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CS109

Section #3
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Section #3 Solutions

1. **Website Visits**: Let X be the number of minutes that a user stays. $X \sim \text{Exp}(\lambda = \frac{1}{5})$.

$$\begin{aligned} P(X > 10) &= 1 - F_X(10) \\ &= 1 - (1 - e^{\lambda 10}) = e^{-2} \approx 0.1353 \end{aligned}$$

2. **Continuous Random Variable**: The number of users that log in B is binomial: $B \sim \text{Bin}(n = 100, p = 0.2)$. It can be approximated with a normal that matches the mean and variance. Let C be the normal that approximates B.

$E[B] = np = 20.$

$Var(B) = np(1-p) = 16$

$C \sim N(\mu = 20, \sigma^2 = 16).$

$$\begin{aligned} P(B > 21) &\approx P(C > 20.5) \\ &= P\Big(\frac{C - 20}{\sqrt{16}} > \frac{20.5 - 20}{\sqrt{16}}\Big) \\ &= P(Z > 0.125) \\ &= 1 - P(Z < 0.125) \\ &= 1 - \phi(0.125) = 1 - 0.5478 = 0.4522 \end{aligned}$$

3. **Continuous Random Variable**:

a. We need $\int_{-\infty}^{\infty} dx\, f_X(x) = 1$.

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f_X(x)dx &= \int_0^1 c(e^{x-1} + e^{-x})dx \\ &= c\left[e^{x-1} - e^{-x}\right]_{x=0}^{1} \\ 1 &= c(e^{1-1} - e^{-1} - (e^{0-1} - e^{-0})) \\ c &= \frac{1}{1 - e^{-1} - (e^{-1} - 1)} = \frac{1}{2 - \frac{2}{e}} \end{aligned}$$

b.

$$\begin{aligned} P(X > 0.75) &= \int_{0.75}^{1} dx\, c(e^{x-1} + e^{-x}) \\ &= -c\left[e^{x-1} - e^{-x}\right]_{x=0.75}^{1} \\ &= \boxed{-c\left(e^{1-1} - e^{-1} - (e^{0.75-1} - e^{-0.75})\right)} \\ &= -c\left(1 - e^{-1} - e^{-0.25} + e^{-0.75}\right) = \frac{1 - e^{-1} - e^{-0.25} + e^{-0.75}}{2 - \frac{2}{e}} \end{aligned}$$

4. Who did it?

We want to compare P(Arrows | Suspect A), P(Arrows | Suspect B), P(Arrows | Suspect C). Let A_1 be the observation of arrow 1 and A_2 be the observation of arrow 2.

Suspect A

$$A_1|\text{Suspect A} \sim N(\mu = 45, \sigma^2 = 9)$$
$$A_2|\text{Suspect A} \sim N(\mu = 88, \sigma^2 = 5)$$

$$\begin{aligned}
P(\text{Arrows}|\text{Suspect A}) &= P(A_1|\text{Suspect A})P(A_2|\text{Suspect A}) \\
&= \epsilon \cdot f_{A_1}(60) \cdot \epsilon \cdot f_{A_2}(94) \\
&= \epsilon^2 \cdot \frac{1}{\sqrt{2\pi \cdot 9}} e^{\frac{-(60-45)^2}{2\cdot 9}} \frac{1}{\sqrt{2\pi \cdot 5}} e^{\frac{-(94-88)^2}{2\cdot 5}} \\
&\approx \epsilon^2 \cdot 0
\end{aligned}$$

Suspect B

$$A_1|\text{Suspect B} \sim N(\mu = 45, \sigma^2 = 10)$$
$$A_2|\text{Suspect B} \sim N(\mu = 88, \sigma^2 = 4)$$

$$\begin{aligned}
P(\text{Arrows}|\text{Suspect B}) &= P(A_1|\text{Suspect B})P(A_2|\text{Suspect B}) \\
&= \epsilon \cdot f_{A_1}(50) \cdot \epsilon \cdot f_{A_2}(86) \\
&= \epsilon^2 \cdot \frac{1}{\sqrt{2\pi \cdot 9}} e^{\frac{-(50-45)^2}{2\cdot 9}} \frac{1}{\sqrt{2\pi \cdot 4}} e^{\frac{-(86-88)^2}{2\cdot 4}} \\
&\approx \epsilon^2 \cdot 0.0044
\end{aligned}$$

Suspect C

$$A_1|\text{Suspect C} \sim N(\mu = 45, \sigma^2 = 11)$$
$$A_2|\text{Suspect C} \sim N(\mu = 88, \sigma^2 = 3)$$

$$\begin{aligned}
P(\text{Arrows}|\text{Suspect C}) &= P(A_1|\text{Suspect C})P(A_2|\text{Suspect C}) \\
&= \epsilon \cdot f_{A_1}(44) \cdot \epsilon \cdot f_{A_2}(84) \\
&= \epsilon^2 \cdot \frac{1}{\sqrt{2\pi \cdot 11}} e^{\frac{-(44-45)^2}{2\cdot 11}} \frac{1}{\sqrt{2\pi \cdot 3}} e^{\frac{-(84-88)^2}{2\cdot 3}} \\
&\approx \epsilon^2 \cdot 0.0018
\end{aligned}$$

Suspect A certainly did not do it. The locations of the arrows are 2.3 times as likely assuming that Suspect B was the culprit than if Suspect C was the culprit.