Algorithmic Analysis and Sorting

Part Three

Announcements

- Link to CS106X website up. Feel free to take a look at their midterm and practice midterms.
- Review Session?
  - Would be a mix of answering questions and showing how I go about solving exam questions (i.e. the stuff from the Study Skills handout)
  - Would be this **Friday 11:00-11:50AM** in Huang Auditorium (where class is)
  - Would be recorded by SCPD

Previously on CS106B...

Big-O Notation

- Notation for summarizing the **long-term growth rate** of some function.
- Useful for analyzing runtime:
  - O(n): The runtime grows linearly.
  - O(n^2): The runtime grows quadratically.
  - O(2^n): The runtime grows exponentially.

Sorting Algorithms

- **Selection Sort**: Find smallest element, then 2nd smallest, then 3rd smallest, …
  - $O(n^2)$
- **Insertion Sort**: Make sure first element is sorted, then first two are sorted, then first three are sorted, …
  - $O(n^2)$
  - "On Average" twice as fast as Selection Sort

Thinking About O(n^2)

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

$T(n)$

2	3	6	7	9	14	15	16

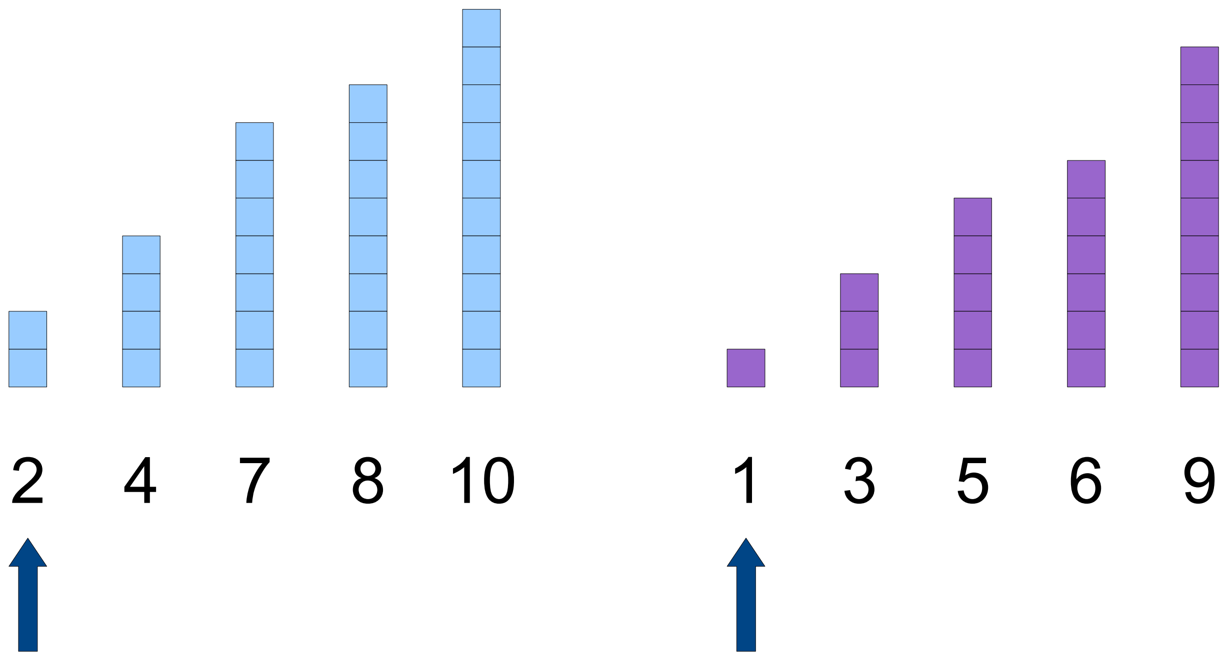
$T(\frac{1}{2}n) \approx \frac{1}{4}T(n)$

1	4	5	8	10	11	12	13

$T(\frac{1}{2}n) \approx \frac{1}{4}T(n)$

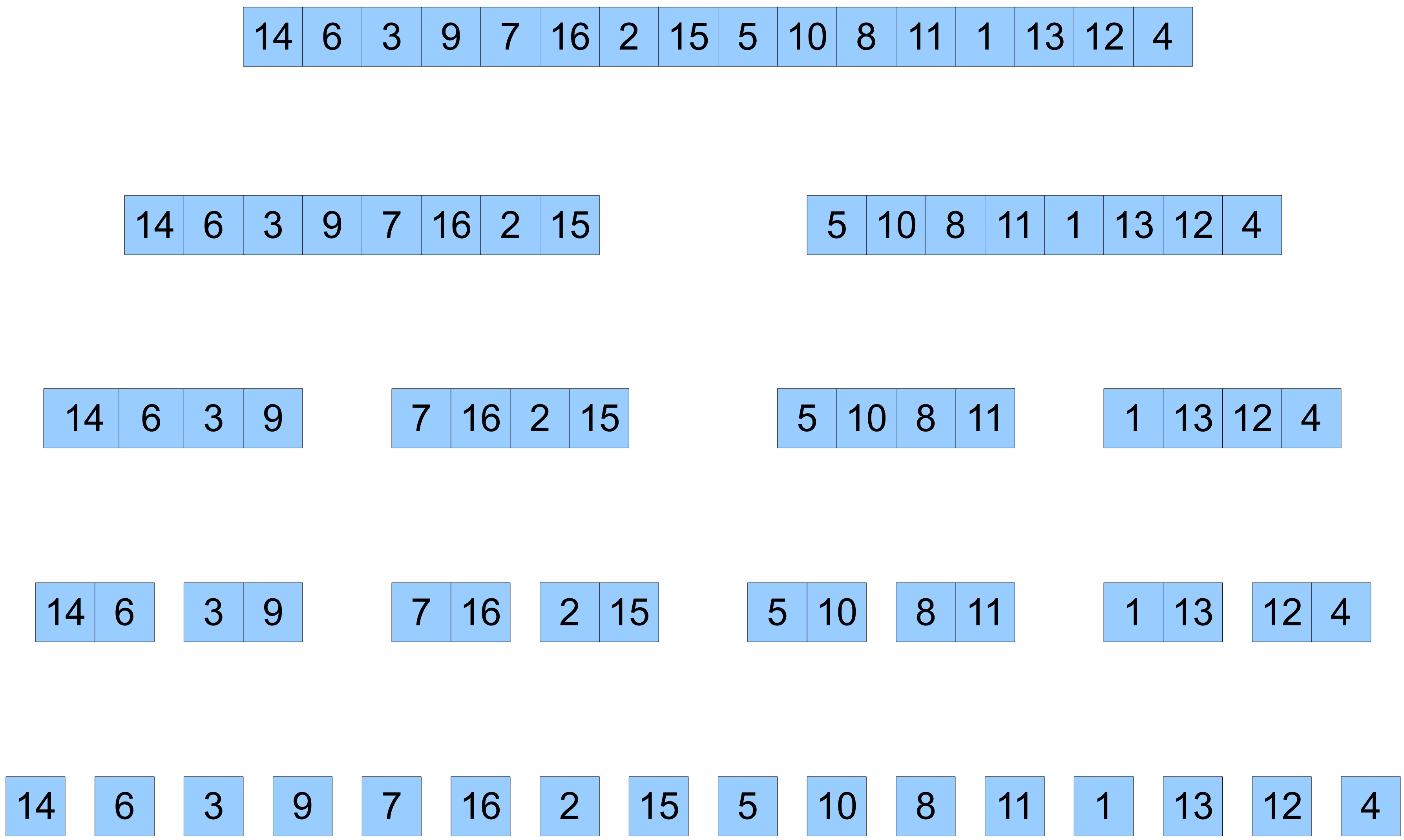
It takes roughly ½T(n) to sort each half separately!

The Key Insight: **Merge**



A Better Idea

- Splitting the input in half and merging halves the work.
- So why not split into four? Or eight?
- **Question**: What happens if we *never stop splitting?*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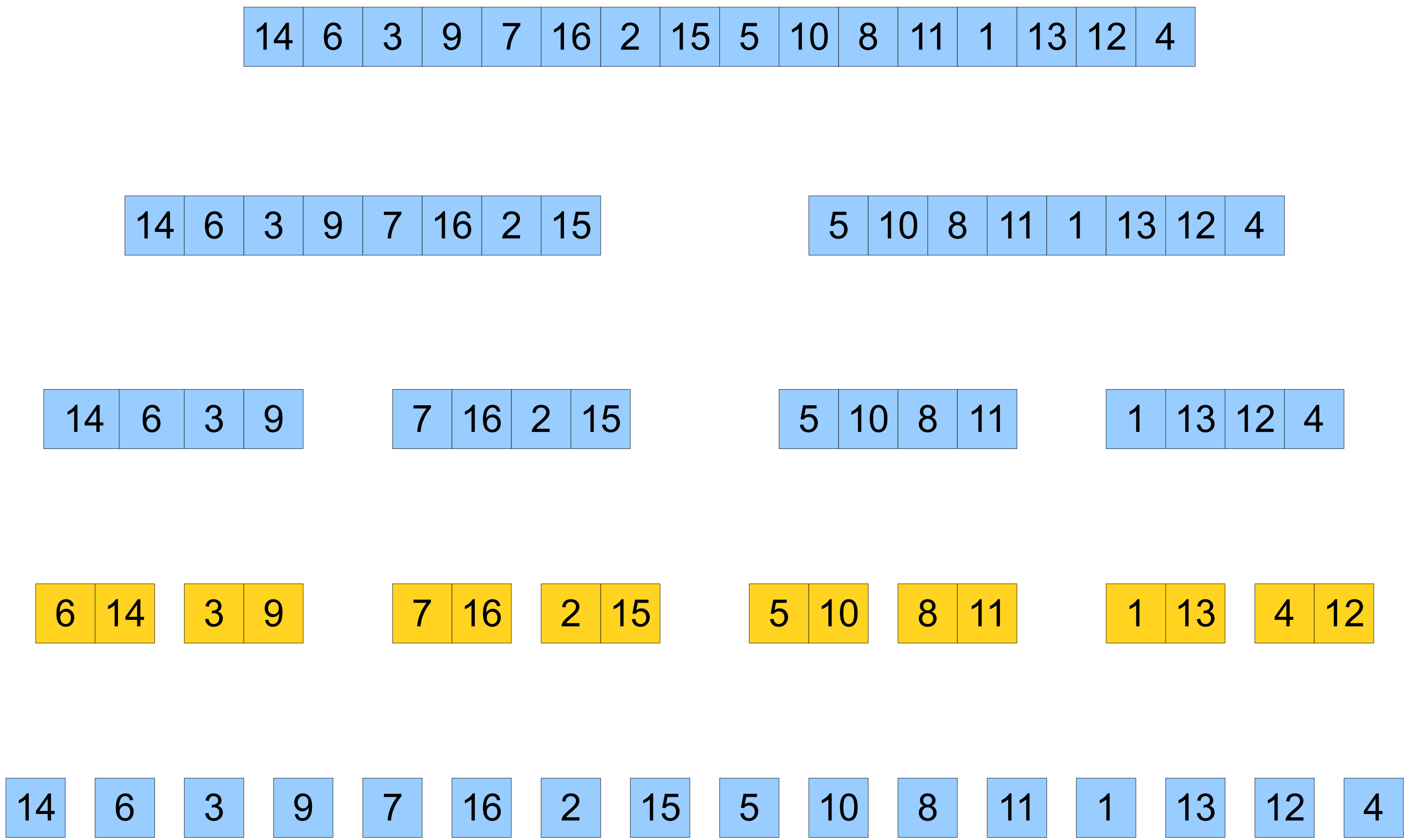
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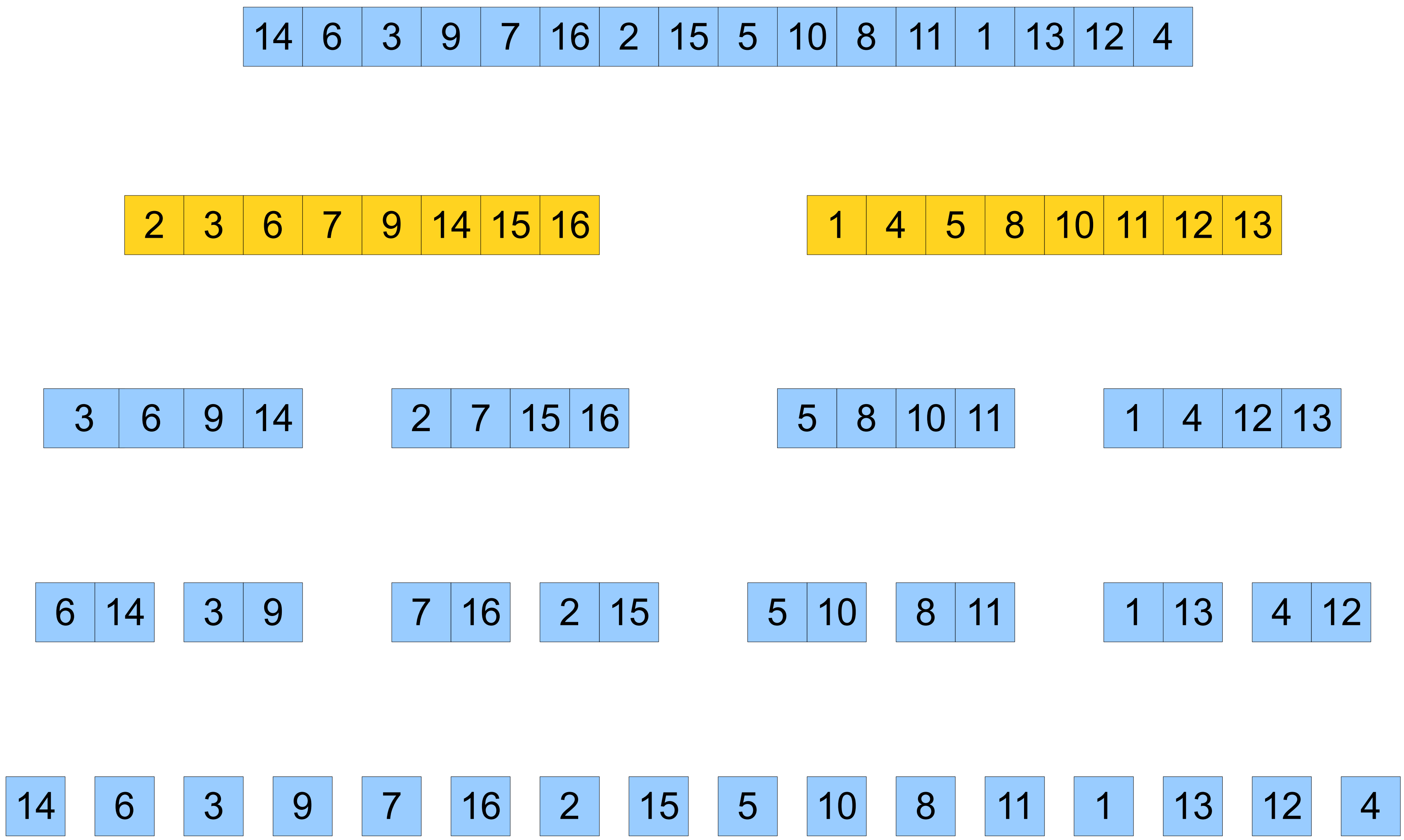
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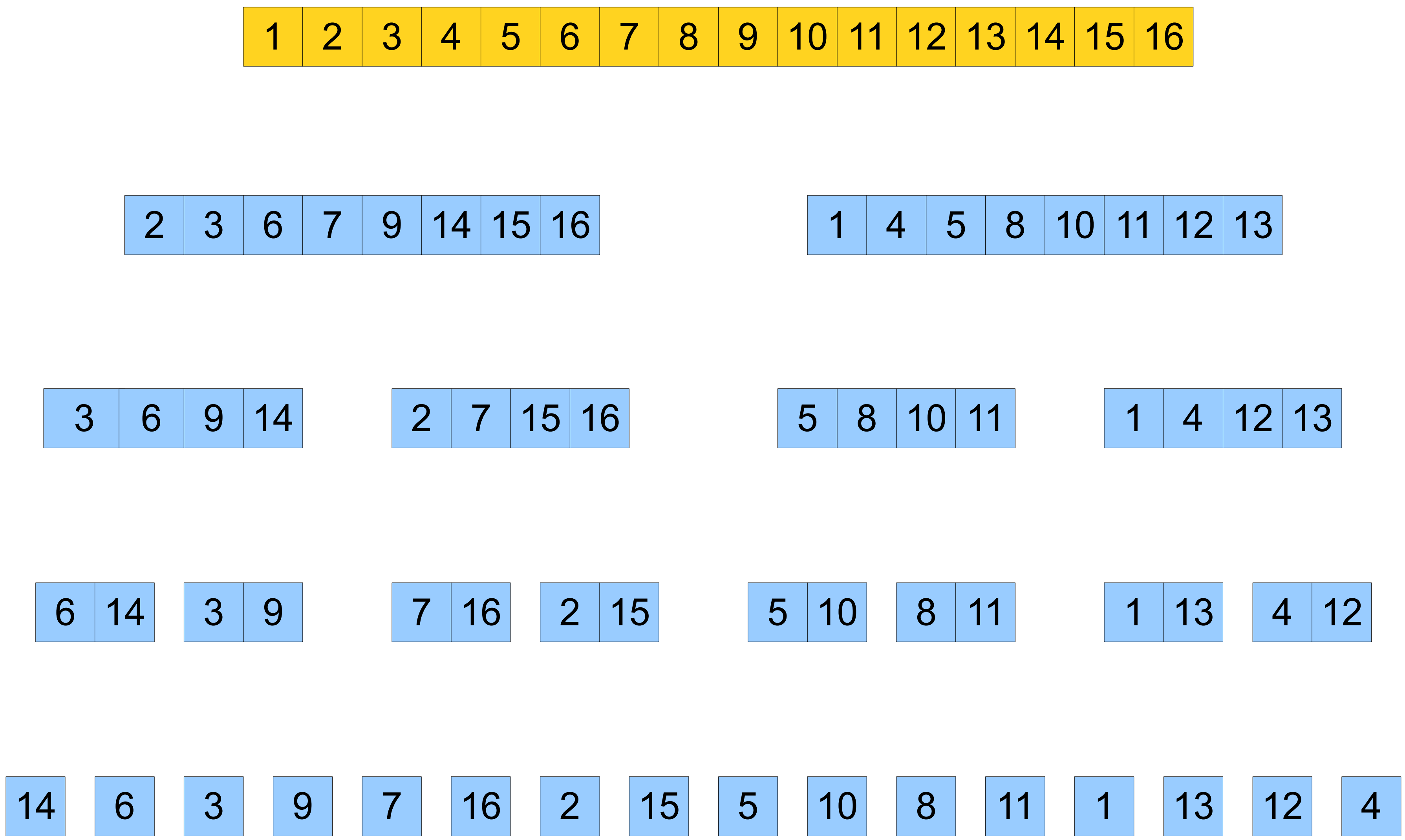
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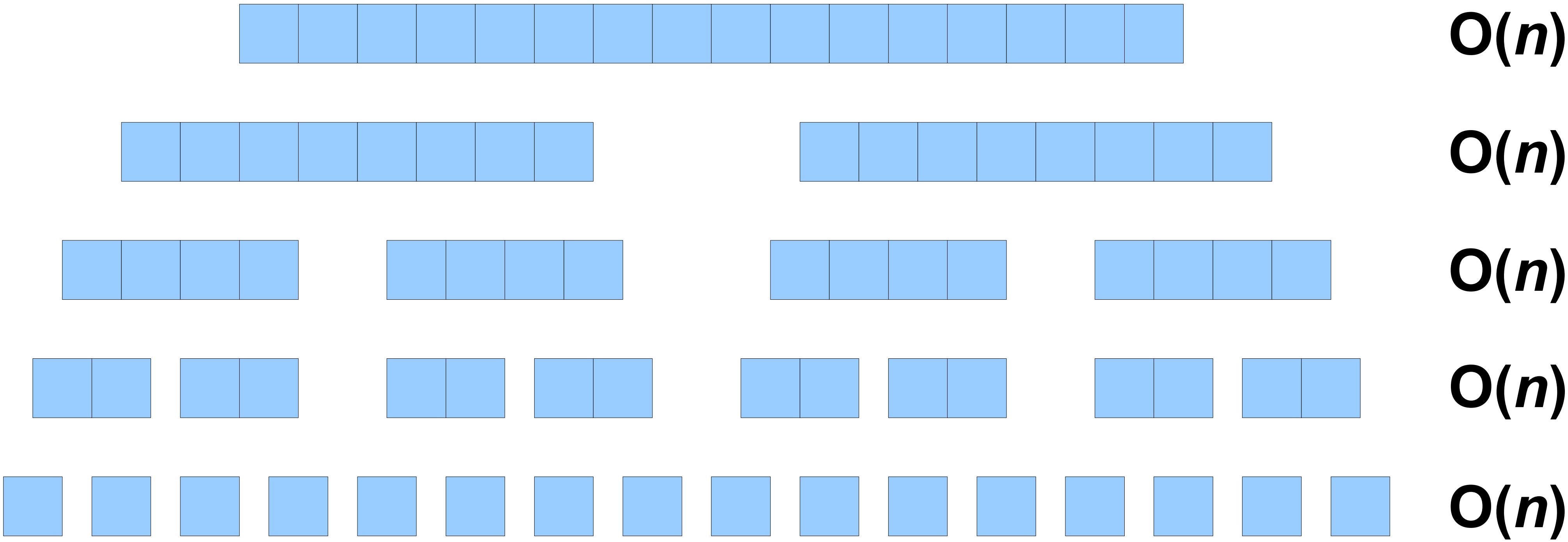
High-Level Idea

- A recursive sorting algorithm!
- **Base Case**:
  - An empty or single-element list is already sorted.
- **Recursive step**:
  - Break the list in half and recursively sort each part.
  - Use **`merge`** to combine them back into a single sorted list.
- This algorithm is called *mergesort*.

A Graphical Intuition



O(n log n)

Mergesort Times

Size	Selection Sort	Insertion Sort	“Split Sort”	Mergesort
10000	0.304	0.160	0.161	0.006
20000	1.218	0.630	0.387	0.010
30000	2.790	1.427	0.726	0.017
40000	4.646	2.520	1.285	0.021
50000	7.395	4.181	2.719	0.028
60000	10.584	5.635	2.897	0.035
70000	14.149	8.143	3.939	0.041
80000	18.674	10.333	5.079	0.042
90000	23.165	12.832	6.375	0.048

Can we do Better?

- Mergesort is $O(n \log n)$.
- This is asymptotically better than $O(n^2)$
- Can we do better?
  - In general, **no**: comparison-based sorts cannot have a worst-case runtime better than $O(n \log n)$.
- **In the worst case, we can only get faster by a constant factor!**

And now... new stuff!

Optimizing Mergesort

- We would like to improve upon Mergesort. But what is there optimize?
  - Let's take a look

```
void mergesort(Vector<int>& v) {
   /* Base case: 0- or 1-element lists are
    * already sorted.
    */
   if (v.size() <= 1) return;

   /* Split v into two subvectors. */
   Vector<int> left, right;
   for (int i = 0; i < v.size() / 2; i++)
       left += v[i];
   for (int i = v.size() / 2; i < v.size(); i++)
       right += v[i];

   /* Recursively sort these arrays. */
   mergesort(left);
   mergesort(right);

   /* Combine them together. */
   merge(left, right, v);
}
```

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}
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Optimizing Mergesort

- We would like to improve upon Mergesort. But what is there optimize?
  - Let's take a look
- It's lame how we have to make these two **`Vector`**s and copy data into them. Is there a way to get around this?

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```
void mergesort(Vector<int>& v, int low, int high) {
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    * already sorted.
    */
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   /* Recursively sort these arrays. */
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   /* Base case: 0- or 1-element lists are
    * already sorted.
    */
   if (v.size() <= 1) return;

   /* Recursively sort these arrays. */
   mergesort(v,low,(low+high)/2);
   mergesort(v,(low+high)/2 + 1, high);

   /* Combine them together. */
   merge(left, right, v);
}
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   /* Combine them together. *
   merge(left, right, v); ????
}
```

We need to change our merge function!!!

A Completely Different Sorting Algorithm…

A Trivial Observation

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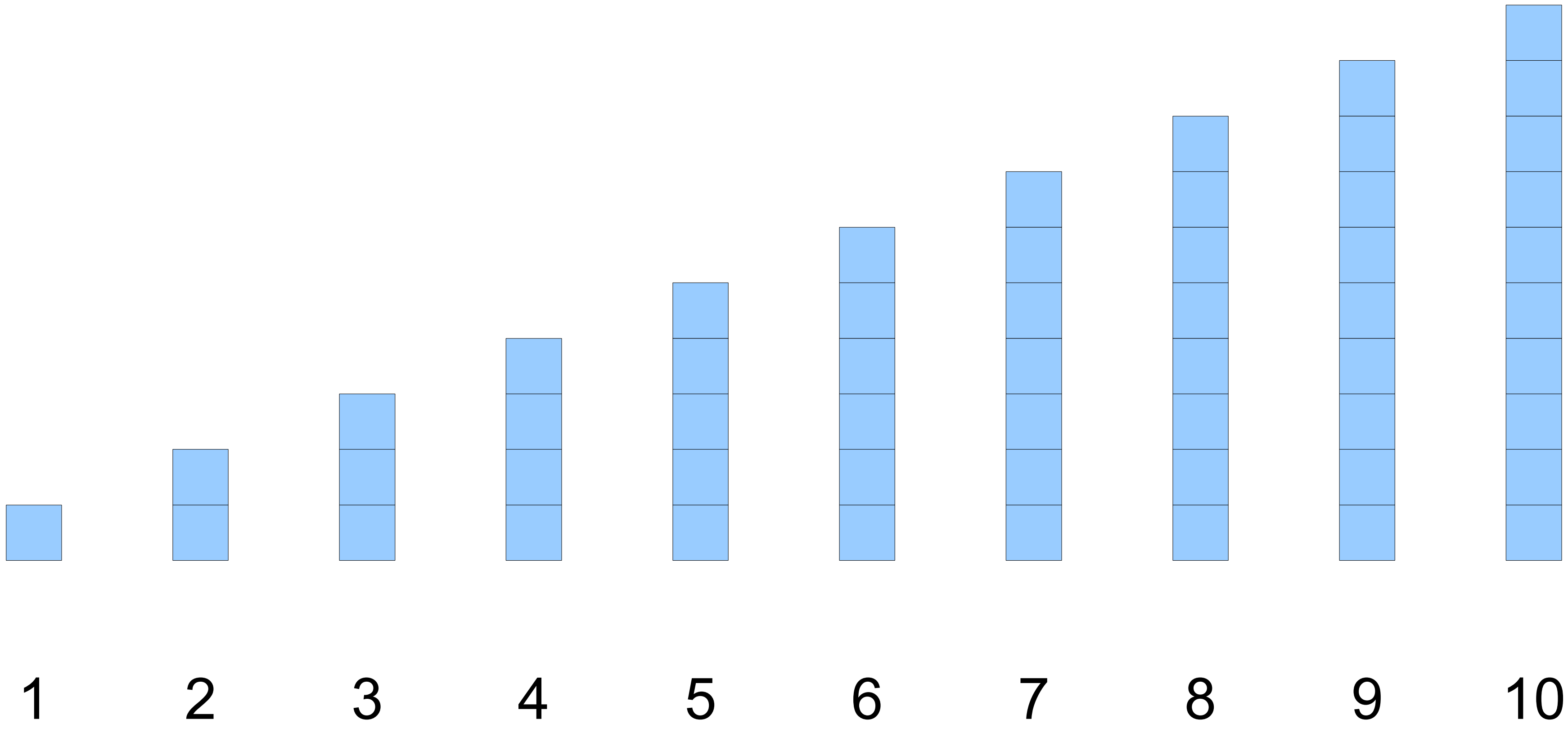

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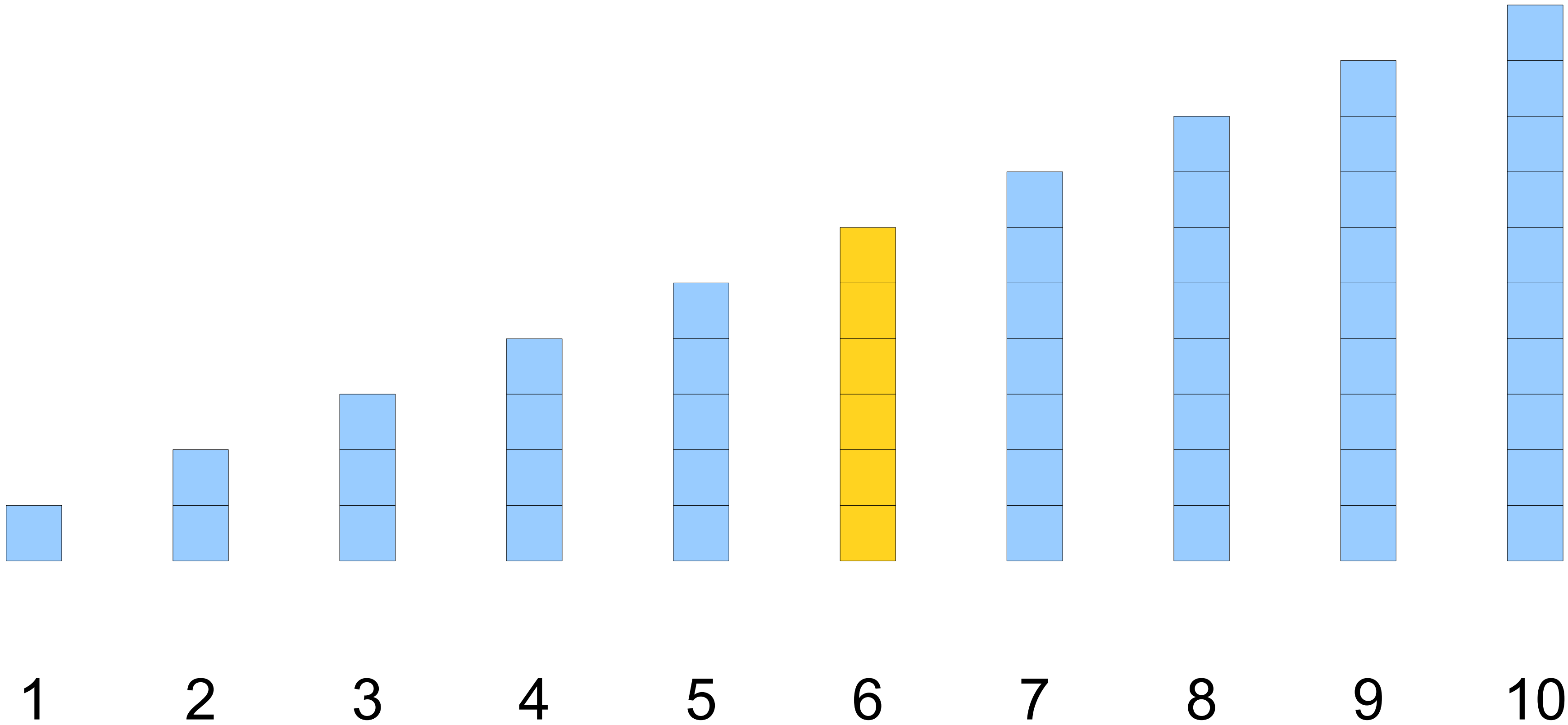

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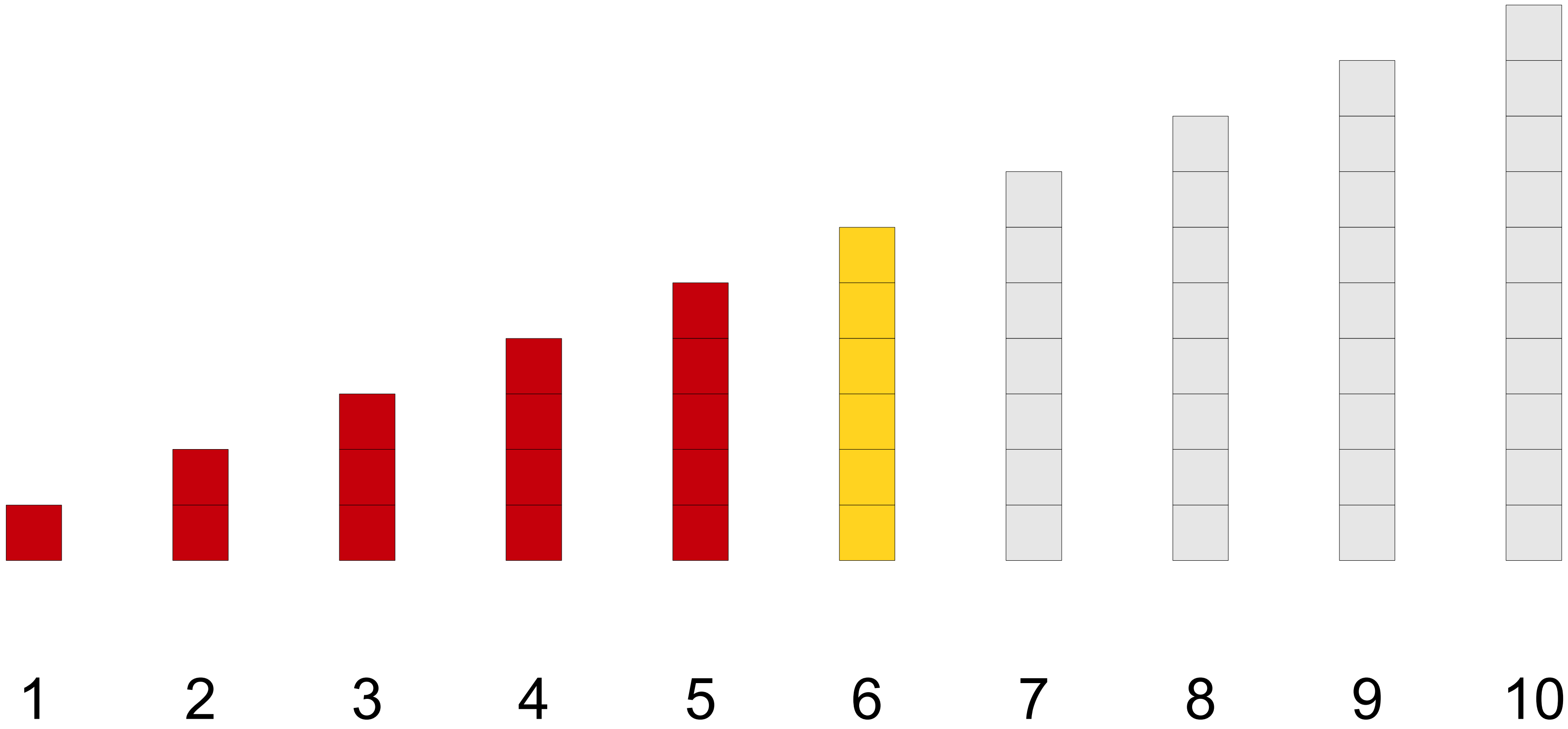

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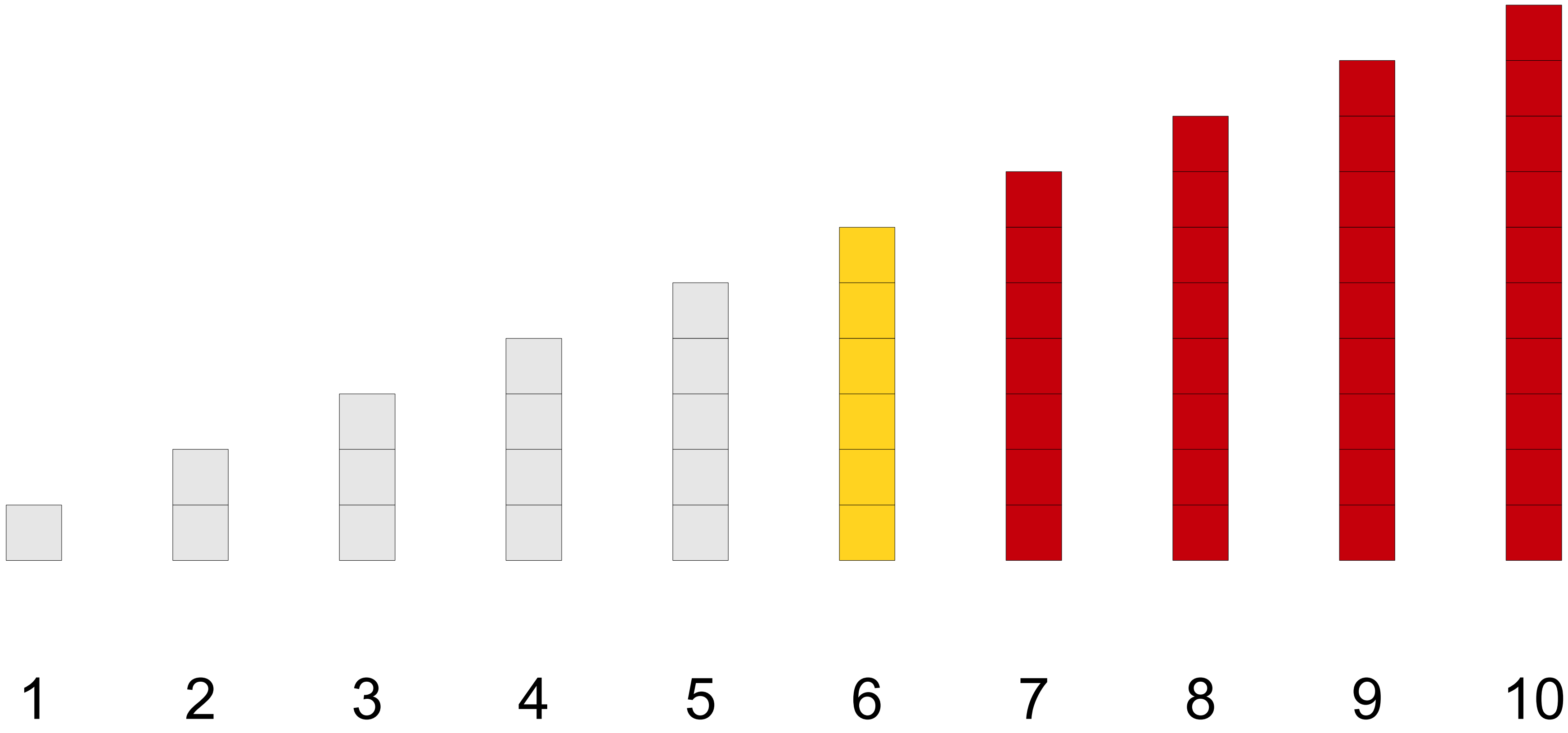


So What?

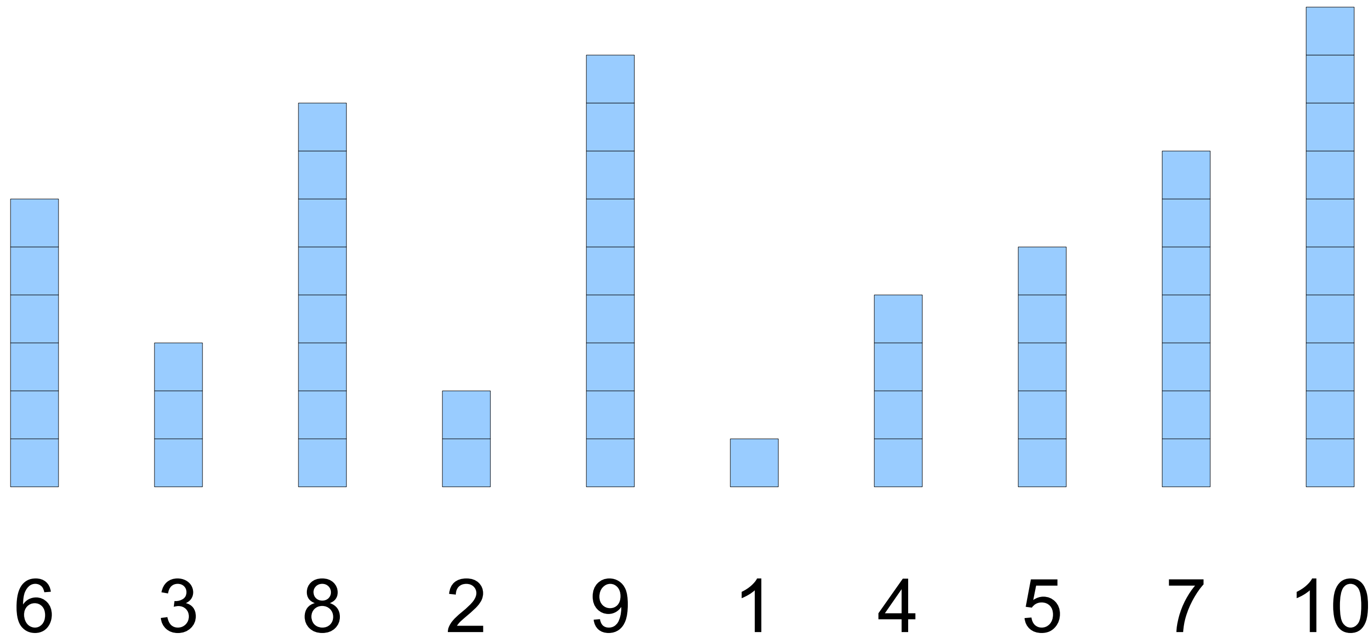
- This idea leads to a particularly clever sorting algorithm that *doesn't* require us to copy data into new **`Vector`**s
- Idea:
  - Pick an element from the array.
  - Put the smaller elements on one side.
  - Put the bigger elements on the other side.
  - Recursively sort each half.
- But how do we do the middle two steps?

Partitioning

- Pick a **pivot element**.
- Move everything less than the pivot to the left of the pivot.
- Move everything greater than the pivot to the right of the pivot.
- Good news: O(n) algorithm exists!
- Bad news: it's a bit tricky...

The Partition Algorithm

The Partition Algorithm



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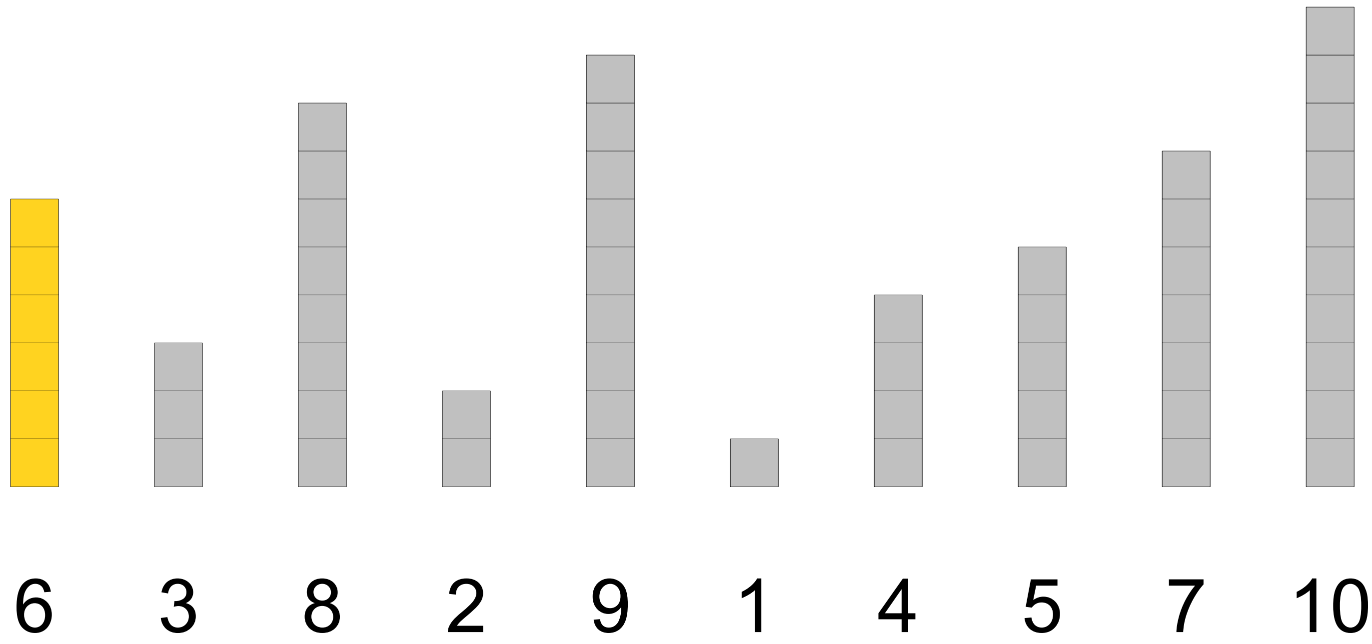

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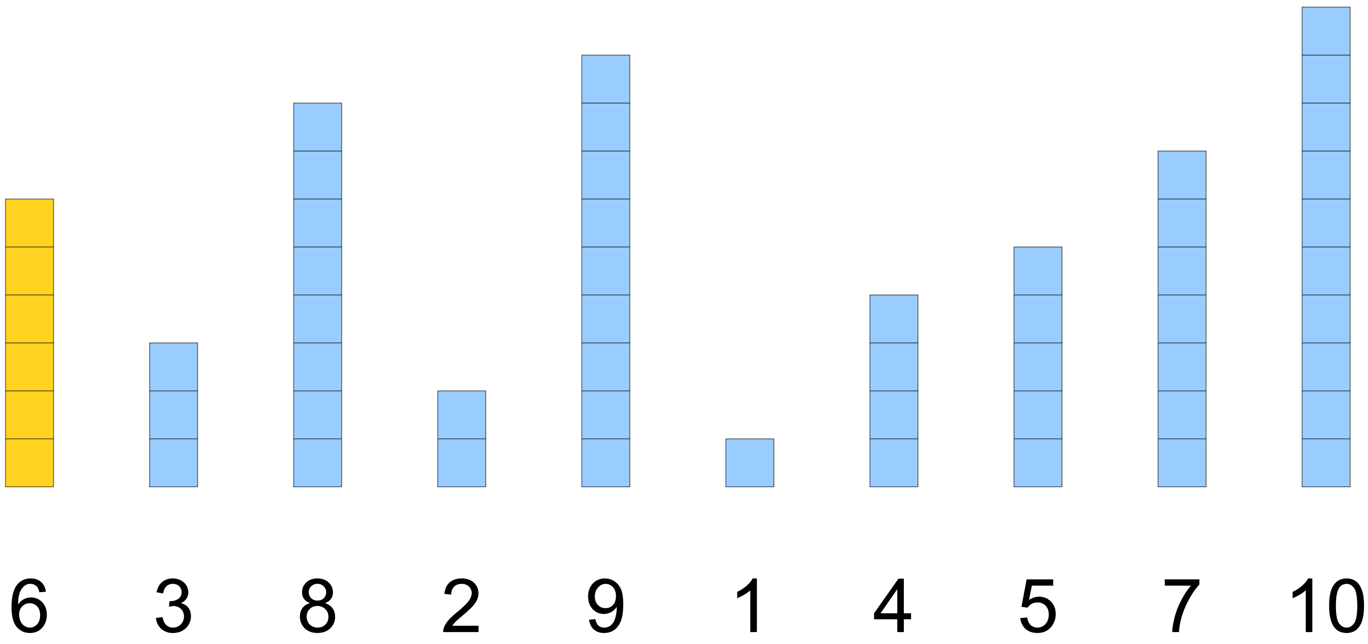

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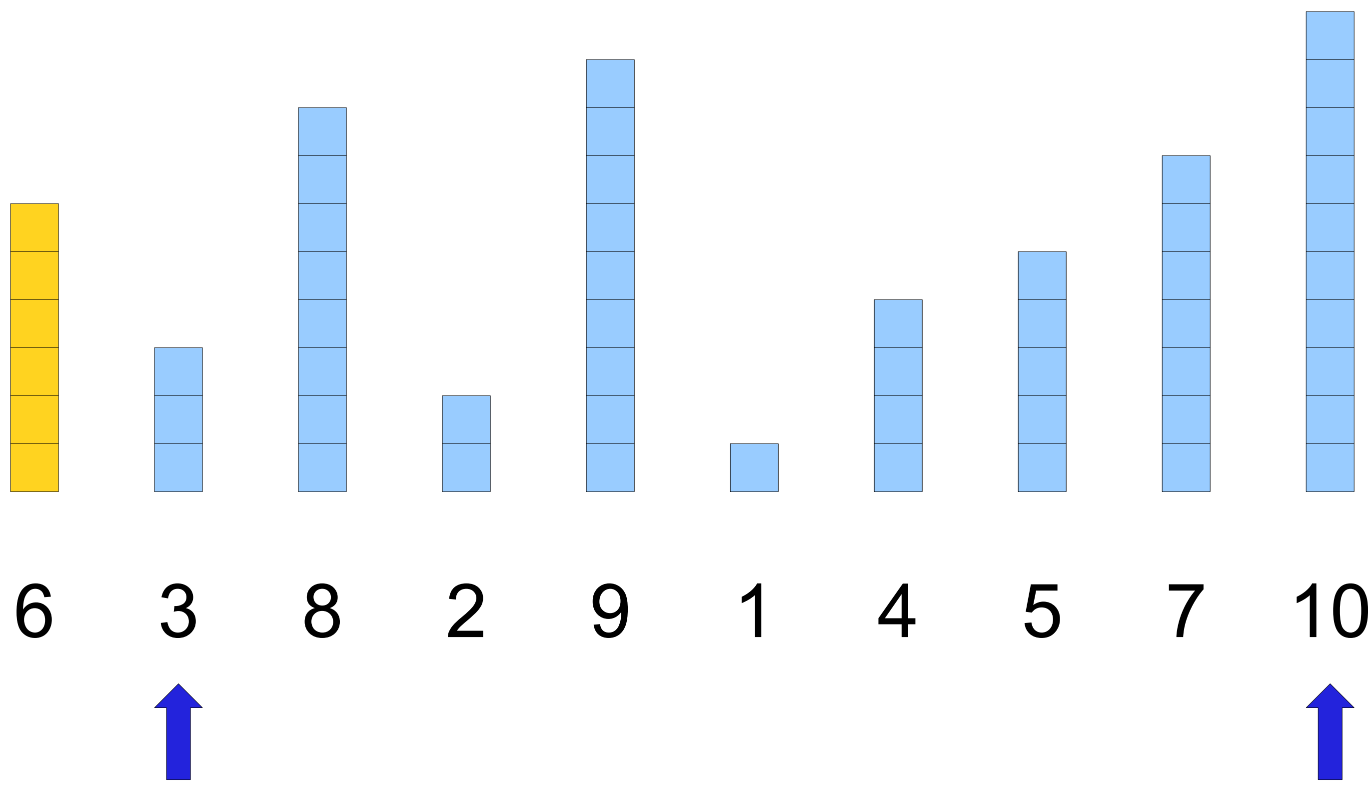

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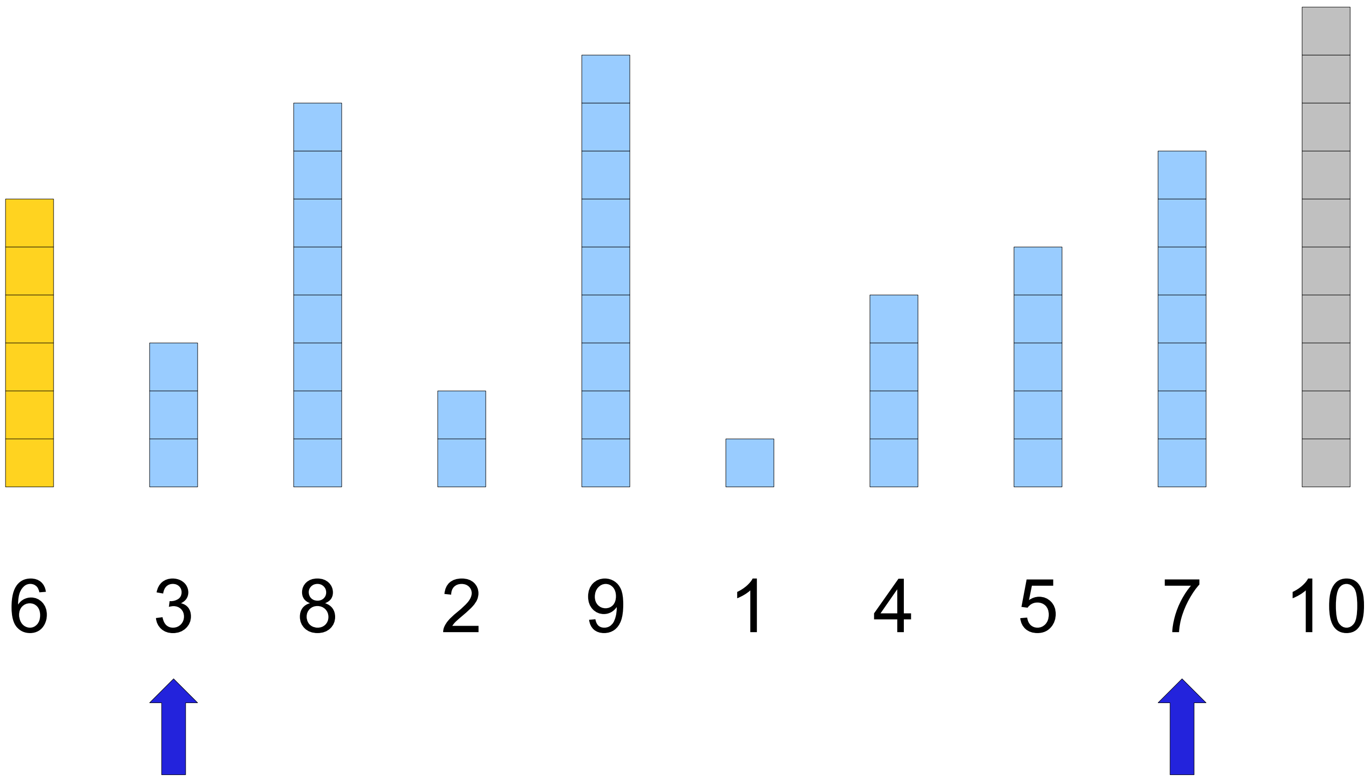

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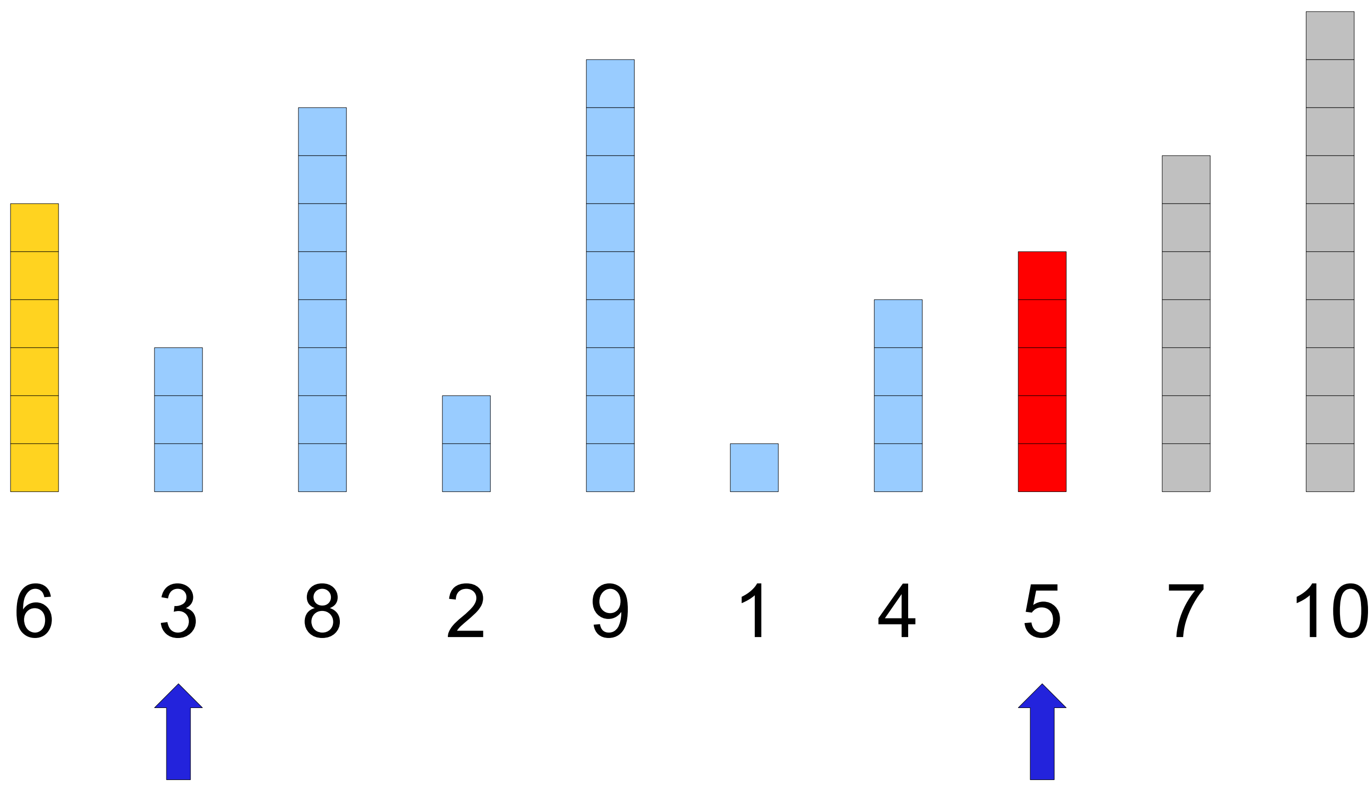

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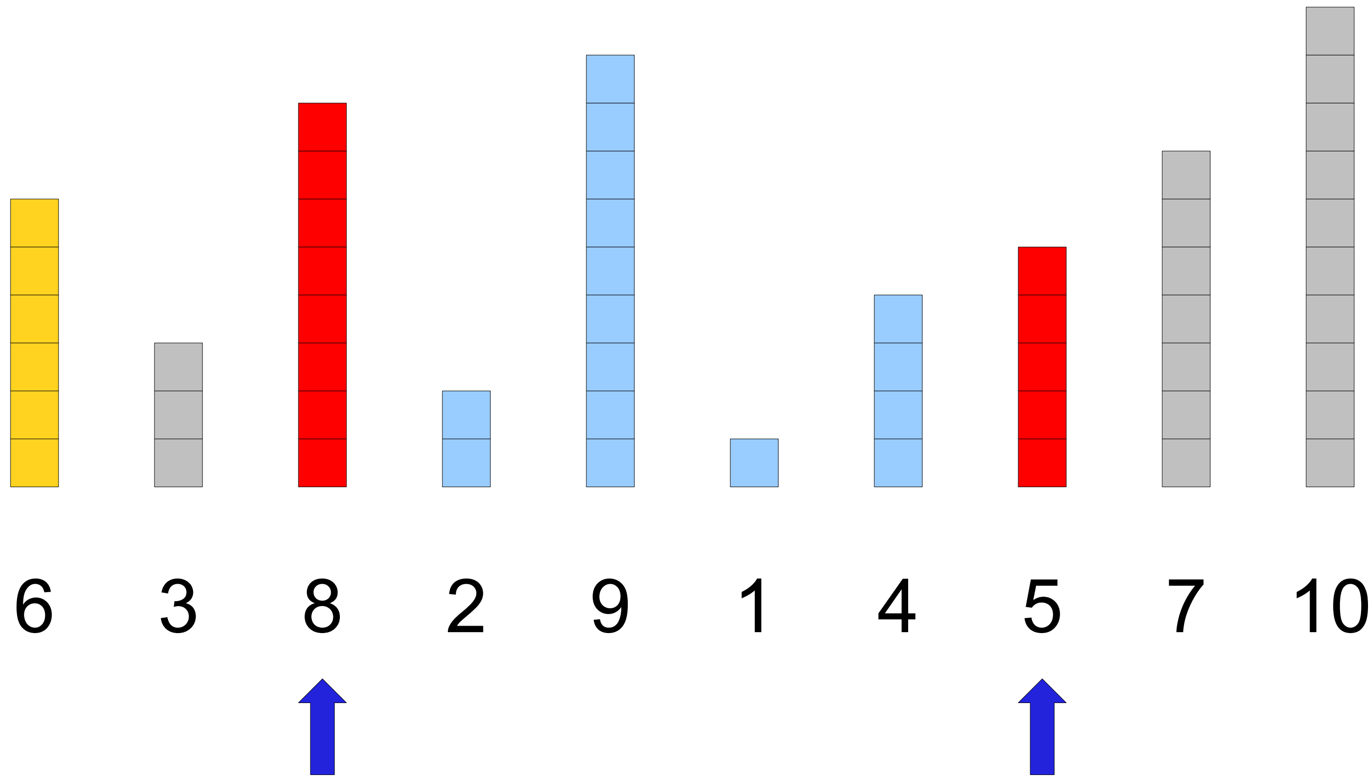

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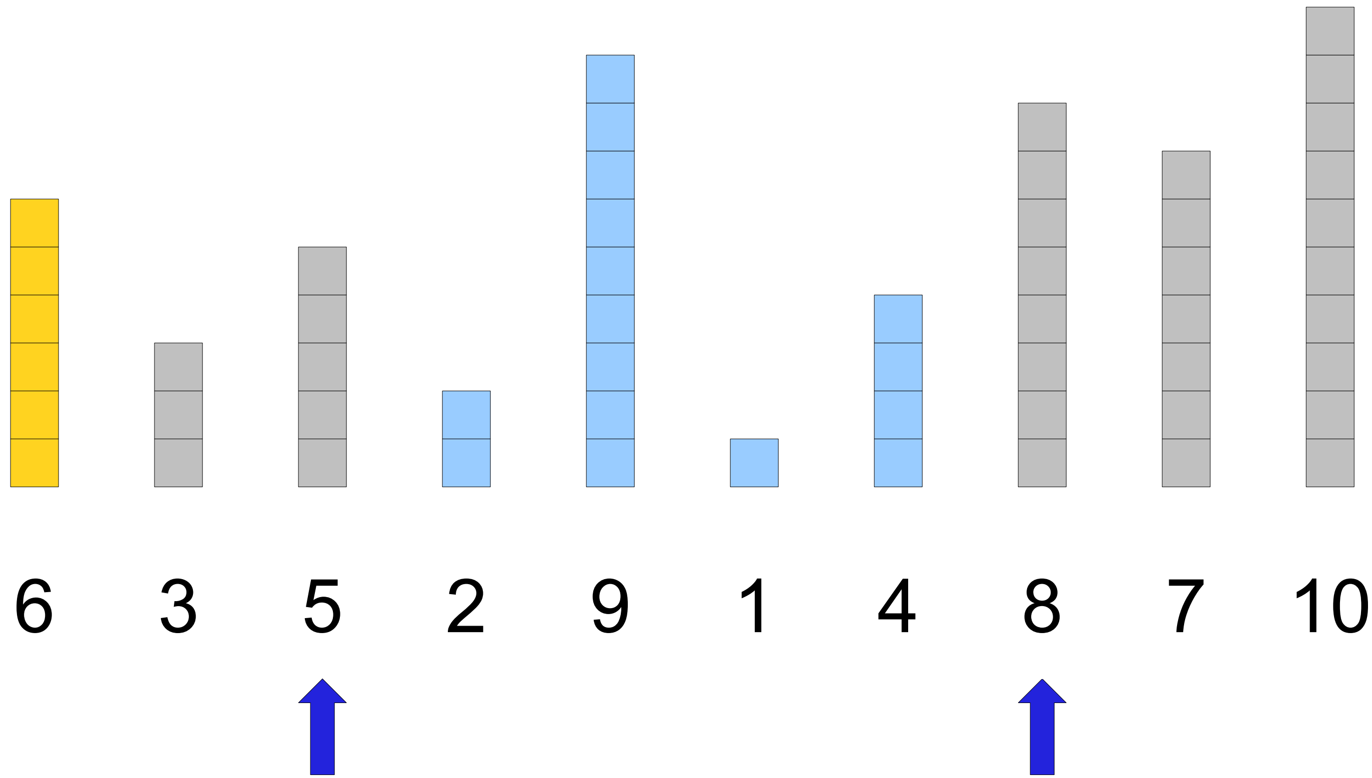

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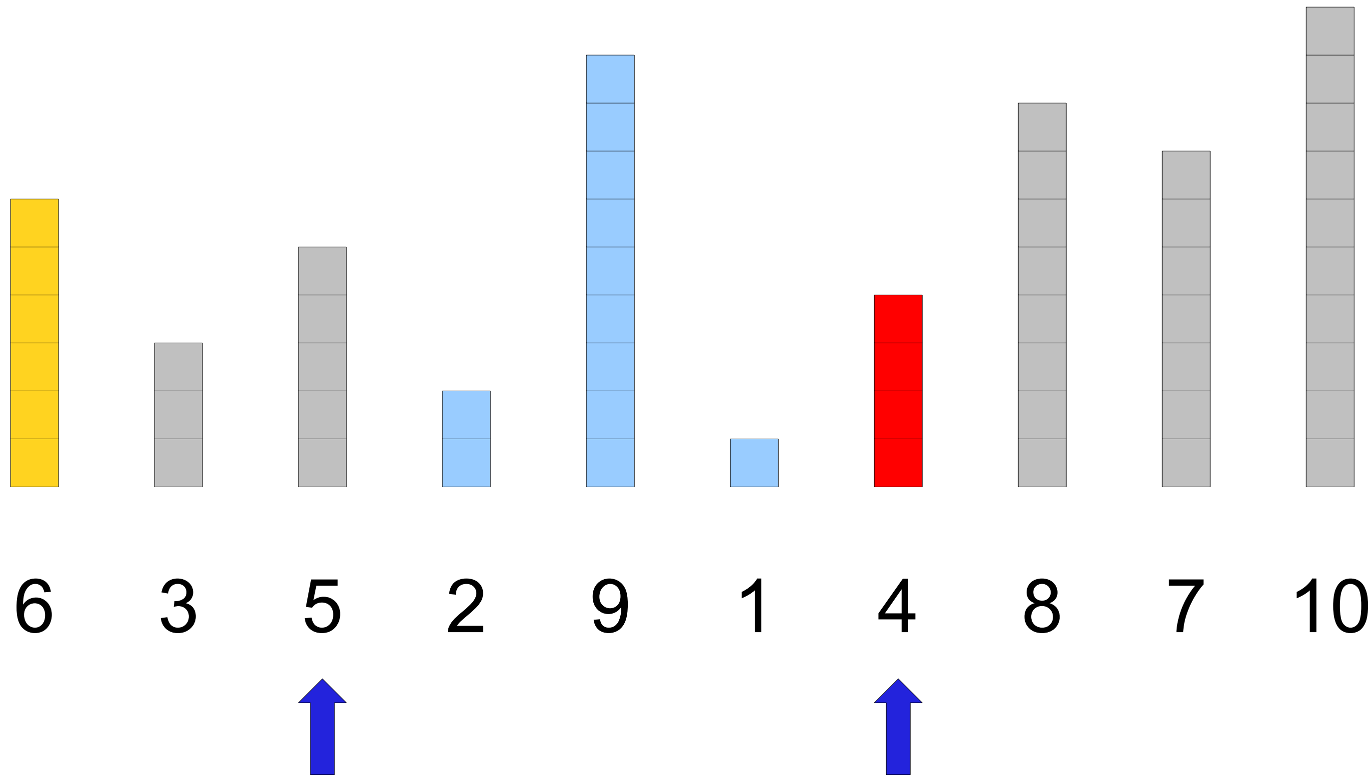

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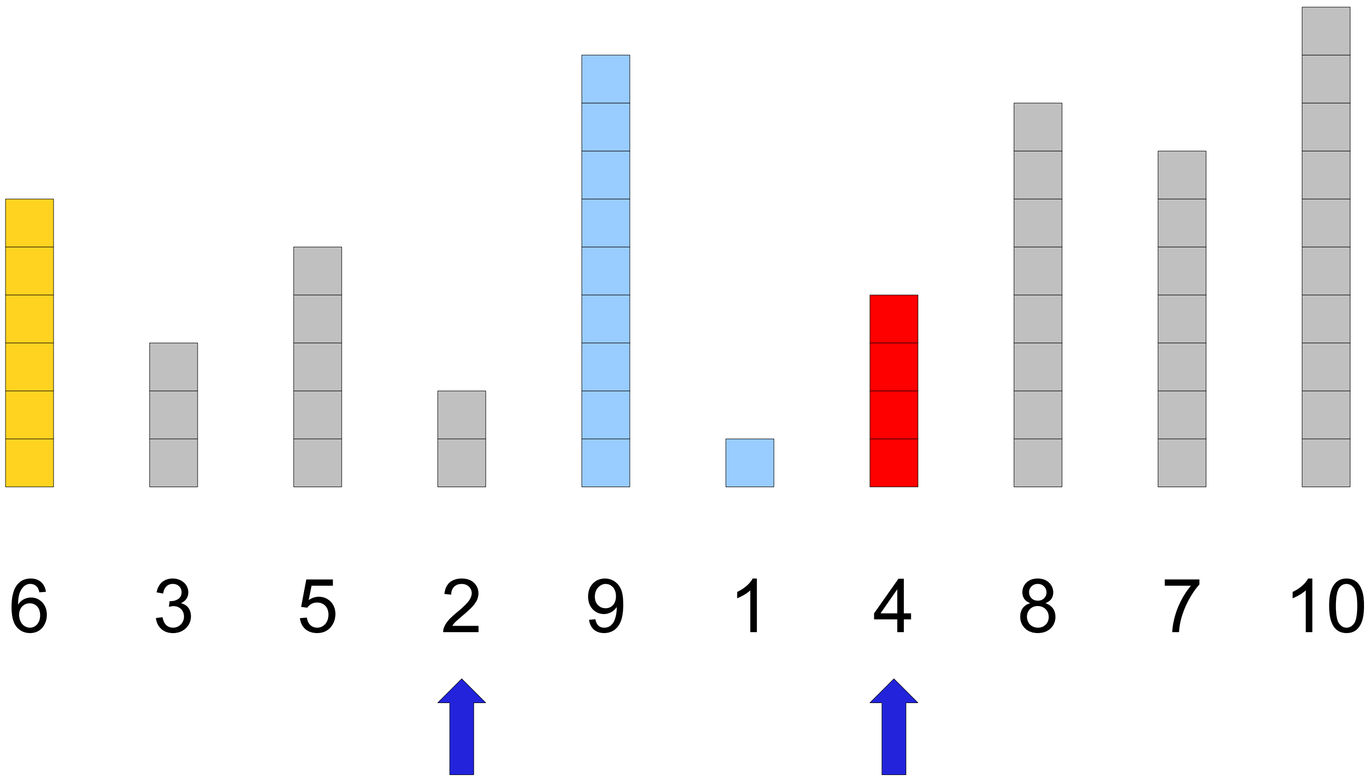

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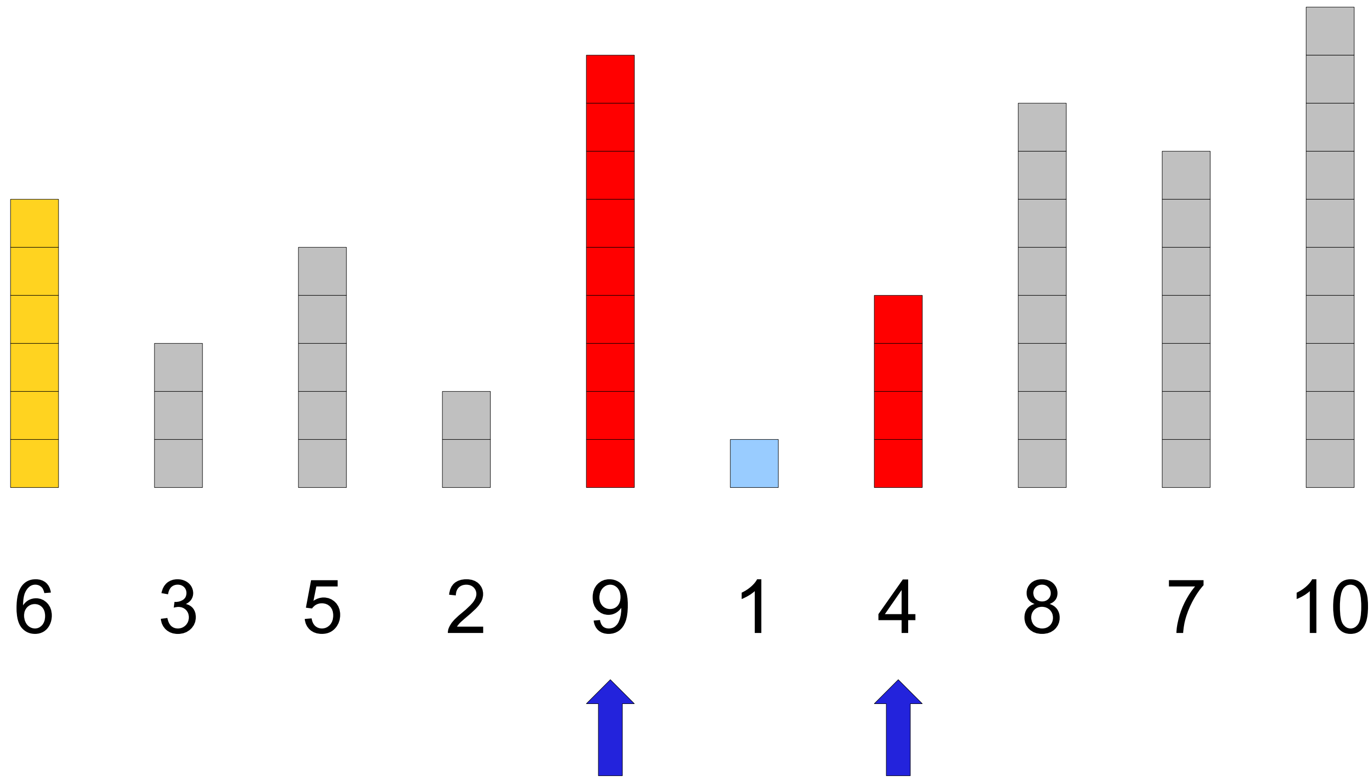

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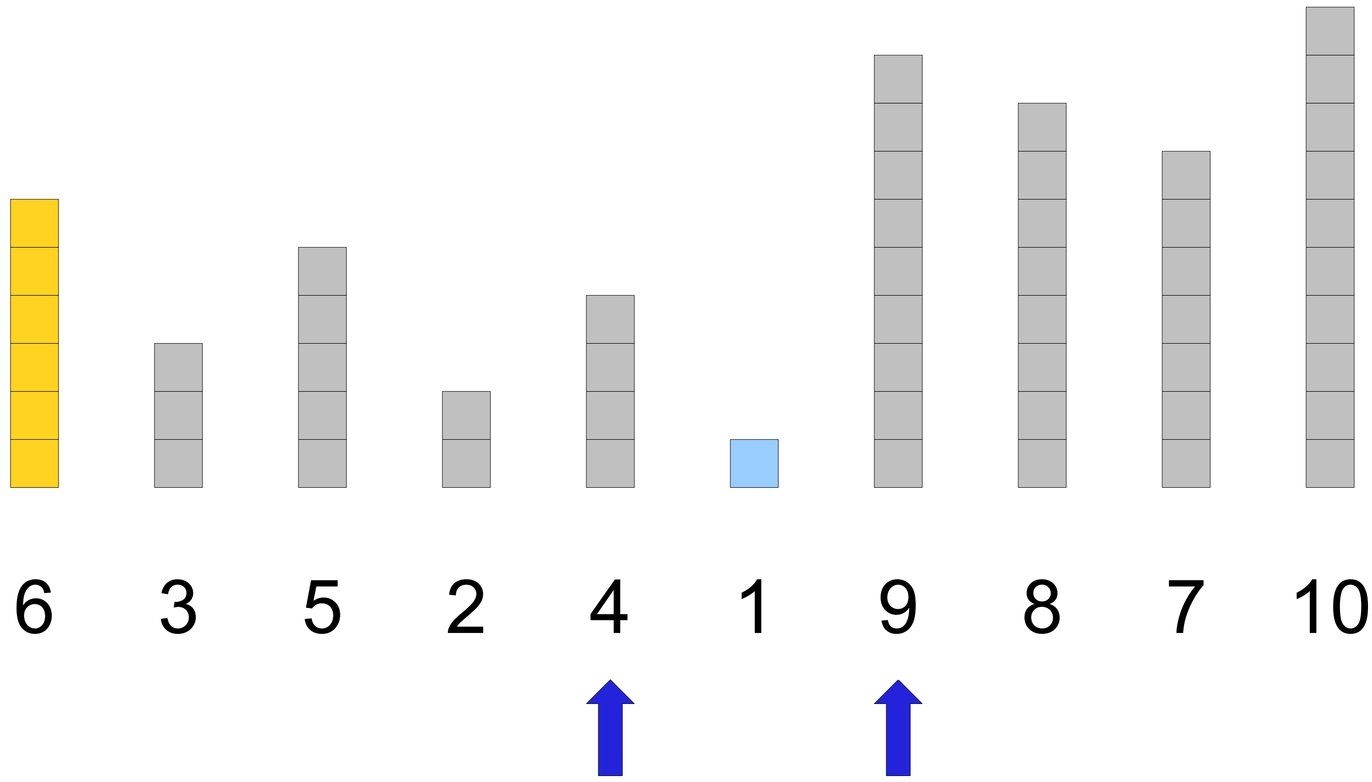

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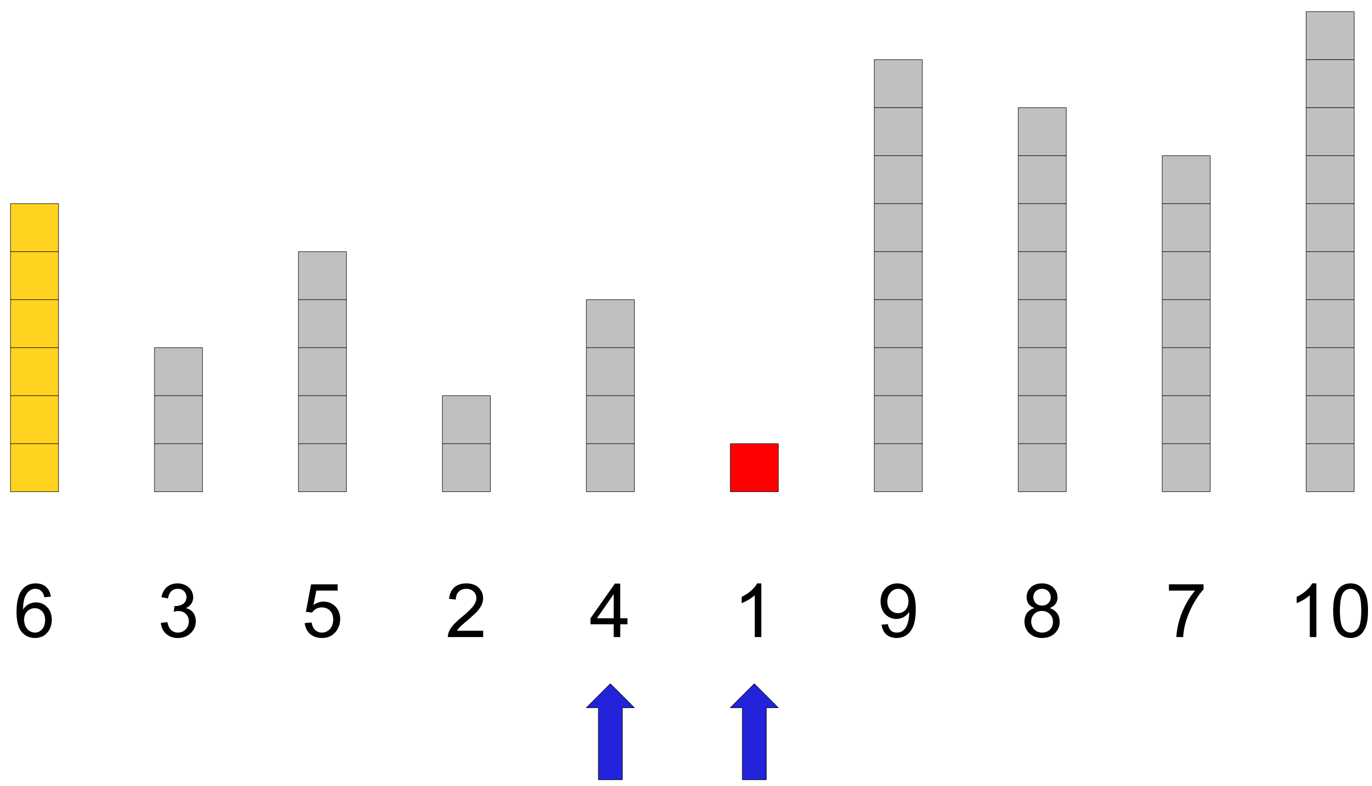

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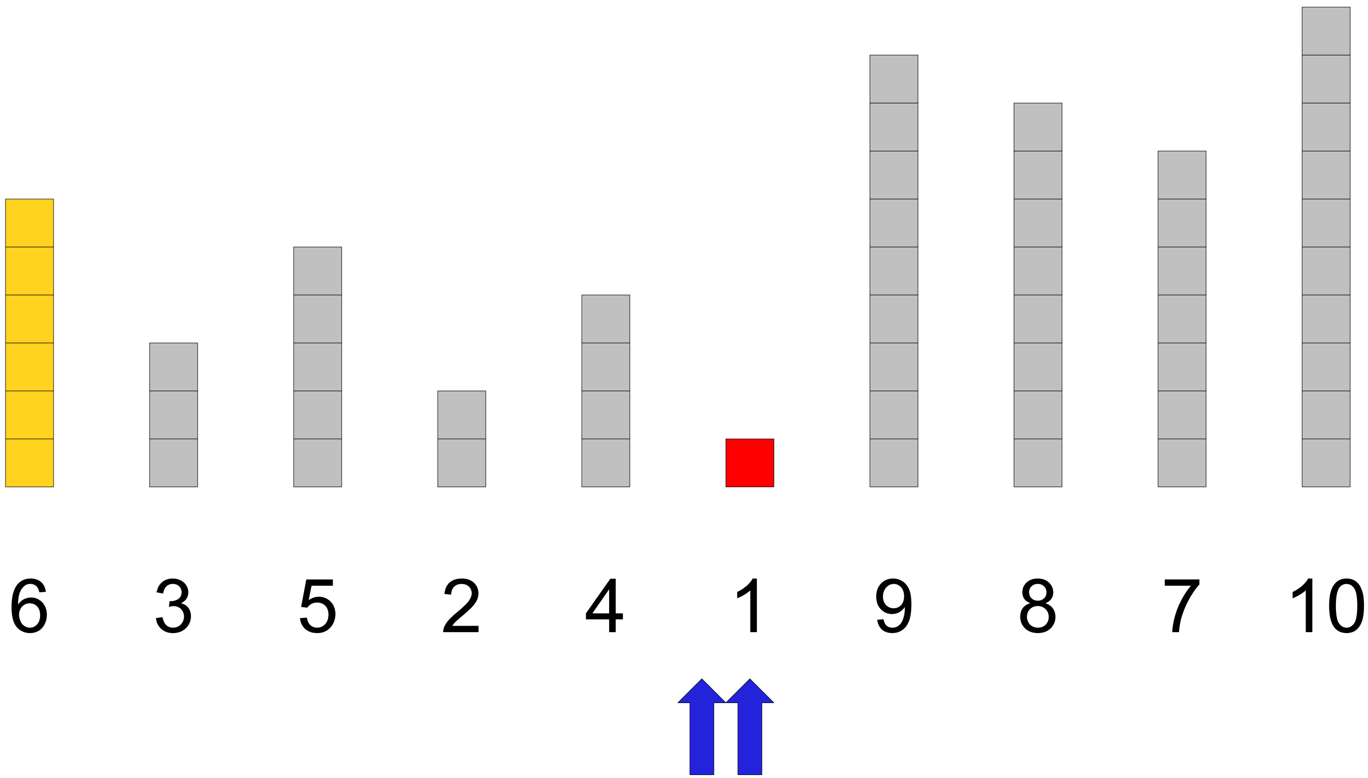

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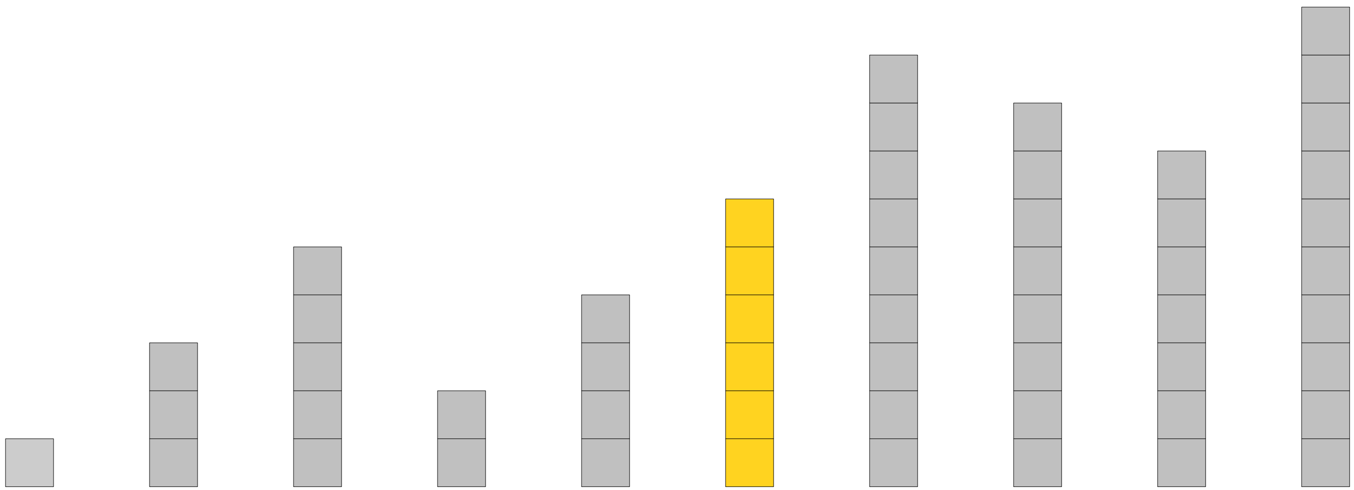

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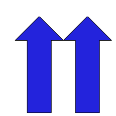

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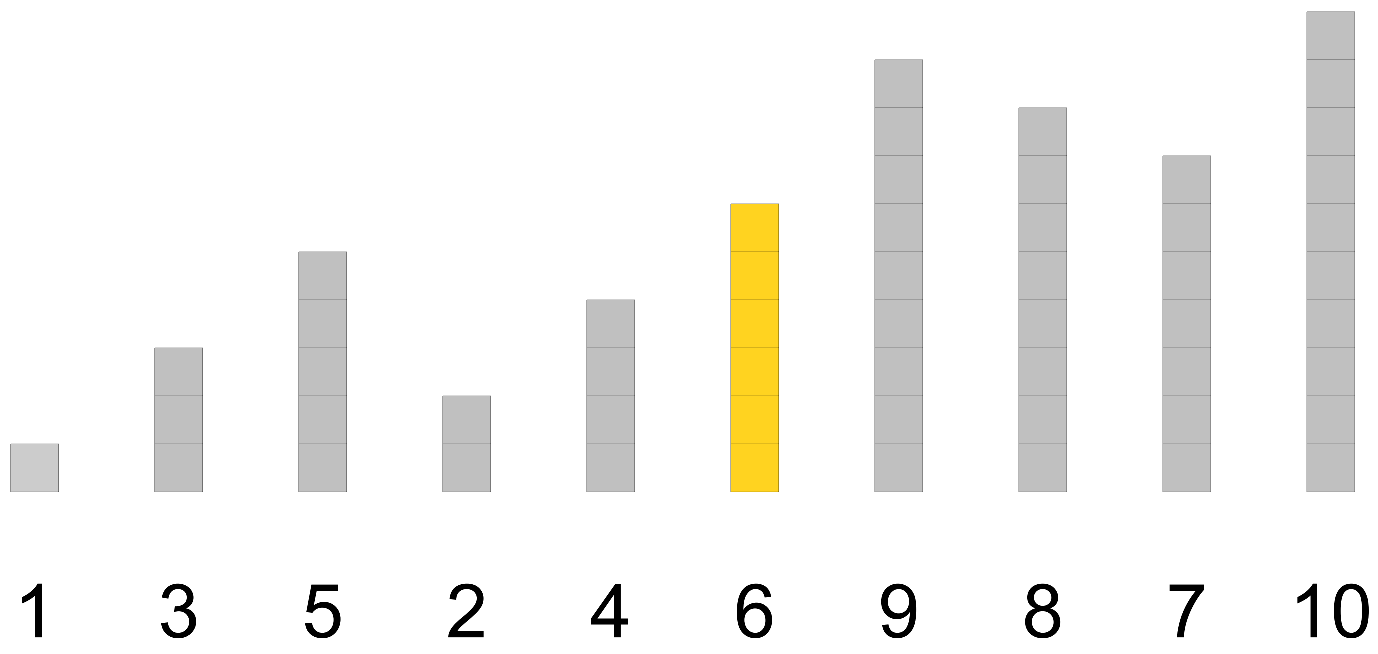

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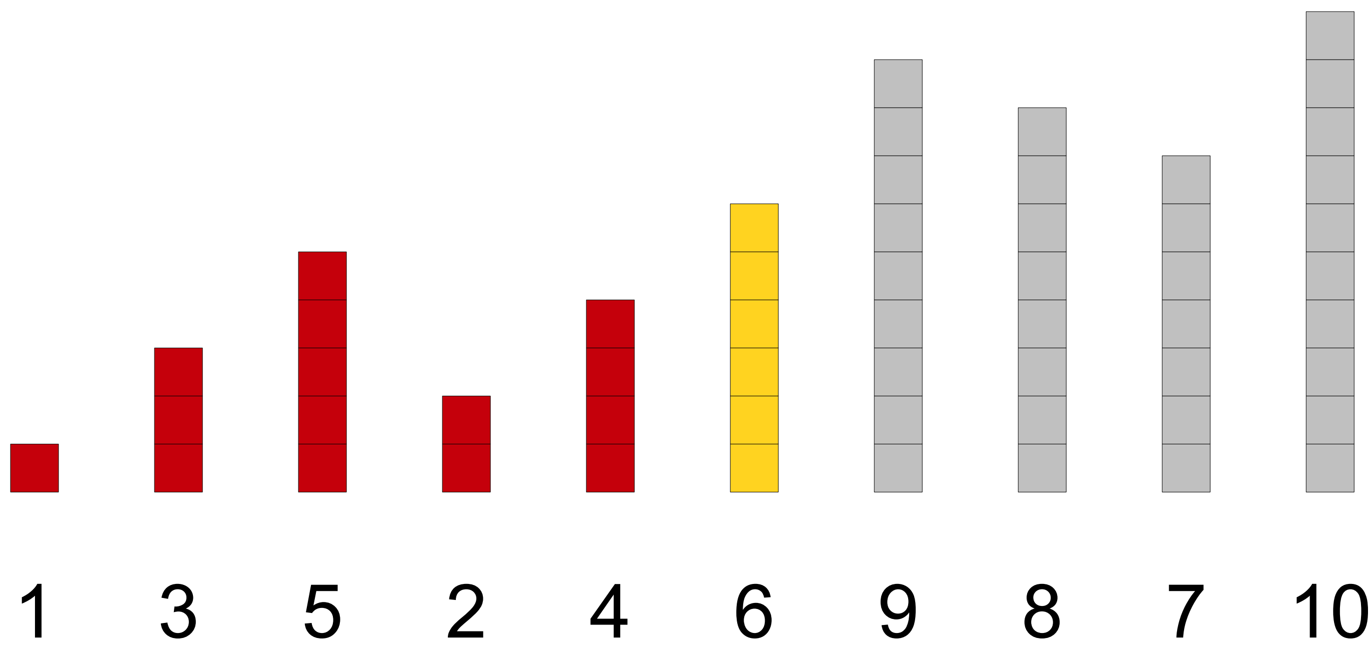

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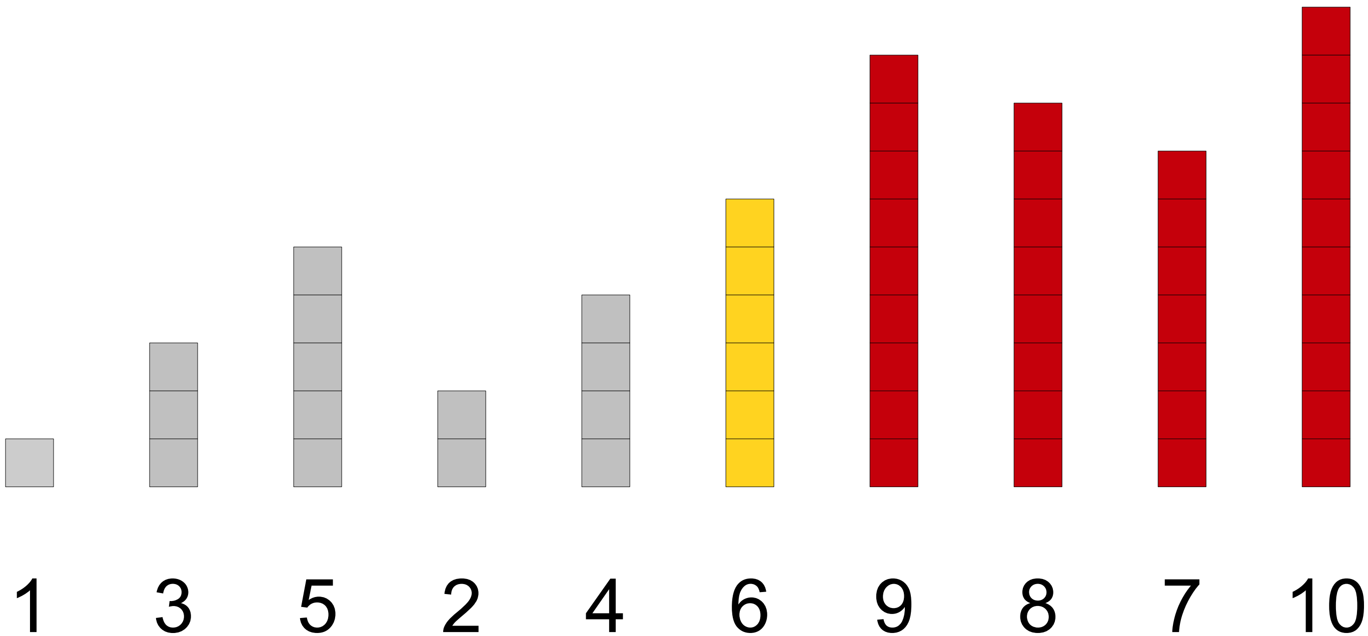

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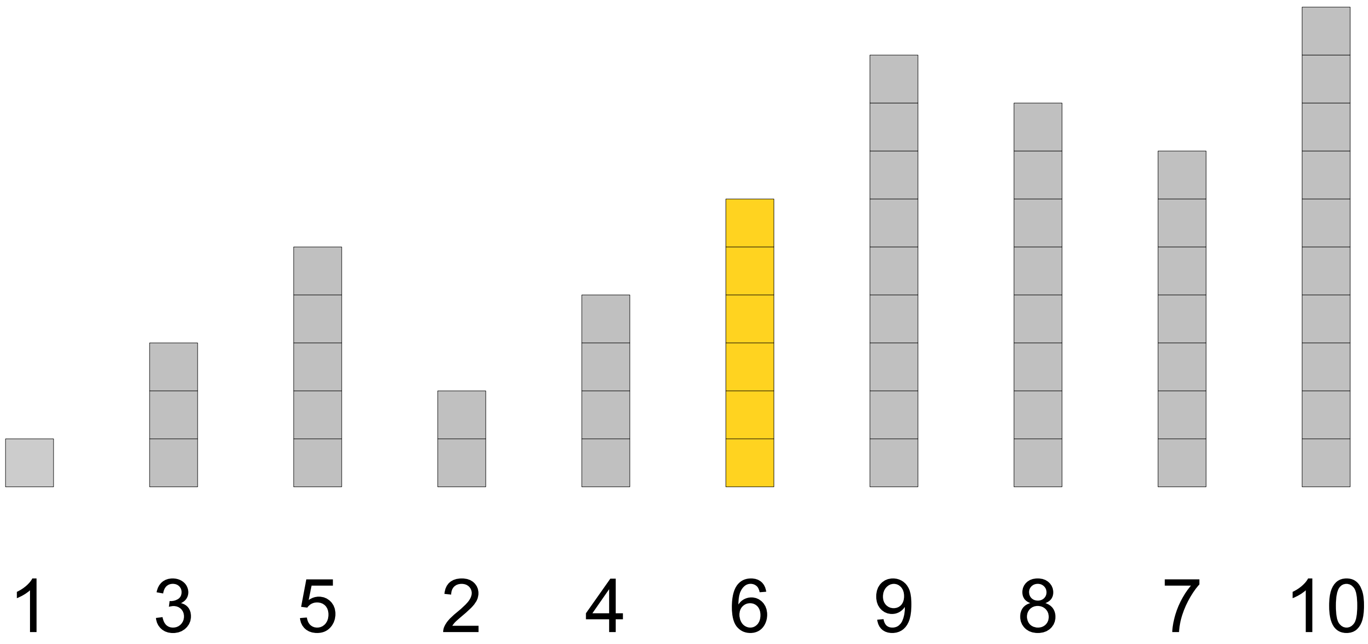


Code for Partition

Code for Partition

```
int partition(Vector<int>& v
  int pivot = v[low];
  int left  = low + 1, right

  while (left < right) {
    while (left < right && v
    while (left < right && v

    if (left < right) swap(v
  }

  if (pivot < v[right]) retu
  swap(v[low], v[right]);
  return right;
}
```

A Partition-Based Sort

- Idea:
  - Partition the array around some element.
  - Recursively sort the left and right halves.
- This works extremely quickly.
- In fact... the algorithm is called ***quicksort***.

Quicksort
(Pseudocode)

Quicksort

Quicksort

```
void quicksort(Vector<int>& v, int low, int high) {
    if (low >= high) return;

    int partitionPoint = partition(v, low, high);
    quicksort(v, low, partitionPoint - 1);
    quicksort(v, partitionPoint + 1, high);
}
```

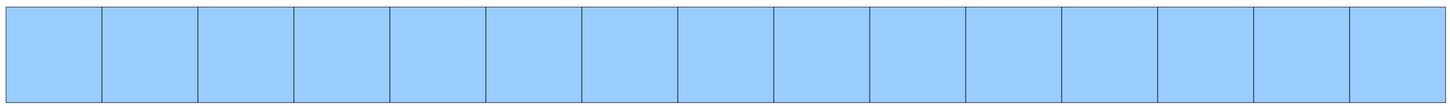
How fast is quicksort?

It depends on our choice of pivot.

What is the perfect pivot?

The median of the elements.

Suppose we get lucky...



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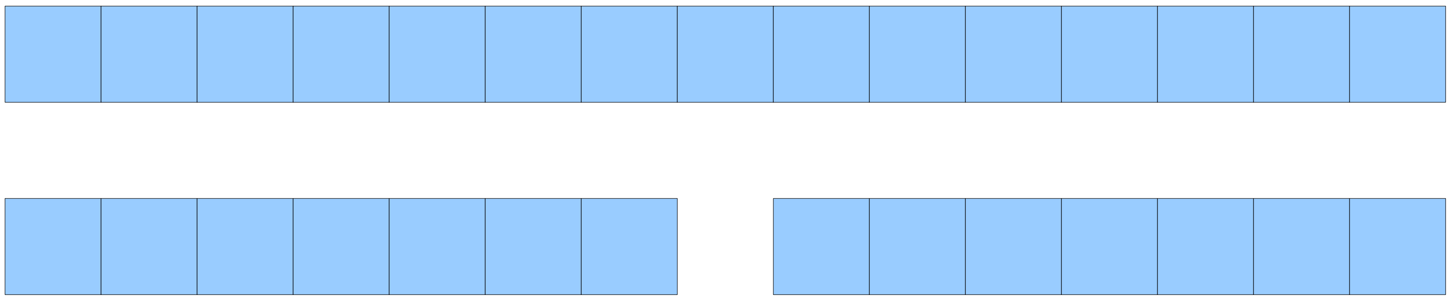

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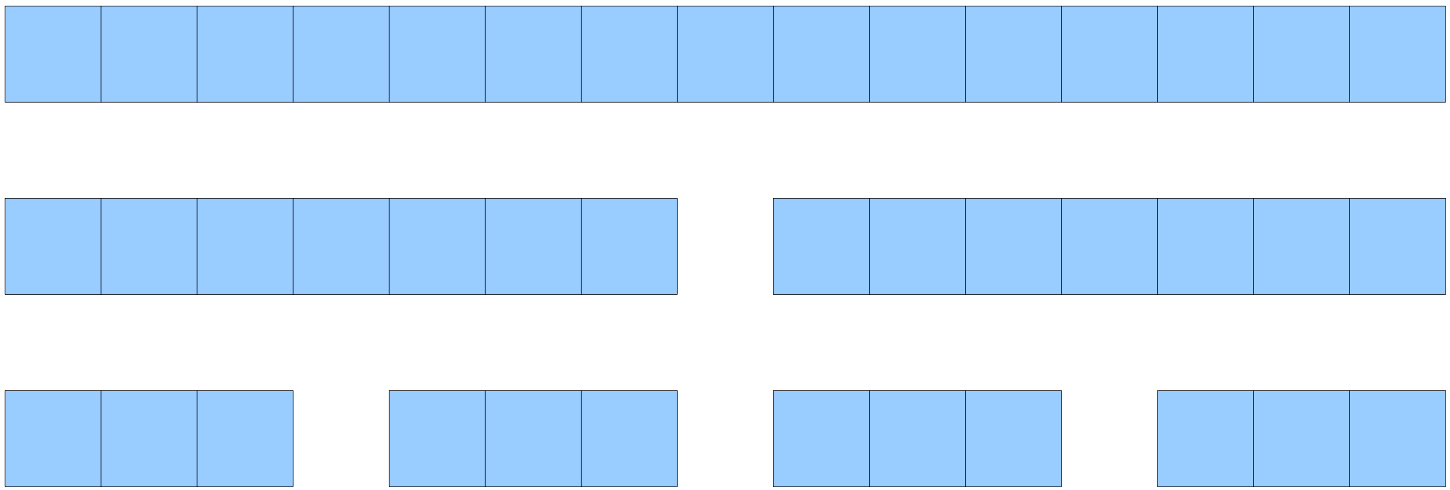

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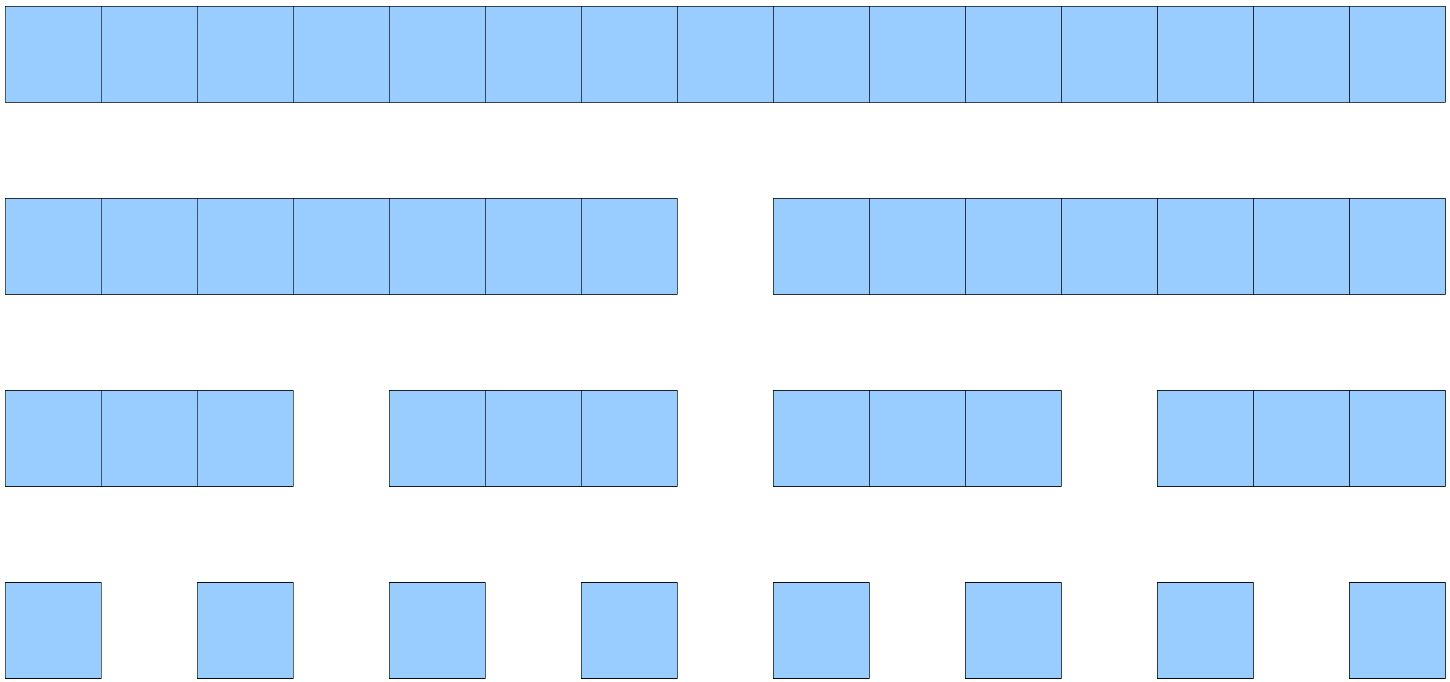

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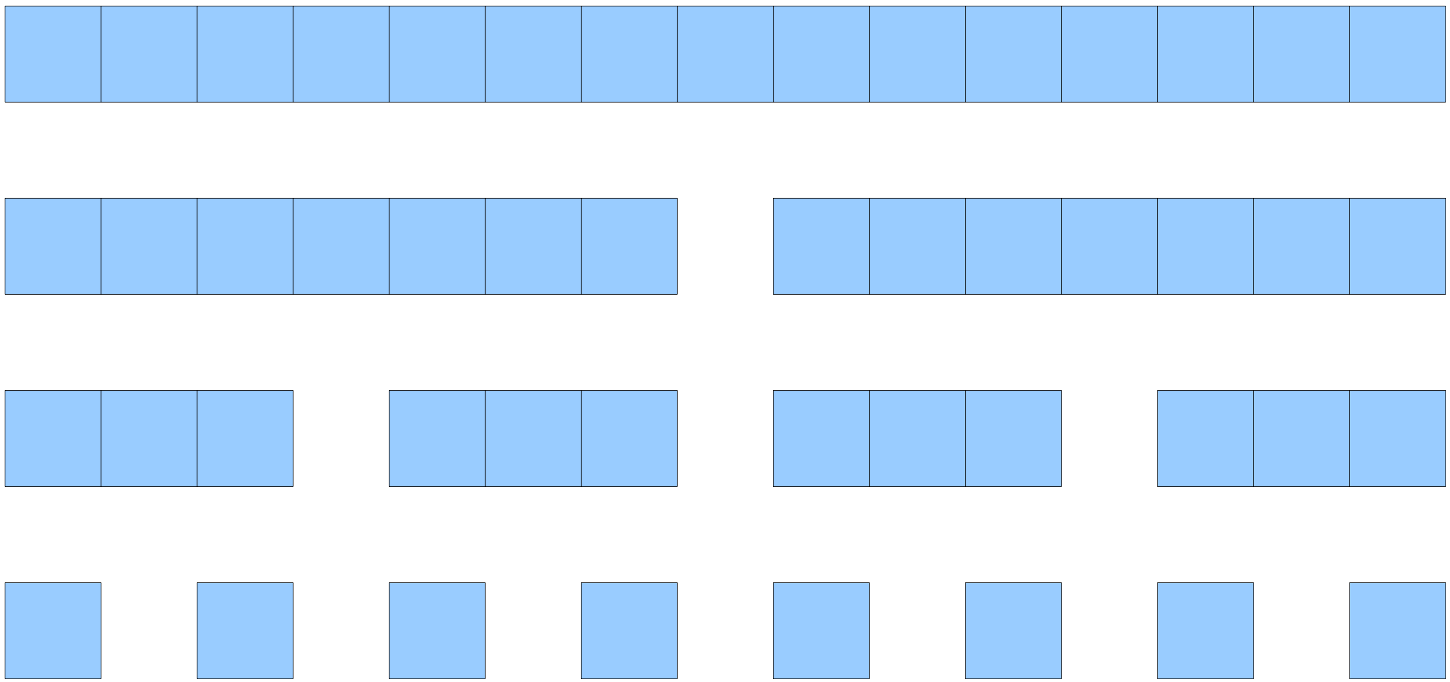
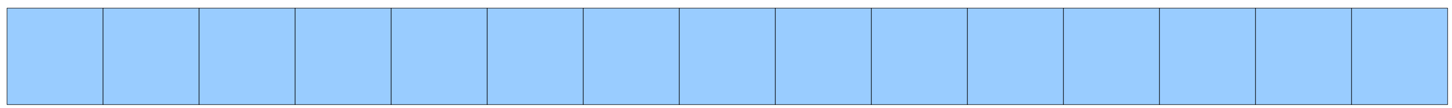


O(*n* log *n*)

Suppose we get *sorta* lucky...



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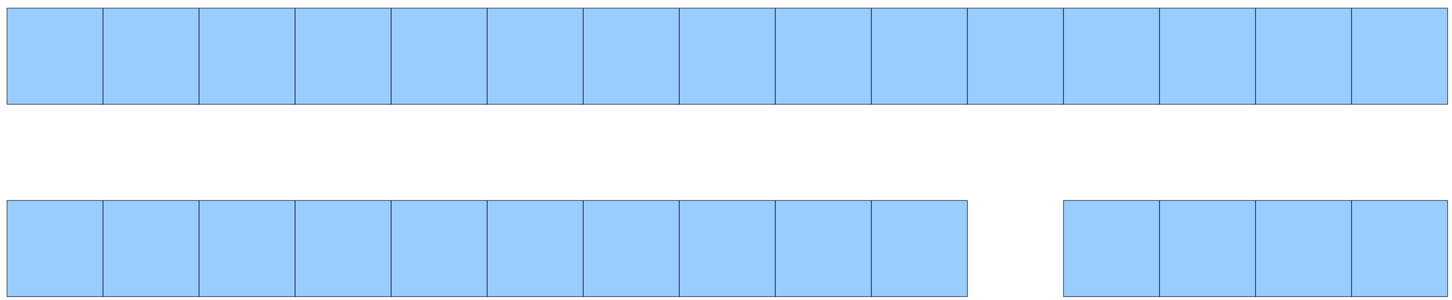

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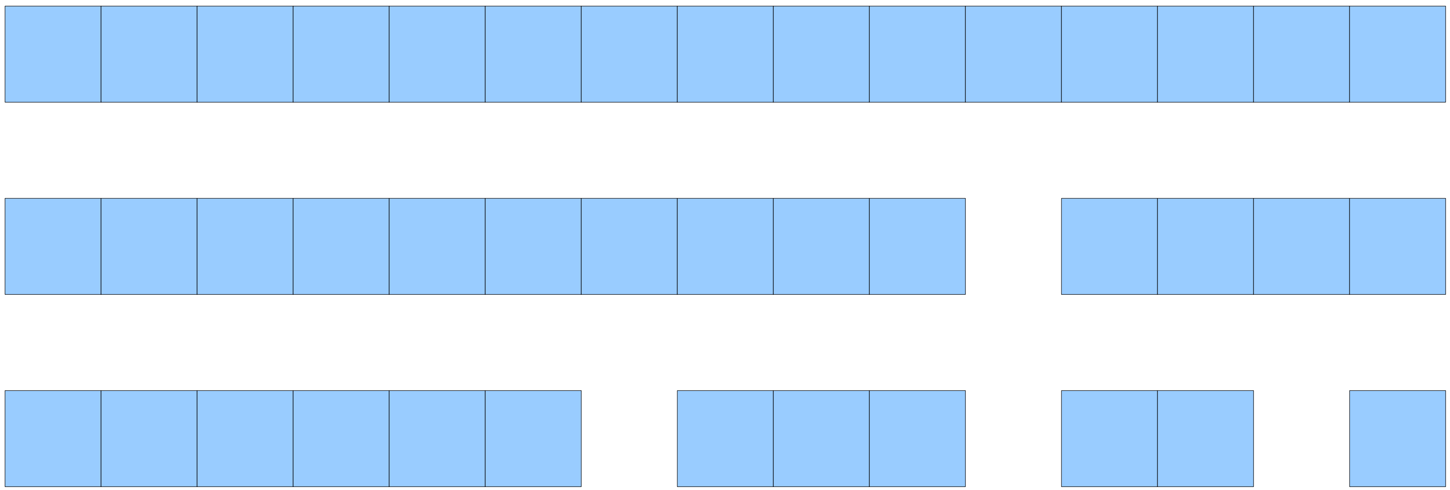

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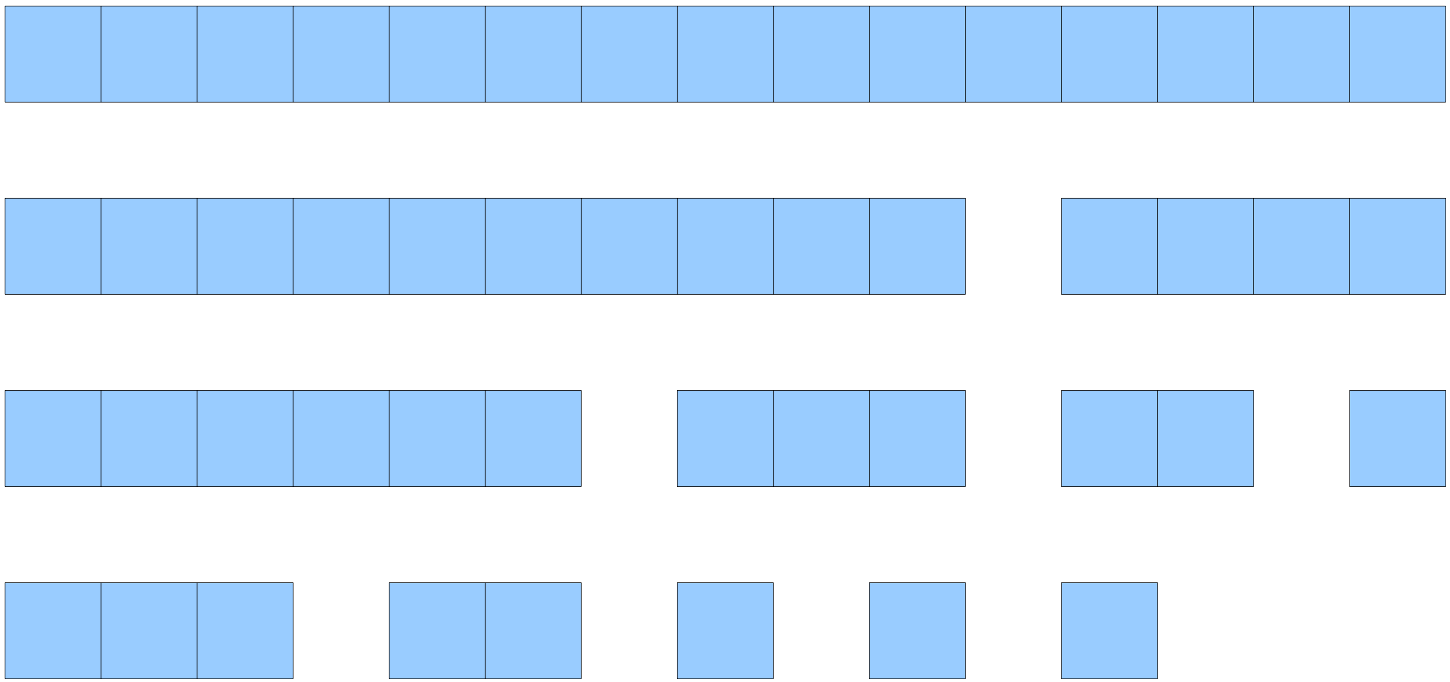

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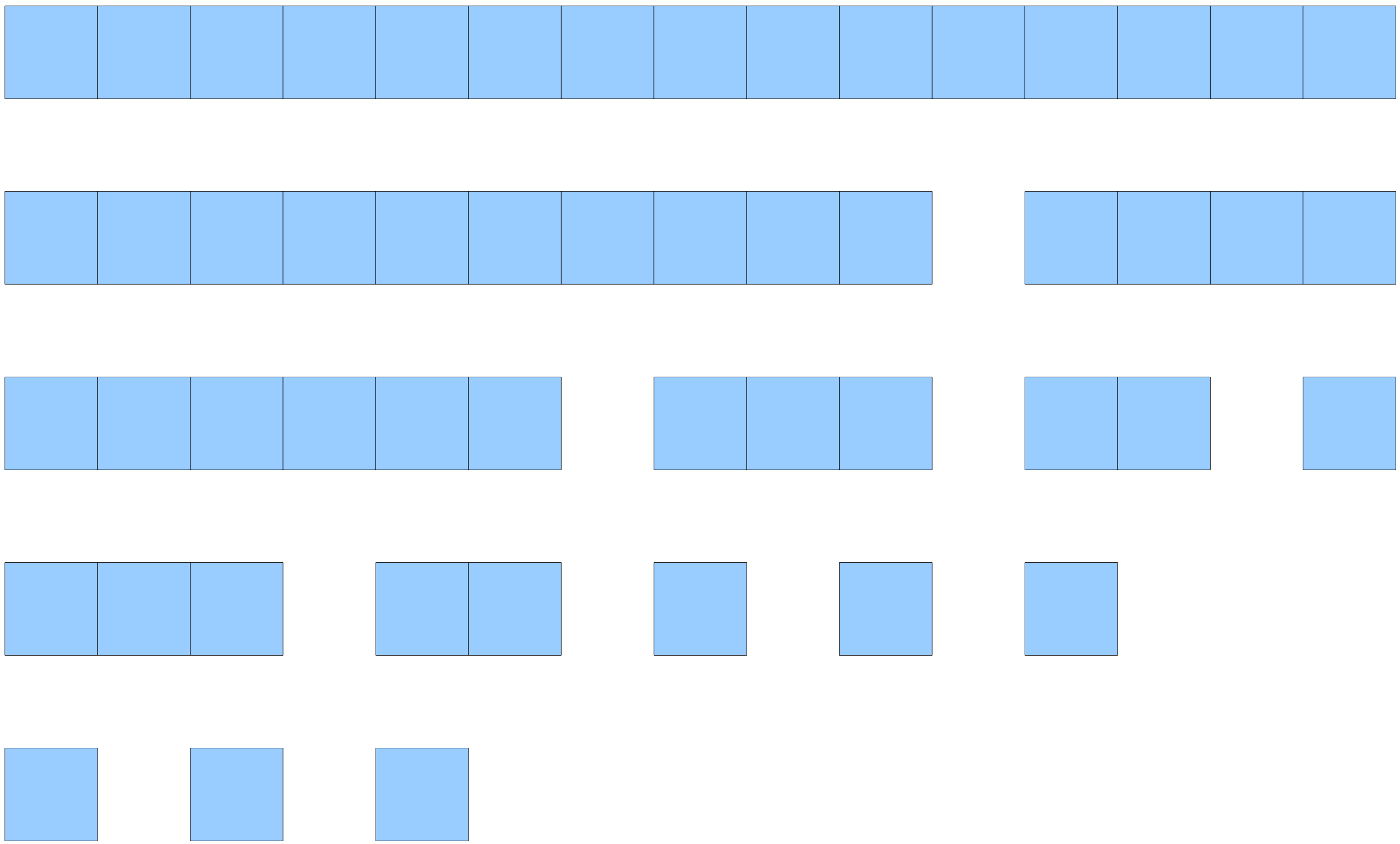

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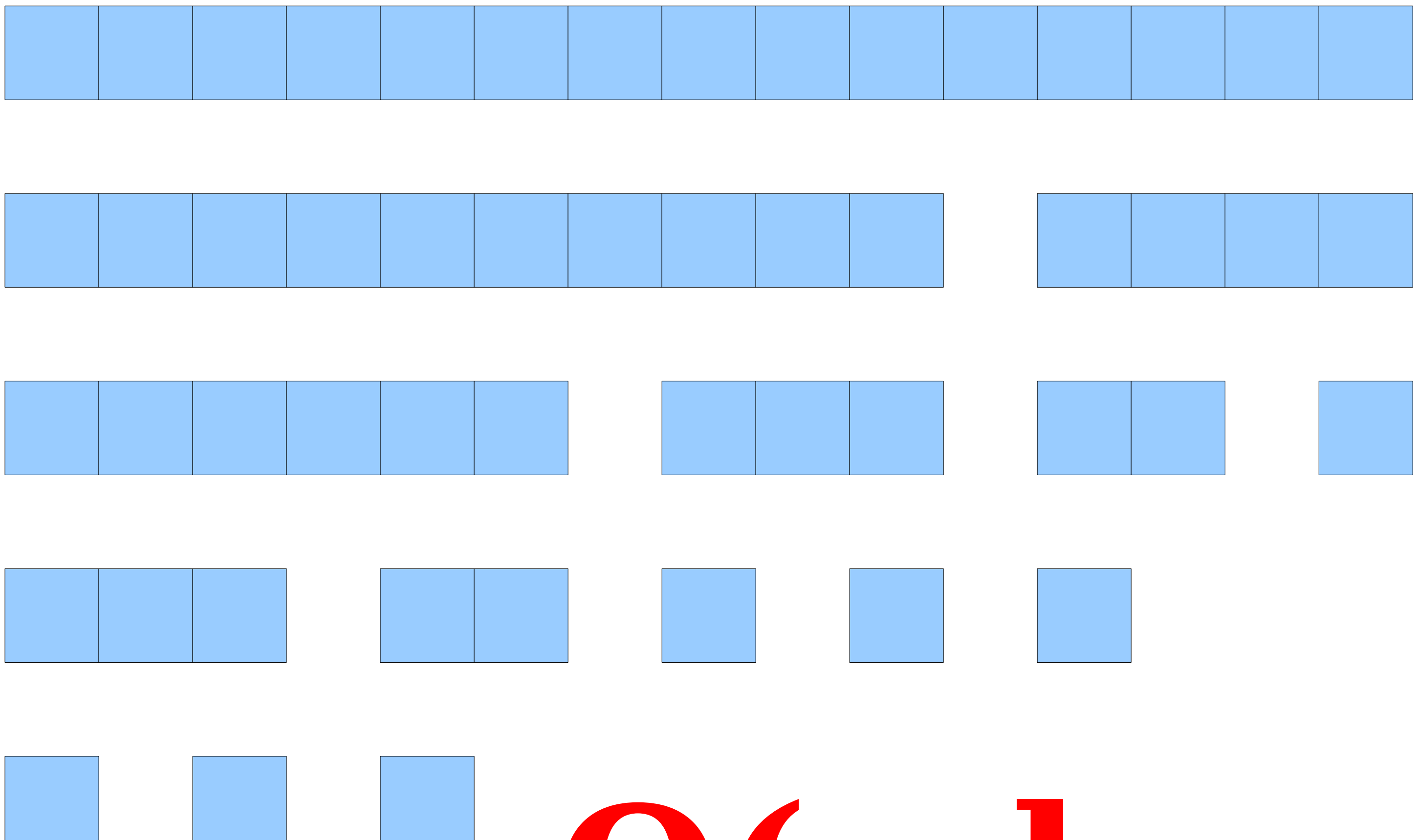
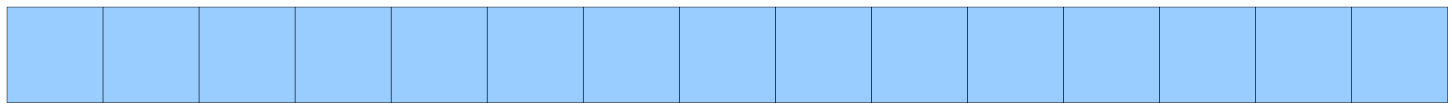
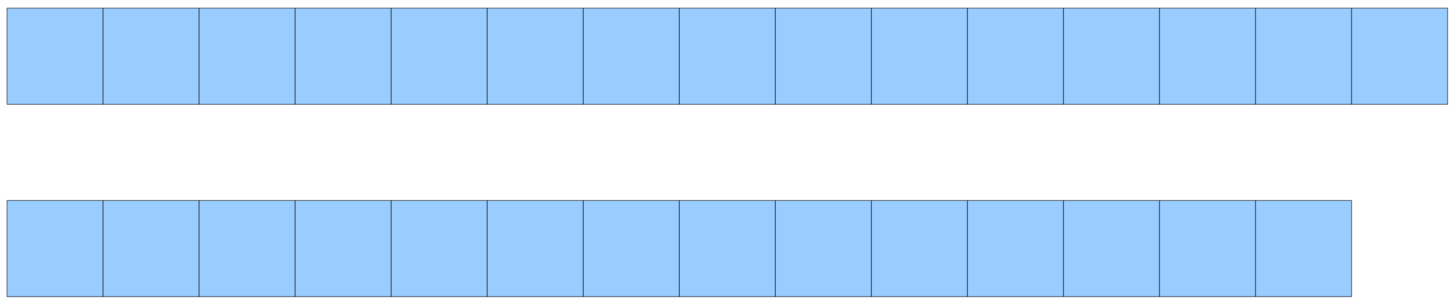


O(*n* log *n*)

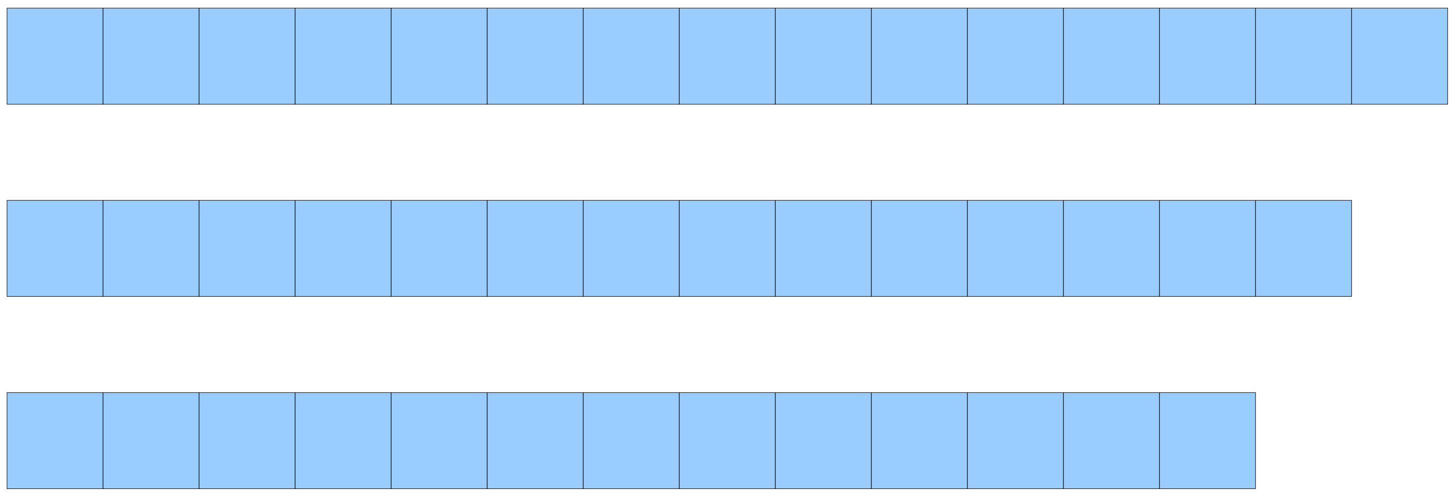
Suppose we get **unlucky**



Suppose we get **unlucky**

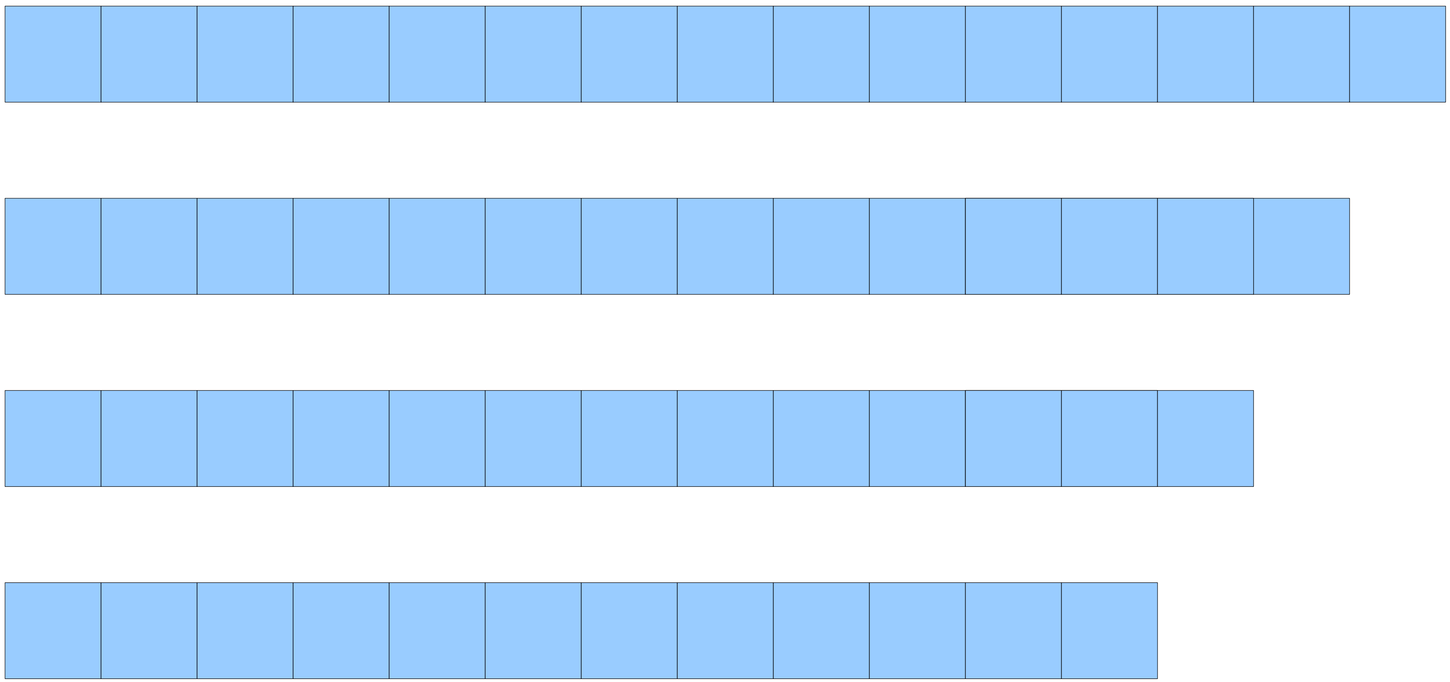


Suppose we get **unlucky**



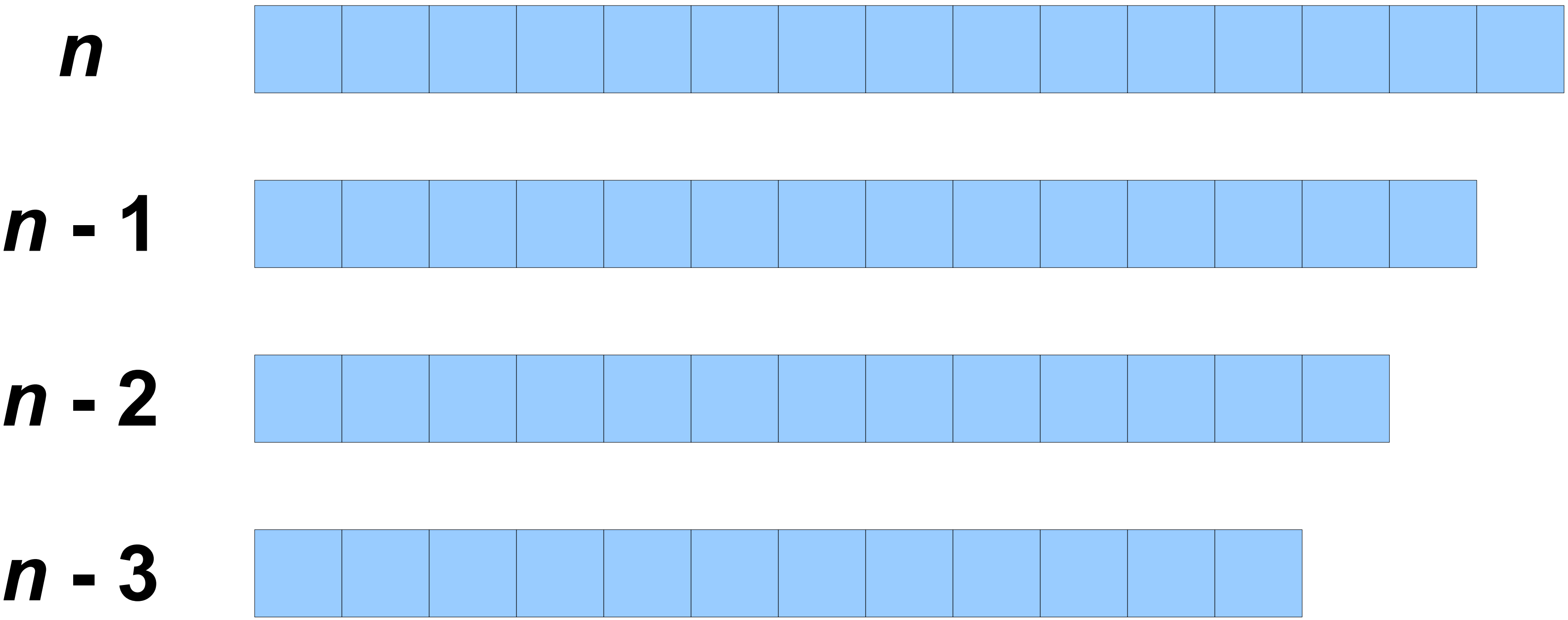
Suppose we get **unlucky**

Suppose we get **unlucky**



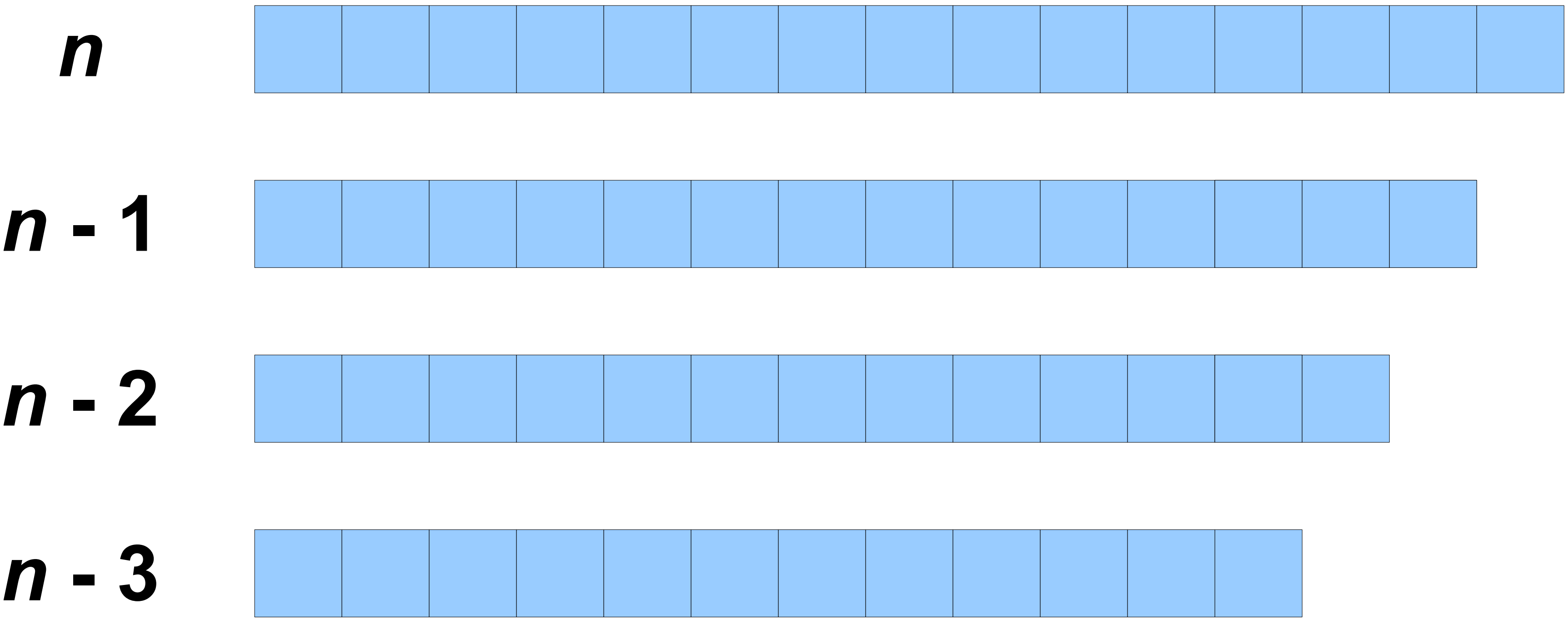
...

Suppose we get **unlucky**



...

Suppose we get **unlucky**



…

$O(n^2)$

Quicksort is Strange

- In most cases, quicksort has runtime O(n log n).
- In the worst case, quicksort has runtime O(n^2).
- How can you avoid this?
- **Pick better pivots!**
  - Pick the median.
    - Can be done in O(n), but *expensive* O(n).
  - Pick the “median-of-three.”
    - Better than nothing, but still can hit worst case.
  - Pick randomly.
    - Extremely low probability of O(n^2).

Quicksort is Fast

- Although quicksort is O(n^2) in the worst case, it is one of the fastest known sorting algorithms.
- O(n^2) behavior is extremely unlikely with random pivots; runtime is usually a very good O(n log n).
- Faster than mergesort because we don't need to copy elements into new **`Vector`**s
- It's hard to argue with the numbers...

Timing Quicksort

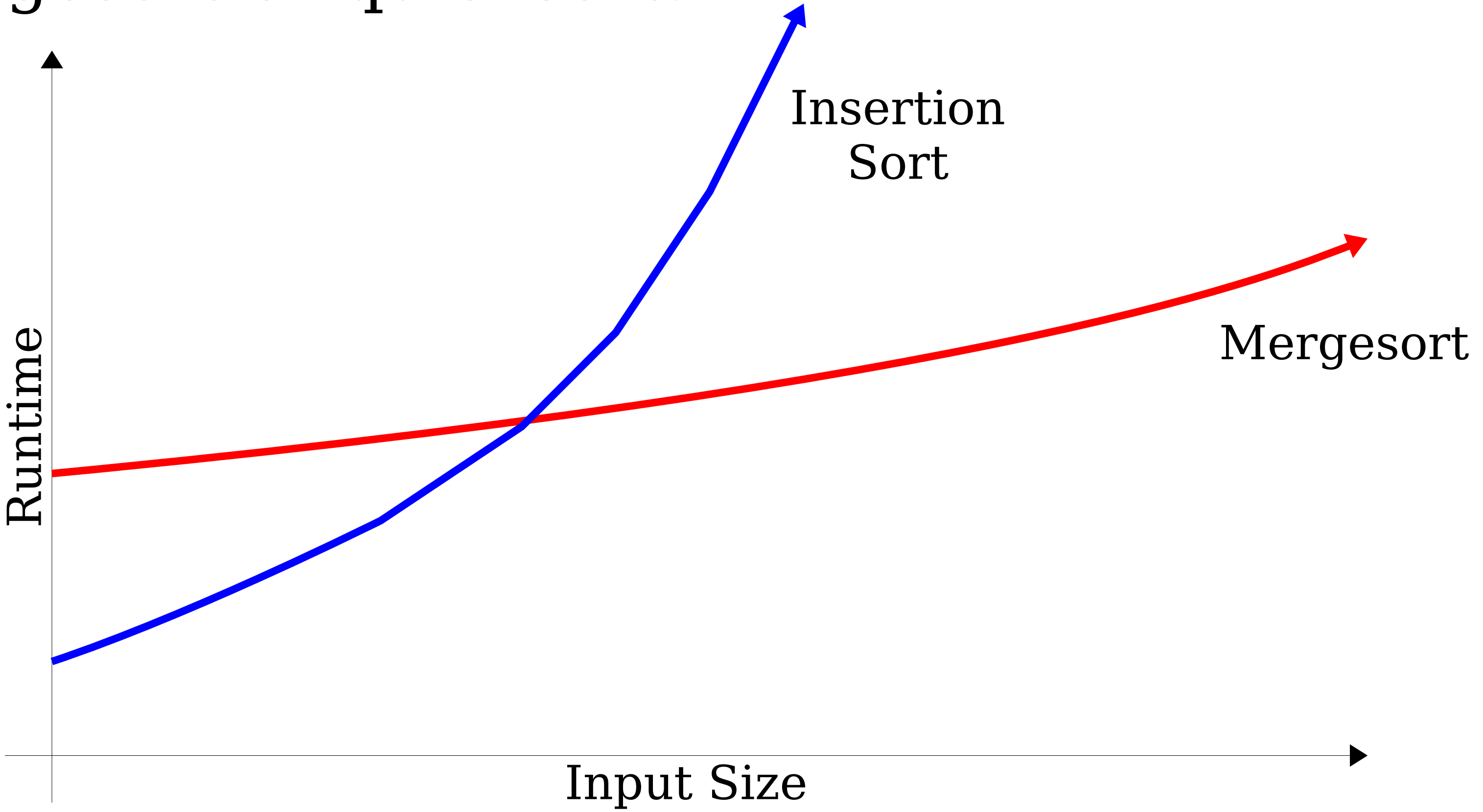
Size	Selection Sort	Insertion Sort	“Split Sort”	Mergesort
10000	0.304	0.160	0.161	0.006
20000	1.218	0.630	0.387	0.010
30000	2.790	1.427	0.726	0.017
40000	4.646	2.520	1.285	0.021
50000	7.395	4.181	2.719	0.028
60000	10.584	5.635	2.897	0.035
70000	14.149	8.143	3.939	0.041
80000	18.674	10.333	5.079	0.042
90000	23.165	12.832	6.375	0.048

Timing Quicksort

Size	Selection Sort	Insertion Sort	"Split Sort"	Mergesort	Quicksort
10000	0.304	0.160	0.161	0.006	0.001
20000	1.218	0.630	0.387	0.010	0.002
30000	2.790	1.427	0.726	0.017	0.004
40000	4.646	2.520	1.285	0.021	0.005
50000	7.395	4.181	2.719	0.028	0.006
60000	10.584	5.635	2.897	0.035	0.008
70000	14.149	8.143	3.939	0.041	0.009
80000	18.674	10.333	5.079	0.042	0.009
90000	23.165	12.832	6.375	0.048	0.012

An Interesting Observation

- Big-O notation talks about long-term growth, but says nothing about small inputs.
- For small inputs, insertion sort can be better than mergesort or quicksort.



Hybrid Sorting Algorithms

- Modify the mergesort algorithm to switch to insertion sort when the input gets sufficiently small.
- This is called a ***hybrid sorting algorithm***.

Hybrid Mergesort (Pseudocode)

Hybrid Mergesort

```
void hybridMergesort(Vector<int>& v) {
   if (v.size() <= kCutoffSize) {
      insertionSort(v);
   } else {
       Vector<int> left, right;
       for (int i = 0; i < v.size() / 2; i++)
           left += v[i];
       for (int i = v.size() / 2; i < v.size(); i++)
           right += v[i];

       hybridMergesort(left);
       hybridMergesort(right);

       merge(left, right, v);
    }
}
```

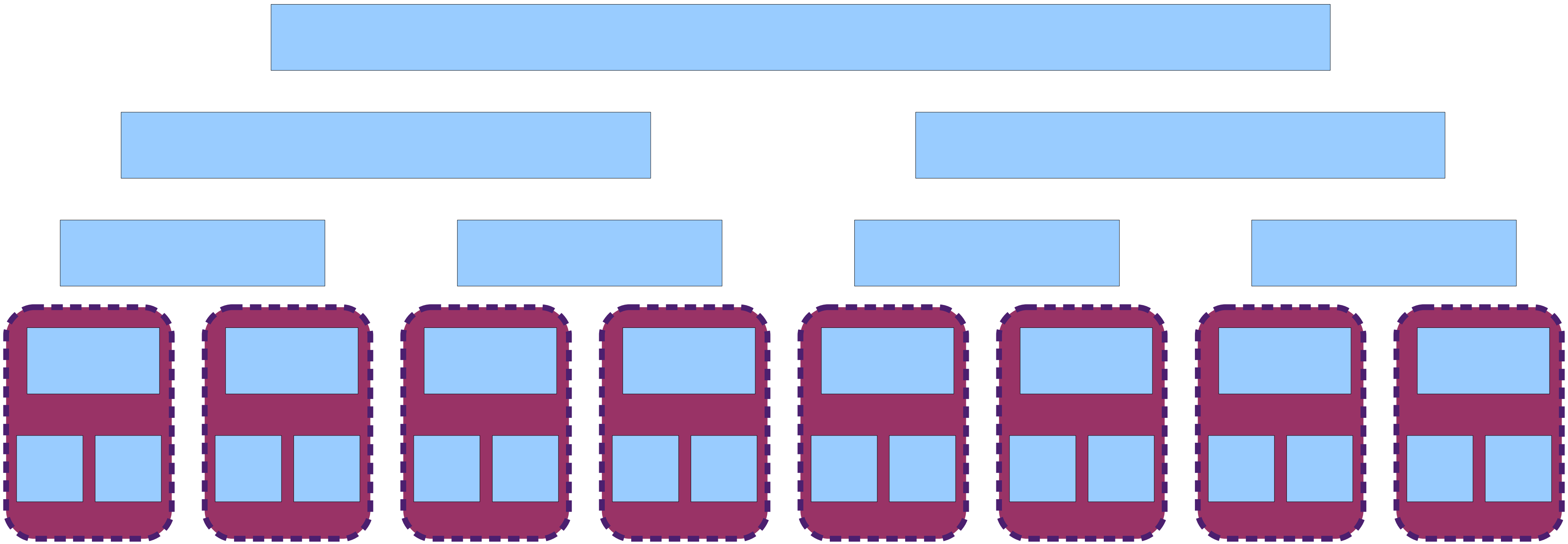
Hybrid Mergesort

```
void hybridMergesort(Vector<int>& v) {
   if (v.size() <= kCutoffSize) {
      insertionSort(v);
   } else {
       Vector<int> left, right;
       for (int i = 0; i < v.size() / 2; i++)
           left += v[i];
       for (int i = v.size() / 2; i < v.size(); i++)
           right += v[i];

       hybridMergesort(left);
       hybridMergesort(right);

       merge(left, right, v);
    }
}
```

Hybrid Sorting Algorithms



Runtime for Hybrid Mergesort

Size	Mergesort	Hybrid Mergesort	Quicksort
100000	0.063	0.019	0.012
300000	0.176	0.061	0.060
500000	0.283	0.091	0.063
700000	0.396	0.130	0.089
900000	0.510	0.165	0.118
1100000	0.608	0.223	0.151
1300000	0.703	0.246	0.179
1500000	0.844	0.28	0.215
1700000	0.995	0.326	0.243
1900000	1.070	0.355	0.274

Hybrid Sorts in Practice

- Introspective Sort (*Introsort)*
  - Based on quicksort, insertion sort, and ***heapsort***.
  - Heapsort is $O(n \log n)$ and a bit faster than mergesort.
  - Uses quicksort, then switches to heapsort if it looks like the algorithm is degenerating to $O(n^2)$.
  - Uses insertion sort for small inputs.
  - Gains the raw speed of quicksort without any of the drawbacks.

Runtime for Introsort

Size	Mergesort	Hybrid Mergesort	Quicksort
100000	0.063	0.019	0.012
300000	0.176	0.061	0.060
500000	0.283	0.091	0.063
700000	0.396	0.130	0.089
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Runtime for Introsort

Size	Mergesort	Hybrid Mergesort	Quicksort	Introsort
100000	0.063	0.019	0.012	0.009
300000	0.176	0.061	0.060	0.028
500000	0.283	0.091	0.063	0.043
700000	0.396	0.130	0.089	0.060
900000	0.510	0.165	0.118	0.078
1100000	0.608	0.223	0.151	0.092
1300000	0.703	0.246	0.179	0.107
1500000	0.844	0.28	0.215	0.123
1700000	0.995	0.326	0.243	0.139
1900000	1.070	0.355	0.274	0.158

We've spent all of our time talking about **fast** and **efficient** sorting algorithms.

However, we have neglected to find **slow** and **inefficient** sorting algorithms.

Sorting the Slow Way: An Analysis of Perversely Awful Randomized Sorting Algorithms

Hermann Gruber[1] and Markus Holzer[2] and Oliver Ruepp[2]

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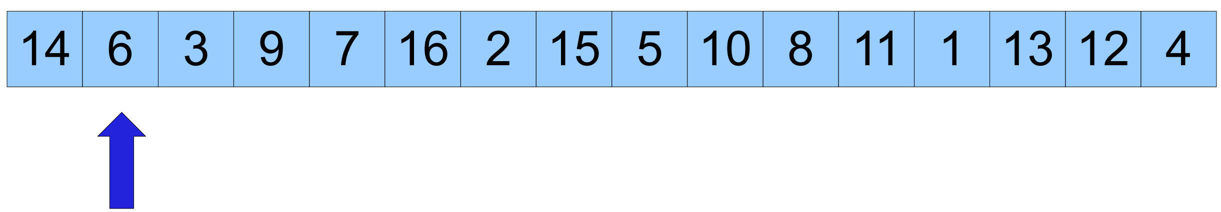
Introducing **Bogosort**

Search

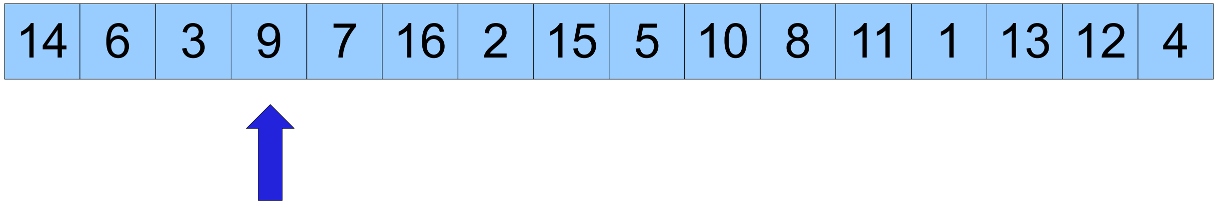
- Now that we're done with sorting, I want to make sure we understand why sorting is so useful.
- Let's say you have a **`Vector<int>`**, and you want to see if it contains some integer **`x`**, how can we do it?

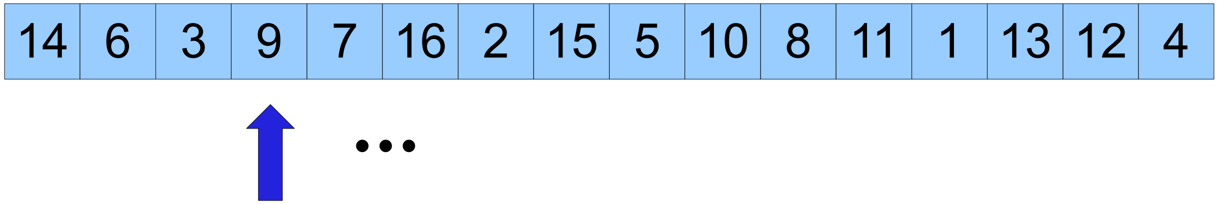
14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4


14 6 3 9 7 16 2 15 5 10 8 11 1 13 12 4

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4


14 6 3 9 7 16 2 15 5 10 8 11 1 13 12 4


14 6 3 9 7 16 2 15 5 10 8 11 1 13 12 4
…

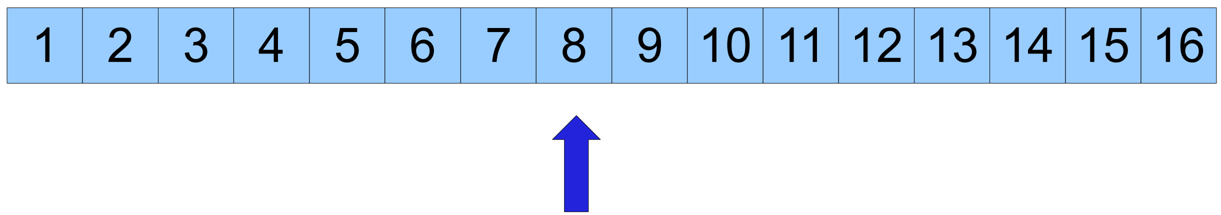
Search

- If your data is unsorted, then the best you can do is **Linear Search**
- What if your data is sorted?

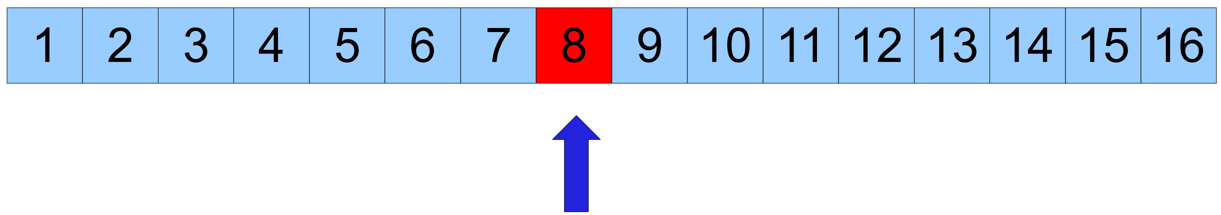
```
x = 6
```

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

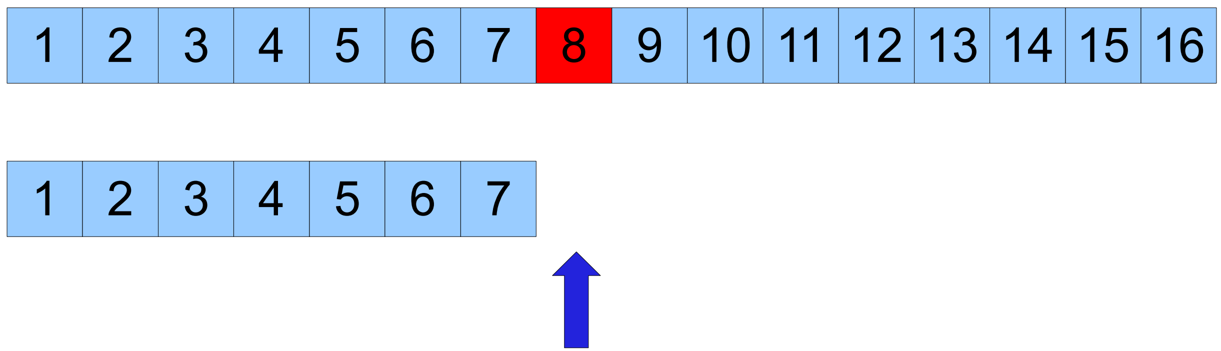
`x = 6`



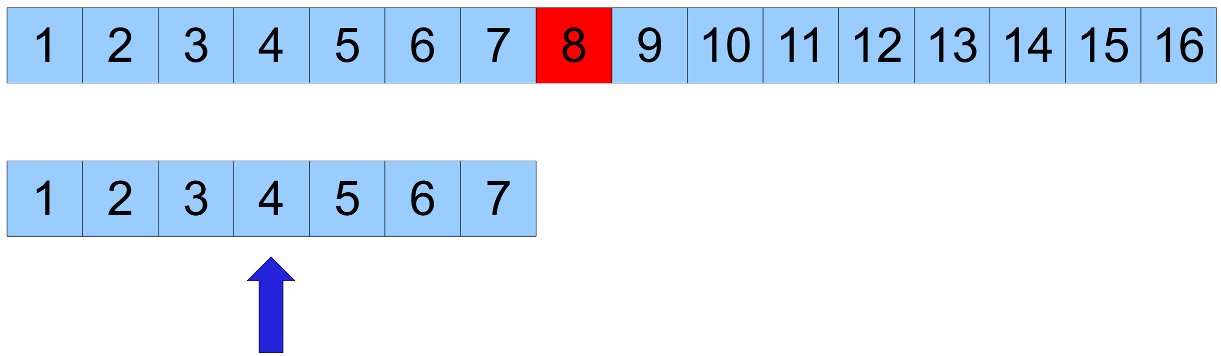
`x = 6`



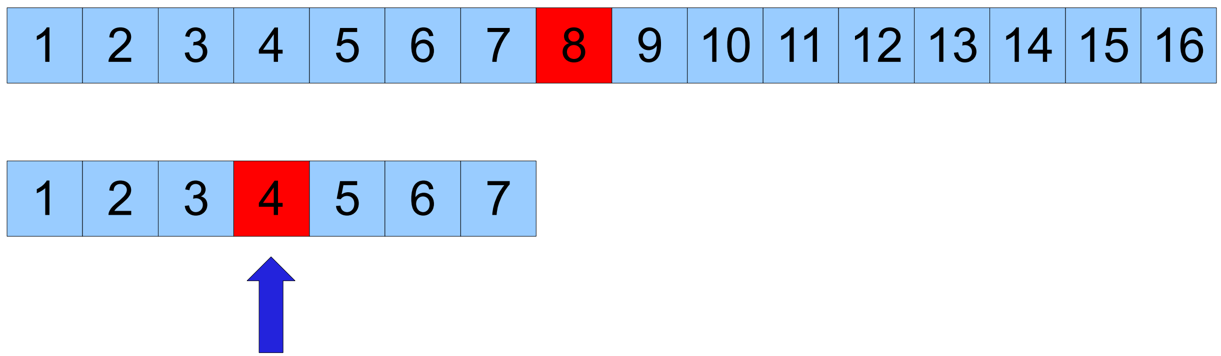
x = 6



x = 6



`x = 6`



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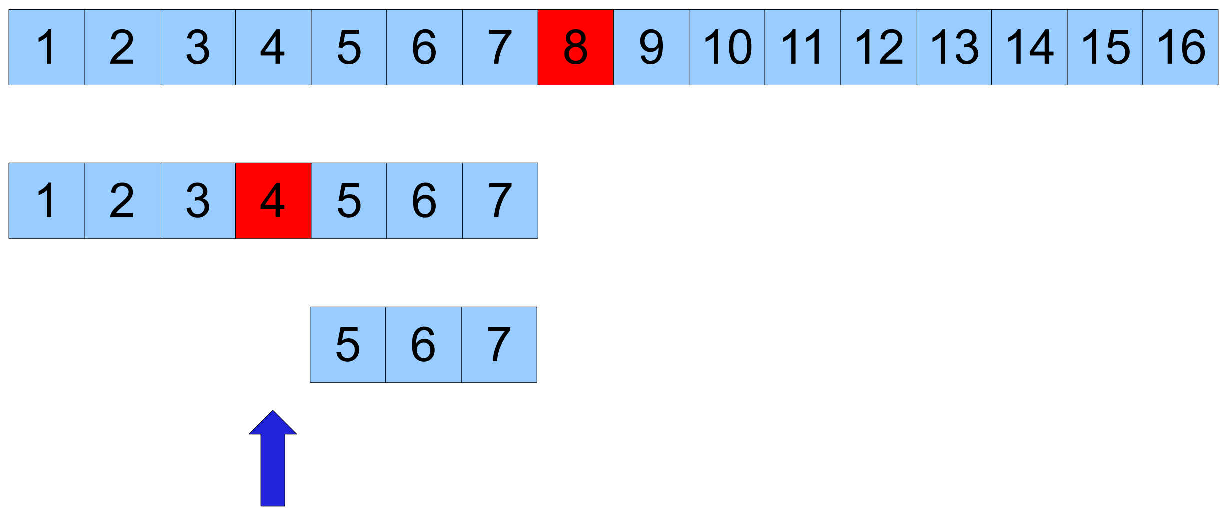

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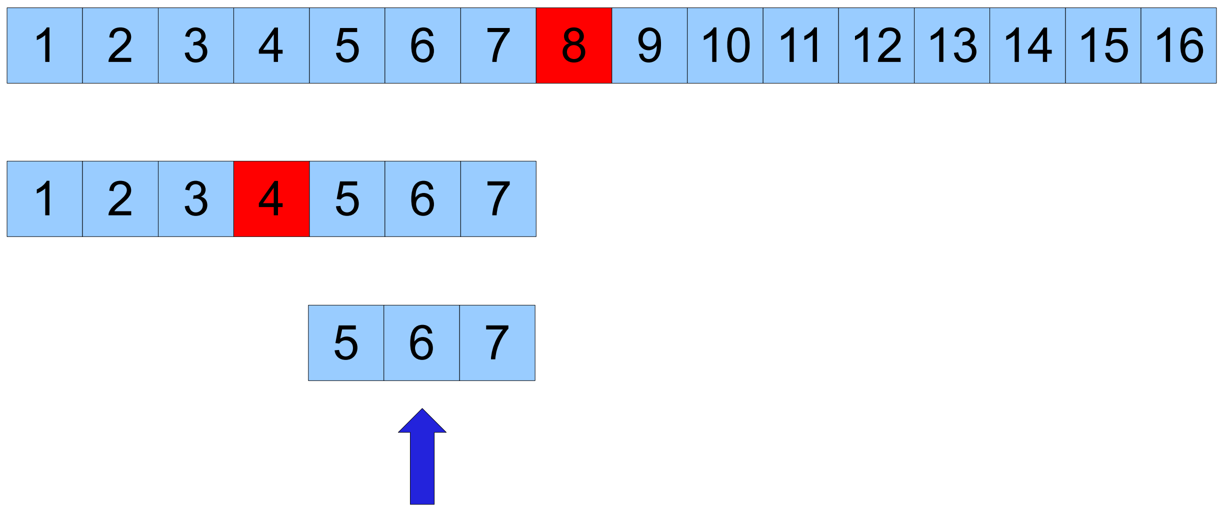
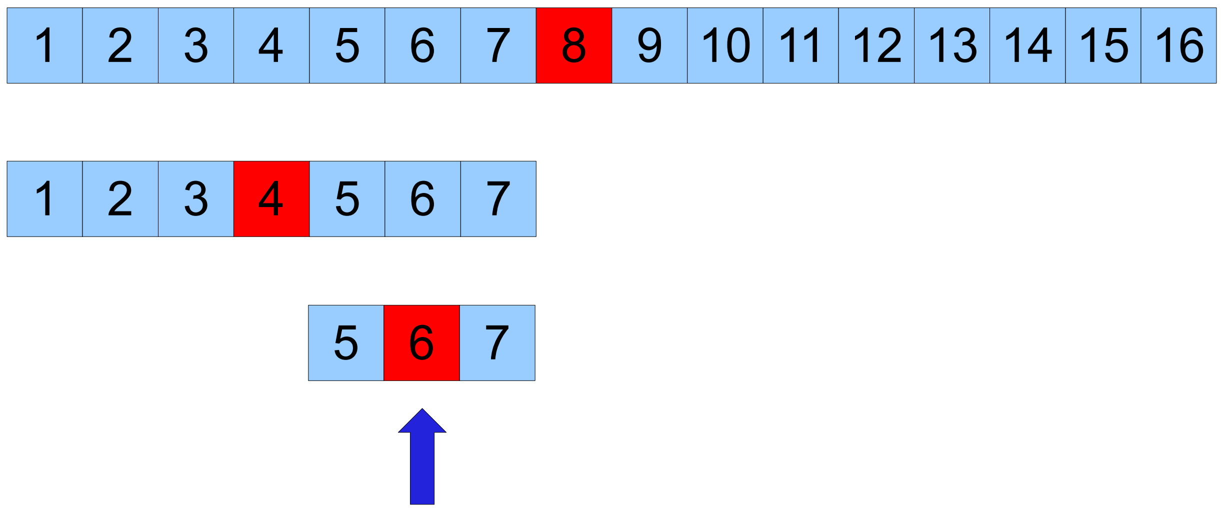
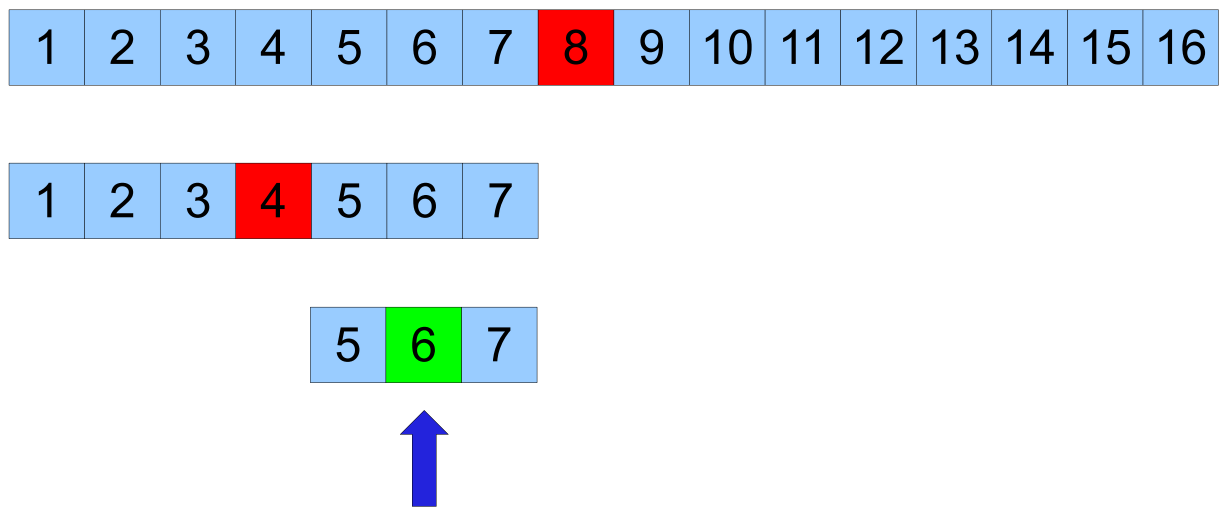


x = 6



x = 6



Search

- If our data is sorted we can run **Binary Search**
- Idea: Keep “discarding” halves of the remaining integers until we either find what we want or run out of elements.

Binary Search (Pseudocode)

Binary Search

```
bool BinarySearch(Vector<int> &v, int x) {
   int leftIndex = 0;
   int rightIndex = v.size() - 1;
   while (leftIndex < rightIndex) {
      int mid = (leftIndex + rightIndex)/2;
      if (v[mid] == x)
         return true;
      if (v[mid] < x)
         leftIndex = mid+1;
      else
         rightIndex = mid-1;
   }
   return false;
}
```

Binary Search

- Linear search runs in **O(n)**
- It turns out that Binary Search runs in **O(log n)**
  - Proof is similar to Mergesort: how many times do we need to divide **n** by 2 until we reach a single element?

Binary Search

- So if we have an unsorted **`Vector`** and want to find a single element, should we sort it first, then run binary search?
  - Sort + Binary Search = **O(n log n)**
  - Linear Search = **O(n)**
- What if we had a way to make sure the Vector was **always** sorted?
  - Our **`Set`** class does something like this.
    - We'll cover this in a couple weeks

Next Time

- **Designing Abstractions**
  - How do you build new container classes?
- **Class Design**
  - What do classes look like in C++?