Problem Set 3

This third problem set explores binary relations, functions, and their properties. We've chosen these problems to help you learn how to reason about these structures, how to write proofs using formal mathematical definitions, and why all this matters in practice.

Before beginning this problem set, we ***strongly*** recommend reading over the following handouts:

- Handout #11, the "Guide to Proofs on Discrete Structures," which explores how to write proofs when definitions are rigorously specified in first-order logic. This handout contains both general guiding principles to follow and some sample proof templates that you're welcome to use here.
- Handout #12, the "Discrete Structures Proofwriting Checklist," which contains some specific items to look for when proofreading your work. We will be applying the items on this checklist when grading your work, so it's worthwhile to apply this checklist to your work before submitting.

We recommend that you take a look at the proofs from this week's lectures to get a sense of what this looks like. The proofs on cyclic relations from Wednesday, or the proofs about injectivity and surjectivity from Friday, are great examples of the style we're looking for.

Good luck, and have fun!

Due Friday, October 9th at 12:00PM noon Pacific.

Problem One: The Gallery

This question is autograded. Download the starter files for Problem Set Three and extract them somewhere convenient on your computer. To answer this question, run the bundled program and choose the problem "The Gallery" from the top-level menu.

In this problem, you'll be presented with six different binary relations. Your task is to determine which of these relations are reflexive, symmetric, transitive, irreflexive, asymmetric, equivalence relations, and/or strict orders. Check the appropriate boxes for each relation at the bottom of the screen.

Our provided starter code will record your answers and store them in the file `res/TheGallery.answers`. ***Please don't modify this file manually – the program will do it for you.*** You can run local tests to check your work, and once you're done you can submit them online through GradeScope by uploading this file.

To help you check your answers, note that

- *three of these six relations are transitive,*
- *at least one relation is both symmetric and asymmetric,*
- *at least one relation is both reflexive and irreflexive, and*
- *at least one relation is asymmetric and irreflexive but not transitive.*

Problem Two: Redefining Equivalence Relations?

Below is a purported proof that every relation that is both symmetric and transitive is also reflexive.

> ***(Incorrect!) Theorem:*** If R is a symmetric and transitive binary relation over a set A, then R is also reflexive.
>
> ***(Incorrect!) Proof:*** Let R be an arbitrary binary relation over a set A such that R is both symmetric and transitive. We need to show that R is reflexive. To do so, consider an arbitrary $x, y \in A$ where xRy. Since R is symmetric and xRy, we know that yRx. Then, since R is transitive, from xRy and yRx we learn that xRx is true. Therefore, R is reflexive, as required. ■

This proof, unfortunately, is incorrect.

i. Run the starter files for PS3 and choose "Relation Editor." Using the editor, open the relation `res/RedefiningEquivalence.relation` and draw a picture the simplest binary relation that is symmetric and transitive but not reflexive. By "simplest," we mean "having the fewest elements in its underlying set out of all the binary relations with this property, and, of those, having the fewest arrows." ***Do not manually edit the file `res/RedefiningEquivalence.relation`***; the program will do that for you.

Your answer to part (i) shows that the proof has to be wrong. But why exactly is that?

ii. Look at the Guide to Proofs on Discrete Structures, and in particular the template proof that a binary relation is reflexive. How are you supposed to set up a proof that a binary relation is reflexive? What does the above proof do? Are these the same as one another?

We wanted you to work through this problem so that you'd see an important class of errors that can arise when writing proofs on discrete structures. It's important to ensure that your proof makes the proper assumptions at the beginning and arrives at the right goal. Otherwise, you risk writing a proof that "from the inside" looks perfectly consistent, but which is wrong "from the outside" because what it does prove doesn't match what it needs to prove.

Problem Three: Power Plays

This question explores a concrete binary relation R over the set $\mathbb{N}$. Specifically, this binary relation R is defined over $\mathbb{N}$ as follows:

$$mRn \qquad \text{if} \qquad \exists a \in \mathbb{Z}.\ m = 2^a \cdot n.$$

As is the convention (see the Guide to Proofs on Discrete Structures), the word "if" in the above statement means "is defined as" and is not an implication. That is, if mRn is true it means that there's an integer a where $m = 2^a \cdot n$, and conversely if there's an integer a where $m = 2^a \cdot n$ then mRn is true.

i. Find two different pairs of natural numbers m and n where mRn and $m = n + 3$. No justification is required.

This is a warm-up to help you get a better sense of what the relation R looks like in practice. Any time you discover a new binary relation, it's worth taking a minute to try some concrete examples. You'll get a much better feel for how the relation works if you do!

ii. Prove that R is an equivalence relation.

Read the Guide to Proofs on Discrete Structures and review the proofs we did in lecture on equivalence relations for an example of how to structure this proof.

iii. Since R is an equivalence relation, it has at least one system of representatives. Fill in the blank below to give a definition of one such system of representatives X. No justification is necessary.

$$X = \{\ k \mid k \in \mathbb{N} \text{ and } k \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } \}.$$

You're not expected to be able to "eyeball" this one just by looking at the definition of R given above. Instead, write out a bunch of natural numbers and see if you spot any patterns.

Remember that a system of representatives needs to have these properties:

- *Every element of ℕ needs to relate to **at least** one element of X.*
- *Every element of ℕ needs to relate to **at most** one element of X.*

So, as a way of checking your work, take the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 and, for each of them, confirm that it's related by R to at least one element of your set X and to at most one element of your set X. (Remember that we consider zero to be a natural number.)

Problem Four: Building Binary Relations

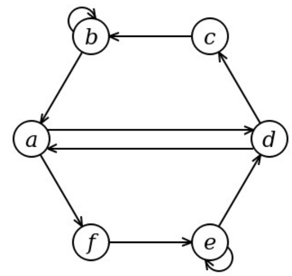
This question explores ways of taking existing binary relations and building new relations from them.

We'll begin by introducing an operation called ***squaring*** that turns one binary relation into another. The definition of squaring is given here: if R is a binary relation over a set A, then the ***square of R***, denoted $\boldsymbol{R^2}$, is a binary relation over A defined as follows:

$$xR^2y \quad \text{if} \quad \exists z \in A.\ (xRz \land zRy).$$

As a reminder, the word "if" here means "is defined as" and is not an implication.

To help you make sense of this definition, let's work through a concrete example.



i. To the right is a picture of a binary relation R over a set A. Using the relation editor from the starter files, open `res/RSquared.relation` and draw a picture of the binary relation R^2. ***Do not manually edit this file***; use the program you downloaded to do this.

Does cR^2a? If so, why? If not, why not? How about cR^2b? How about dR^2c?

Squares of relations play nicely with equivalence relations.

ii. Prove that if R is an equivalence relation over a set A, then so is R^2. To do so, fill in the blanks in the proof template given below.

> ***Theorem:*** If R is an equivalence relation over a set A, then R^2 is also an equivalence relation over A.
>
> ***Proof:*** Let R be an arbitrary equivalence relation over a set A. We will prove that R^2 is an equivalence relation by proving that it is _____, _____, and _____.
>
> To prove that R^2 is ______, consider an arbitrary $a \in A$. We will prove that ______. To do so, we need to show that there is a $b \in A$ where _____ and _____. Specifically, pick b = _____. Then ____________________________, as required.
>
> To prove that R^2 is _____, consider an arbitrary $a \in A$ and $b \in A$ where aR^2b. We will prove that _____. To do so, note that since aR^2b is true, we know there is a $c \in A$ where _____. Then, since ____ and R is symmetric, we know that _____. Similarly, since _____, we know that _____. Therefore, ____________________, as required.
>
> To prove that R^2 is _____, ____. ■

As before, the relative sizes of the blanks do not indicate how much you need to write in each section. Feel free to expand the final blank into as many sentences as are necessary.

Now, one last definition. If R is a binary relation over A, the ***undirected component of R***, denoted $\boldsymbol{E_R}$, is a binary relation over A defined as follows:

$$xE_Ry \quad \text{if} \quad xRy \land yRx.$$

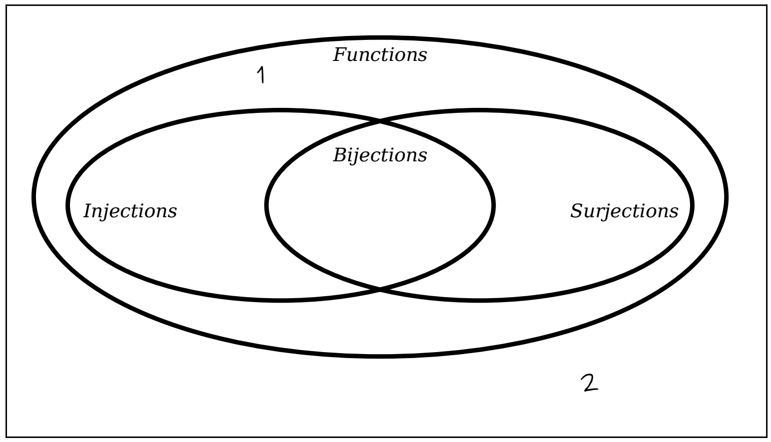
iii. Use the relation editor, open `res/ER.relation` and draw a picture of the relation E_R, where R is the binary relation given in the picture above.

iv. Prove that if R is a reflexive, transitive relation over a set A, then E_R is an equivalence relation.

E_R is defined in terms of R, but you can't assume E_R is reflexive and transitive just because R is. You need to prove that. Go slowly. To prove E_R is transitive, what would you assume, and what do you need to show?

Problem Five: Properties of Functions

Below is a list of purported functions. For each of those purported functions, determine where in Venn diagram that object goes. To get you started, we've shown you where functions 1 and 2 go.



To submit your answers, run the starter files for PS3 and choose the "Properties of Functions" option. This program will write your answers to `res/PropertiesOfFunctions.answers`. You can use the provided test cases to check your work, and when you're ready you can submit this file to Gradescope. ***Do not manually edit this file***; use the program instead.

1. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = 137$.
2. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = -137$.

Make sure you can explain why these first two items go where they do in this diagram!

3. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = \frac{n+|n|}{2}$.

The notation |n| means "the absolute value of n." For example, $|\pi| = |-\pi| = \pi$.

4. $f : \mathbb{Z} \to \mathbb{N}$ defined as $f(n) = \frac{n+|n|}{2}$.
5. $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = \frac{n+|n|}{2}$.
6. $f : \mathbb{R} \to \mathbb{N}$ defined as $f(n) = \frac{n+|n|}{2}$.
7. $f : \mathbb{N} \to \mathbb{R}$ defined as $f(n) = \frac{n+|n|}{2}$.
8. $f : \mathbb{R} \to \mathbb{R}$ defined as $f(n) = \sqrt{n}$.

The notation $\sqrt{n}$ denotes the ***principal square root*** *of n, the nonnegative one.*

9. $f : \mathbb{R} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
10. $f : \{ x \in \mathbb{R} \mid x \geq 0 \} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
11. $f : \{ x \in \mathbb{R} \mid x \geq 0 \} \to \mathbb{R}$ defined as $f(n) = \sqrt{n}$.
12. $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n^2 + 2 & \text{if } n < 137 \\ n^2 - 2 & \text{if } n > 137 \end{cases}$$

13. $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} 2 - n & \text{if } n \leq 2 \\ n - 2 & \text{if } n \geq 2 \end{cases}$$

14. $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n^2 - 3n + 2 & \text{if } n \leq 2 \\ n & \text{if } n \geq 2 \end{cases}$$

15. $f : \mathbb{N} \to \mathbb{Z}$ defined as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

Problem Six: Odd and Even Functions

Let's suppose that we have a function $f : \mathbb{R} \to \mathbb{R}$. We say that f is ***even*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = f(x).$$

i. Fill in the blank to give an example of an even function $p : \mathbb{R} \to \mathbb{R}$. No justification is necessary.

$$p(x) = \underline{\hspace{3em}}.$$

ii. Prove that if f is an even function, then f is ***not*** a bijection.

If f isn't a bijection, then either it isn't injective, or it isn't surjective, or both. So write out the negations of the statements "f is an injection" and "f is a surjection" in first-order logic and simplify them as much as possible. Then, see if you can prove that either of those statements are true.

When proving an existential statement of the form ∃x. [something], we prefer it if you give a concrete, specific choice of x (e.g. x = 137 or x = Ø) rather than giving a broad class of options that work for x. This makes things easier for your proof – you can just verify that your one specific choice of x works – and it makes it less likely that your proof has an error (if you say "pick any object meeting criteria A, B, and C, you have to then prove why there even are objects meeting criteria A, B, and C).

A function $f : \mathbb{R} \to \mathbb{R}$ is called ***odd*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = -f(x).$$

iii. Fill in the blank to give an example of an odd function $q : \mathbb{R} \to \mathbb{R}$. No justification is necessary.

$$q(x) = \underline{\hspace{3em}}.$$

iv. Prove that if f and g are odd functions, then $g \circ f$ is also odd.

Every integer is either even or odd (but not both). This is not the case with functions.

v. Prove that there is a function from $\mathbb{R}$ to $\mathbb{R}$ that is neither odd nor even.

We are asking for a formal proof here. Is this a universal or an existential statement? Check the Guide to Proofs on Discrete Structures for information about how to define functions inside of proofs.

vi. Prove that there is a function from $\mathbb{R}$ to $\mathbb{R}$ that is both even and odd.

Now, let's define what it means to "add" two functions. Given two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we can define the ***sum of f and g***, denoted $\boldsymbol{f + g}$, as a function $f + g : \mathbb{R} \to \mathbb{R}$ where

$$(f + g)(x) = f(x) + g(x).$$

Odd and even numbers behave in predictable ways when you add them. To what extent is that also true of odd and even functions?

vii. Prove or disprove: for any functions f and g, if f and g are even, then $f + g$ is even.

This is a prove-or-disprove problem, so your first task is to figure out whether this statement is true or false. Notice that the statement here is a universally-quantified statement ("for any functions f and g, …"). To prove this statement, you need to show the claim holds regardless of which f and g you pick.

This statement's negation is an existentially-quantified statement ("there exist functions f and g where …"). If you want to prove this statement is false, give concrete choices of f and g for which it's not true, then prove that your choices have all the right properties.

viii. Prove or disprove: for any functions f and g, if f and g are odd, then $f + g$ is even.

ix. Prove or disprove: for any functions f and g, if f is odd and g is even, then $f + g$ is odd.

Problem Seven: Left, Right, and True Inverses

In lecture, we briefly touched on the idea of inverse functions. It turns out that the notion of what an inverse function is is a bit more nuanced than it appears. Specifically, there are several different notions of what an inverse can be, each of which behaves in a slightly different way. This question explores three different notions of inverse functions, along with their properties.

Let $f : A \to B$ be a function. A function $g : B \to A$ is called a ***left inverse*** of f if the following is true:

$$\forall a \in A.\ g(f(a)) = a.$$

i. Find examples of a function f and two *different* functions g and h such that both g and h are left inverses of f. This shows that left inverses don't have to be unique. (Two functions g and h are different if there is some x where $g(x) \neq h(x)$.) Express your answer by drawing pictures along the lines of what we did in class: draw ovals representing the sets A and B, add dots to those ovals to denote their elements, then express f, g, and h by drawing arrows between those dots.

If you draw A and B as sets, then arrows from A to B represent applying the function f, and arrows from B back to A represent applying the function g. So look back at what you found when you expanded out the definition. Can you express that in terms of arrows going left and right between these sets?

ii. Prove that if f is a function that has a left inverse, then f is injective.

As a hint on this problem, look back at the proofs we did with injections in lecture. To prove that a function is an injection, what should you assume about that function, and what will you end up proving about it?

Let $f : A \to B$ be a function. A function $g : B \to A$ is called a ***right inverse*** of f if the following is true:

$$\forall b \in B.\ f(g(b)) = b.$$

iii. Find examples of a function f and two different functions g and h such that both g and h are right inverses of f. This shows that right inverses don't have to be unique. As in part (i), express your answer by drawing pictures of f, g, and h along the lines of what we did in lecture.

iv. Prove that if f is a function that has a right inverse, then f is surjective.

If $f : A \to B$ is a function, then a ***true inverse*** (often just called an ***inverse***) of f is a function g that's simultaneously a left and right inverse of f. In parts (i) and (iii) of this problem you saw that functions can have several different left inverses or right inverses. However, a function can only have a single true inverse.

v. Prove that if $f : A \to B$ is a function and both $g_1 : B \to A$ and $g_2 : B \to A$ are inverses of f, then $g_1(b) = g_2(b)$ for all $b \in B$.

vi. Explain why your proof from part (v) doesn't work if g_1 and g_2 are just *left* inverses of f, not full inverses. Be specific – you should point at a claim in your proof that is no longer true in this case.

vii. Explain why your proof from part (v) doesn't work if g_1 and g_2 are just *right* inverses of f, not full inverses. Be specific – you should point at a claim in your proof that is no longer true in this case.

Left and right inverses have some surprising applications. We'll see one of them next week!

Optional Fun Problem: Infinity Minus Two

Let $[0, 1]$ denote the set $\{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \}$ and $(0, 1)$ denote the set $\{ x \in \mathbb{R} \mid 0 < x < 1 \}$. That is, the set $[0, 1]$ is the set of all real numbers between 0 and 1, *inclusive*, and the set $(0, 1)$ is the set of all real numbers between 0 and 1, *exclusive*. These sets differ only in that the set $[0, 1]$ includes 0 and 1 and the set $(0, 1)$ excludes 0 and 1.

Give the definition of bijection $f : [0, 1] \to (0, 1)$ via an explicit rule (i.e. writing out $f(x)$ = __________ or defining f via a piecewise function), then prove that your function is a bijection.