Problem Set 4

This fourth problem set explores set cardinality and graph theory. It serves as tour of the infinite (through set theory) and the finite (through graphs and their properties) and will give you a better sense for how discrete mathematical structures connect across these domains. Plus, you'll get to see some pretty pictures and learn about why all this matters in the first place.

Some of the questions on this problem set will assume you've read the online Guide to Cantor's Theorem, which goes into more detail about the mechanics of the proof of Cantor's theorem as well as some auxiliary definitions.

Good luck, and have fun!

Due Friday, October 16th at 12:00PM noon Pacific

Problem One: Cartesian Products and Set Cardinalities

If A and B are sets, the ***Cartesian product*** of A and B, denoted $\boldsymbol{A \times B}$, is the set

$$\{ (x, y) \mid x \in A \land y \in B \}.$$

Intuitively, $A \times B$ is the set of all ordered pairs you can make by taking one element from A and one element from B, in that order. For example, the set $\{1, 2\} \times \{u, v, w\}$ is

$$\{ (1, u), (1, v), (1, w), (2, u), (2, v), (2, w) \}.$$

For the purposes of this problem, let's have ★ and ☺ denote two arbitrary objects where ★ ≠ ☺. Over the course of this problem, we're going to ask you to prove that $|\mathbb{N} \times \{★, ☺\}| = |\mathbb{N}|$.

i. Define a bijection $f : \mathbb{N} \times \{★, ☺\} \to \mathbb{N}$. The inputs to this function are elements of $\mathbb{N} \times \{★, ☺\}$, so you can define your function by writing

$$f(n, x) = \underline{\hspace{5cm}}$$

where $n \in \mathbb{N}$ and $x \in \{★, ☺\}$.

You might want to draw some pictures of the set $\mathbb{N} \times \{★, ☺\}$ so that you can get a better visual intuition.

As a hint, start off by drawing a picture showing a way to pair off the elements of $\mathbb{N} \times \{★, ☺\}$ with the elements of $\mathbb{N}$ so that no elements of either set are uncovered or paired with multiple elements. Then, find a way of encoding your drawing symbolically.

In defining this function, you cannot assume ★ or ☺ are numbers, since they're arbitrary values out of your control. See if you can find a way to define this function that doesn't treat ★ and ☺ algebraically. You may find it helpful to use piecewise functions.

ii. Prove that the function you came up with in part (ii) is a bijection.

The result you've proved here shows that $2\aleph_0 = \aleph_0$. Isn't infinity weird?

Problem Two: Understanding Diagonalization

Proofs by diagonalization are tricky and rely on nuanced arguments. In this problem, we'll ask you to review the formal proof of Cantor's theorem to help you better understand how it works.

(Please read the Guide to Cantor's Theorem before attempting this problem.)

i. Consider the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as $f(n) = \varnothing$. Trace through our formal proof of Cantor's theorem with this choice of f in mind. In the middle of the argument, the proof defines some set D in terms of f. Given that $f(n) = \varnothing$, what is that set D? Provide your answer without using set-builder notation. Make sure you see why $f(n) \neq D$ for any $n \in \mathbb{N}$, though you don't need to explain this in your answer.

Make sure you can determine what the set D is both by using the visual intuition behind Cantor's theorem and by symbolically manipulating the formal definition of D given in the proof.

ii. Let f be the function from part (i). Find a set $S \subseteq \mathbb{N}$ such that $S \neq D$, but $f(n) \neq S$ for any $n \in \mathbb{N}$. Justify your answer. This shows that while the diagonalization proof will always find *some* set D that isn't covered by f, it won't find *every* set with this property.

iii. Repeat part (i) of this problem using the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as

$$f(n) = \{ m \in \mathbb{N} \mid m \geq n \}.$$

iv. Repeat part (ii) of this problem using the function f from part (iii).

v. Give a function $f : \mathbb{N} \to \wp(\mathbb{N})$ such that the set D obtained from the proof of Cantor's theorem is the set $\{ n \in \mathbb{N} \mid n \text{ is even} \}$. Briefly justify your answer.

Problem Three: Simplifying Cantor's Theorem?

Below is a purported proof that $|S| \neq |\wp(S)|$ that doesn't use a diagonal argument:

> ***Theorem:*** If S is a set, then $|S| \neq |\wp(S)|$.
>
> ***(Incorrect!) Proof:*** Let S be any set and consider the function $f : S \to \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.
>
> Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ such that $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we see that $\{x_1\} = \{x_2\}$. Since two sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required.
>
> However, f is not surjective. Notice that $\varnothing \in \wp(S)$, since $\varnothing \subseteq S$ for any set S, but that there is no x such that $f(x) = \varnothing$; this is because $\varnothing$ contains no elements and $f(x)$ always contains one element. Since f is not surjective, it is not a bijection. Thus $|S| \neq |\wp(S)|$. ■

Unfortunately, this argument is incorrect. What's wrong with this proof? Justify your answer by pointing to a specific incorrect claim that's made here and explaining why it's incorrect.

*As a note, the issue here is **not** that this proof breaks down in the case where S is the empty set. You can define a function whose domain is the empty set via any rule you'd like, since that function can't be evaluated.*

Problem Four: Linkage Graphs

Graphs come in all shapes and sizes. This problem concerns a particular type of graphs called *linkage graphs* and some of their properties.

Let's begin with a definition. An undirected graph $G = (V, E)$ is called a ***linkage graph*** if it satisfies the following requirement:

For any nodes $u, v \in V$ where $u \neq v$, there is ***exactly one*** node $z \in V$ where $\{u, z\} \in E$ and $\{v, z\} \in E$.

As a reminder, undirected graphs cannot have edges from nodes back to themselves.

This definition is surprisingly compact, and yet it tells us a *lot* about the structure of linkage graphs. In the remainder of this problem, you'll explore some properties of linkage graphs and eventually get a feeling for what shapes they have.

i. Prove that if $G = (V, E)$ is a linkage graph, then G does not contain any simple cycles of length four.

This is a great spot to draw a bunch of pictures. See if you can intuit this one visually, then translate what you've drawn out into a formal written argument.

In lecture, we talked about triangles in graphs. As a refresher, a ***triangle*** in a graph is a simple cycle of length three.

ii. Let $G = (V, E)$ be a linkage graph with more than one node. Prove that for every node $v \in V$, there's a triangle that contains v.

Again, draw pictures.

iii. Download the starter files for Problem Set Four, extract them somewhere convenient, then run the provided starter program and choose the "Graph Editor" option. Use the provided program to draw a linkage graph with exactly seven nodes in `res/Linkage.graph`. You can check your answer using the "Run Tests" button. Submit this file online via Gradescope.

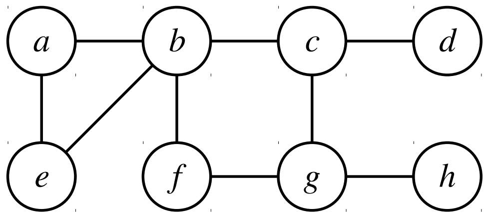
Problem Five: Independent and Dominating Sets

An ***independent set*** in a graph $G = (V, E)$ is a set $I \subseteq V$ with the following property:

$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

Let's begin with a quick warm-up about independent sets.

i. Consider the graph shown below. Give two different independent sets of this graph, each of which has cardinality three or greater. No justification is necessary.



Now, a new definition. A ***dominating set*** in G is a set $D \subseteq V$ with the following property:

$$\forall v \in V.\ (v \notin D \rightarrow \exists u \in D.\ \{u, v\} \in E)$$

As above, it's good to play around with this definition a bit before moving on.

ii. Give two different examples of dominating sets of the above graph, each of which has cardinality four or less. No justification is necessary.

iii. Let $G = (V, E)$ be a graph with the following property: every node in G is adjacent to at least one other node in G. Prove that if I is an independent set in G, then $V - I$ is a dominating set in G.

Notice that we're asking you to show that $V - I$ is a dominating set, not that I is a dominating set. Also, we recommend drawing some pictures here to get a sense of how this works. After all, you have a couple of examples of independent sets from part (i) of this problem!

Use the formal definitions to guide your proofs. If you proceed via a direct proof or via contrapositive, what, exactly, will you be assuming, and what will you be proving? If you write this as a proof by contradiction, what specifically is it that you're assuming for the sake of contradiction?

An independent set I in a graph G is a ***maximal independent set*** in G if there is no independent set I' in G where $I \subsetneq I'$. (Here, $I \subsetneq I'$ denotes that I is a strict subset of I').

iv. Find independent sets I and J of the graph from part (i) of this problem such that I is maximal but $|I| < |J|$. No justification is necessary.

Yes, this is possible. The definition of a maximal independent set is meant to be taken literally.

v. Prove that if I is a maximal independent set in $G = (V, E)$, then I is a dominating set of G.

You can build a great intuition for this result by drawing some pictures and thinking about what has to happen for a set of nodes to be an independent set and for a set of nodes to be a dominating set. When it comes time to write out your proof, however, you'll need to use the formal first-order definitions of independent sets, maximal independent sets, and dominating sets.

Problem Six: Highly Irregular Graphs

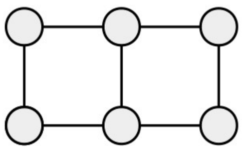
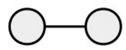
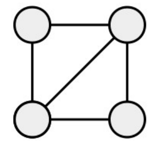
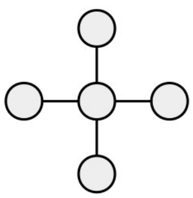
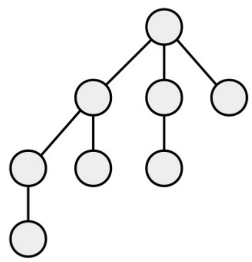
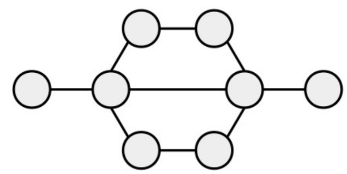
As a refresher, the ***degree*** of a node in a graph G, denoted $\boldsymbol{deg(v)}$, is the number of nodes that v is adjacent to. Equivalently, it's the number of edges touching v.

Now, a new definition. A graph $G = (V, E)$ is called ***highly irregular*** if this first-order formula is true:

$$\forall v \in V.\ \forall x \in V.\ \forall y \in V.\ (x \neq y\ \land\ \{v, x\} \in E\ \land\ \{v, y\} \in E\ \to\ deg(x) \neq deg(y)).$$

That definition might look like a mouthful, but it's actually not that bad once you get the hang of it.

i. Below is a collection of graphs. Which ones are highly irregular? No justification is necessary.

Graph 1	***Graph 2***	***Graph 3***
		
Graph 4	***Graph 5***	***Graph 6***
		

Since the definition of highly irregular graphs depends on the degrees of the nodes, it's probably not a bad idea to annotate each node with its degree.

If G is a graph with at least one node, then $\boldsymbol{\Delta(G)}$ denotes the maximum degree of any of the nodes in G. Although graphs in general can have exactly one degree of node $\Delta(G)$, highly irregular graphs cannot.

ii. Prove that if G is a highly irregular graph and $\Delta(G) \geq 2$, then G has at least two nodes of degree $\Delta(G)$.

As a hint, proceed by contradiction. Look back at the pictures above that you identified as highly irregular graphs – do you notice anything about where the nodes of degree $\Delta(G)$ are? See if you can use that to build out your proof.

As a refresher, a ***triangle*** in a graph is a simple cycle of length three.

iii. Run the provided starter program and choose the "Graph Editor" option. Use the provided program to draw a highly-irregular graph that contains a triangle in `res/HighlyIrregular.graph`.. For full credit, your solution should have ten nodes or fewer.

Something to think about: if G is highly irregular and G contains a triangle, what is the smallest possible value for $\Delta(G)$? Based on that and your observations from part (ii) of this problem, what does that tell you about the shape of the graph? Use that to guide your search.

Problem Seven: Friends, Strangers, Enemies, and Hats

In lecture, we proved the Theorem on Friends and Strangers, which says that if you have a group of six people where, for each pair of people, those people either know one another (they're friends) or they don't know each other (they're strangers), you can always find three mutual friends or three mutual strangers. Here, "three mutual friends" means "three people where each two of them are friends," and "three mutual strangers" means "three people where each two of them are strangers."

This is one of many different results about what must happen when you get a sufficiently large number of people together. The rest of this problem explores some other results in that vein.

i. There's a party with 36 attendees. Each person is wearing a hat, and there are seven possible hat colors: aureolin, bole, chartreuse, drab, ecru, fulvous, and gamboge. (Yes, those are all colors.) As in the Theorem on Friends and Strangers, for any pair of people at the party, either the pair are friends or the pair are strangers.

   Prove that you can always find three mutual friends all wearing the same color hat or three mutual strangers all wearing the same color hat.

ii. There's a party with 17 attendees. This time, things are a bit more complicated. For each pair of people at the party, either those people are strangers, those people are friends, or those people are enemies. Fortunately, none of them are wearing hats.

   Prove that you can always find three mutual friends, or three mutual strangers, or three mutual enemies.

Find problems like these interesting? Take Math 107 (Graph Theory) or Math 108 (Combinatorics)!

Optional Fun Problem: How Many Functions Are There?

If A and B are sets, we can define the set $\boldsymbol{B^A}$ to be the set of all functions from A to B. Formally speaking:

$$B^A = \{\ f \ \mid\ f : A \to B\ \}$$

Prove that $|\mathbb{N}| < |\mathbb{N}^{\mathbb{N}}|$. This shows that $\aleph_0 < {\aleph_0}^{\aleph_0}$, even though as you proved in Problem One, $\aleph_0 = 2{\cdot}\aleph_0$. Isn't infinity weird?