16: Continuous Joint Distributions

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Continuous joint distributions
Remember target?

\[
\frac{59}{309} = 0.191
\]

Good times...
The CS109 logo was created by throwing 500,000 darts according to a joint distribution.

If we throw another dart according to the same distribution, what is $P(\text{dart hits within } r \text{ pixels of center})$?

Quick check: What is the probability that a dart hits at (456.2344132343, 532.1865739012)?
CS109 logo with darts

P(dart hits within $r$ pixels of center)?

1 pixel = 1 dart thrown at screen

Possible dart counts (in 100x100 boxes)
CS109 logo with darts

$P(\text{dart hits within } r \text{ pixels of center})$?

Possible dart counts (in 50x50 boxes)
CS109 logo with darts

P(dart hits within $r$ pixels of center)?

Possible dart counts
(in infinitesimally small boxes)
Continuous joint probability density functions

If two random variables $X$ and $Y$ are jointly continuous, then there exists a joint probability density function $f_{X,Y}$ defined over $-\infty < x, y < \infty$ such that:

$$P(a_1 \leq X \leq a_2, \ b_1 \leq Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y) dy \ dx$$
From one continuous RV to jointly continuous RVs

Single continuous RV $X$

- PDF $f_X$ such that $\int_{-\infty}^{\infty} f_X(x) \, dx = 1$
- Integrate to get probabilities

Jointly continuous RVs $X$ and $Y$

- PDF $f_{X,Y}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1$
- Double integrate to get probabilities

Probability for jointly continuous RVs is **volume** under a surface.
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

Write down the definite double integral that must integrate to 1:
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.
• Support: $0 \leq X \leq 1, \ 0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

Write down the definite double integral that must integrate to 1:

$$
\int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy = 1 \quad \text{or} \quad \int_{x=0}^{1} \int_{y=0}^{2} xy \, dy \, dx = 1
$$

(used in next slide)
Double integrals without tears

Let $X$ and $Y$ be two continuous random variables.
• Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over $X$ and $Y$?

0. Set up integral:
$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, dx \, dy = \int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy$$

1. Evaluate inside integral by treating $y$ as a constant:
$$\int_{y=0}^{2} \left( \int_{x=0}^{1} xy \, dx \right) \, dy = \int_{y=0}^{2} y \left( \int_{x=0}^{1} x \, dx \right) \, dy = \int_{y=0}^{2} y \left[ \frac{x^2}{2} \right]_{0}^{1} \, dy = \int_{y=0}^{2} \frac{1}{2} \, dy$$

2. Evaluate remaining (single) integral:
$$\int_{y=0}^{2} \frac{1}{2} \, dy = \left[ \frac{y^2}{4} \right]_{y=0}^{2} = 1 - 0 = 1$$

Yes, $g(x, y)$ is a valid joint PDF because it integrates to 1.
Marginal distributions

Suppose $X$ and $Y$ are continuous random variables with joint PDF:

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \, dx = 1
$$

The marginal density functions (marginal PDFs) are therefore:

$$
f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) dy \quad \quad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b) dx
$$
Back to darts!

Match $X$ and $Y$ to their respective marginal PDFs:

[top-down view]

[side view]
Back to darts!

Match $X$ and $Y$ to their respective marginal PDFs:
If you want more practice with double integrals, I’ve included two exercises at the end of this lecture.
Joint CDFs
An observation: Connecting CDF to PDF

For a continuous random variable $X$ with PDF $f$, the CDF (cumulative distribution function) is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

The density $f$ is therefore the derivative of the CDF, $F$:

$$f(a) = \frac{d}{da}F(a)$$

(Fundamental Theorem of Calculus)
Joint cumulative distribution function

For two random variables $X$ and $Y$, there can be a joint cumulative distribution function $F_{X,Y}$:

$$ F_{X,Y}(a, b) = P(X \leq a, Y \leq b) $$

For discrete $X$ and $Y$:

$$ F_{X,Y}(a, b) = \sum_{x \leq a} \sum_{y \leq b} p_{X,Y}(x, y) $$

For continuous $X$ and $Y$:

$$ F_{X,Y}(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y) \, dy \, dx $$

$$ f_{X,Y}(a, b) = \frac{\partial^2}{\partial a \, \partial b} F_{X,Y}(a, b) $$
Single variable CDF, graphically

\[
\lim_{x \to -\infty} F_X(x) = 0 \\
\lim_{x \to +\infty} F_X(x) = 1
\]

\[F_X(x) = P(X \leq x)\]
Joint CDF, graphically

\[
\lim_{x,y \to -\infty} F_{X,Y}(x, y) = 0
\]

\[
\lim_{x,y \to +\infty} F_{X,Y}(x, y) = 1
\]

\[
f_{X,Y}(x, y)
\]

\[
F_{X,Y}(x, y) = P(X \leq x, Y \leq y)
\]
Independent Continuous RVs
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y) \quad \forall x, y$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \quad \forall x, y$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Proof of PDF:

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_X(x)F_Y(y)$$
$$= \frac{\partial}{\partial x} \frac{\partial}{\partial y} F_X(x)F_Y(y) = \frac{\partial}{\partial x} F_X(x)\frac{\partial}{\partial y} F_Y(y)$$
$$= f_X(x)f_Y(y)$$
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

More generally, $X$ and $Y$ are independent if joint density factors separately:

$$f_{X,Y}(x, y) = g(x)h(y), \text{ where } -\infty < x, y < \infty$$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$  
   where $0 < x, y < \infty$

2. $f_{X,Y}(x, y) = 4xy$  
   where $0 < x, y < 1$

3. $f_{X,Y}(x, y) = 24xy$  
   where $0 < x + y < 1$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$

2. $f_{X,Y}(x, y) = 4xy$
   where $0 < x, y < 1$

3. $f_{X,Y}(x, y) = 24xy$
   where $0 < x + y < 1$

Separable functions:
- $g(x) = 3e^{-3x}$
- $h(y) = 2e^{-2y}$

Separable functions:
- $g(x) = 2x$
- $h(y) = 2y$

Cannot capture constraint on $x + y$ into factorization!

If you can factor densities over all of the support, you have independence.
Bivariate Normal Distribution
Bivariate Normal Distribution

$X_1$ and $X_2$ follow a bivariate normal distribution if their joint PDF $f$ is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

Can show that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

(Ross chapter 6, example 5d)

Often written as:

- Vector $\mathbf{X} = (X_1, X_2)$
- Mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)$, Covariance matrix: $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

Recall correlation: $\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$

We will focus on understanding the shape of a bivariate Normal RV.
Back to darts

These darts were actually thrown according to a bivariate normal distribution:

\[(X, Y) \sim \mathcal{N}(\mu, \Sigma)\]

\[
\mu = \left( \begin{array}{c}
450 \\
600 \\
\end{array} \right)
\]

\[
\Sigma = \begin{bmatrix}
900^2/4 & 0 \\
0 & 900^2/25 \\
\end{bmatrix}
\]

Marginal PDFs:

\[X \sim \mathcal{N}\left(450, \frac{900^2}{4}\right)\]

\[Y \sim \mathcal{N}\left(600, \frac{900^2}{25}\right)\]
A diagonal covariance matrix

Let $X = (X_1, X_2)$ follow a bivariate normal distribution $X \sim \mathcal{N}(\mu, \Sigma)$, where

$$\mu = (\mu_1, \mu_2), \quad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Are $X_1$ and $X_2$ independent?

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$  \hspace{1cm} \text{(Note covariance: $\rho\sigma_1\sigma_2 = 0$)}

$X_1$ and $X_2$ are independent with marginal distributions

$X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$
16: Continuous Joint Distributions (I)

Slides by Lisa Yan
July 24, 2020
Jointly continuous RVs

\( X \) and \( Y \) are jointly continuous if they have a joint PDF:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1
\]

Most things we’ve learned about discrete joint distributions translate:

Marginal distributions

\[
p_X(a) = \sum_y p_{X,Y}(a, y)
\]

\[
f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) \, dy
\]

Independent RVs

\[
p_{X,Y}(x, y) = p_X(x) p_Y(y)
\]

\[
f_{X,Y}(x, y) = f_X(x) f_Y(y)
\]

Expectation (e.g., LOTUS)

\[
E[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)
\]

\[
E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) \, dy \, dx
\]

...etc.
Think

Slide 35 has a question to go over by yourself.

Post any clarifications here!

https://us.edstem.org/courses/109/discussion/60584

Think by yourself: 2 min
Warmup exercise

\(X\) and \(Y\) have the following joint PDF:

\[ f_{X,Y}(x, y) = 3e^{-3x} \]

where \(0 < x < \infty, 1 < y < 2 \)

1. Are \(X\) and \(Y\) independent?

2. What is the marginal PDF of \(X\)? Of \(Y\)?

3. What is \(E[X + Y]\)?
Warmup exercise

\( X \) and \( Y \) have the following joint PDF:

\[
 f_{X,Y}(x, y) = 3e^{-3x}
\]
where \( 0 < x < \infty, 1 < y < 2 \)

1. Are \( X \) and \( Y \) independent? ✅

2. What is the marginal PDF of \( X \)? Of \( Y \)?

3. What is \( E[X + Y] \)?

\[ g(x) = 3Ce^{-3x}, \ 0 < x < \infty \quad C \text{ is a constant} \]

\[ h(y) = \frac{1}{C}, \ 1 < y < 2 \]

\[ f_X(x) = \int_{1}^{2} f_{X,Y}(x, y) \, dy \]

\[ f_Y(y) = \int_{0}^{\infty} f_{X,Y}(x, y) \, dx \]
Warmup exercise

X and Y have the following joint PDF:

\[ f_{X,Y}(x, y) = 3e^{-3x} \]
where \( 0 < x < \infty, 1 < y < 2 \)

1. Are \( X \) and \( Y \) independent? \( \square \)

2. What is the marginal PDF of \( X \)? Of \( Y \)?

\[ g(x) = 3Ce^{-3x}, \quad 0 < x < \infty \quad \text{C is a constant} \]
\[ h(y) = \frac{1}{C}, \quad 1 < y < 2 \]

3. What is \( E[X + Y] \)?
Breakout Rooms

Check out the question on the next slide. Post any clarifications here!

https://us.edstem.org/courses/667/discussion/94992

Breakout rooms: 4 min. Introduce yourself!
The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define \( X = \) # minutes past 12pm that person 1 arrives. \( X \sim \text{Unif}(0, 30) \)
\( Y = \) # minutes past 12pm that person 2 arrives. \( Y \sim \text{Unif}(0, 30) \)

What is the probability that the first to arrive waits >10 mins for the other?

Compute: \( P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y) \) (by symmetry)

1. What is “symmetry” here?
2. How do we integrate to compute this probability?
The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define

\[ X = \text{# minutes past 12pm that person 1 arrives. } X \sim \text{Unif}(0, 30) \]
\[ Y = \text{# minutes past 12pm that person 2 arrives. } Y \sim \text{Unif}(0, 30) \]

What is the probability that the first to arrive waits >10 mins for the other?

Compute:

\[ P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y) \quad \text{(by symmetry)} \]

\[ = 2 \cdot \int_{x+10 < y} f_{X,Y}(x,y) \, dx \, dy = 2 \cdot \int_{x+10 < y, \, 0 \leq x,y, \leq 30} (1/30)^2 \, dx \, dy \quad \text{(independence)} \]

\[ = \frac{2}{30^2} \int_{10}^{30} \int_{0}^{y-10} \, dx \, dy = \frac{2}{30^2} \int_{10}^{30} (y - 10) \, dy = \ldots = \frac{4}{9} \]
Interlude for jokes/announcements
Announcements

Grades
Pset 3 by: Tonight
Midterm by: End of weekend

Mid-quarter feedback form
Link (Stanford account login)
Open until: Wednesday

Problem Set 4 due soon
Due: Monday 7/27 1pm

Problem Set 5 out tonight
Due: Monday 8/3 1pm
Covers: Up to Wed’s lecture (16)
Interesting probability news

What That Election Probability Means

Even when you shift the probability far left or far right, the opposing candidate still gets some wins. That doesn’t mean a forecast was wrong. That’s just randomness and uncertainty at play. The probability estimates the percentage of times you get an outcome if you were to do something multiple times.

https://flowingdata.com/2016/07/28/what-that-election-probability-means/
“Suppose there is a remote stretch of coastline with two small villages, A and B, each with a small number of houses.”

“In any particular week, the probability distribution over the number of houses C impacted by power outages in each village is as follows:”

\[
P_A(C = c) = \begin{cases} 
0.6 & c = 0 \\ 
0.4 & c = 2 \\ 
0 & \text{otherwise}
\end{cases} \quad P_B(C = c) = \begin{cases} 
0.3 & c = 0 \\ 
0.7 & c = 3 \\ 
0 & \text{otherwise}
\end{cases}
\]

Suppose you are trying to assign generators to these villages permanently.

Utilization: \( E[\text{# of generators that are actually used}] \)

Fairness: \( E[\text{village A houses in need get generators}] \approx E[\text{village B houses in need get generators}] \)

How do you choose an allocation that optimizes utilization subject to our fairness constraint?

=> With probability :-)

Bivariate Normal Distribution

The bivariate normal distribution of $X = (X_1, X_2)$:

$$X \sim \mathcal{N} (\mu, \Sigma)$$

- Mean vector $\mu = (\mu_1, \mu_2)$
- Covariance matrix: $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$ \quad Cov($X_1, X_2$) = Cov($X_2, X_1$) = $\rho \sigma_1 \sigma_2$
- Marginal distributions: $X_1 \sim \mathcal{N} (\mu_1, \sigma_1^2), X_2 \sim \mathcal{N} (\mu_2, \sigma_2^2)$
- For bivariate normals in particular, Cov($X_1, X_2$) = 0 implies $X_1, X_2$ independent.

We will focus on understanding the shape of a bivariate Normal RV.
Check out the question on the next slide. Post any clarifications here!

https://us.edstem.org/courses/667/discussion/94992

Breakout rooms: 3 min. Introduce yourself!
(X, Y) Matching (all have $\mu = (0, 0)$)
$(X, Y)$ Matching (all have $\mu = (0,0)$)

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$
Proportions from joint CDFs

Recall for a single RV $X$ with CDF $F_X$:

CDF: $P(X \leq x) = F_X(x)$

$$P(a < X \leq b) = F_X(b) - F(a)$$

For two RVs $X$ and $Y$ with joint CDF $F_{X,Y}$:

Joint CDF: $P(X \leq x, Y \leq y) = F_{X,Y}(x, y)$

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$

$$F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) - F_{X,Y}(a_2,b_1) + F_{X,Y}(a_1,b_1)$$

Note strict inequalities; these properties hold for both discrete and continuous RVs.
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =
F_{X,Y}(a_2,b_2) - F_{X,Y}(a_1,b_2) - F_{X,Y}(a_2,b_1) + F_{X,Y}(a_1,b_1)
\]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

\[ P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \\
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \]
Probability with Instagram!

(for next time)

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
In a Gaussian blur, for every pixel:

- Weight each pixel by the probability that $X$ and $Y$ are both within the pixel bounds
- The weighting function is a Bivariate Gaussian (Normal) standard deviation parameter $\sigma$

Gaussian blurring with $\sigma = 3$:

$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2 \cdot 3^2}$$

What is the weight of the center pixel?

$$P(-0.5 < X \leq 0.5, -0.5 < Y \leq 0.5) =$$

$$= 0.206$$
Next time:
More Cont. Joint and Central Limit Theorem
Extra
1. Integral practice

Let $X$ and $Y$ be two continuous random variables with joint PDF:

$$f(x, y) = \begin{cases} 4xy & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X \leq Y)$?

$$P(X \leq Y) = \int_{x \leq y, \ 0 \leq x, y \leq 1} \int_{y=0}^{1} \int_{x=0}^{y} 4xy \, dx \, dy$$

$$= \int_{y=0}^{1} 4y \left[ \frac{x^2}{2} \right]_{0}^{y} \, dy = \int_{y=0}^{1} 2y^3 \, dy = \left[ \frac{2}{4}y^4 \right]_{0}^{1} = \frac{1}{2}$$
2. How do you integrate over a circle?

P(dart hits within $r = 10$ pixels of center)?

$$P(x^2 + y^2 \leq 10^2) = \int \int f_{X,Y}(x,y) \, dy \, dx$$

$x^2 + y^2 \leq 10^2$

Let’s try an example that doesn’t involve integrating a Normal RV 😊
2. Imperfection on Disk

You have a disk surface, a circle of radius $R$. Suppose you have a single point imperfection uniformly distributed on the disk.

What are the marginal distributions of $X$ and $Y$? Are $X$ and $Y$ independent?

$$f_{X,Y}(x, y) = \begin{cases} 
\frac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\
0 & \text{otherwise}
\end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy = \frac{1}{\pi R^2} \int_{x^2+y^2\leq R^2} dy$$

$$= \frac{1}{\pi R^2} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$$

$$f_Y(y) = \frac{2\sqrt{R^2-y}}{\pi R^2}$$

No, $X$ and $Y$ are dependent. $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$.