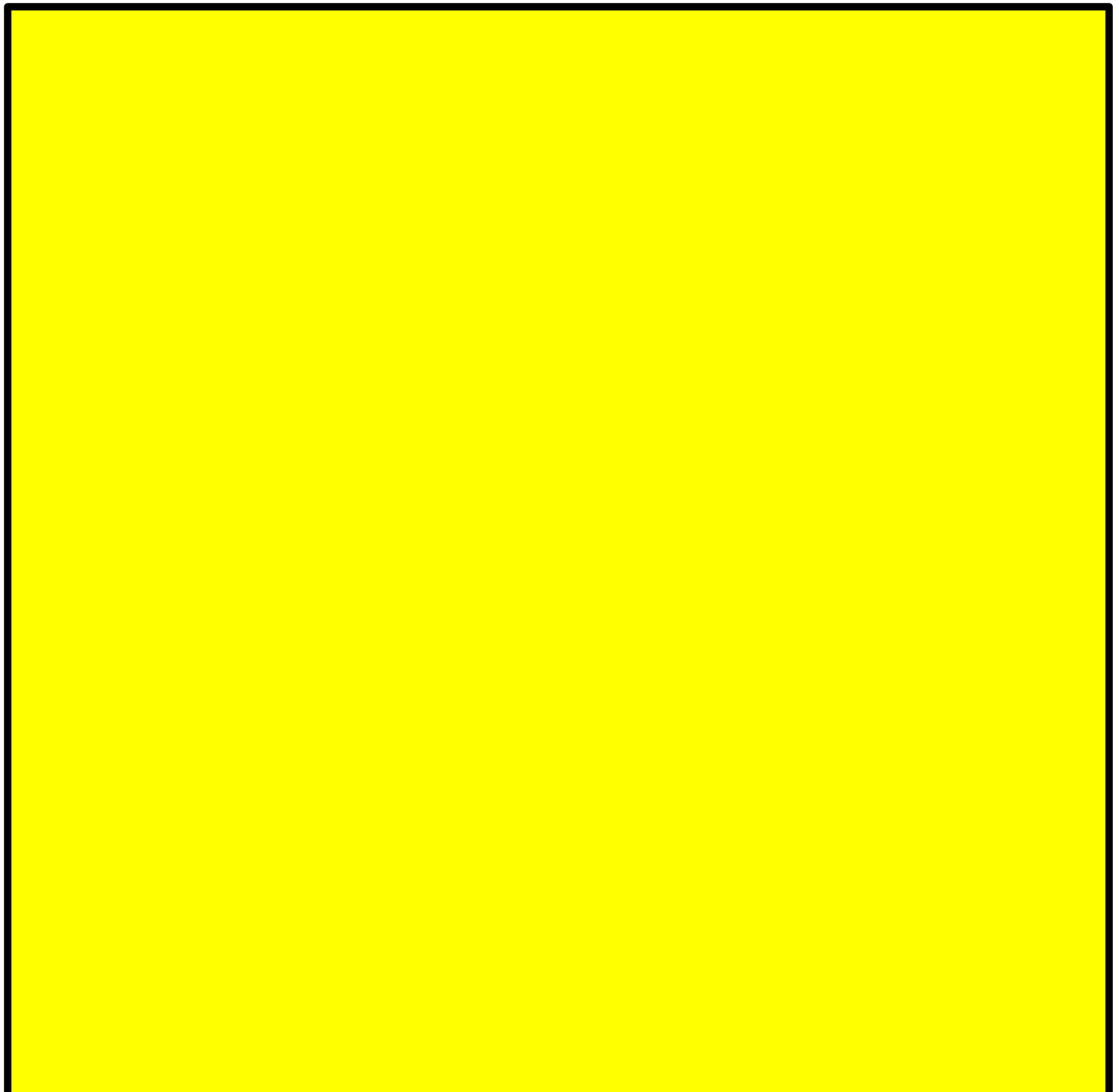

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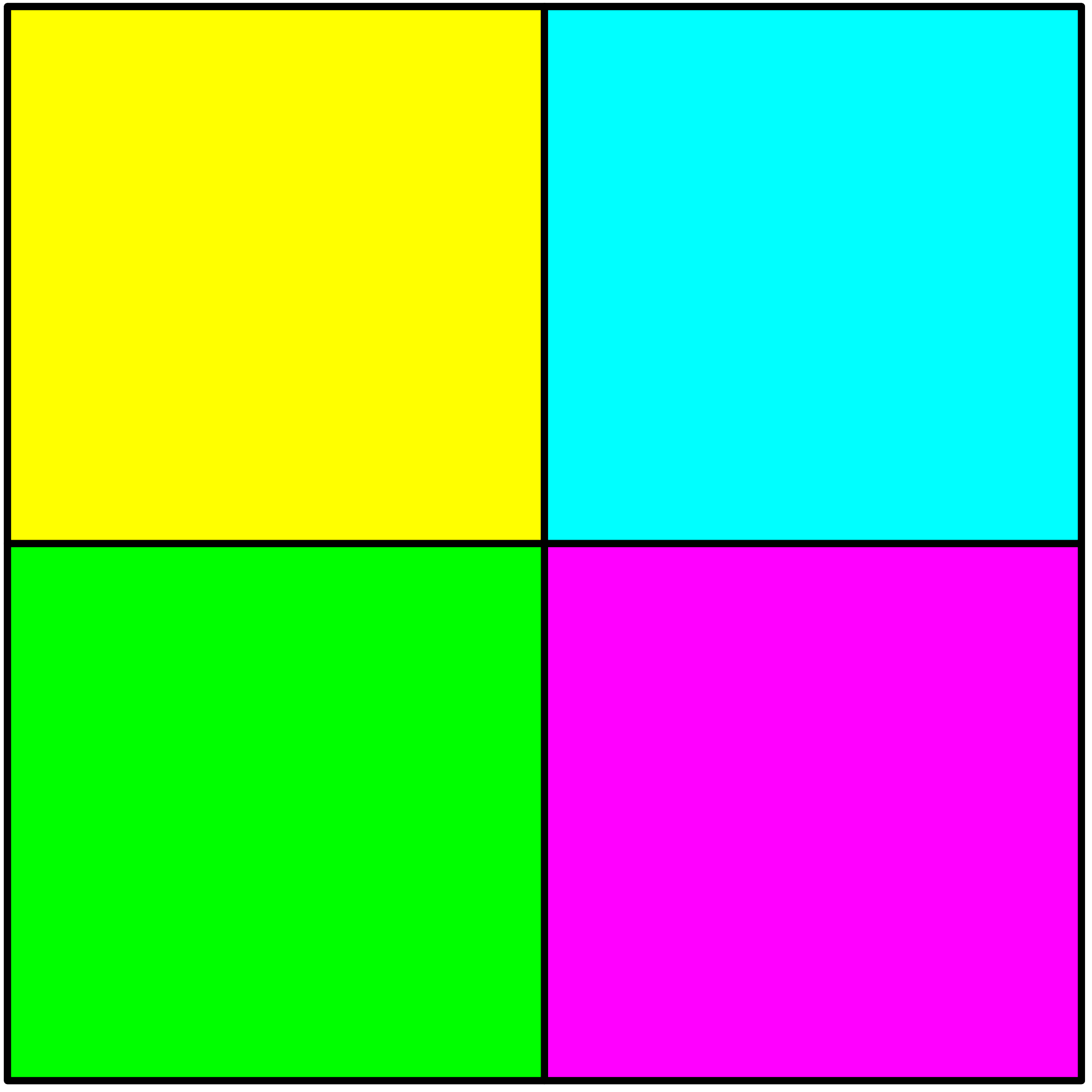

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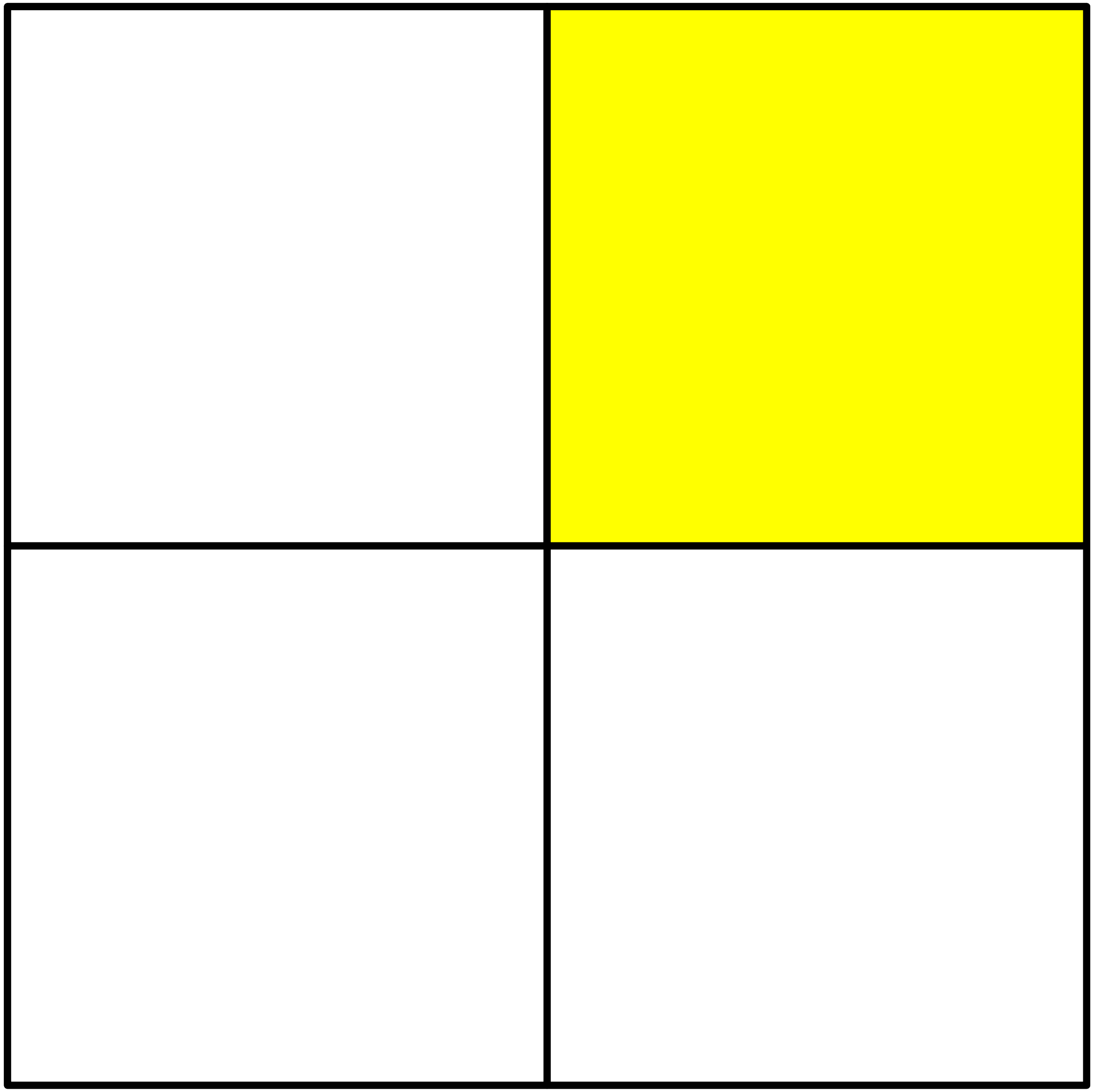

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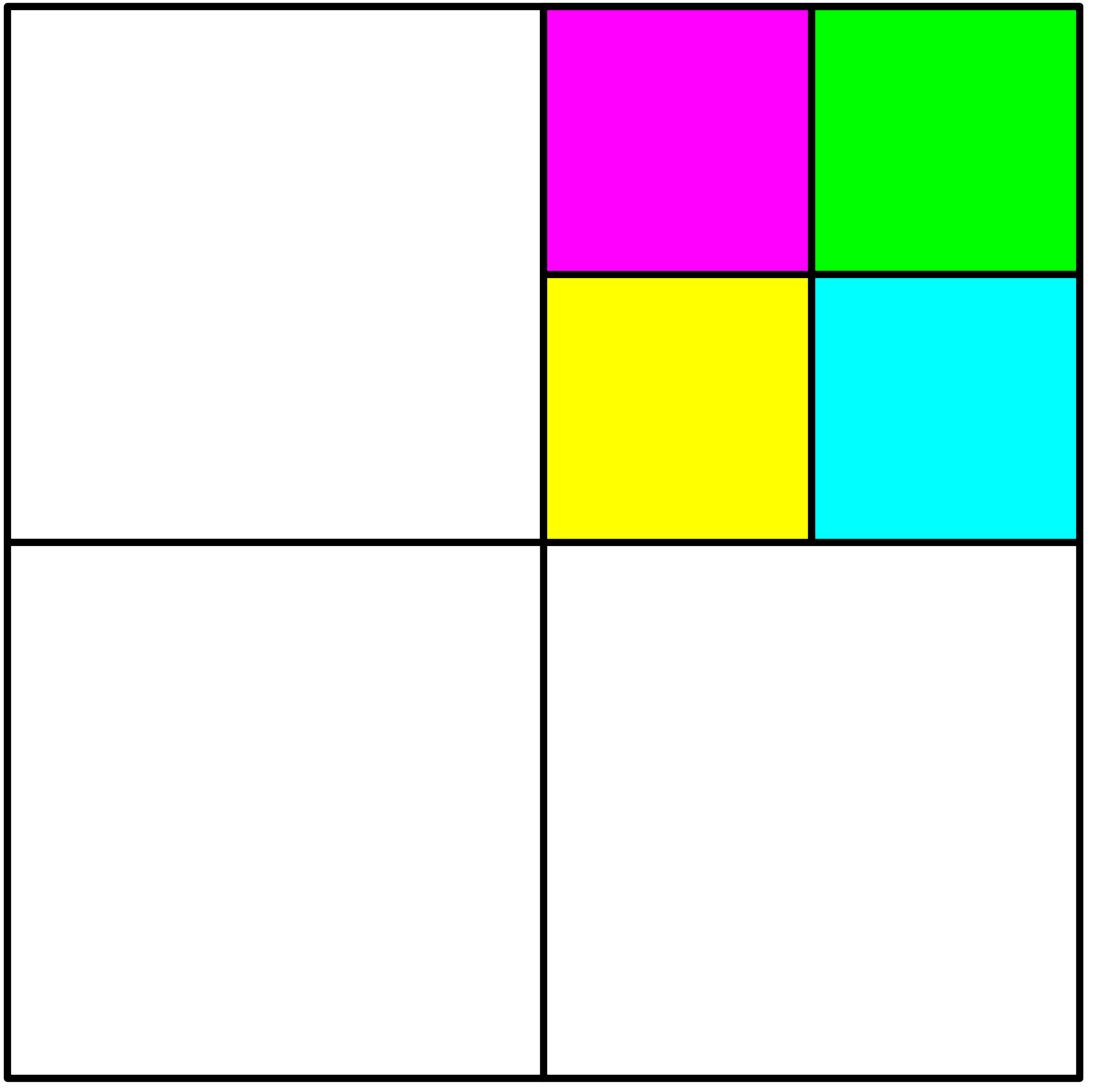

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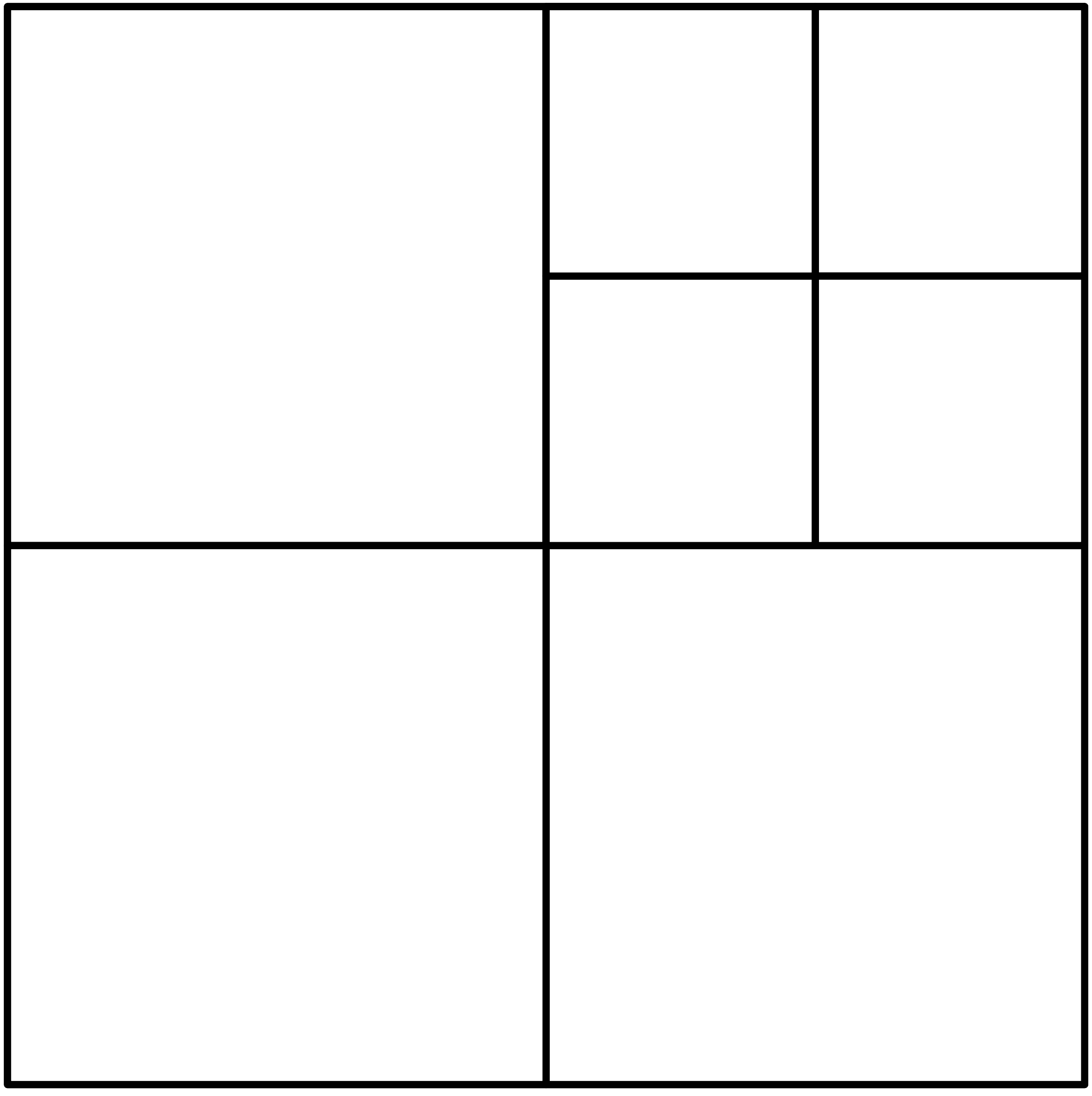

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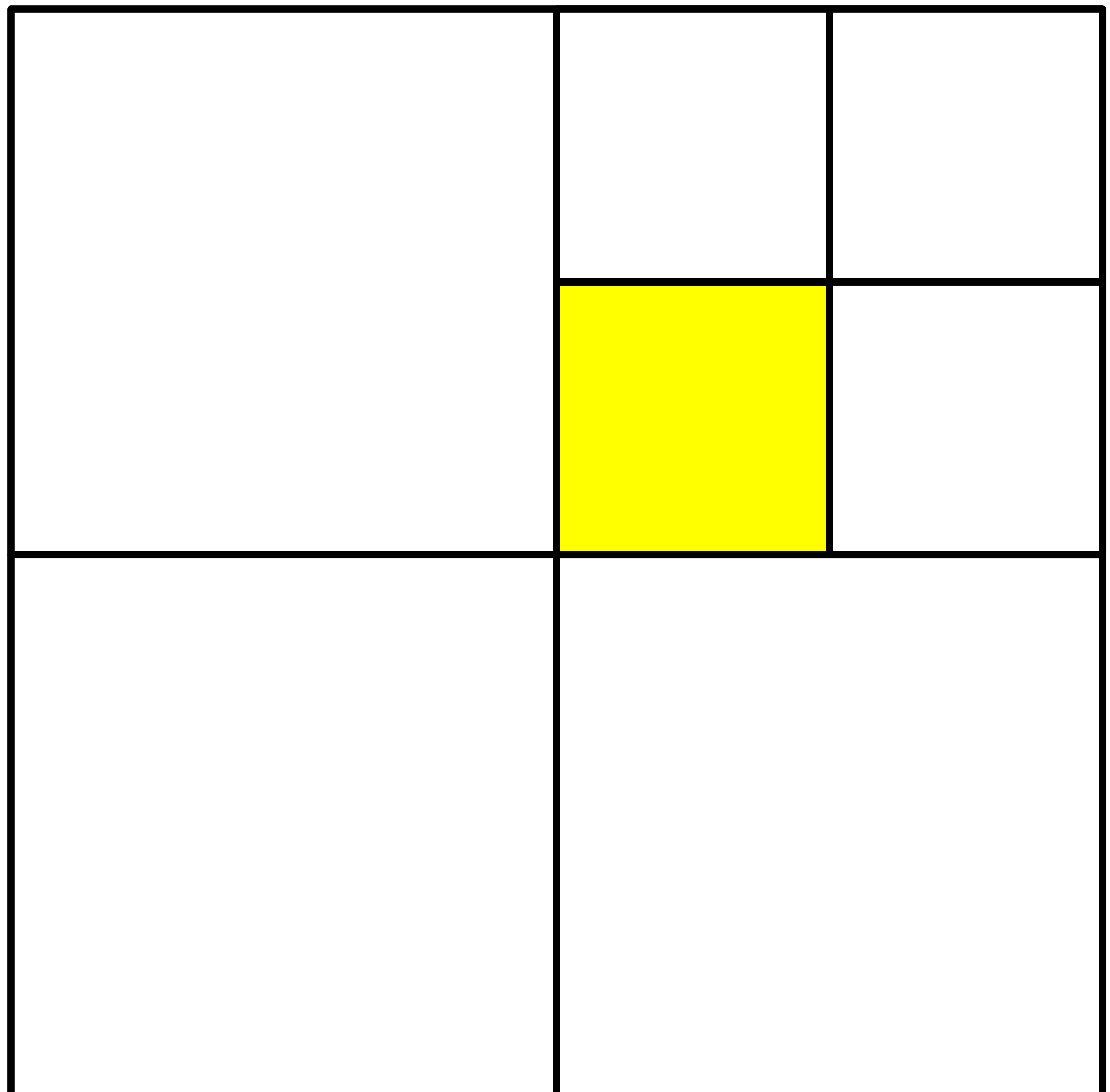

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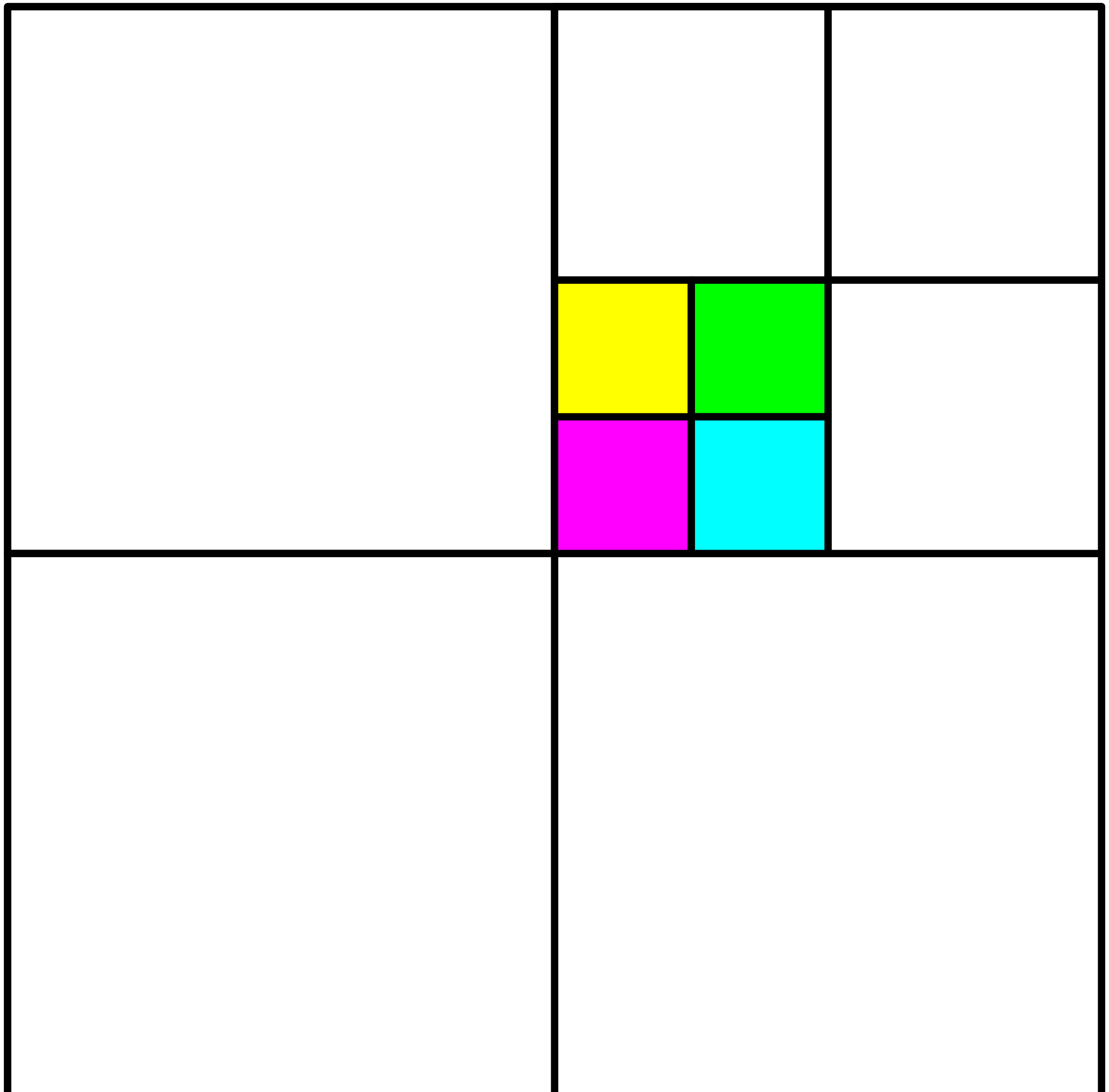

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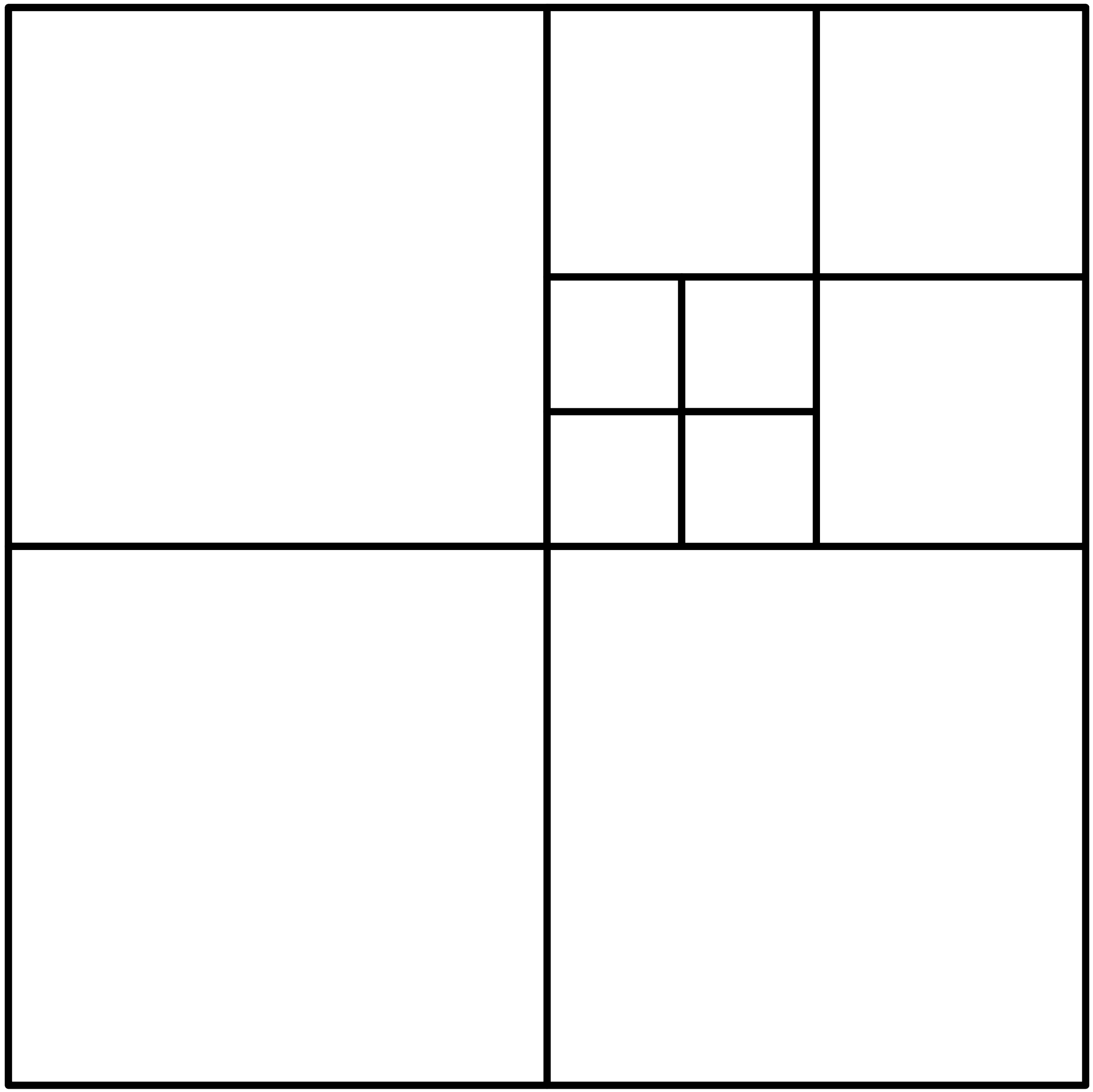

The Key Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
- For multiples of three, start with 6 and keep adding three squares until n is reached.
- For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
- For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, it is possible to subdivide a square into n smaller squares.

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As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares.

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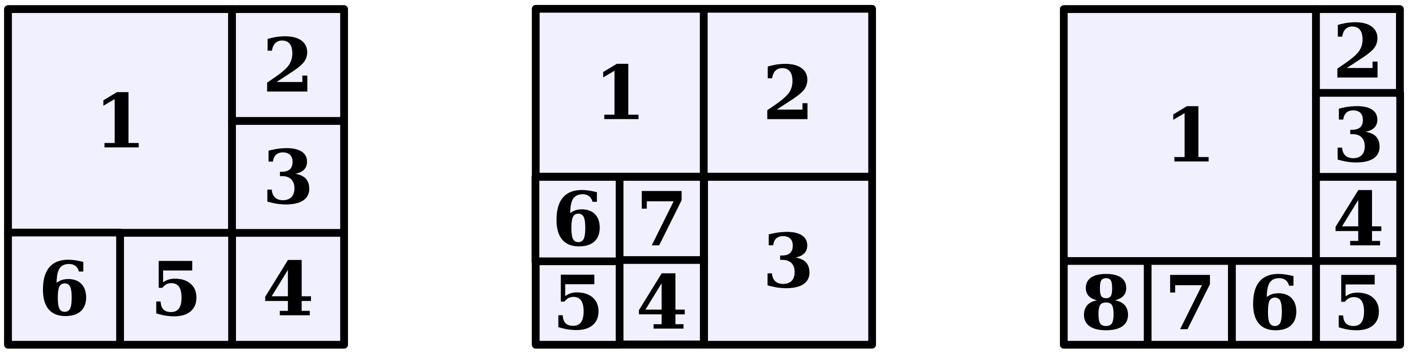

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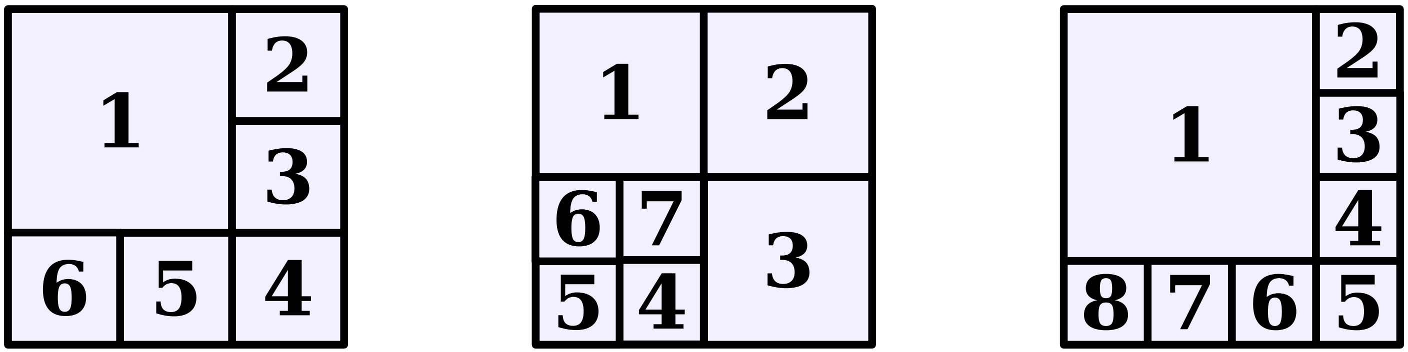

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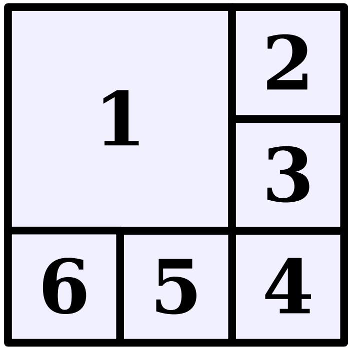

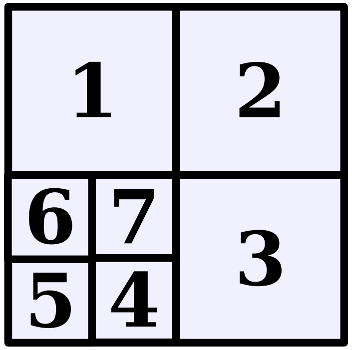

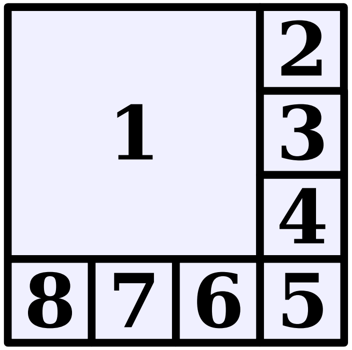

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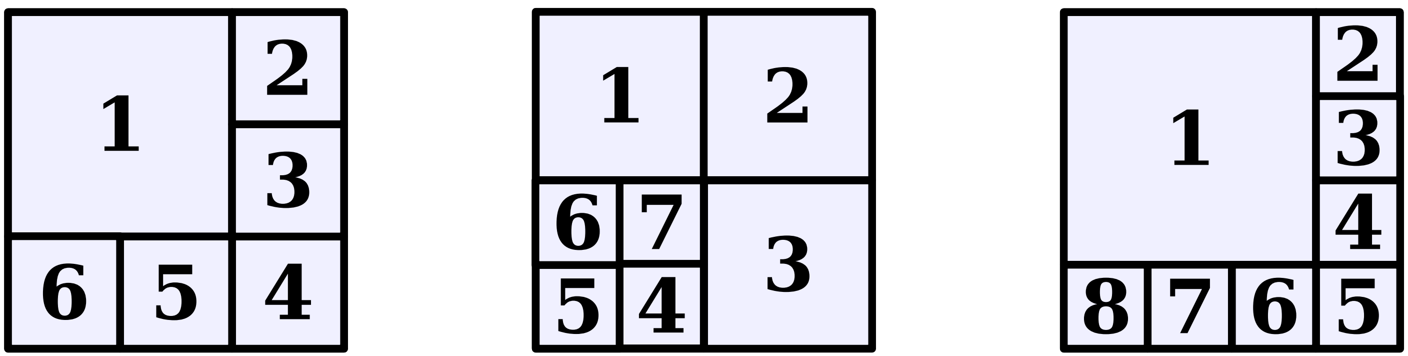

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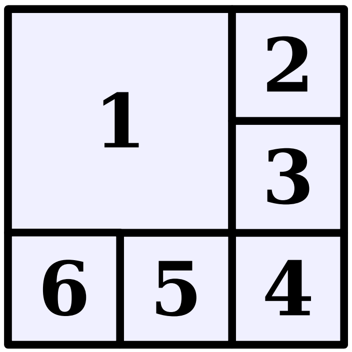

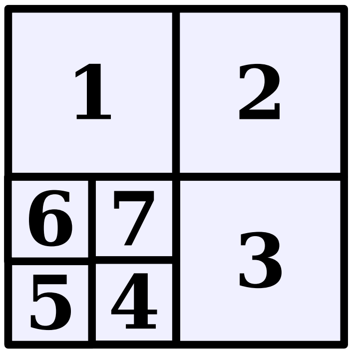

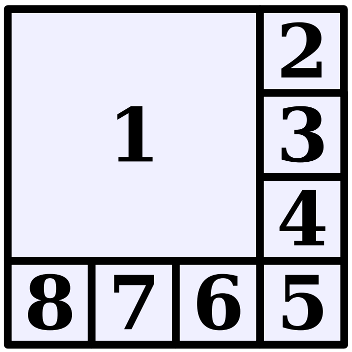

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Theorem: For any $n \geq 6$, it is possible to subdivide a square into n smaller squares.

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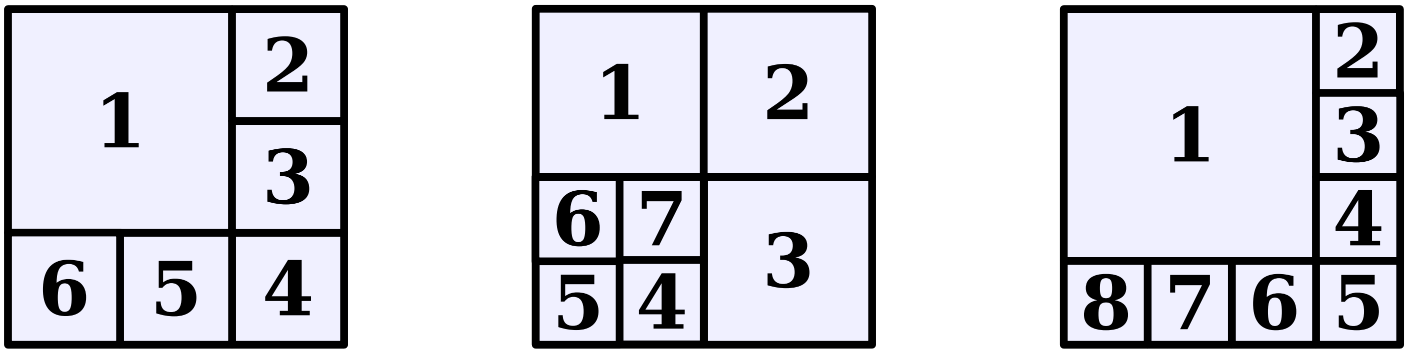

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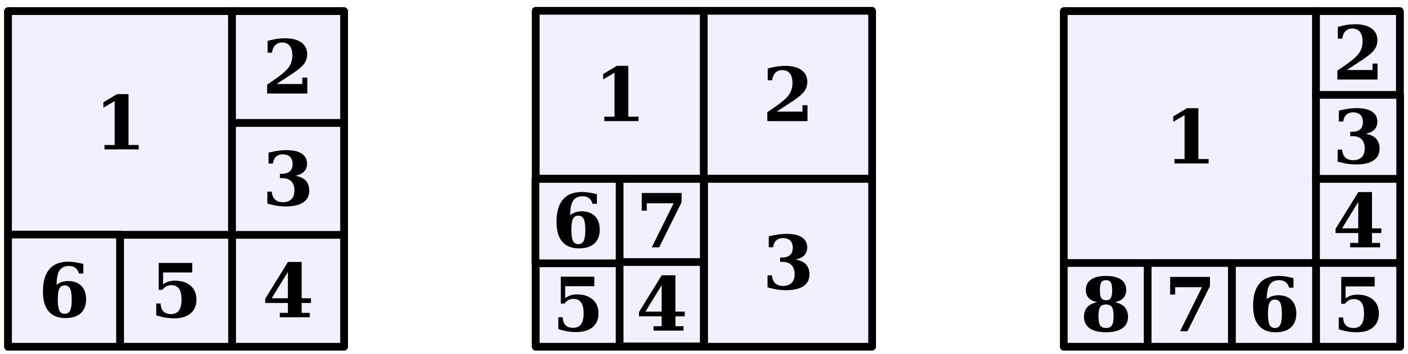

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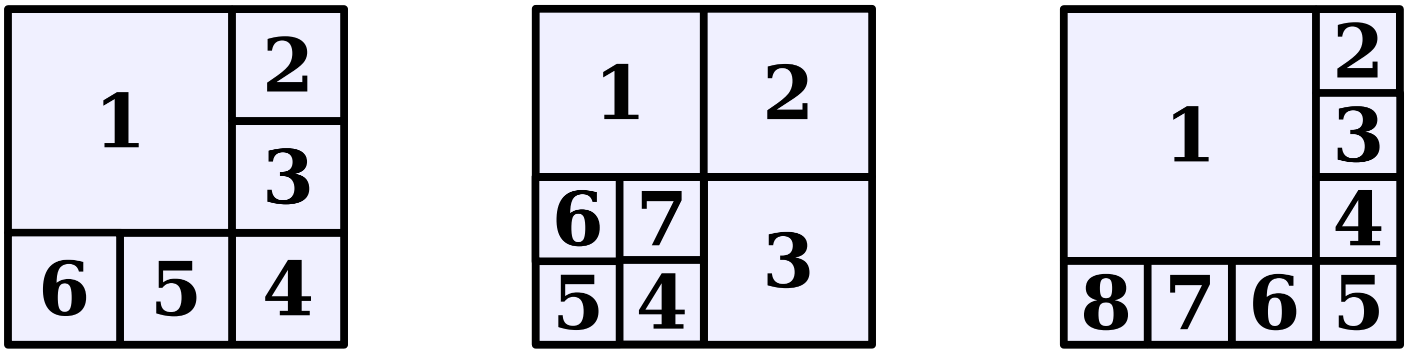

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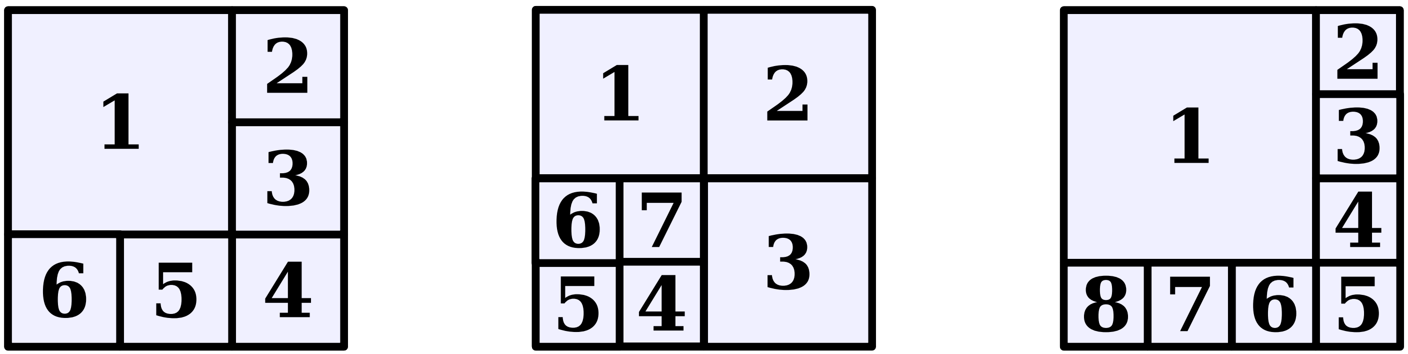

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Fun and totally optional exercise: you can also prove this directly!

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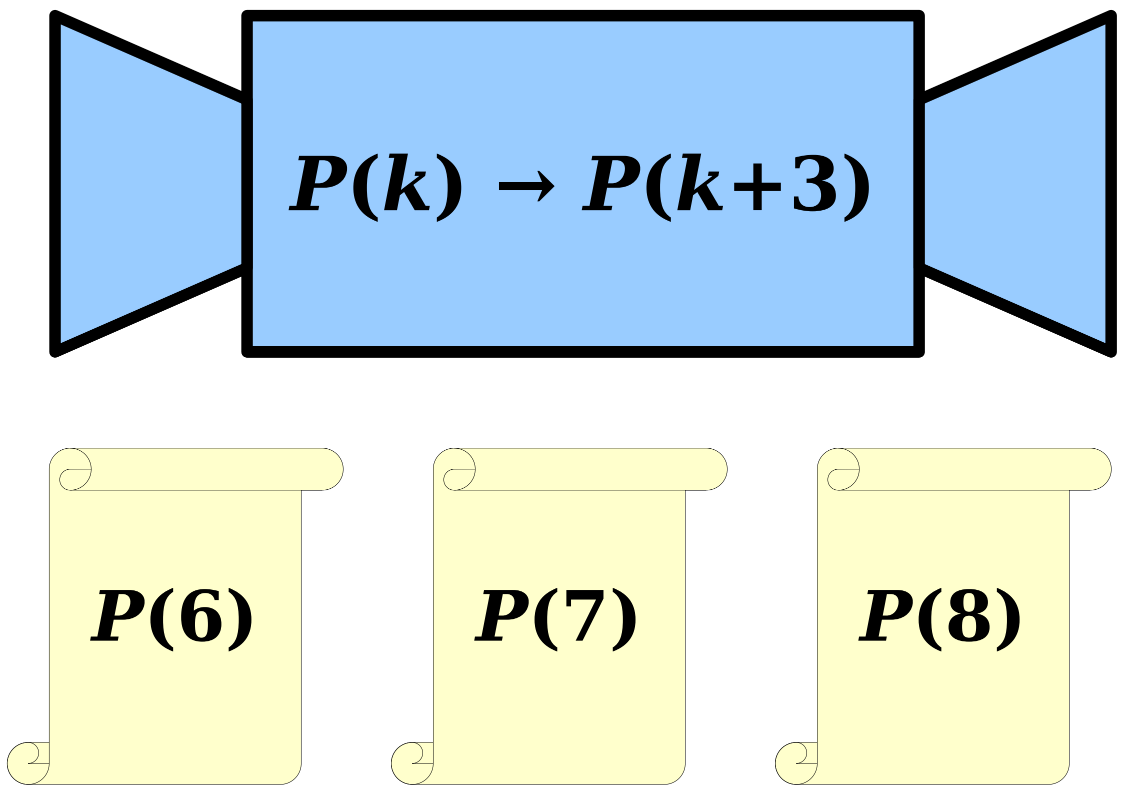
This induction has three consecutive base cases and takes steps of size three.

Thinking back to our "induction machine" analogy:



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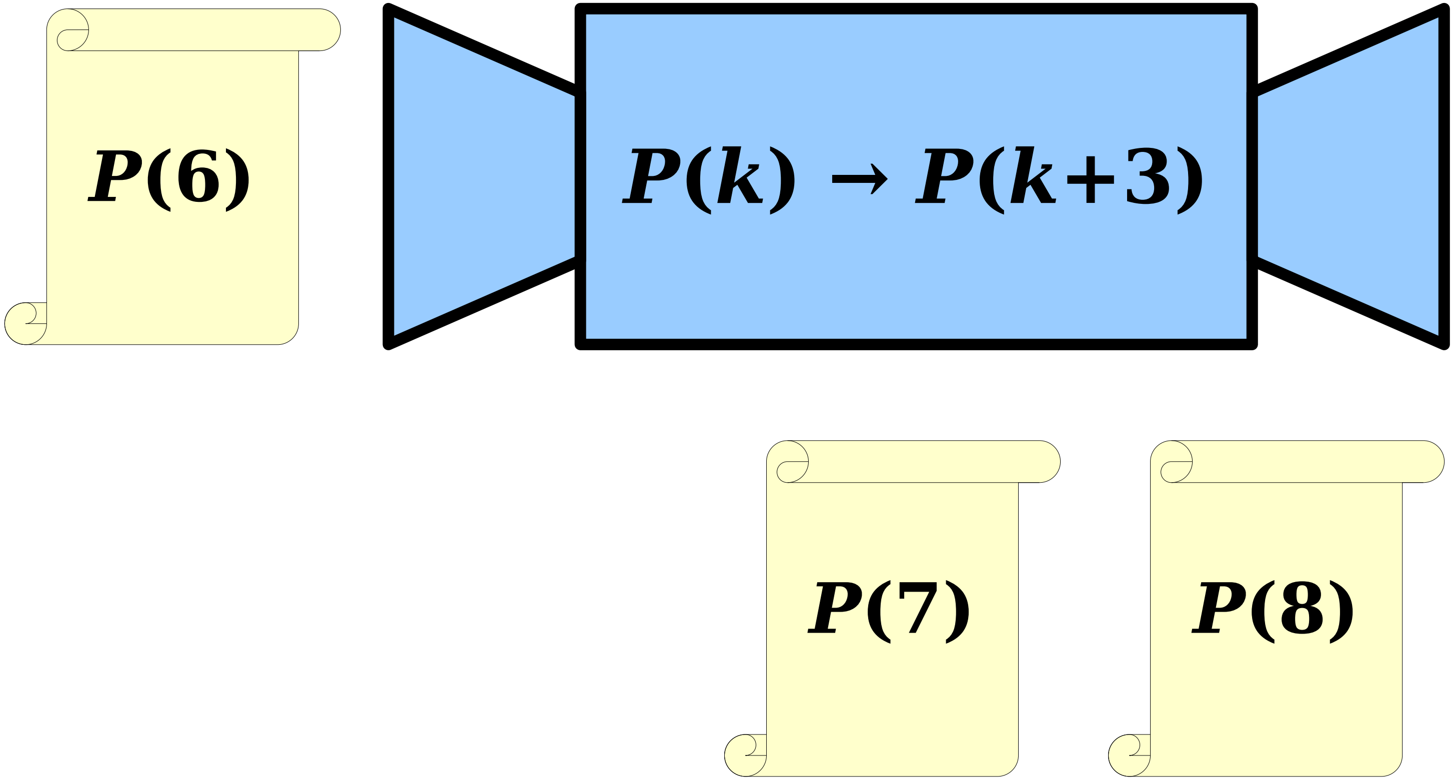

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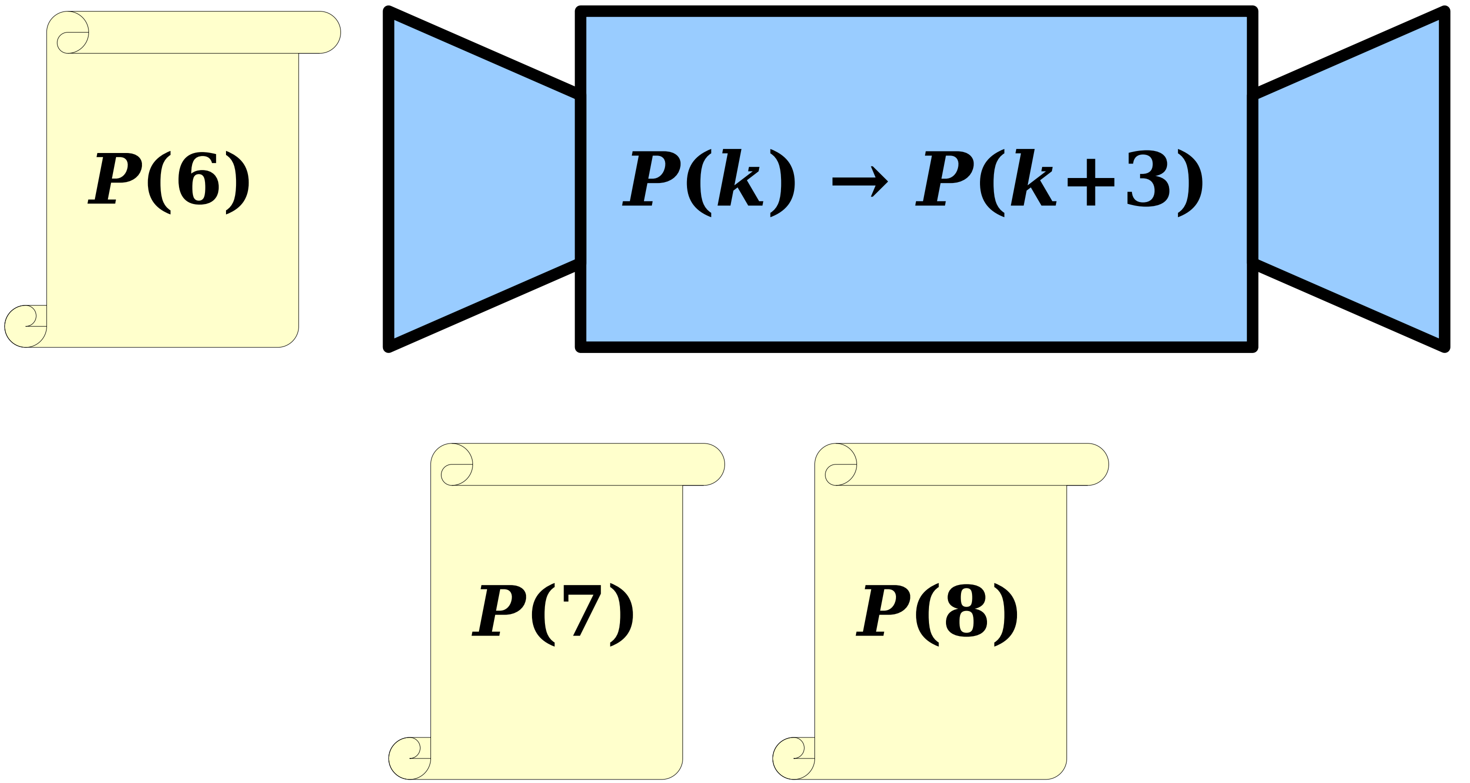

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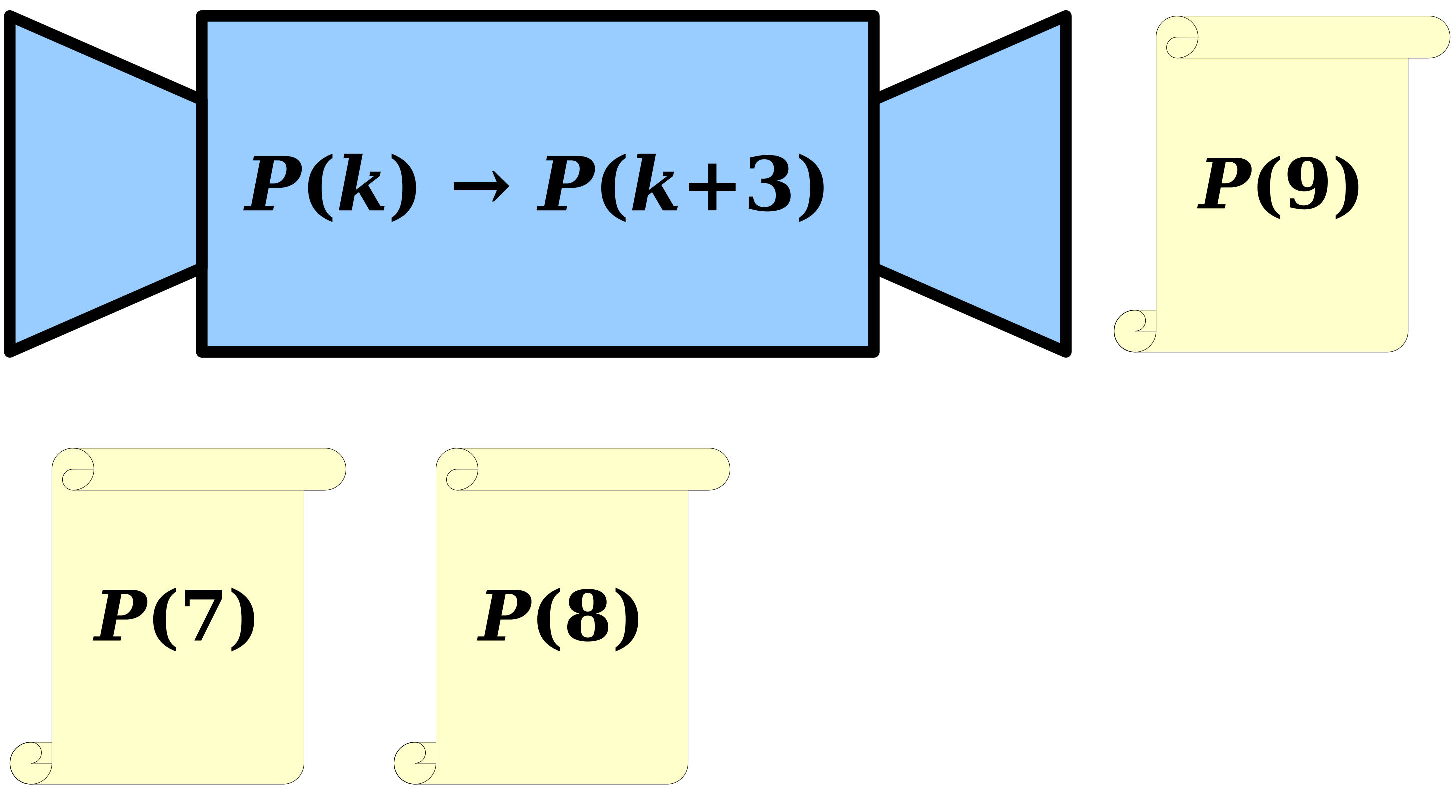

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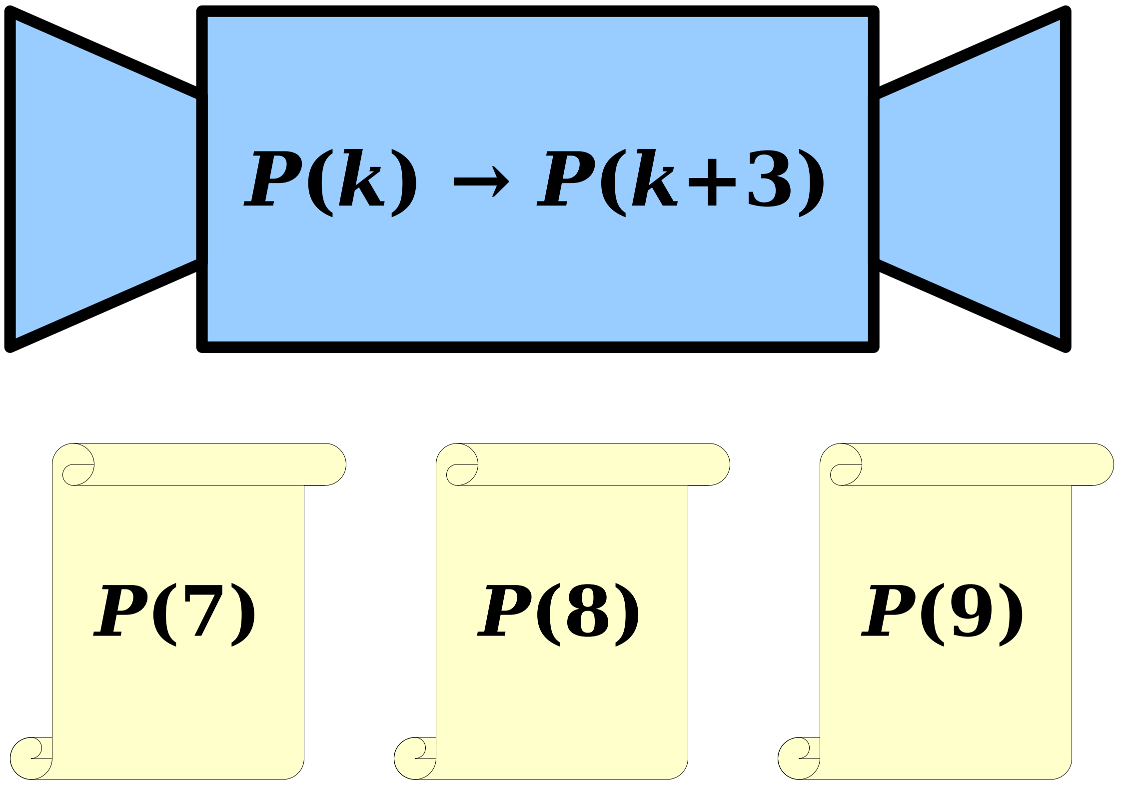

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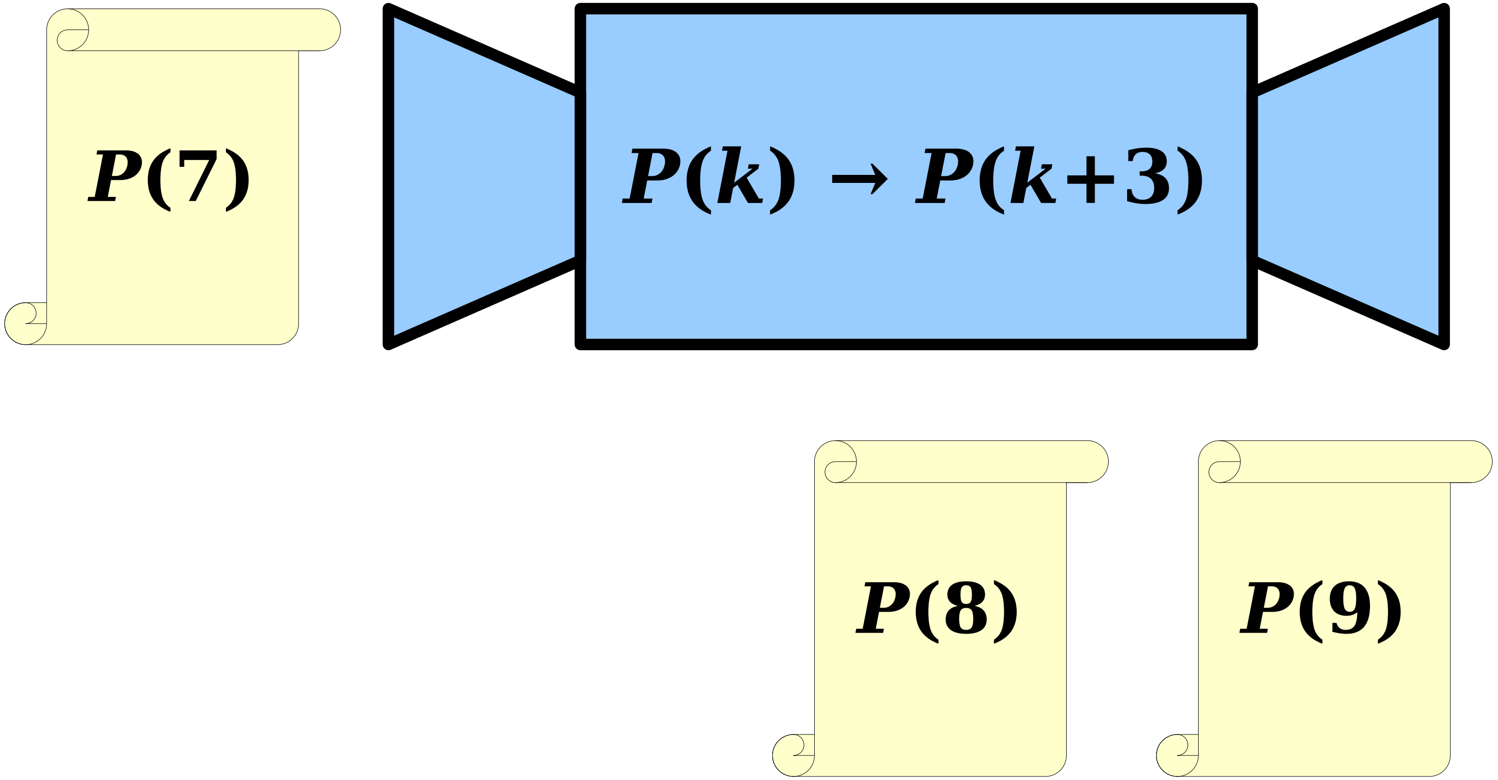

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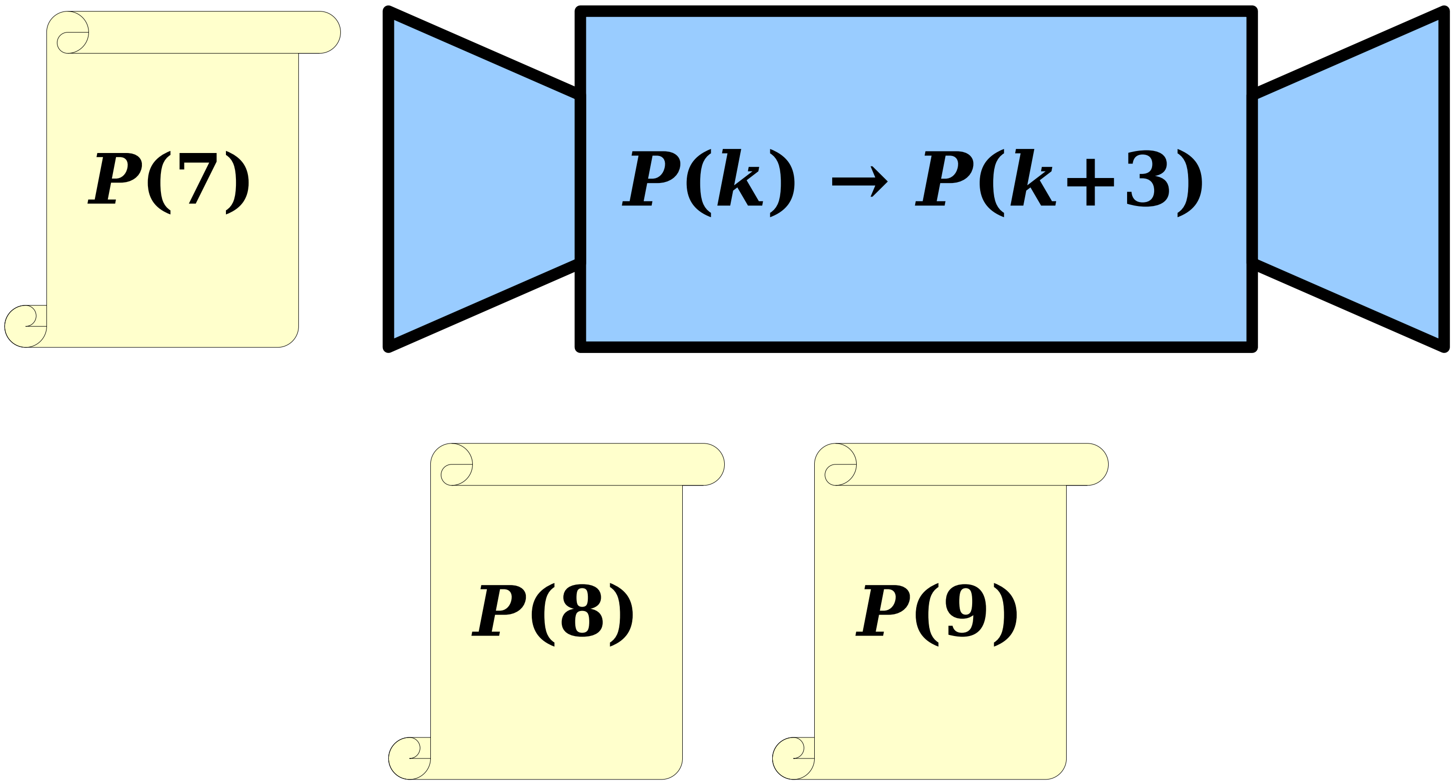

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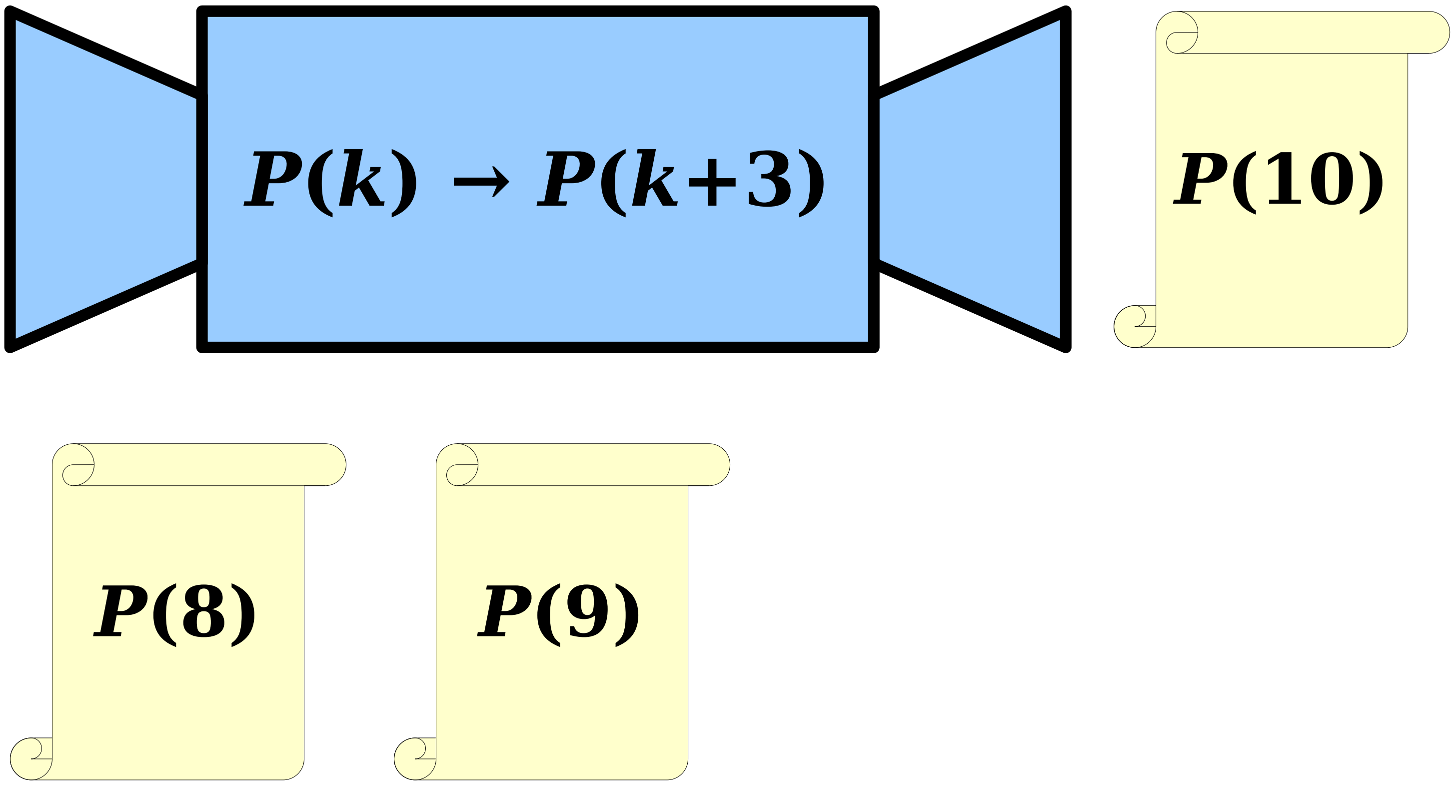

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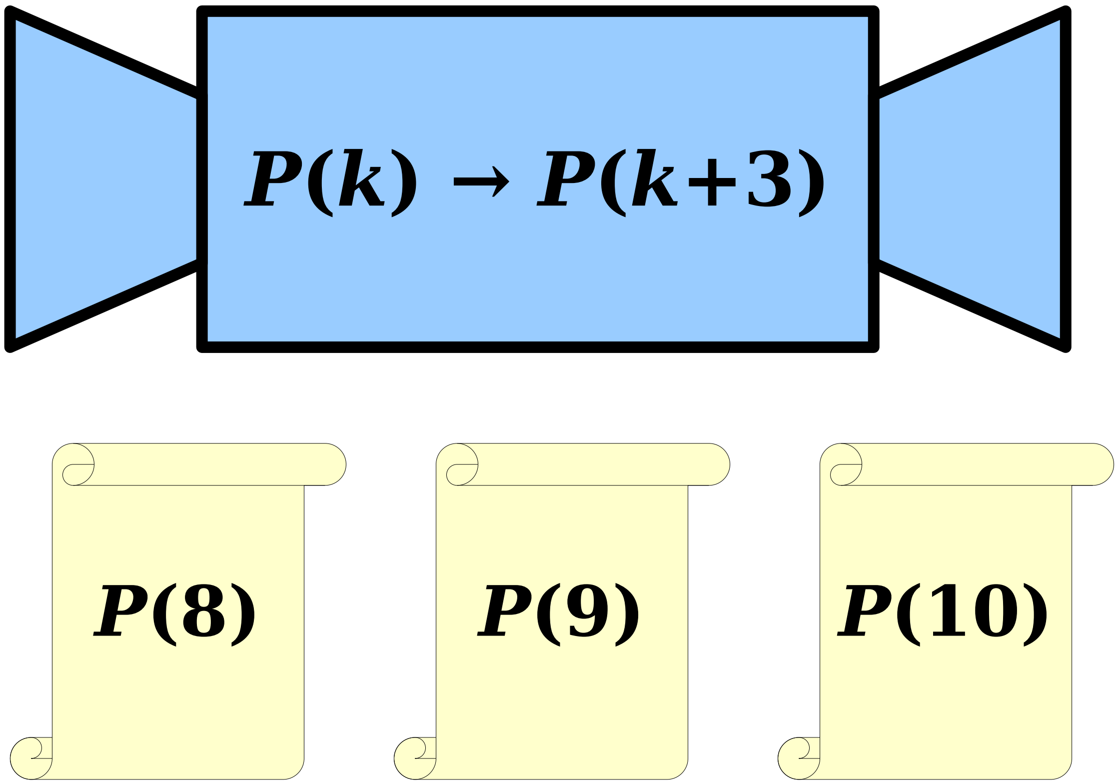

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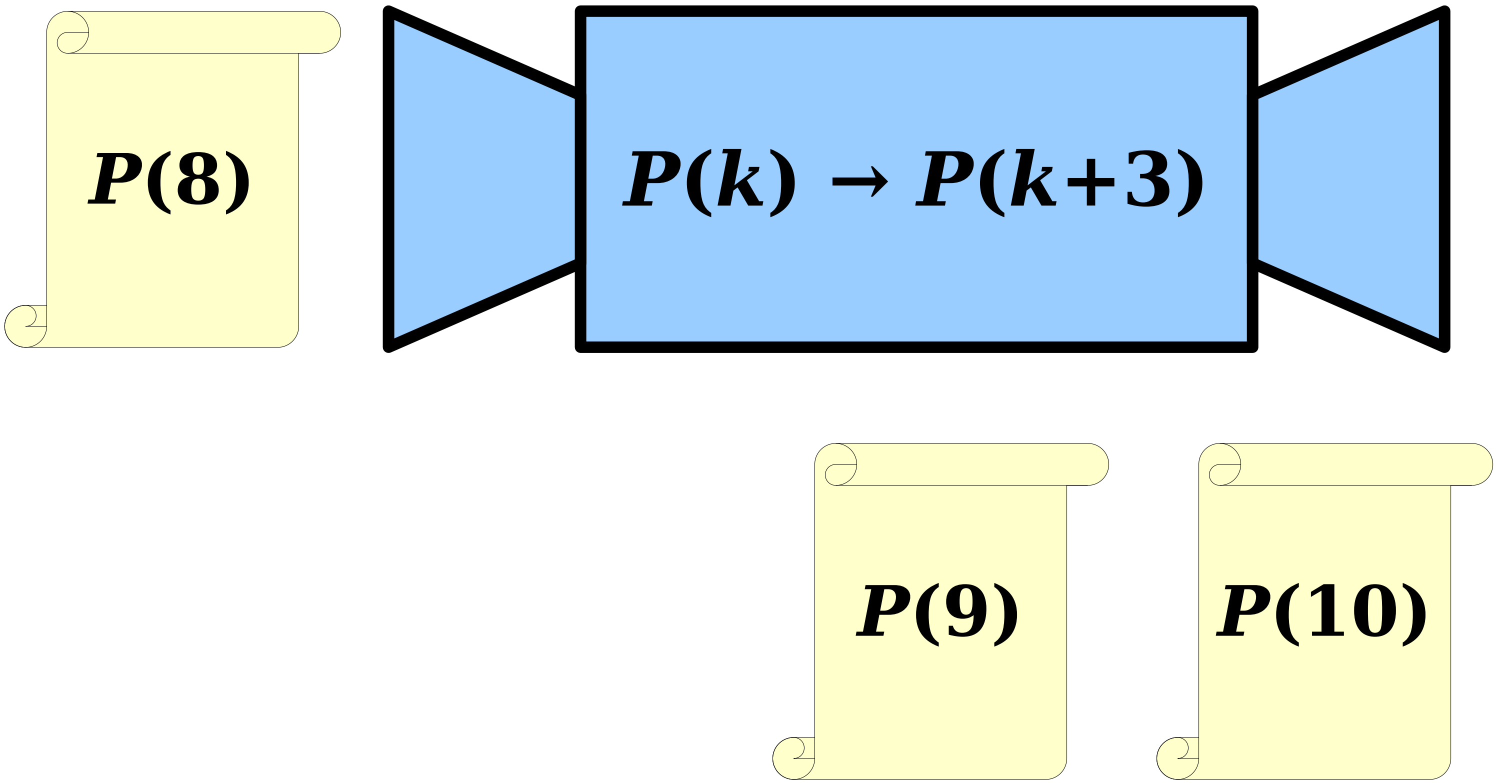

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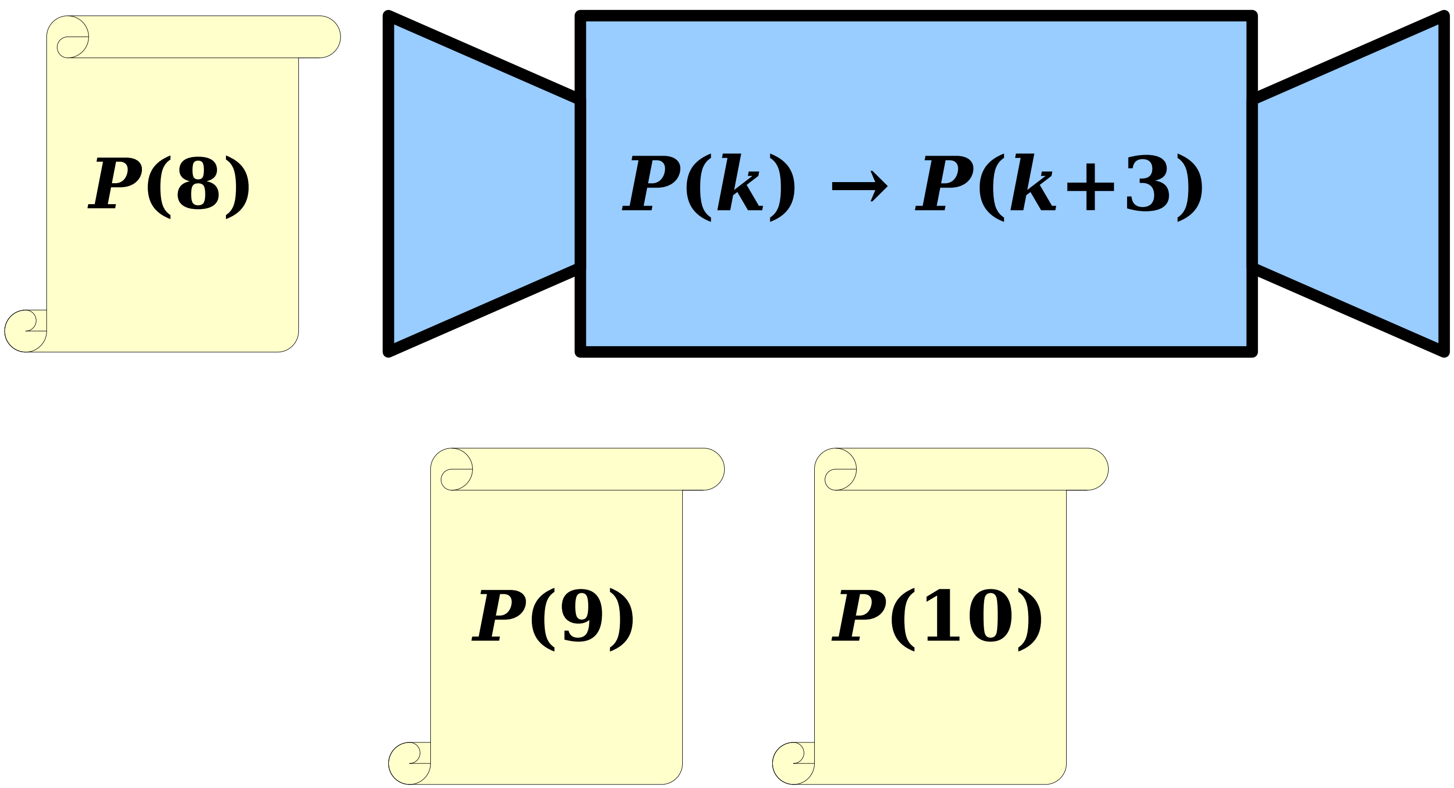

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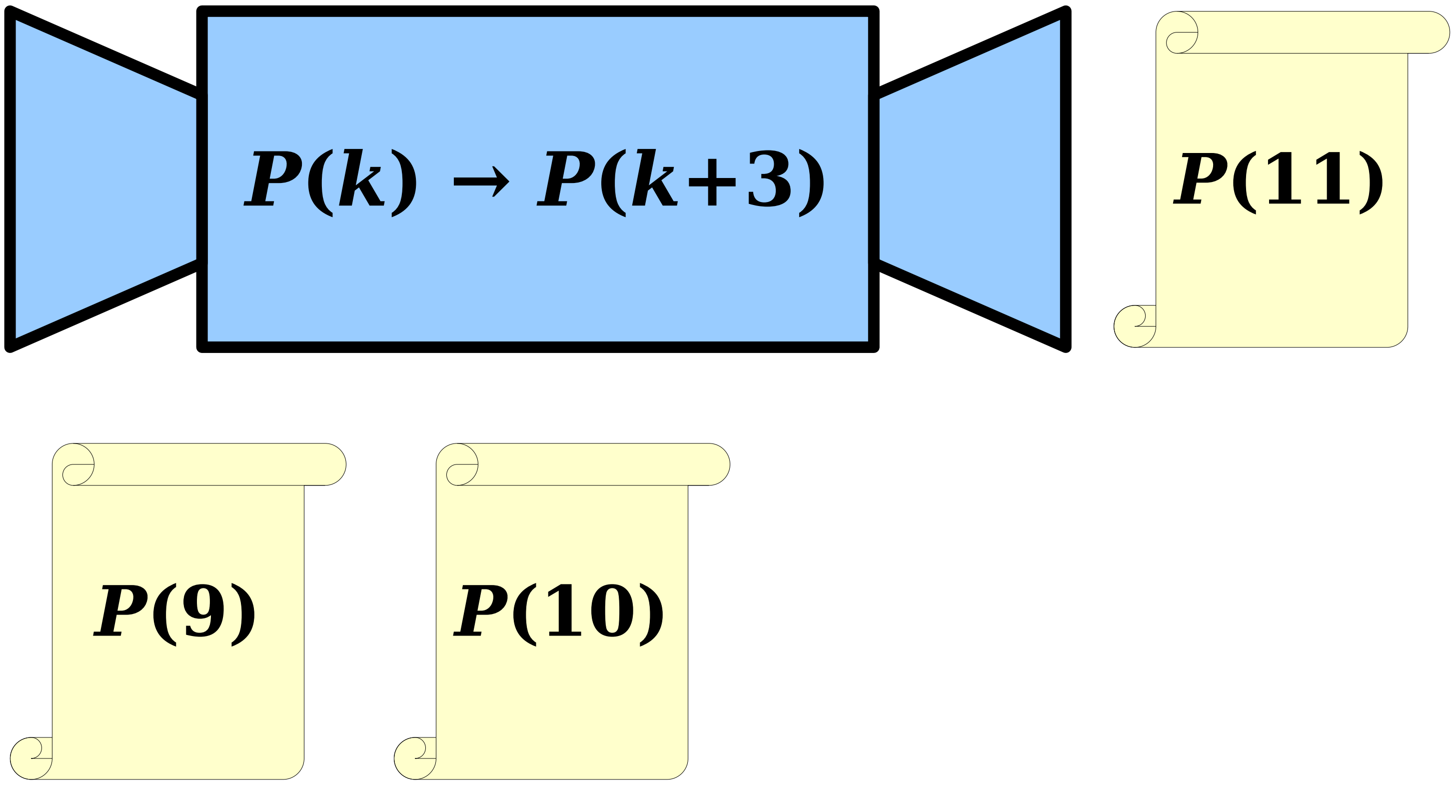

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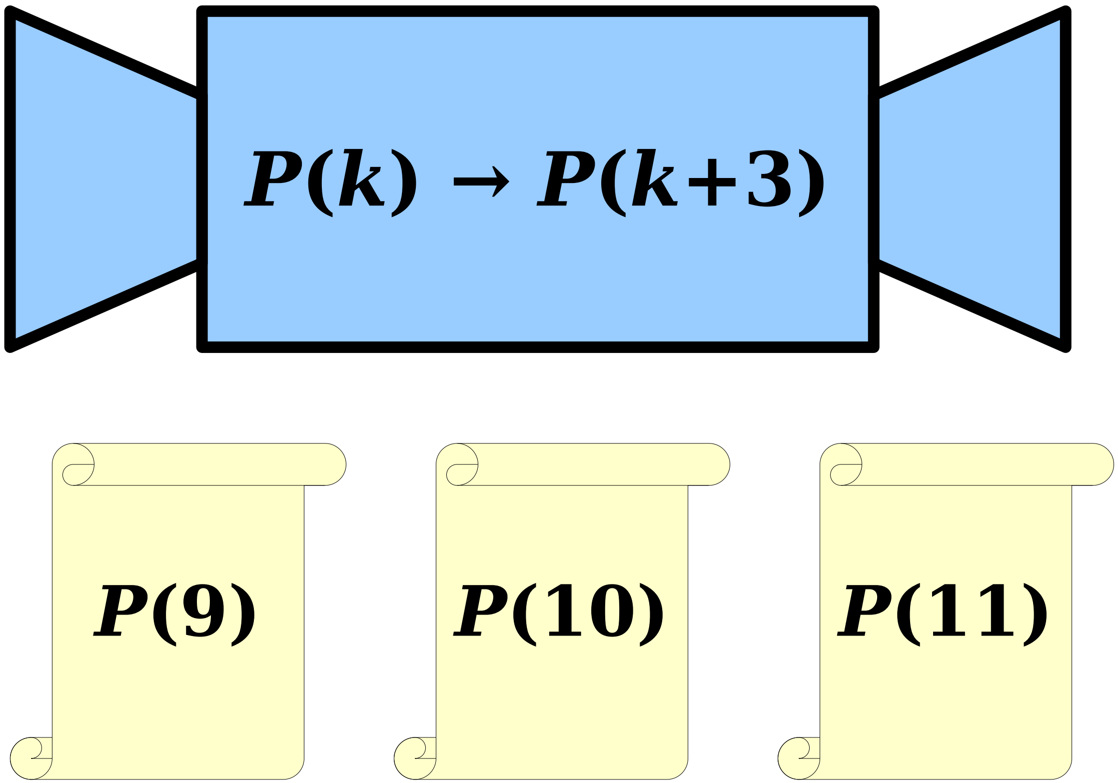

Generalizing Induction

When doing a proof by induction,

- feel free to use multiple base cases, and
- feel free to take steps of sizes other than one.
- Just be sure you actually need more than one base case!
- And be careful to make sure you cover all the numbers you think that you're covering!

We won't require that you prove you've covered everything, but it doesn't hurt to double-check!

More on Square Subdivisions

There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.

In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.

Good starting resource: this Numberphile video on ***Squaring the Square***.

How Not To Induct, Part 2

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

All Horses are the Same Color

$P(0)$ = "All groups of 0 horses always have the same color"

Vacuously true!

Base case: $n = 0$

All Horses are the Same Color

Assume $P(k)$ = "All groups of k horses always have the same color"

Inductive hypothesis: $n = k$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

Inductive hypothesis: $n = k+1$

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Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

By $P(k)$, these k horses have the same color

Inductive hypothesis: $n = k+1$

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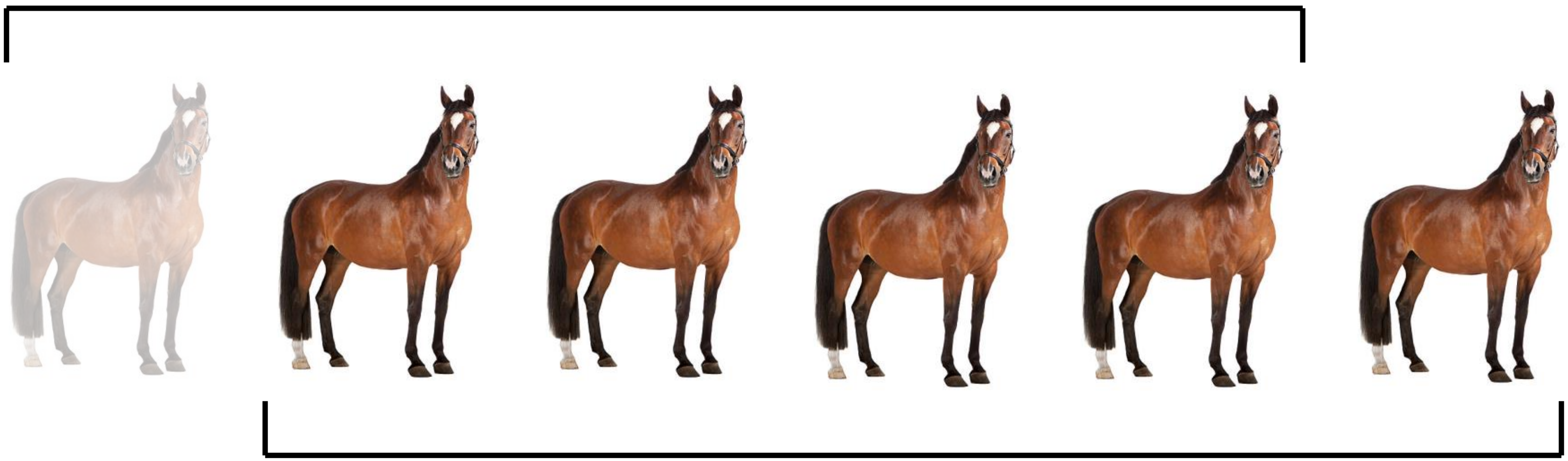

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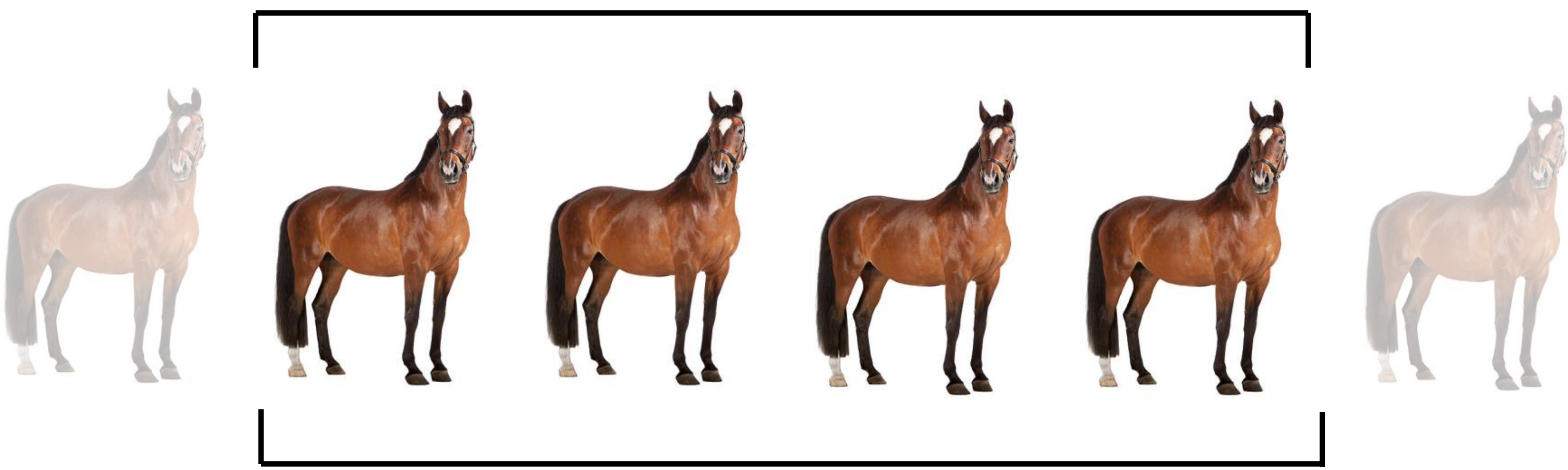
Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

These horses in the middle were in both sets



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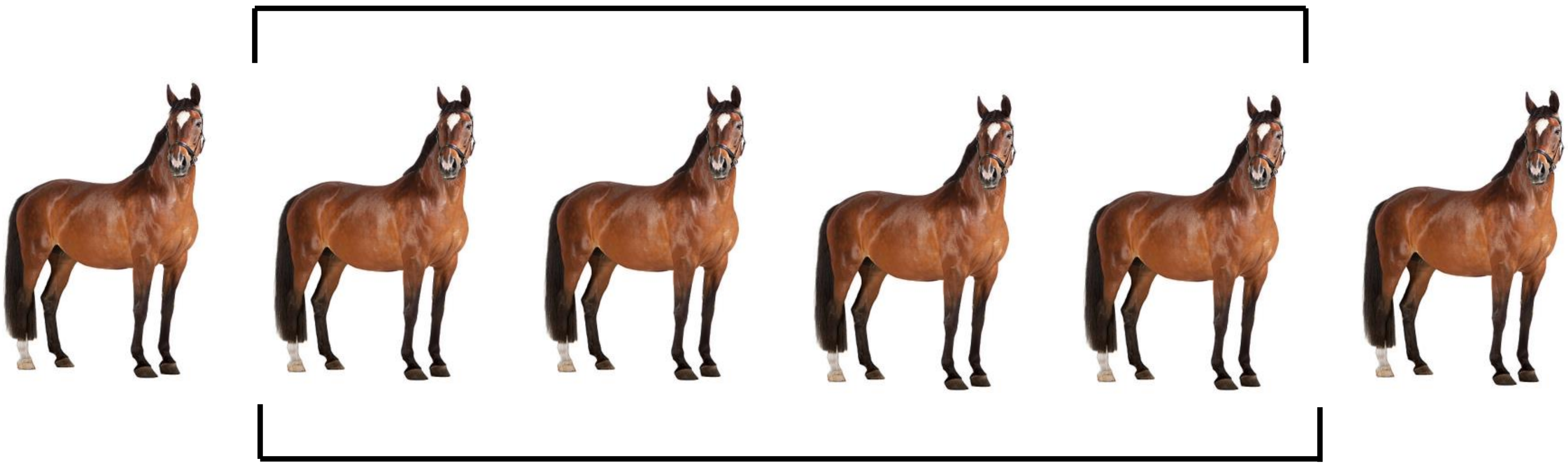

And we said that both horses on the ends are the same co

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

So all $k+1$ horses have the same color!

Inductive hypothesis: $n = k+1$

Theorem: All horses are the same color.

Proof: Let $P(n)$ be the statement "all groups of n horses are the same color." We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

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What's wrong with this proof?

Next Time

Variations on Induction

- Complete induction

Thought for the Weekend:

If you don’t know what the problem was, you haven’t fixed it.