Chris Piech
CS109

Section #2
October 13, 2017

Section #2 Solutions

---

1. **Breaking Vegas**: Let X be the number of dollars that your earn.

   The possible values of x are from the outcomes of: winning on your first bet, winning on your second bet, and so on.

$$\begin{aligned} E[X] &= \frac{18}{38} + \frac{20}{38}\frac{18}{28}(2-1) + \left(\frac{20}{38}\right)^2\frac{18}{28}(4-2-1) + \dots \\ &= \sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i\left(\frac{18}{38}\right)\left(2^i - \sum_{j=0}^{i-1}2^j\right) \\ &= \left(\frac{18}{38}\right)\sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i \\ &= \left(\frac{18}{38}\right)\frac{1}{1-\frac{20}{38}} = 1 \end{aligned}$$

   Real games have maximum bet amounts. You have finite money and casinos can kick you out. But, if you had no betting limits and infinite money, then go for it! (and tell me which planet you are living on).

2. **Sending Bits to Space**: Hamming codes are super interesting. It's worth looking up if you haven't seen them before! All these problems could be approached using a binomial distribution (or from first principles).

   a. Let Y be the number of bits corrupted. $Y \sim \text{Bin}(n = 4, p = 0.01)$.

$$P(Y = 0) = \binom{4}{0}0.9^4 = 0.656$$

   b. Let Z be the number of bits corrupted. $Z \sim \text{Bin}(n = 7, p = 0.01)$. A correctable message is received if Z equals 0 or 1:

$$\begin{aligned} P(\text{correctable}) &= P(Z = 0) + P(Z = 1) \\ &= \binom{7}{0}(0.1)^0(0.9)^7 + \binom{7}{1}(0.1)^1(0.9)^6 = 0.850 \end{aligned}$$

   That is a 30% improvement!

c. Let X_i be the number of copies of bit i which are not corrupted. We can represent each as a Binomial Random Variable: $X_i \sim \text{Bin}(n = 100, p = 0.9)$.

$$\begin{aligned}
P(\text{correctable}) &= \prod_{i=1}^{4} P(X_i > 50) \\
&= \prod_{i=1}^{4} \sum_{j=51}^{100} P(X_i = j) \\
&= \prod_{i=1}^{4} \sum_{j=51}^{100} \binom{100}{j} (0.9)^j (0.1)^{100-j} \\
&= \Big(\sum_{j=51}^{100} \binom{100}{j} (0.9)^j (0.1)^{100-j} \Big)^4 > 0.999
\end{aligned}$$

But now you need to send 400 bites, instead of the 7 required by hamming codes :-).

3. Launch: Let X be the number of requests received in a six hour period. The observed rate is $\frac{1}{6}$ messages per six hour period. Thus: $X \sim \text{Poi}(\lambda = \frac{1}{6})$

$$\begin{aligned}
P(\text{at least 1 message}) &= 1 - P(X = 0) \\
&= 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda} \\
&= 1 - e^{-\frac{1}{6}} = 0.154
\end{aligned}$$

$\text{Var}(X) = \lambda = \frac{1}{6}$

4. Slowing down hackers:

a. $p = \frac{999}{1000}^{i-1} \frac{1}{1000}$

b. Let Z be the number of seconds before the hacker guesses correctly. We want $\text{E}[Z]$:

$$\begin{aligned}
E[Z] &= (\text{first try}) + (\text{second try}) + (\text{third try}) + \ldots \\
&= \frac{1}{1000} 2^0 + \frac{999}{1000} \frac{1}{1000} 2^1 + \frac{999}{1000}^2 \frac{1}{1000} 2^2 + \ldots \\
&= \frac{1}{1000} \sum_{i=0}^{\infty} \Big(\frac{999}{1000} \Big)^i 2^i \\
&= \frac{1}{1000} \sum_{i=0}^{\infty} \Big(\frac{999 \cdot 2}{1000} \Big)^i = \infty
\end{aligned}$$

The last sum term is the sum of a geometric series where the base is greater than 1.