Problem Set 9

What problems are beyond our capacity to solve? Why are they so hard? And why is anything that we've discussed this quarter at all practically relevant? In this problem set – the last one of the quarter! – you'll explore the absolute limits of computing power.

Before attempting any of the problems on this problem set, we strongly recommend reading over the ***Guide to Self-Reference*** and ***Guide to the Lava Diagram*** that are available on Canvas, which provide a ton of extra background that you might find useful here.

As always, please feel free to drop by office hours or ask questions on Ed if you have any questions. We'd be happy to help out.

Good luck, and have fun!

Due Friday, November 20th at 12:00PM noon Pacific

Problem One: Isn't Everything Undecidable?

(We recommend reading the Guide to Self-Reference on the course website before attempting this problem.)

In lecture, we proved that A_{TM} and the halting problem are undecidable – that, in some sense, they're beyond the reach of algorithmic problem-solving. The proofs we used involved the nuanced technique of self-reference, which can seem pretty jarring and weird the first time you run into it. The good news is that with practice, you'll get the hang of the technique pretty quickly!

One of the most common questions we get about self-reference proofs is why you can't just use a self-reference argument to prove that *every* language is undecidable. As is often the case in Theoryland, the best way to answer this question is to try looking at some of the ways you might use self-reference to prove that every language is undecidable, then see where those arguments break down.

To begin with, consider this proof:

> ***Theorem:*** All languages are undecidable.
>
> ***Proof:*** Suppose for the sake of contradiction that there is a decidable language L. This means there's a decider for L; call it `inL`.
>
> Now, consider the following program, which we'll call P:
>
> ```
> int main() {
>     string input = getInput();
>
>     /* Do the opposite of what's expected. */
>     if (inL(input)) {
>         reject();
>     } else {
>         accept();
>     }
> }
> ```
>
> Now, given any input w, either $w \in L$ or $w \notin L$. If $w \in L$, then the call to `inL(input)` will return true, at which point P rejects w, a contradiction! Otherwise, if $w \notin L$, then the call to `inL(input)` will return false, at which point P accepts w, a contradiction!
>
> In both cases we reach a contradiction, so our assumption must have been wrong. Therefore, no languages are decidable. ■

This proof has to be wrong because we know of many decidable languages.

i. What's wrong with this proof? Be as specific as possible.

Go one sentence at a time. For each claim that's made, ask yourself – why, specifically, is this statement correct?

Here's another incorrect proof that all languages are undecidable:

> ***Theorem:*** All languages are undecidable.
>
> ***Proof:*** Suppose for the sake of contradiction that there is a decidable language L. This means that there is some decider D for the language L, which we can represent in software as a method `willAccept`. Then we can build the following self-referential program, which we'll call P:
>
> ```
> int main() {
>     string me = mySource();
>     string input = getInput();
>
>     /* See whether we'll accept, then do the opposite. */
>     if (willAccept(me, input)) {
>         reject();
>     } else {
>         accept();
>     }
> }
> ```
>
> Now, given any input w, program P either accepts w or it does not accept w. If P accepts w, then the call to `willAccept(me, input)` will return true, at which point P rejects w, a contradiction! Otherwise, we know that P does not accept w, so the call to `willAccept(me, input)` will return false, at which point P accepts w, a contradiction!
>
> In both cases we reach a contradiction, so our assumption must have been wrong. Therefore, no languages are decidable. ■

It's a nice read, but this proof isn't correct.

ii. What's wrong with this proof? Be as specific as possible.

Pull up the proof that A_{TM} is undecidable and compare this proof and that one side-by-side, going one sentence at a time if you need to.

Many of the examples we've seen of undecidable languages involve checking for properties of Turing machines or computer programs, which might give you the sense that *every* question you might want to ask about TMs or programs is undecidable. That isn't the case, though, and this question explores why.

Consider the following language L:

$$L = \{ \langle P \rangle \mid P \text{ is a syntactically valid C++ program } \}$$

Below is a purported proof that L is undecidable:

> ***Theorem:*** The language L is undecidable.
>
> ***Proof:*** Suppose for the sake of contradiction that L is decidable. That means that there's some decider D for L, which we can represent in software as a function `isSyntactically-Valid` that takes as input a program and then returns whether that program has correct syntax. Given this function, consider the following program P:
>
> ```
> int main() {
>     string me = mySource();
>
>     /* Execute a line based on whether our syntax is right. */
>     if (isSyntacticallyValid(me)) {
>         oops, this line of code isn't valid C++!
>     } else {
>         int num = 137; // Perfectly valid syntax!
>     }
> }
> ```
>
> Now, either this program P is syntactically valid or it is not. If P has valid syntax, then when P is run on any input, it will get its own source code, determine that it is syntactically valid, then execute a syntactically invalid line of code – a contradiction! Otherwise, if P is not syntactically valid, then when P is run on any input, it will get its own source code, determine that it is not syntactically valid, at which point it executes a perfectly valid line of C++ code – a contradiction!
>
> In either case we reach a contradiction, so our assumption must have been incorrect. Therefore, L is undecidable. ■

This proof, unfortunately, is incorrect.

iii. What's wrong with this proof? Be as specific as possible.

Problem Two: Password Checking

(We recommend reading the Guide to Self-Reference on the course website before attempting this problem.)

When you log onto a website with a password, you have the presumption that your password is the only possible password that will log you in. There shouldn't be a "master key" password that can unlock any account, since that would be a huge security vulnerability. But how could you tell? If you had the source code to the password checking system, could you figure out whether your password was the only password that would grant you access to the system?

Let's frame this question in terms of Turing machines. If we wanted to build a TM password checker, "entering your password" would correspond to starting up the TM on some string, and "gaining access" would mean that the TM accepts your string. Let's suppose your password is the string `iheartquokkas`. We'll say that a ***password checker*** is a TM M where

$$\mathscr{L}(M) = \{\texttt{iheartquokkas}\};$$

that is, the TM accepts your password `iheartquokkas`, and it doesn't accept anything else. Given a TM, is there some way you could tell whether the TM was a password checker?

Consider the following language L:

$$L = \{ \langle M \rangle \mid M \text{ is a TM and } M \text{ is a password checker } \}.$$

Your task in this problem is to prove that L is undecidable (that is, $L \notin \mathbf{R}$). This means that there's no algorithm that can mechanically check whether a TM is suitable as a password checker. Rather than dropping you headfirst into this problem, we've split this problem apart into a few smaller pieces.

Let's suppose for the sake of contradiction that $L \in \mathbf{R}$. That means that there is some function

```
bool isPasswordChecker(string program)
```

with the following properties:

- If `program` is the source of a program that accepts just the string `iheartquokkas`, then calling `isPasswordChecker(program)` will return `true`.
- If `program` is not the source of a program that accepts just the string `iheartquokkas`, then calling `isPasswordChecker(program)` will return `false`.

We can try to build a self-referential program that uses the `isPasswordChecker` function to obtain a contradiction. Here's a first try:

```
int main() {
    string me = mySource();
    string input = getInput();

    if (isPasswordChecker(me)) {
        reject();
    } else {
        accept();
    }
}
```

This code is, essentially, a (minimally) modified version of the self-referential program we used to get a contradiction for the language A_{TM}.

i. Prove that the above program P is not a password checker.

What is the definition of a password checker? Based on that, what do you need to prove to show that P is not a password checker?

(Continued on the next page.)

ii. Suppose that this program is ***not*** a password checker. Briefly explain why no contradiction arises in this case – no formal justification is necessary.

A good question to think about in the course of answering part (ii) of this problem: this program is very close to the one from the proof that A_{TM} is not decidable. Why do you get a contradiction in the original proof that A_{TM} is undecidable? Why doesn't that same contradiction work here?

Ultimately, the goal of building a self-referential program here is to have the program cause a contradiction regardless of whether or not it's a password checker. As you've seen in part (ii), this particular program does not cause a contradiction if it isn't a password checker. Consequently, if we want to prove that $L \notin \mathbf{R}$, we need to modify it so that it leads to a contradiction in the case where it is not a password checker.

iii. Modify the above code so that it causes a contradiction regardless of whether it's a password checker. Then, briefly explain why your modified program is correct. No formal proof is necessary.

Follow the advice from the Guide to Self-Reference. Write out a specification of what your self-referential program is trying to do. Based on that, craft code for each of the two cases.

Problem Three: Double Verification

This problem explores the following beautiful and fundamental theorem about the relationship between the **R** and **RE** languages:

If L is a language, then $L \in \mathbf{R}$ if and only if $L \in \mathbf{RE}$ and $\overline{L} \in \mathbf{RE}$

This theorem has a beautiful intuition: it says that a language L is decidable ($L \in \mathbf{R}$) precisely if for every string in the language, it's possible to prove it's in the language ($L \in \mathbf{RE}$) and, simultaneously, for every string not in the language, it's possible to prove that the string is not in the language ($\overline{L} \in \mathbf{RE}$). In this problem, we're going to ask you to prove one of the two directions of this theorem.

Let L be a language where $L \in \mathbf{RE}$ and $\overline{L} \in \mathbf{RE}$. This means that there's a verifier V_{yes} for L and a verifier V_{no} for $\overline{L}$. In software, you could imagine that V_{yes} and V_{no} correspond to methods with these signatures:

```
bool checkInL(string w, string c)
bool checkInLBar(string w, string c)
```

Prove that $L \in \mathbf{R}$ by writing pseudocode for a function

```
bool isInL(string w)
```

that accepts as input a string w, then returns true if $w \in L$ and returns false if $w \notin L$. Then, write a brief proof explaining why your pseudocode meets these requirements. You don't need to write much code here. If you find yourself writing ten or more lines of pseudocode, you're probably missing something.

The theorem you proved in this problem is extremely useful for building an intuition for what languages are decidable. You'll see this in the next problem.

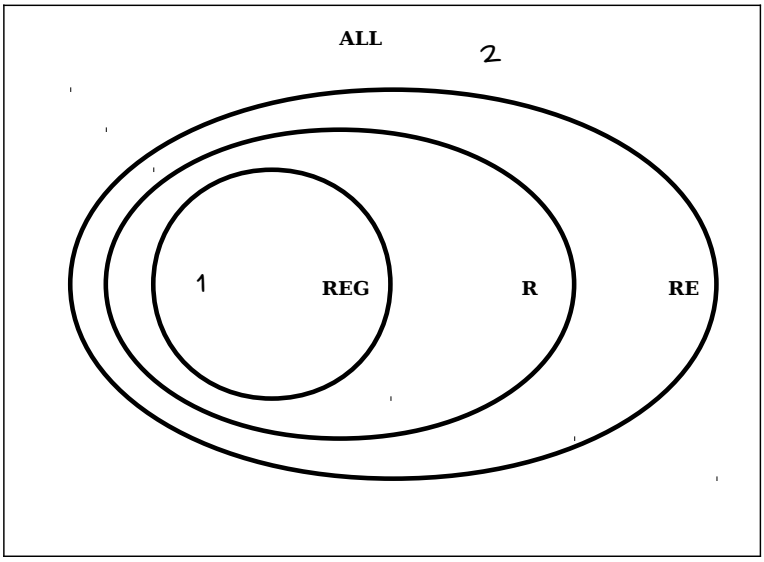
What other constructions have we done on verifiers? How did they work?

Problem Four: The Lava Diagram

(We ***strongly*** recommend reading over the Guide to the Lava Diagram before starting this problem.)

Download the starter files for Problem Set 9 and extract them somewhere convenient. Run the provided program and choose "The Lava Diagram." You'll be presented with a Venn diagram showing the overlap of different classes of languages we've studied so far. We have also provided you a list of twelve numbered languages. For each of those languages, draw where in the Venn diagram that language belongs. As an example, we've indicated where Language 1 and Language 2 should go. No proofs or justifications are necessary – the purpose of this problem is to help you build a better intuition for what makes a language regular, **R**, **RE**, or none of these.

Use the provided to test your solution locally, then submit the file `res/LavaDiagram.answers` to Gradescope when you're done.



1. Σ^*
2. L_D
3. $\{ \mathtt{a}^n \mid n \in \mathbb{N} \}$
4. $\{ \mathtt{a}^n \mid n \in \mathbb{N}$ and is a multiple of 137 $\}$
5. $\{ \mathtt{1}^n\mathtt{+}\mathtt{1}^m\mathtt{=}\mathtt{1}^{n+m} \mid m, n \in \mathbb{N} \}$
6. $\{ \langle M \rangle \mid M$ is a Turing machine and $\mathscr{L}(M) \neq \varnothing \}$
7. $\{ \langle M \rangle \mid M$ is a Turing machine and $\mathscr{L}(M) = \varnothing \}$
8. $\{ \langle M \rangle \mid M$ is a Turing machine and $\mathscr{L}(M) = L_D \}$

Make sure you can clearly explain the answers to the previous three parts of this problem before moving on.

9. $\{ \langle M, n \rangle \mid M$ is a TM, $n \in \mathbb{N}$, and M ***accepts*** all strings in its input alphabet of length at most $n \}$
10. $\{ \langle M, n \rangle \mid M$ is a TM, $n \in \mathbb{N}$, and M ***rejects*** all strings in its input alphabet of length at most $n \}$
11. $\{ \langle M, n \rangle \mid M$ is a TM, $n \in \mathbb{N}$, and M ***loops*** on all strings in its input alphabet of length at most $n \}$
12. $\{ \langle M_1, M_2, M_3, w \rangle \mid M_1, M_2$, and M_3 are TMs, w is a string, and at least two of M_1, M_2, and M_3 accept $w. \}$

Problem Five: co-RE, Disjoint Unions, and Terrifyingly Difficult Problems

When we first saw that A_{TM} is undecidable, we could at least take consolation in the fact that A_{TM} is recognizable. Then we discovered that L_D is unrecognizable, and up to now we haven't had a comforting fact to fall back on. But there is something about L_D we can console ourselves with: the complement of L_D, the language $\overline{L}_D$, is an **RE** language.* In other words, while there's no general way to prove that some TM *will not* accept its own encoding, there *is* a general way to prove that some TM *will* accept its own encoding.

The fact that L_D isn't an **RE** language while $\overline{L}_D$ is still in **RE** suggests that there might be a bit more to the computability landscape than just **R** and **RE**. And indeed there is: the same way that A_{TM} is the poster child of an **RE** language, the language L_D is the poster child of a ***co-RE language***. Formally speaking, the class co-**RE** consists of all the languages whose *complement* is an **RE** language:

$$\text{co-}\mathbf{RE} = \{ L \mid \overline{L} \in \mathbf{RE} \}.$$

Intuitively speaking, the co-**RE** languages are languages where if you have a string that *isn't* in the language, there's some easy way to prove that it isn't in the language.

i. Prove that $A_{TM} \notin$ co-**RE**.

*You haven't seen the term "co-**RE**" before, but you have seen something like this somewhere.*

At this point, we have an **RE** language that's not in co-**RE** (A_{TM}) and a co-**RE** languages that's not in **RE** (L_D). As a coda to our treatment of unsolvable problems, let's see a language that's neither in **RE** nor co-**RE**. In other words, there's no general way to prove that strings in that language are indeed in that language, nor is there way to prove that strings not in that language are indeed not in that language. Yikes!

In what follows, let's assume all languages are over the alphabet $\Sigma = \{\texttt{0}, \texttt{1}\}$. Given two languages A and B over Σ, the ***disjoint union*** of A and B, denoted $A \uplus B$, is the language

$$A \uplus B = \texttt{0}A \cup \texttt{1}B.$$

For example, if $A = \{ \texttt{1}, \texttt{10}, \texttt{100}, \texttt{1000} \}$ and $B = \{ \varepsilon, \texttt{0}, \texttt{1}, \texttt{00}, \texttt{01}, \texttt{10}, \texttt{11} \}$, then $A \uplus B$ is the language

$$A \uplus B = \{ \texttt{01}, \texttt{010}, \texttt{0100}, \texttt{01000}, \texttt{1}, \texttt{10}, \texttt{11}, \texttt{100}, \texttt{101}, \texttt{110}, \texttt{111} \}$$

Notice how each string in $A \uplus B$ is tagged with which language it originated in. Any string that starts with `0` came from A, and any string that starts with `1` came from B.

ii. Let A and B be languages where $A \uplus B \in$ **RE**. Show that $A \in$ **RE**. To do so, assume you have code for a recognizer for $A \uplus B$ and use it to write code for a recognizer for A. Briefly explain how your code works, but no formal proof is required.

iii. Let A and B be languages where $A \uplus B \in$ co-**RE**. Show that $A \in$ co-**RE**, structuring your answer along the lines of what you did in part (ii) of this problem.

*If you have a language in co-**RE**, what can you say about that language? If you want to prove that a language is in co-**RE**, what do you need to prove about that language?*

The results you proved in parts (ii) and (iii) of this problem can be extended to show that if $A \uplus B \in$ **RE**, then $B \in$ **RE**, and if $A \uplus B \in$ co-**RE**, then $B \in$ co-**RE**, though you don't need to prove this.

iv. Let L be an undecidable language. Prove $L \uplus \overline{L} \notin$ **RE** and $L \uplus \overline{L} \notin$ co-**RE**.

Your result from part (iv) shows there are problems harder than A_{TM} and L_D; one example is $L_D \uplus \overline{L}_D$. But even this nightmarish problem sits within a class of problems that can still be touched by computing power, if only in a very weak sense. Specifically, $L_D \uplus \overline{L}_D$ is what's called a Δ_2^0 language, where Δ_2^0 is a class of languages that contains all of **RE** and co-**RE**, plus a bunch of other problems. And Δ_2^0 itself sits inside even larger classes of languages, stretching outward to infinity. Want to learn more? Take Phil 152!

* While we're not going to ask you to formally prove that $\overline{L}_D$ is an **RE** language, we would *definitely* recommend taking a few minutes to ponder how you'd build a recognizer or verifier for it.

Optional Fun Problem: Quine Relays

Write four different C++ programs with the following properties:

- Running the first program prints the complete source code of the second program.
- Running the second program prints the complete source code of the third program.
- Running the third program prints the complete source code of the fourth program.
- Running the fourth program prints the complete source code of the first program.
- None of the programs perform any kind of file reading.

In other words, we'd like a collection of four different programs, each of which prints the complete source of the next one in the sequence, wrapping back around at the end.

You can download a set of starter files for this Optional Fun Problem from Canvas. There's a separate autograder for this problem that you can submit to on Gradescope.

This is actually a really fun problem to try. Once you figure out the trick, it's not that hard to code it up.

Grand Challenge Problem: P $\stackrel{?}{=}$ NP (Worth an A+, $1,000,000, and a Ph.D)

Prove or disprove: **P = NP**.

Take fifteen minutes and try this. Seriously. And if you can't crack this problem, feel free to submit your best effort, or the silliest answer you can think of.

Congratulations on finishing the last problem set of the quarter!

We're extremely impressed with how much progress you've made since the start of the quarter. Take care, stay safe, and enjoy the break!