A former student of mine (Montse Cordero) is studying different methods of teaching how to divide fractions.

A key step in her analysis is looking at the computational complexity of the different approaches using big-O notation!
Recap from Last Time
Big-O Notation

- **Big-O notation** is a quantitative way to describe the runtime of a piece of code.
- Works by dropping constants and low-order growth terms.
- For example, this code runs in time $O(n)$:
  ```c++
  for (int i = 0; i < vec.size(); i++) {
    cout << vec[i] << endl;
  }
  ```
Big-O Notation

- **Big-O notation** is a quantitative way to describe the runtime of a piece of code.
- Works by dropping constants and low-order growth terms.
- For example, this code runs in time $O(n^2)$:

```cpp
for (int i = 0; i < vec.size(); i++) {
    for (int j = 0; j < vec.size(); j++) {
        cout << (vec[i] + vec[j]) << endl;
    }
}
```
Sorting Algorithms

• The *sorting problem* is to take in a list of things (integers, strings, etc.) and rearrange them into sorted order.

• Last time, we saw *selection sort*, an algorithm that runs in time $O(n^2)$.

• This means that doubling the objects of elements to sort will (roughly) quadruple the time required.

• Our question for today: can we sort numbers faster than this?
New Stuff!
Another Idea: *Insertion Sort*
Another Idea: *Insertion Sort*

7 4 2 1 6
Another Idea: **Insertion Sort**

```
7
4
2
1
6
```
Another Idea: **Insertion Sort**

7
4
2
1
6
Another Idea: Insertion Sort

7 4 2 1 6
Another Idea: \textit{Insertion Sort}
Another Idea: **Insertion Sort**
Another Idea: Insertion Sort
Another Idea: **Insertion Sort**
Another Idea: *Insertion Sort*
Another Idea: **Insertion Sort**

```
4  2  7  1  6
```
Another Idea: **Insertion Sort**
Another Idea: **Insertion Sort**
Another Idea: **Insertion Sort**
Another Idea: **Insertion Sort**
Another Idea: *Insertion Sort*
Another Idea: *Insertion Sort*
Another Idea: **Insertion Sort**

2

1

4

7

6
Another Idea: \textit{Insertion Sort}
Another Idea: *Insertion Sort*
Another Idea: *Insertion Sort*
Another Idea: *Insertion Sort*
Another Idea: **Insertion Sort**
Another Idea: **Insertion Sort**
Another Idea: **Insertion Sort**

1  2  4  6  7
/**
 * Sorts the specified vector using insertion sort.
 *
 * @param v The vector to sort.
 */

void insertionSort(Vector<int>& v) {
    for (int i = 0; i < v.size(); i++) {
        /* Scan backwards until either (1) there is no
         * preceding element or the preceding element is
         * no bigger than us.
         */
        for (int j = i - 1; j >= 0; j--) {
            if (v[j] <= v[j + 1]) break;
            /* Swap this element back one step. */
            swap(v[j], v[j + 1]);
        }
    }
}
How Fast is Insertion Sort?

1 2 4 6 7
How Fast is Insertion Sort?
How Fast is Insertion Sort?
How Fast is Insertion Sort?
How Fast is Insertion Sort?

1 2 4 6 7
How Fast is Insertion Sort?
How Fast is Insertion Sort?
How Fast is Insertion Sort?

1

2

4

6

7
How Fast is Insertion Sort?
How Fast is Insertion Sort?
How Fast is Insertion Sort?

1  2  4  6  7
How Fast is Insertion Sort?

1 2 4 6 7
How Fast is Insertion Sort?

1 2 4 6 7
How Fast is Insertion Sort?
How Fast is Insertion Sort?

Work done: $O(n)$
How Fast is Insertion Sort?
How Fast is Insertion Sort?

7
6
4
2
1
How Fast is Insertion Sort?
How Fast is Insertion Sort?
How Fast is Insertion Sort?

6

7

4

2

1
How Fast is Insertion Sort?
How Fast is Insertion Sort?

6

7

4

2

1
How Fast is Insertion Sort?

6  7  4  2  1
How Fast is Insertion Sort?

6
4
7
2
1
How Fast is Insertion Sort?

6

4

7

2

1
How Fast is Insertion Sort?

4  6  7  2  1
How Fast is Insertion Sort?

4  6  7  2  1
How Fast is Insertion Sort?

4 6 7 2 1
How Fast is Insertion Sort?
How Fast is Insertion Sort?

4  6  2  7  1
How Fast is Insertion Sort?
How Fast is Insertion Sort?

4  2  6  7  1
How Fast is Insertion Sort?
How Fast is Insertion Sort?

2 4 6 7 1
How Fast is Insertion Sort?
How Fast is Insertion Sort?

2  4  6  7  1
How Fast is Insertion Sort?

2 4 6 7 1
How Fast is Insertion Sort?

2  4  6  1  7
How Fast is Insertion Sort?
How Fast is Insertion Sort?

2 4 1 6 7
How Fast is Insertion Sort?

2  4  1  6  7
How Fast is Insertion Sort?

2 1 4 6 7
How Fast is Insertion Sort?
How Fast is Insertion Sort?

1  2  4  6  7
How Fast is Insertion Sort?

1  2  4  6  7
How Fast is Insertion Sort?

Work done: $O(n^2)$
<table>
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<tr>
<td>90000</td>
<td>23.165</td>
<td>12.832</td>
</tr>
</tbody>
</table>
Of insertion sort and selection sort:

1. Which algorithm runs does more work at the start?
2. Which algorithm runs does more work at the end?
Thinking About $O(n^2)$

$T(n)$
Thinking About $O(n^2)$
Thinking About $O(n^2)$

$T(n)$

$T(\frac{1}{2}n)$

$T(\frac{1}{2}n)$
Thinking About $O(n^2)$

$$T(n) = T\left(\frac{1}{2}n\right) + (n/2)^2 = T\left(\frac{1}{2}n\right) + n^2/4$$
Thinking About O(n^2)

\[ T(\frac{1}{2}n) \approx \frac{1}{4}T(n) \]

\[ (n / 2)^2 = n^2 / 4 \]
Thinking About $O(n^2)$

$O(n^2) = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$
The Key Insight: *Merge*
The Key Insight: *Merge*
The Key Insight: **Merge**
The Key Insight: **Merge**
The Key Insight: *Merge*
The Key Insight: **Merge**
The Key Insight: *Merge*

![Diagram showing the process of merging two sorted lists into a single sorted list.](image-url)
The Key Insight: **Merge**
The Key Insight: *Merge*
The Key Insight: **Merge**
The Key Insight: **Merge**
The Key Insight: **Merge**

![Diagram showing the merge process with numbers 7, 8, 10 on the left and 5, 6, 9 on the right. The process involves merging the smaller lists to create the final sorted list.](image-url)
The Key Insight: **Merge**
The Key Insight: \textit{Merge}
The Key Insight: **Merge**
The Key Insight: \textit{Merge}
The Key Insight: **Merge**
The Key Insight: \textbf{Merge}
The Key Insight: *Merge*
The Key Insight: **Merge**
The Key Insight: *Merge*
The Key Insight: *Merge*
The Key Insight: **Merge**

Each step makes a single comparison and reduces the number of elements by one.

If there are $n$ total elements, this algorithm runs in time $O(n)$. 

1 2 3 4 5 6 7 8 9 10
/**
 * Given two queues of elements in sorted order, merges them together
 * into a single sorted sequence.
 *
 * This can easily be adapted to work with Vectors or other types of
 * sequences, but I thought it was easiest using Queues.
 *
 * @param one The first sorted queue.
 * @param two The second sorted queue.
 * @return A single sorted queue holding all elements of one and two.
 */
Queue<int> merge(Queue<int>& one, Queue<int>& two) {
    Queue<int> result;
    /* Keep comparing the first elements of each queue against one
     * another until one queue is exhausted.
     */
    while (!one.isEmpty() && !two.isEmpty()) {
        if (one.peek() < two.peek()) result.enqueue(one.dequeue());
        else result.enqueue(two.dequeue());
    }
    /* Add all remaining elements of the other queue to the result. */
    while (!one.isEmpty()) result.enqueue(one.dequeue());
    while (!two.isEmpty()) result.enqueue(two.dequeue());
    return result;
}
“Split Sort”
void splitSort(Vector<int>& v) {
  /* Split the vector in half */
  Vector<int> left, right;
  for (int i = 0; i < v.size() / 2; i++) {
    left += v[i];
  }
  for (int j = v.size() / 2; j < v.size(); j++) {
    right += v[i];
  }

  /* Sort each half. */
  insertionSort(left);
  insertionSort(right);

  /* Merge them back together. */
  merge(left, right, v);
}
## Performance Comparison

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## Performance Comparison

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</table>
Time-Out for Announcements!
Assignment 4

● **Reminder**: Assignment 4 is due one week from Friday.

● Following the assignment timetable we suggested, you should try to be done with the Doctors Without Orders problem by the end of the evening and start working on Disaster Planning.

● Again, remember that the midterm is coming up right after the assignment due date, so this is probably not a good place to use late days.
Practice Midterm Exam

- We’ll be holding a practice midterm exam on **Monday, February 13th** from **7PM - 10PM**, in **Hewlett 200**.

- It’s held under realistic conditions and is a fantastic way to prepare for the exam. **You should plan to attend this unless you have a compelling reason not to do so.**

- We’ll have more details about the midterm exam on Friday.
Honor Code Reminder

- The good news is that the *overwhelming majority* of you are honest and hardworking.
- The bad news is that some of you turned in Assignment 2 submissions that are clearly not your own work.
- *We take the Honor Code seriously in this course.* We’ll be writing up these cases and submitting them to the Office of Community Standards.
  - Standard sanction for a first offense is many hours of community service, an academic integrity seminar, and a possible suspension.
- Please ask for help if you need it. That’s what we’re here for!
More Assorted Sorts of Sorts!
A Better Idea

- We can speed up insertion sort by almost a factor of two by splitting the array in half, sorting each part independently, and merging the results together.
- So why not split into fourths? That would give a 4× improvement.
- So why not split into eighths? That would give an 8× improvement.
- **Question:** What happens if we *never stop* splitting?
High-Level Idea

• A recursive sorting algorithm!

• **Base Case:**
  • An empty or single-element list is already sorted.

• **Recursive step:**
  • Break the list in half and recursively sort each part.
  • Use `merge` to combine them back into a single sorted list.

• This algorithm is called `mergesort`.
void mergesort(Vector<int>& v) {
    /* Base case: 0- or 1-element lists are already sorted. */
    if (v.size() <= 1) return;

    /* Split v into two subvectors. */
    Vector<int> left, right;
    for (int i = 0; i < v.size() / 2; i++) {
        left += v[i];
    }
    for (int i = v.size() / 2; i < v.size(); i++) {
        right += v[i];
    }

    /* Recursively sort these arrays. */
    mergesort(left);
    mergesort(right);

    /* Combine them together. */
    merge(left, right, v);
}
What is the complexity of mergesort?
void mergesort(Vector<int>& v) {
    /* Base case: 0- or 1-element lists are already sorted. */
    if (v.size() <= 1) return;

    /* Split v into two subvectors. */
    Vector<int> left, right;
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        right += v[i];
    }

    /* Recursively sort these arrays. */
    mergesort(left);
    mergesort(right);

    /* Combine them together. */
    merge(left, right, v);
}
```c
void mergesort(Vector<int>& v) {
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    if (v.size() <= 1) return;

    /* Split v into two subvectors. */
    Vector<int> left, right;
    for (int i = 0; i < v.size() / 2; i++) {
        left += v[i];
    }
    for (int i = v.size() / 2; i < v.size(); i++) {
        right += v[i];
    }

    /* Recursively sort these arrays. */
    mergesort(left);
    mergesort(right);

    /* Combine them together. */
    merge(left, right, v);
}
```

$O(n)$ work

$O(n)$ work
void mergesort(Vector<int>& v) {
    /* Base case: 0- or 1-element lists are
     * already sorted.
     */
    if (v.size() <= 1) return;

    /* Split v into two subvectors. */
    Vector<int> left, right;
    for (int i = 0; i < v.size() / 2; i++) {
        left += v[i];
    }
    for (int i = v.size() / 2; i < v.size(); i++) {
        right += v[i];
    }

    /* Recursively sort these arrays. */
    mergesort(left);
    mergesort(right);

    /* Combine them together. */
    merge(left, right, v);
}
A Graphical Intuition

O(n)

O(n)

O(n)

O(n)

O(n)

O(n)
A Graphical Intuition

How many levels are there?
Slicing and Dicing

- After zero recursive calls: $n$
- After one recursive call: $n / 2$
- After two recursive calls: $n / 4$
- After three recursive calls: $n / 8$
  ...
- After $k$ recursive calls: $n / 2^k$
Cutting in Half

- After $k$ recursive calls, there are $n / 2^k$ elements left.
- Mergesort stops recursing when there are zero or one elements left.
- Solving for the number of levels:
  \[ n / 2^k = 1 \]
  \[ n = 2^k \]
  \[ \log_2 n = k \]
- So mergesort recurses $\log_2 n$ levels deep.
A Graphical Intuition

O(n)

O(n)

O(n)

O(n)

O(n log n)
## Mergesort Times

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## Mergesort Times

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</table>
Can we do Better?

• Mergesort is $O(n \log n)$, which is faster than insertion sort’s $O(n^2)$ runtime.

• Can we do better than this?
  • In general, no: comparison-based sorts cannot have a worst-case runtime better than $O(n \log n)$.

• *In the worst case, we can only get faster by a constant factor!*

• What might that look like?
An Interesting Observation

- Big-O notation talks about long-term growth, but says nothing about small inputs.
- For small inputs, insertion sort can be faster than mergesort.
Hybrid Sorting Algorithms

• Modify the mergesort algorithm to switch to insertion sort when the input gets sufficiently small.

• This is called a hybrid sorting algorithm.
void hybridMergesort(Vector<int>& v) {
    if (v.size() <= kCutoffSize) {
        insertionSort(v);
    } else {
        Vector<int> left, right;
        for (int i = 0; i < v.size() / 2; i++) {
            left += v[i];
        }
        for (int i = v.size() / 2; i < v.size(); i++) {
            right += v[i];
        }
        hybridMergesort(left);
        hybridMergesort(right);
        merge(left, right, v);
    }
}
void hybridMergesort(Vector<int>& v) {
    if (v.size() <= kCutoffSize) {
        insertionSort(v);
    } else {
        Vector<int> left, right;
        for (int i = 0; i < v.size() / 2; i++) {
            left += v[i];
        }
        for (int i = v.size() / 2; i < v.size(); i++) {
            right += v[i];
        }
        hybridMergesort(left);
        hybridMergesort(right);
        merge(left, right, v);
    }
}
# Runtime for Hybrid Mergesort

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Hybrid Sorts in Practice

- Introspective Sort (*Introsort*)
  - Based on three sorting algorithms: quicksort, heapsort, and insertion sort.
  - Quicksort runs in time $O(n \log n)$ on average, but in the worst case runs in time $O(n^2)$.
  - Heapsort runs in time $O(n \log n)$ and is very memory-efficient. You’ll see it on Assignment 5!
  - Uses insertion sort for small inputs.
## Runtime for Introsort

<table>
<thead>
<tr>
<th>Size</th>
<th>Mergesort</th>
<th>Hybrid Mergesort</th>
<th>Introsort</th>
</tr>
</thead>
<tbody>
<tr>
<td>100000</td>
<td>0.063</td>
<td>0.019</td>
<td>0.009</td>
</tr>
<tr>
<td>300000</td>
<td>0.176</td>
<td>0.061</td>
<td>0.028</td>
</tr>
<tr>
<td>500000</td>
<td>0.283</td>
<td>0.091</td>
<td>0.043</td>
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<tr>
<td>700000</td>
<td>0.396</td>
<td>0.130</td>
<td>0.060</td>
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<tr>
<td>900000</td>
<td>0.510</td>
<td>0.165</td>
<td>0.078</td>
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<td>0.092</td>
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<tr>
<td>1900000</td>
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<td>0.158</td>
</tr>
</tbody>
</table>
Why All This Matters

- Big-O notation gives us a *quantitative way* to predict runtimes.
- Those predictions provide a *quantitative intuition* for how to improve our algorithms.
- Understanding the nuances of big-O notation then leads us to design algorithms that are better than the sum of their parts.
Next Time

• *Designing Abstractions*
  • How do you build new container classes?

• *Class Design*
  • What do classes look like in C++?