1) Review relevant concepts
2) Practice problems
The variance of $X$ is $\nu$

Compute $\text{Var}(2X)$ in terms of $\nu$

$\text{Var}(2X) = 4 \text{Var}(X)$

$4\nu$

$X_1$ & $X_2$ have the same distribution. Variance is also $\nu$. Also independent.

Compute $\text{Var}(X_1 + X_2)$ in terms of $\nu$

$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$

$\nu + \nu = 2\nu$
Random variable is a function from sample space to $\mathbb{R}$

Experiment: Dice roll

$Y(\square)$
\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)} \]

**Bayes Theorem**

**Definition of Conditional Probability**

\[ E[x] = \sum_x x P(x) \]

\[ y = g(x) \]

\[ E[y] = \sum_y y P(y) = \sum_x g(x) P(x) \leftarrow \text{This is easier} \]
\[ P(A) = ? \]

\[ P(A) = P(A \cap B) + P(A \cap B^c) \]

\[ P(A) = P(A \cap B) + P(A \cap C) + P(A \cap D) \]
\[ E[x+y] = E[x] + E[y] \]  
\[
\text{Linearity of expectation}
\]
\[ E[xy] = E[x]E[y] \]  
\[
\text{Do not need to be independent}
\]
\[ \text{MUST be independent} \]
\[ \text{Var}(x+y) = \text{Var}(x) + \text{Var}(y) \]
\[ P(A \cap B \cap C) = P(A | B \cap C) P(B | C) P(C) \]

\[ = P(A | B \cap C) P(B | C) P(C) \]

\[ = P(C | B \cap A) P(B \cap A) \]

\[ = P(C | B \cap A) P(B | A) P(A) \]

*Chain Rule*
$X \sim N(\mu, \sigma^2)$

Defined on $(-\infty, \infty)$

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

\[ \uparrow \text{hard to integrate} \]

Mean = Median = Mode
$Z \sim N(0, 1)$

$X \sim N(\mu_1, \sigma_1^2)$

$\frac{X - \mu_1}{\sigma_1} \sim N(0, 1)$

$\gamma \sim N(\mu_2, \sigma_2^2)$

$\frac{\gamma - \mu_2}{\sigma_2} \sim N(0, 1)$

$X \neq \gamma$
\[ x \sim \mathcal{B}(n, \theta) \]

\[ x \sim y \sim \mathcal{N}(np, np(1-\theta)) \]

\[ p(x \geq 65) \approx p(y \geq 64.5) \]

\[ \text{does not matter if } \theta \approx \frac{1}{2} \]
\[ p_{xy}(x, y) = P(X = x, Y = y) \]  \text{Joint}

\[ p_x(x) = \sum_y p_{xy}(x, y) \]  \text{Marginal}

Given marginals \( p_x(x) \) & \( p_y(y) \) is it possible to compute \( p_{xy}(x, y) \)?  \text{False}

Given joint \( p_{xy}(x, y) \) \& marginals \( p_x(x) \) & \( p_y(y) \)?  \text{True}
Multinomial

\[ p(x_1 = c_1, x_2 = c_2, \ldots) = \binom{n}{c_1, c_2, \ldots} \prod_{i=1}^{m} p_{c_i} \]

\[ \sum_{i=1}^{m} c_i = n \]

\[ \sum_{i=1}^{m} p_{c_i} = 1 \]
Question: Compute $a$ & $b$, assume $X$ & $Y$ are independent.

Joint table

$X = -3 \quad X = 9.076$

$Y = 112 \quad 0.1 \quad 0.3$

$Y = 147 \quad a \quad b$

$p_{xy}(x, y) = p_x(x)p_y(y) = \left(\sum p_{xy}(x, j)\right)\left(\sum p_{x, y}(i, y)\right)_{i \in x, j \in \text{ran}(Y)}$

$0.1 = (0.1 + 0.3)(0.1 + a)$

$0.3 = (0.1 + 0.3)(0.3 + b)$
Combining Random Variables

Independent

Convolution

\[ p(x + y = n) = \sum_{k \in \text{Range}(x)} p(x = k) \cdot p(y = n - k) \]
\[ X \sim \text{Bin}(n_1, p) \]
\[ Y \sim \text{Bin}(n_2, p) \]

\[ (X + Y) \sim \text{Bin}(n_1 + n_2, p) \]

\[ X \sim \text{Ber}(p) \]
\[ Y \sim \text{Ber}(p) \]

\[ X + Y \sim \text{Bin}(2, p) \]
$X \sim Pois(\lambda_1)$

$Y \sim Pois(\lambda_2)$

$X + Y \sim Pois(\lambda_1 + \lambda_2)$

$X \sim Geo(p)$

$Y \sim Geo(p)$

$X + Y \sim NegBin(2, p)$

Must be Independent
X \sim N(\mu_1, \sigma_1^2)

Y \sim N(\mu_2, \sigma_2^2)

\text{Must be independent}

X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)

\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]

= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]
Correlation measures linear relationship

\[ \rho(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} \]

\[-1 \leq \rho \leq 1\]
$y = x$
\[ E \left[ \frac{X}{Y = y} \right] = \sum_x x \cdot P(x = x | Y = y) = \sum_x \rho_{x|y}(y,y) \]

\[ E \left[ X \mid Y \right] \]

\[ \uparrow \text{ does not make sense} \]
\[ E[X] = E[E[X|Y]] = \sum_{y} P(y \mid y) \ E[X \mid y = y] \]

Law of Total Probability
Bayesian Networks - each node is a conditional distribution

\[ P(\text{Flu} = 1) = 0.1 \]

\[ P(\text{Fever} \mid \text{Flu} = 1) = 0.9 \]
\[ P(\text{Flu} = 1 \land \text{Fever} = 1) = 0.05 \]

\[ P(\text{Fever} = 0, \text{Fever} = 0, \text{Sick} = 1, \text{Tired} = 1) \]
\[ P(\text{Tired} \mid \text{Fever}, \text{Fever}, \text{Sick}) P(\text{Fever}, \text{Fever}, \text{Sick}) \]
\[ P(\text{Tired} \mid \text{Fever}, \text{Sick}) \] because \( \text{Tired} \) and \( \text{Fever} \) are conditionally independent on parents of \( \text{Tired} \)

\[ \text{define as } P(\text{Tired} \mid \text{Flu}, \text{Sick}) \]
Two things to remember

1) Each variable is conditionally independent of its non-descendants given its parents

\[ P(\text{child} \mid \text{Parent}_1, \text{Parent}_2, \text{non-descendant}) \]

\[ = P(\text{child} \mid \text{Parent}_1, \text{Parent}_2) \]
2) Apply chain rule starting with variables with more arrows (good rule of thumb)

\[ P(FL, FE, U, T) \]
\[ \Rightarrow P(T \mid FL, FE, U) P(FL, FE, U) \]
\[ \downarrow \text{ Good} \]
\[ = P(FL \mid FE, U, T) P(FE, U, T) \]
\[ \times \text{ Bad because no table} \]
Continuos PDF

\[ P \left( a_1 \leq x \leq a_2, b_1 \leq y \leq b_2 \right) \]

\[ \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x,y) \, dy \, dx \]

CDF

\[ F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x,y) \, dy \, dx \]
10 machines of type W

\[ W_i = \text{usage of } W \sim \text{Poi}(4) \]

\[ X_i = \text{usage of } X \sim \mathcal{N}(5,3) \]

\[ Y \text{ is usage of select machines} \]

\[ E[Y] = ? \]

\[ Y = AW + BX \]

\[ E[Y] = E[AW + BX] \]

\[ = E[AW] + E[BX] \]


\[ = (10)(0.2)(4) + (10)(0.2)(5) \]

\[ = 18 \]
\[ P(Y \geq 20 \mid 3w_s, 0x_s) \]

\[ C = W_1 + W_2 + W_3 \]

\[ C \sim \text{Poi}(4 + 4 + 4) \]

\[ P(C \geq 20) = \sum_{i=20}^{\infty} \frac{e^{-12} 12^i}{i!} \]

\[ = 1 - \sum_{i=1}^{19} \frac{e^{-12} 12^i}{i!} \]
\[ P(Y \geq 20 | 3 X_3, 0 \text{ w.s.}) \]

\[ C = X_1 + X_2 + X_3 \]

\[ C \sim N(5.3, 3.3) \]

\[ N(15, 9) \]

\[ P(C \geq 20) = 1 - P(C < 20) \]

\[ = 1 - P\left( \frac{C - 15}{3} < \frac{20 - 15}{3} \right) \]

\[ = 1 - \phi(1.67) \]

\[ \Rightarrow 1 - \Phi(1.67) \]

\[ = 0.0475 \]