Problem Set 5

This problem set – the last one purely on discrete mathematics – is designed as a cumulative review of the topics we've covered so far and a proving ground to try out your newfound skills with mathematical induction. The problems here span all sorts of topics – higher dimensions, tiling problems, and games – and we hope that it serves as a fitting coda to our whirlwind tour of discrete math!

We recommend that you ***read Handout #16, "Guide to Induction," before starting this problem set***. It contains a lot of useful advice about how to approach problems inductively, how to structure inductive proofs, and how to not fall into common inductive traps. Additionally, before submitting, be sure to ***read over Handout #17, the "Induction Proofwriting Checklist,"*** for a list of specific things to watch for in your solutions before submitting.

As a note on this problem set – normally, you're welcome to use any proof technique you'd like to prove results in this course. On this problem set, we've specifically asked on some problems that you prove a result inductively. For those problems, you should prove those results using induction or complete induction, even if there is another way to prove the result. (If you'd like to use induction in conjunction with other techniques like proof by contradiction or proof by contrapositive, that's perfectly fine.)

Good luck, and have fun!

Due Friday, October 23rd at 12:00PM noon Pacific

Problem One: Induction Proof Critiques

Below are a collection of proofs by induction. For each proof, critique its style in regards to the Induction Proofwriting Checklist and Guide to Induction. Then critique its correctness and identify any logic errors. You do not need to rewrite these proofs – just give us your feedback on the items we mentioned.

i. Critique the following proof about sums of natural numbers.

> ***Theorem:*** The sum of the first n natural numbers is $n(n - 1) / 2$.
>
> ***Proof:*** Let $P(n) = n(n - 1) / 2$. We will prove by induction on n that $P(n)$ holds for all $n \in \mathbb{N}$, from which theorem follows.
>
> As our base cases, we prove $P(0)$ and $P(1)$. First we'll prove $P(0)$, that the sum of the first zero natural numbers is $0(0 - 1) / 2$. The sum of no numbers is the empty sum (0), and we see that $0(0 - 1) / 2 = 0$, so $P(0)$ is true. Next, we'll prove $P(1)$. The sum of the first natural number is 0 (since 0 is the smallest natural number), and we note that $1(1 - 1) / 2 = 0$, so $P(1)$ is true.
>
> For our inductive step, assume for all natural numbers k that $P(k)$ is true. This means
>
> $$0 + 1 + 2 + \ldots + (k - 1) = k(k - 1) / 2.$$
>
> We will prove $P(k+1)$, that the sum of the first $k+1$ natural numbers is equal to $(k + 1)k / 2$. Starting with the sum of the first $k+1$ natural numbers, we see that
>
> $$\begin{aligned} 0 + 1 + 2 + \ldots + (k - 1) + k &= (k+1)k / 2 \\ 0 + 1 + 2 + \ldots + (k - 1) &= (k + 1)k / 2 - k \\ 0 + 1 + 2 + \ldots + (k - 1) &= (k + 1)k / 2 - 2k / 2 \\ 0 + 1 + 2 + \ldots + (k - 1) &= (k + 1 - 2)k / 2 \\ 0 + 1 + 2 + \ldots + (k - 1) &= k(k - 1) / 2. \end{aligned}$$
>
> This last equation is equation (1), which we know to be true. Thus $P(k+1)$ holds, completing the induction. ■

ii. Critique the following proof about directed graphs.

> ***Theorem:*** No directed graphs have any cycles.
>
> ***Proof:*** Let $P(n)$ be the statement "for any $n \in \mathbb{N}$, no directed graph with n nodes contains a cycle." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.
>
> As a base case, we prove $P(0)$, that no directed graphs with 0 nodes contain a cycle. To see this, consider any directed graph G with no nodes. Since G has no nodes, it has no paths, since all paths contain at least one node. Therefore, G has no cycles, as required.
>
> For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true and no directed graph with k nodes has any cycles. We will prove that $P(k+1)$ is true, namely, that no directed graphs with $k+1$ nodes have any cycles.
>
> Consider a directed graph G with k nodes. By our inductive hypothesis, we know that G has no cycles. Now, consider the graph G' formed by adding a new node v to G, then adding edges from v to each node in G. We claim that this directed graph has no cycles. To see this, note that any cycle in G' must involve v, since G has no cycles on its own. But this is impossible, since any path that leaves v can never return to it. Therefore, G' has no cycles, so $P(k+1)$ holds, completing the induction. ■

Problem Two: Recurrence Relations

A ***recurrence relation*** is a way of defining an infinitely long sequence (usually, a sequence of numbers). A recurrence relation specifies the value of the first term or terms of the sequence, then defines the remaining entries from the previous terms. For example, here's a simple recurrence relation:

$$a_0 = 1 \qquad\qquad a_{n+1} = 2a_n$$

The first terms of this sequence are given as follows:

- $a_0 = 1$, since that's what the first rule says.
- $a_1 = 2$, since the second rule says that $a_1 = 2a_0 = 2 \cdot 1 = 2$.
- $a_2 = 4$, since the second rule says that $a_2 = 2a_1 = 2 \cdot 2 = 4$.
- $a_3 = 8$, since the second rule says that $a_3 = 2a_2 = 2 \cdot 4 = 8$.

Extending further, this sequence starts off with the numbers

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \ldots,$$

which all happen to be powers of two. It turns out that this isn't a coincidence – this recurrence relation perfectly describes the powers of two.

i. Prove by induction that for any $n \in \mathbb{N}$, we have $a_n = 2^n$.

In case you're wondering what you're asked to prove here, you can think of this recurrence relation as a mathematical way of writing out this recursive function:

```
int a(int n) {
    if (n == 0) return 1;
    return 2 * a(n - 1);
}
```

For any $n \in \mathbb{N}$, you can compute `a(n)` *by just running this code, and after doing some computation it will return the value of a_n. What we're asking you to do is the mathematical equivalent of showing that the value returned by* `a(n)` *is always 2^n. While it might help to think about things in terms of this analogy, your proof should not reference this code and should just use the definitions given in the problem statement.*

Perhaps the most famous recurrence relation is the ***Fibonacci sequence***, which is defined as follows:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

The first terms of this sequence are given as follows:

- $F_0 = 0$, since that's what the first rule says.
- $F_1 = 1$, since that's what the second rule says.
- $F_2 = 1$, since the third rule says that $F_2 = F_0 + F_1 = 0 + 1 = 1$.
- $F_3 = 2$, since the third rule says that $F_3 = F_1 + F_2 = 1 + 1 = 2$.
- $F_4 = 3$, since the third rule says that $F_4 = F_2 + F_3 = 1 + 2 = 3$.

The first ten terms of the Fibonacci sequence are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34. *(Make sure you see why!)*

If you pull out a calculator and compute ratios of consecutive Fibonacci numbers, you'll find that the ratio tends toward 1.6180339… . This number is the ***golden ratio***, denoted **φ** (the Greek letter phi). Its exact value is $\varphi = \frac{1+\sqrt{5}}{2}$, and φ has nice property that $\varphi^2 = 1 + \varphi$.

ii. Prove, by induction, that $\varphi^{n+1} = \varphi \cdot F_{n+1} + F_n$ for all natural numbers n.

While you can solve this problem by substituting $\varphi = \frac{1+\sqrt{5}}{2}$ and doing a bunch of algebra, you might find it more useful to instead just use the fact that $\varphi^2 = 1 + \varphi$.

Want to learn more about recurrences and Fibonacci numbers? Take Math 108!

Problem Three: The Four Square Theorem

In this problem, we're going to ask you to prove that for every $n \in \mathbb{N}$, there are natural numbers a, b, c, and d such that

$$103^n = a^2 + b^2 + c^2 + d^2.$$

To do so, we'd like you to use this predicate $P(n)$:

> $P(n)$ is the statement "there are natural numbers a, b, c, and d such that $103^n = a^2 + b^2 + c^2 + d^2$."

Before proceeding, read the Guide to Induction and Induction Proofwriting Checklist on Canvas.

i. Is $P(n)$ a universally-quantified statement or an existentially-quantified statement? Based on that, will you "induct up," or will you "induct down?"

ii. Prove by induction that $P(n)$ is true for all natural numbers n.

As a hint, use multiple base cases and take steps of size two.

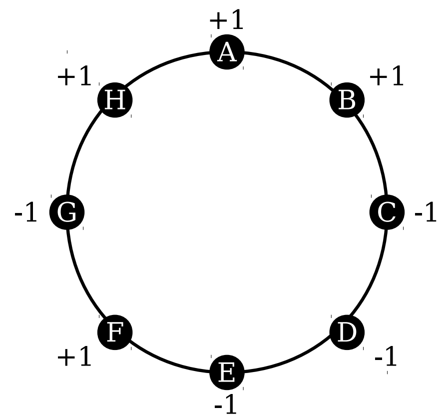
Fun fact: there's a theorem called the ***four square theorem*** that says that *every* natural number can be written as the sum of four squares. Isn't that surprising? If you're curious to learn more about why this is, take Math 152!

Problem Four: The Circle Game

You have a circle with $2n$ arbitrarily-chosen points on its circumference for some natural number $n \geq 1$. Of the $2n$ points, n are labeled +1, and the remaining n are labeled -1. One sample circle with eight points, of which four are labeled +1 and four are labeled -1, is shown below.

Here's a game you can play. Pick one of the $2n$ points as your starting point, then move clockwise around the circle. You lose the game if at any point on you pass through more -1 points than +1 points. You win the game if you get all the way back to your starting point without losing. For example, if you start at point A, the game would go like this:

Start at A: +1.
Pass through B: +2.
Pass through C: +1.
Pass through D: 0.
Pass through E: -1. *(You lose.)*



If you started at point G, the game would go like this:

Start at G: -1 *(You lose.)*

However, if you started at point F, the game would go like this:

Start at F: +1.
Pass through G: 0.
Pass through H: +1.
Pass through A: +2.
Pass through B: +3.
Pass through C: +2.
Pass through D: +1.
Pass through E: +0.
Return to F. *(You win!)*

There's a remarkable theorem about this game:

Theorem: For any $n \geq 1$, if n points labeled +1 are placed on the boundary of the circle and n points labeled -1 are placed on the boundary of the circle, there's always some point from which you can start and win the circle game.

We're going to ask you to prove this using a proof by induction with this predicate $P(n)$:

$P(n)$ is the predicate "for any circle with n points labeled +1 and n points labeled -1 on its boundary, there is a starting point from which you can win the circle game."

Before proceeding, let's confirm you see the general structure of the proof.

i. Is $P(n)$ a universally-quantified statement or an existentially-quantified statement? Based on that, will you "induct up," or will you "induct down?"

There are multiple quantifiers in $P(n)$. We care about the first one.

ii. Prove by induction that $P(n)$ is true for all natural numbers $n \geq 1$.

Problem Five: It'll All Even Out

Our very first proof by induction was the proof that for any natural number n, we have that

$$2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1.$$

This result is still true for the case where $n = 0$, since in that case the sum on the left-hand side of the equation is the *empty sum* of zero numbers, which is by definition equal to zero. It's also true for the case where $n = 1$; in that case, the sum on the left-hand side of the equality just has a single term in it (2^0) and the right-hand side has the same value.

Below is a proof by complete induction of an incorrect statement about what happens when you sum up zero or more real numbers:

> ***(Incorrect!) Theorem:*** The sum of any number of real numbers is even.
>
> ***(Incorrect!) Proof:*** Let $P(n)$ be the statement "the sum of any collection of n real numbers is even." We will prove by complete induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.
>
> As a base case, we prove $P(0)$, that the sum of any collection of 0 real numbers is even. The sum of any zero numbers is the empty sum and is by definition equal to 0, which is even. Thus $P(0)$ holds.
>
> For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(0)$, …, and $P(k)$ are true. We will prove that $P(k+1)$ is true, meaning that the sum of any collection of $k+1$ real numbers is even. To do so, let x_1, x_2, …, x_k, and x_{k+1} be arbitrary real numbers and consider the sum
>
> $$x_1 + x_2 + \ldots + x_k + x_{k+1}.$$
>
> We can group the first k terms and the last term independently to see that
>
> $$x_1 + x_2 + \ldots + x_k + x_{k+1} = (x_1 + x_2 + \ldots + x_k) + (x_{k+1}).$$
>
> Now, consider the sum $x_1 + x_2 + \ldots + x_k$ of the first k terms. This is the sum of k real numbers, so by our inductive hypothesis that $P(k)$ is true we know that this sum must be even. Similarly, consider the sum x_{k+1} consisting of just the single term x_{k+1}. By our inductive hypothesis that $P(1)$ is true, we know that this sum must be even.
>
> Overall, we have shown that $x_1 + x_2 + \ldots + x_k + x_{k+1}$ can be written as the sum of two even numbers (namely, $x_1 + x_2 + \ldots + x_k$ and x_{k+1}), so $x_1 + x_2 + \ldots + x_k + x_{k+1}$ is even. Thus $P(k+1)$ is true, completing the induction. ■

Of course, this theorem isn't true, so the proof must be wrong. But what specifically did it do wrong?

i. The proof defines a predicate $P(n)$, then uses complete induction to prove $P(n)$ holds for all $n \in \mathbb{N}$. Is $P(n)$ actually a predicate? Does it pass the Induction Proofwriting Checklist? Is it actually the case that, if $P(n)$ is true for all $n \in \mathbb{N}$, then the theorem in question is true? If any of your answers are "no," explain why, pointing out, specifically, what the proof does wrong.

ii. Is $P(0)$ true? Is the base case of this proof written correctly? If not, point out a specific claim it makes that's incorrect and explain why it's incorrect.

We aren't looking for "sins of omission" here in the sense of "the proof should have also done X in addition to what it already did." Rather, we're looking for "sins of commission" in sense of "the proof does X, and X is incorrect."

iii. Is $P(1)$ true? Is the inductive step of this proof written correctly? If not, point out a specific claim it makes that's incorrect and explain why it's incorrect.

Problem Six: For All The Marbles

Consider the following game. There are two bags of marbles shared between two players. The players alternate taking turns removing any nonzero number of marbles from any single bag of their choice. If at the start of a player's turn both bags are empty, that player loses the game.

Prove by induction that if the two bags start with the same number of marbles in them, then the second player can always win the game if she plays correctly.

Play this game with a partner until you can find a winning strategy. Once you spot the pattern, see if you can find a way to formalize it using induction. Be wary of writing statements of the form "and so on" or "by repeating this;" induction is the proper way to formalize those sorts of ideas.

Something to think about – you know that the number of marbles in each bag will be decreasing. Can you say how much that number will decrease by? Based on that, what style of proof should you use here?

Induction is often used to study games and find winning strategies. Do a quick Google search for "solved games" and see what pops up – you'd be amazed what we know and how we know it!

Problem Seven: Factorials! Together!

If k is a natural number, the number ***k factorial***, denoted ***k*!**, is defined as follows:

$$0! = 1 \qquad\qquad (k+1)! = (k+1) \cdot k!$$

For reference, the first few factorials are

$$0! = 1 \quad 1! = 1 \quad 2! = 2 \quad 3! = 6 \quad 4! = 24 \quad 5! = 120 \quad 6! = 720 \quad 7! = 5040$$

$59! \approx$ the number of protons in the observable universe

Intuitively speaking, $k!$ is the product of all the natural numbers from 1 to k inclusive. While that's a great intuition to have, if you're working with factorials in the context of a proof, you'll want to use the definition given above.

In this problem, we're going to ask you to prove the following theorem:

Theorem: For any natural numbers m and n, we have $m! \cdot n! \le (m + n)!$.

Since $m!$ and $n!$ are defined inductively, we'll need to use a proof by induction to establish this result. What makes this problem interesting is that, in this case, there are two independent natural numbers m and n whose factorials are being computed, but induction only works on a single variable. But not to worry – turns out there's a nice technique for making this proof work. In particular, consider this predicate $P(m)$:

$P(m)$ is the statement "for all $n \in \mathbb{N}$, we have $m! \cdot n! \le (m + n)!$."

Before proceeding, take a minute to make sure you see why this works.

i. Briefly explain why proving $P(m)$ is true for all $m \in \mathbb{N}$ proves the theorem.

ii. Prove the theorem using a proof by induction and the predicate $P(m)$ given above.

As a hint, you should only need to use induction on m. You can work with n simply by picking it arbitrarily.

Factorials show up all the time in computer science. In CS109 they're used in formulas to count the number of objects of various types. In CS161 they're used to prove limits on the speed with which computers can sort lists of values.

Problem Eight: Dedekind-Infinite Sets

Suppose you have a function $f : A \to A$ from some set A to itself that is injective but ***not*** surjective. Knowing nothing more than this, you can conclude that A has to be infinite. This question explores why.

i. Since $f : A \to A$ is not a surjection, the following first-order logic statement about f is ***not*** true:

$$\forall x \in A.\ \exists y \in A.\ f(y) = x$$

Since the above formula is ***not*** true, its negation must be true. Negate the above first-order logic statement and simplify it as much as possible.

The negation that you came up with in part (i) of this problem tells us that there's an element $x \in A$ with certain properties. We can use this element x to define the following recurrence relation:

$$\begin{aligned} e_0 &= x\text{, where } x \text{ is the element of } A \text{ singled out above.} \\ e_{n+1} &= f(e_n) \end{aligned}$$

This recurrence relation defines a series of elements e_0, e_1, e_2, e_3, … that continues outward to infinity. Amazingly, each element in the sequence is different from the rest. The rest of this question explores why.

ii. Prove that $e_1 \neq e_0$, that $e_2 \neq e_0$, and that $e_2 \neq e_1$.

To prove that no two terms in this sequence are equal, we're going to ask you to prove this theorem:

Theorem: For any natural numbers m and n where $m < n$, we have $e_m \neq e_n$.

If this theorem is true, it means, for example, that $e_{137} \neq e_{42}$, since we can plug in $m = 42$ and $n = 137$. Similarly, we know that $e_{103} \neq e_{166}$, which would follow from plugging in $m = 103$ and $n = 166$.

iii. Prove, by induction on n, that this predicate $P(n)$ is true for all $n \in \mathbb{N}$:

$P(n)$ is the statement "for any $m \in \mathbb{N}$ where $m < n$, we have $e_n \neq e_m$."

Try generalizing your answers to part (ii) of this problem.

There are a lot of variables to keep track of, so be careful to scope and introduce them properly. The inductive step of this problem, in particular, would be a ***great*** *place to write out two columns, one of the things you're assuming, and one of the things that you're proving. The most common class of mistake we tend to see on this problem is mixing up arbitrarily-chosen values with placeholders. One specific thing to keep an eye on:* ***make specific claims about specific variables****, and, specifically,* ***be very careful to make sure you aren't using placeholder variables****.*

Is this a problem where you'll induct up? Or induct down?

You've just shown that there must be infinitely many elements in A, since the sequence e_0, e_1, e_2, … stretches on forever. And all that follows just from the fact that f is injective but not surjective!

A set A with a function $f : A \to A$ that is injective but not surjective is called ***Dedekind-infinite***. In the early days of set theory, the question arose of how to define what "infinite" meant without referring to natural numbers, and Richard Dedekind proposed this definition, hence the name. Later, it was discovered that this question was far more nuanced than anyone had expected. Want to hear more? Take Math 161!

Optional Fun Problem: Egyptian Fractions

The Fibonacci sequence mentioned in Problem Two is named after Leonardo Fibonacci, an eleventh-century Italian mathematician who is credited with introducing Hindu-Arabic numerals (the number system we use today) to Europe in his book *Liber Abaci*. This book also contained an early description of the Fibonacci sequence, from which the sequence takes its name.

Liber Abaci also described a method of writing out fractions called ***Egyptian fractions***, which has been employed since ancient times; the Rhind Mathematical Papyrus, composed about 3,500 years ago in Thebes, includes several tables of fractions written out this way.

An Egyptian fraction is a sum of ***distinct*** fractions whose numerators are all 1 (these fractions are called ***unit fractions***). For example, here are some sample Egyptian fraction representations:

$$\frac{2}{3}=\frac{1}{2}+\frac{1}{6} \qquad \frac{2}{15}=\frac{1}{10}+\frac{1}{30}$$

$$\frac{7}{15}=\frac{1}{3}+\frac{1}{8}+\frac{1}{120} \qquad \frac{2}{85}=\frac{1}{51}+\frac{1}{255}$$

Egyptian fractions are useful for divvying up objects fairly. For example, suppose you have two cakes to distribute to fifteen people – that is, everyone should get a $^2/_{15}$ fraction of those cakes. Begin by slicing each cake into tenths and giving each person one ($^1/_{10}$). Now, take the remaining tenths you haven't distributed and cut them into thirds, giving thirtieths of the original cake. Each person then takes one of those ($^1/_{30}$). Because $^1/_{10} + {}^1/_{30} = {}^2/_{15}$, everyone gets their fair share. Pretty cool, isn't it?

One way of finding an Egyptian fraction representation of a rational number is to use a ***greedy algorithm*** that works by finding the largest unit fraction at any point that can be subtracted out from the rational number. For example, to compute the fraction for $^{42}/_{137}$, we would start off by noting that ¼ is the largest unit fraction less than $^{42}/_{137}$. We then say that

$$\frac{42}{137}=\frac{1}{4}+\left(\frac{42}{137}-\frac{1}{4}\right)=\frac{1}{4}+\frac{31}{548}$$

We then repeat this process by finding the largest unit fraction less than $^{31}/_{548}$ and subtracting it out. This number is $^1/_{18}$, so we get

$$\frac{42}{137}=\frac{1}{4}+\left(\frac{42}{137}-\frac{1}{4}\right)=\frac{1}{4}+\frac{1}{18}+\left(\frac{31}{548}-\frac{1}{18}\right)=\frac{1}{4}+\frac{1}{18}+\frac{5}{4{,}932}$$

The largest unit fraction we can subtract from $^5/_{4932}$ is $^1/_{987}$:

$$\frac{42}{137}=\frac{1}{4}+\frac{1}{18}+\left(\frac{5}{4{,}932}-\frac{1}{987}\right)=\frac{1}{4}+\frac{1}{18}+\frac{1}{987}+\frac{1}{1{,}622{,}628}$$

And at this point we're done, because the leftover fraction is itself a unit fraction.

Prove that the greedy algorithm for Egyptian fractions always terminates for any rational number r in the range $0 < r < 1$ and always produces a valid Egyptian fraction. (A ***rational number*** is a real number that can be written as $r = {}^p/_q$ for some integers p and q where $q \neq 0$.) That is, the sum of the unit fractions should be the original number, there should only be finitely many fractions, and no unit fraction should be repeated. This shows that every rational number in the range $0 < r < 1$ has at least one Egyptian fraction representation.