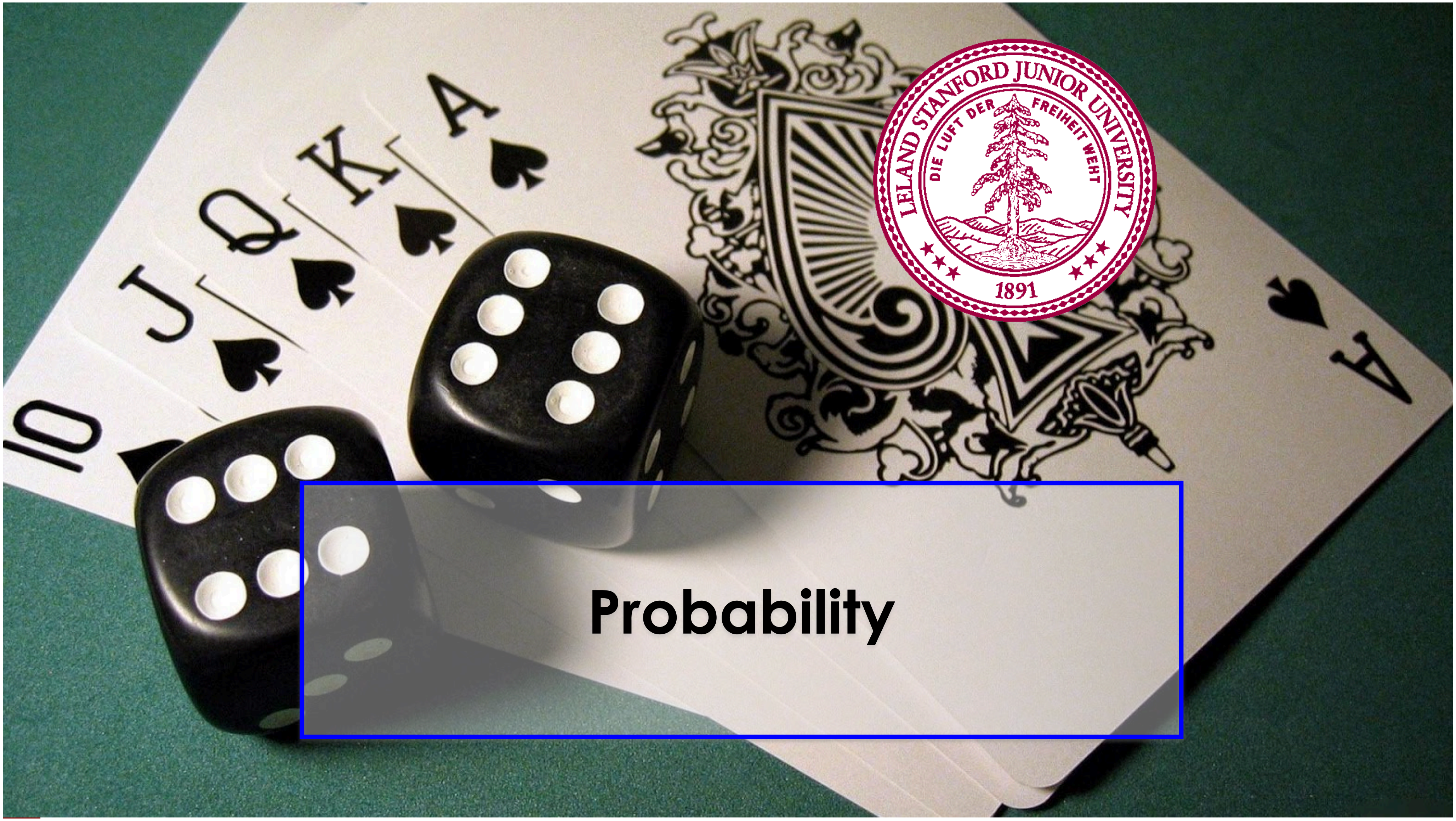

Probability

Counting Rules

Counting operations on *n* objects

- **Sort, order matters (perms)**
  - Distinct: $n!$
  - Some Distinct: $\frac{n!}{n_1!n_2!\ldots}$
- **Choose *k* (combinations)**
  - Distinct: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- **Put in *r* buckets**
  - Distinct: r^n
  - None Distinct: $\frac{(n+r-1)!}{n!(r-1)!}$

Counting Rules

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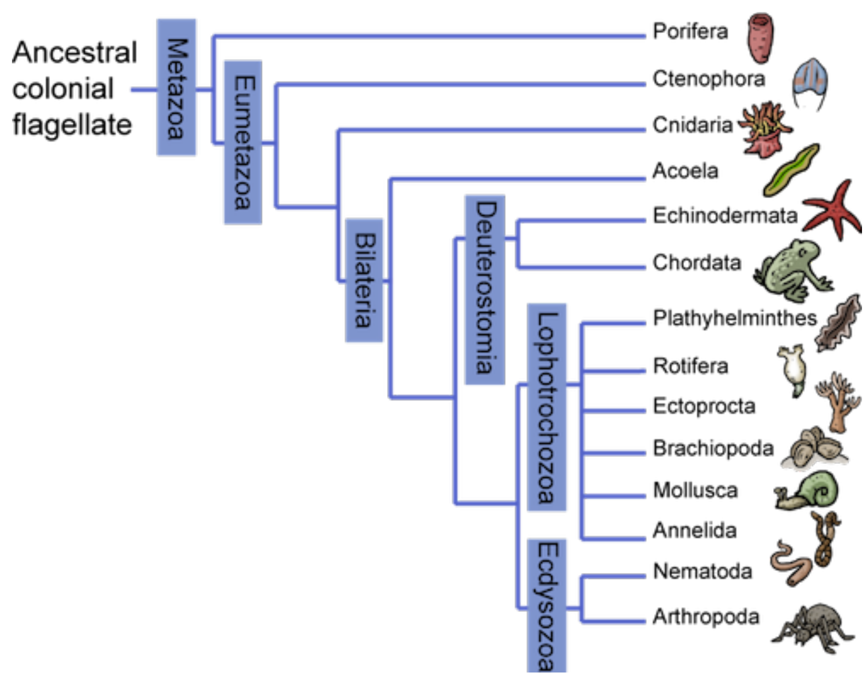
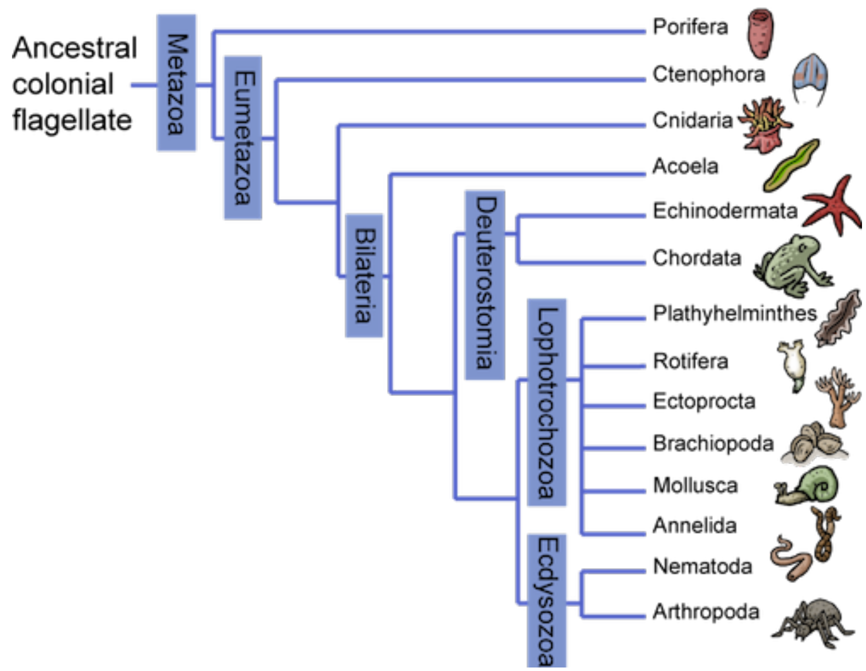
Counting Review

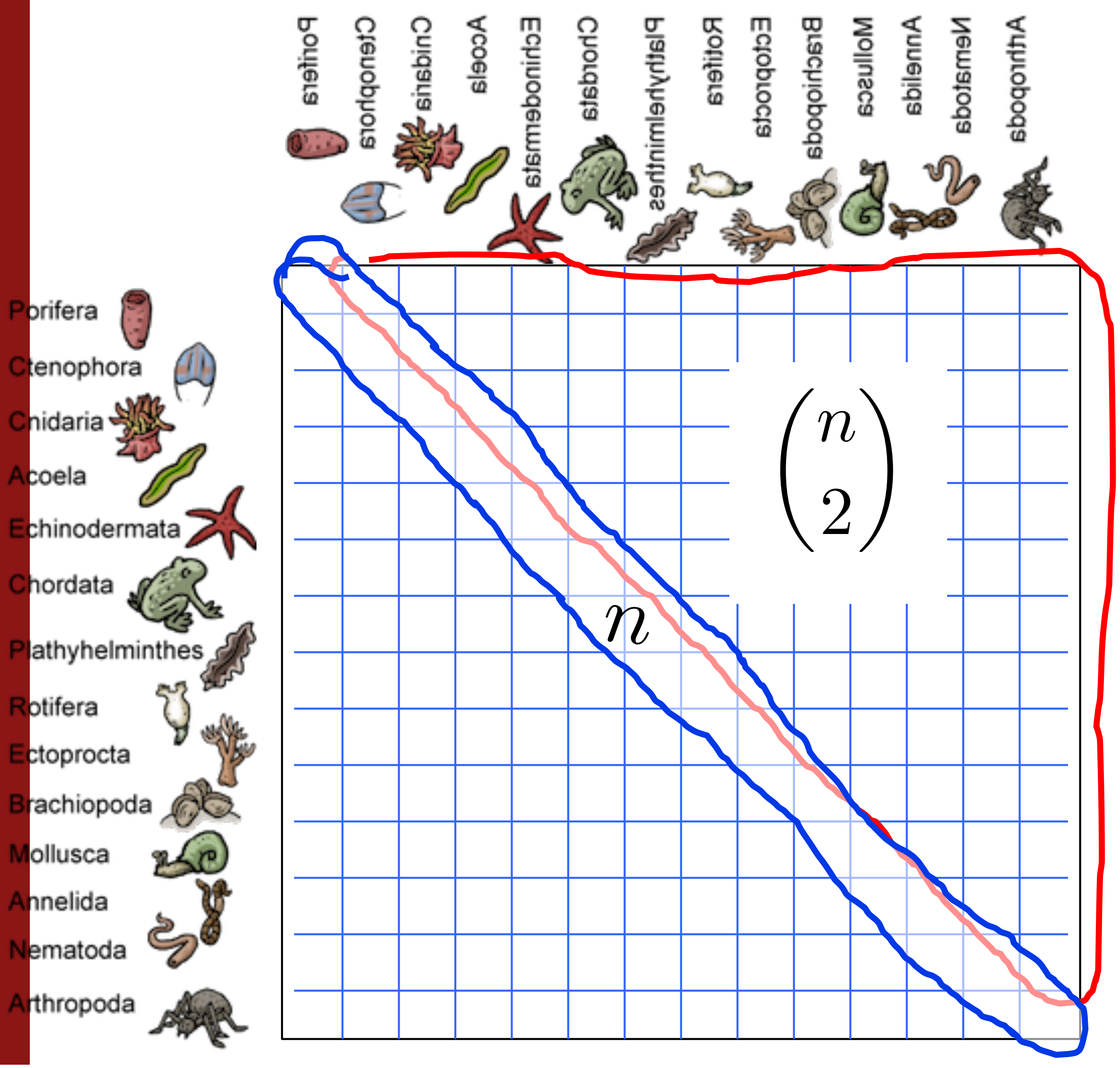
For a DNA tree we need to calculate the DNA distance between each pair of animals. How many calculations are needed?

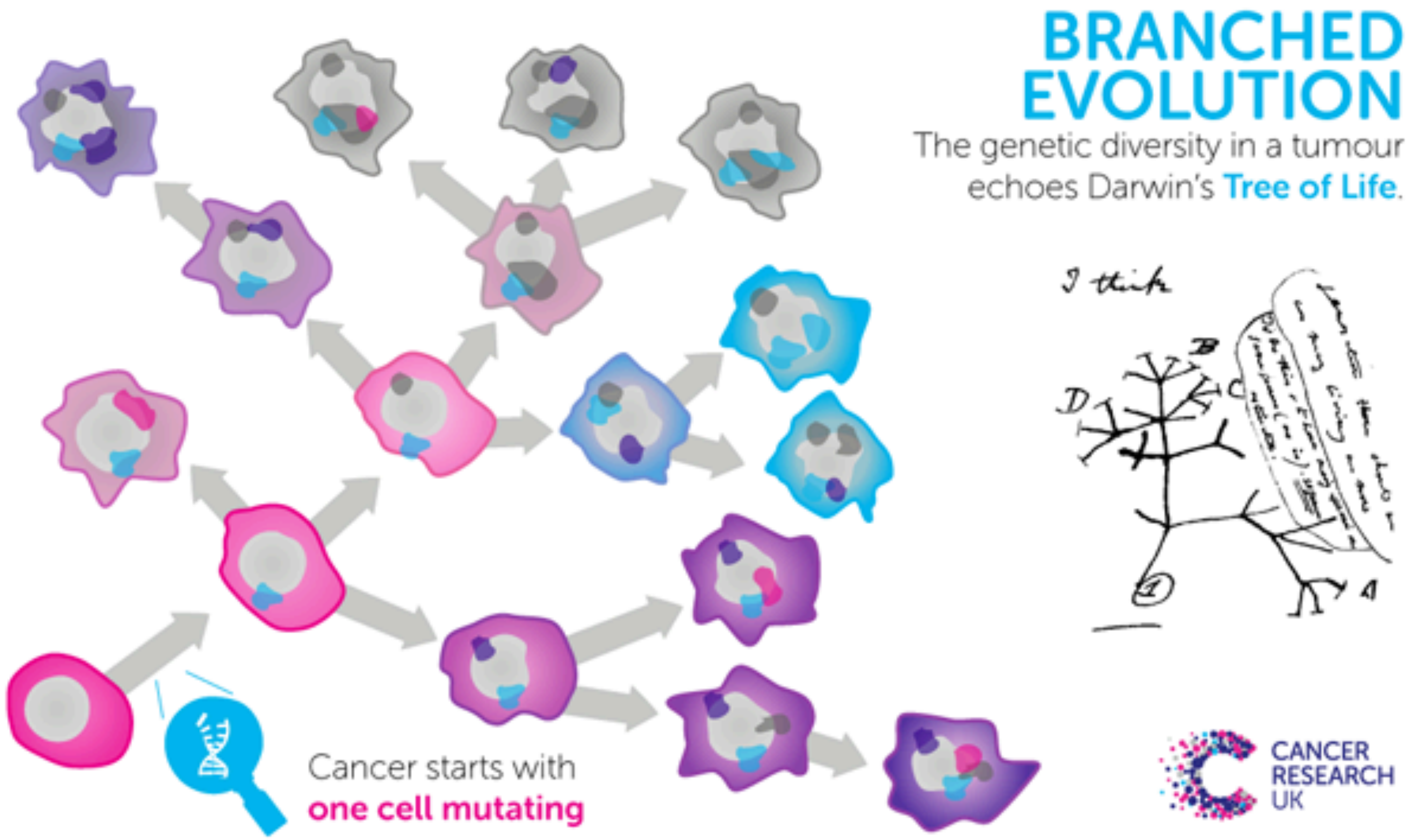


Counting Review

Q: There are *n* animals.
How many distinct pairs of animals are there?




Porifera
Ctenophora
Cnidaria
Acoela
Echinodermata
Chordata
Plathyhelminthes
Rotifera
Ectoprocta
Brachiopoda
Mollusca
Annelida
Nematoda
Arthropoda
$\binom{n}{2}$
n


BRANCHED EVOLUTION
The genetic diversity in a tumour echoes Darwin's Tree of Life.
I think
A
B
C
D
1
Cancer starts with one cell mutating
CANCER RESEARCH UK

End Review

Sample Space

- **Sample space**, S, is set of all possible outcomes of an experiment
  - Coin flip: S = {Head, Tails}
  - Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
  - Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}
  - # emails in a day: $S = \{x \mid x \in \mathbf{Z}, x \geq 0\}$ (non-neg. ints)
  - YouTube hrs. in day: $S = \{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$

Event Space

- **Event**, E, is some subset of S $(E \subseteq S)$
  - Coin flip is heads: $E = \{\text{Head}\}$
  - ≥ 1 head on 2 coin flips: $E = \{(H, H), (H, T), (T, H)\}$
  - Roll of die is 3 or less: $E = \{1, 2, 3\}$
  - # emails in a day ≤ 20: $E = \{x \mid x \in \mathbf{Z}, 0 \leq x \leq 20\}$
  - Wasted day (≥ 5 YT hrs.): $E = \{x \mid x \in \mathbf{R}, 5 \leq x \leq 24\}$

Note: When Ross uses: $\subset$, he really means: $\subseteq$

Event Space

Sample Space, S

- Coin flip
  $S = \{\text{Heads}, \text{Tails}\}$
- Flipping two coins
  $S = \{(\text{H,H}), (\text{H,T}), (\text{T,H}), (\text{T,T})\}$
- Roll of 6-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in a day
  $S = \{x \mid x \in \mathbb{Z}, \ x \geq 0\}$
- TikTok hours in a day
  $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$

Event, E

- Flip lands heads
  $E = \{\text{Heads}\}$
- ≥ 1 head on 2 coin flips
  $E = \{(\text{H,H}), (\text{H,T}), (\text{T,H})\}$
- Roll is 3 or less:
  $E = \{1, 2, 3\}$
- Low email day (≤ 20 emails)
  $E = \{x \mid x \in \mathbb{Z}, \ 0 \leq x \leq 20\}$
- Wasted day (≥ 5 TT hours):
  $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$

What is a probability?

Number between 0 and 1

A number to which we ascribe meaning

$$P(E)$$

* Our belief that an event E occurs

A number to which we ascribe meaning

$$\Pr(E)$$

* Our belief that an event E occurs

What is a Probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

What is a Probability?

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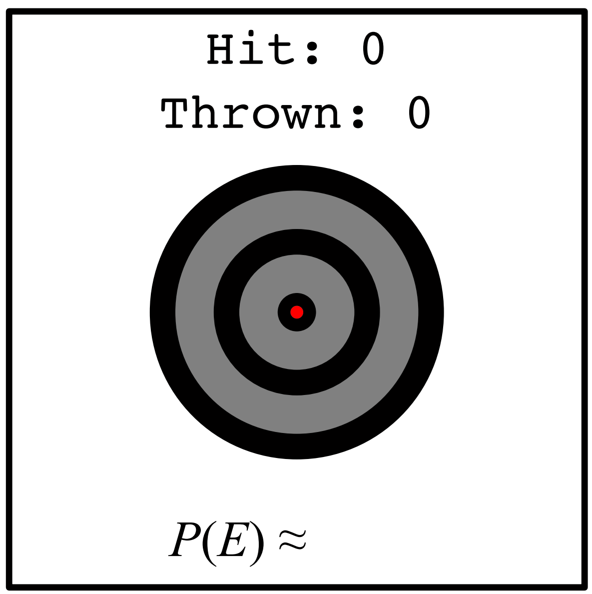
n is the number of trails

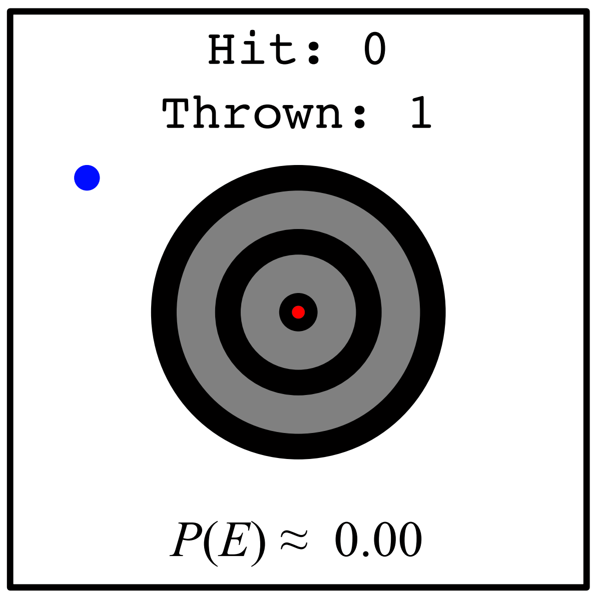


The "event" *E* is that you hit the target

What is a Probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n is the number of trails



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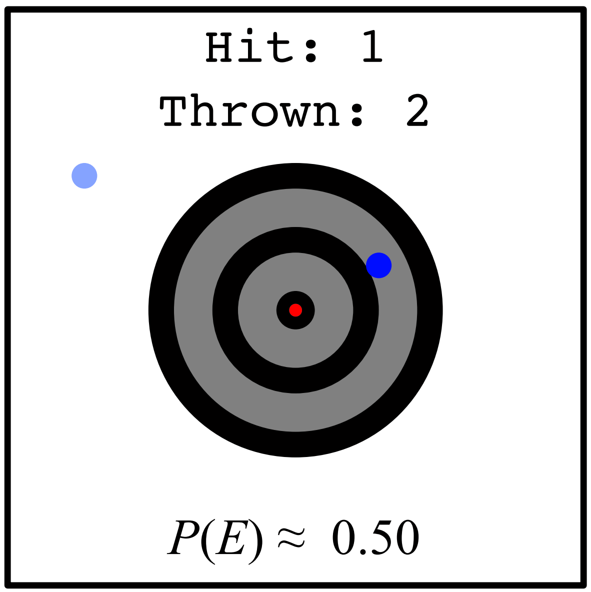
What is a Probability?

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

n is the number of trails



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$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

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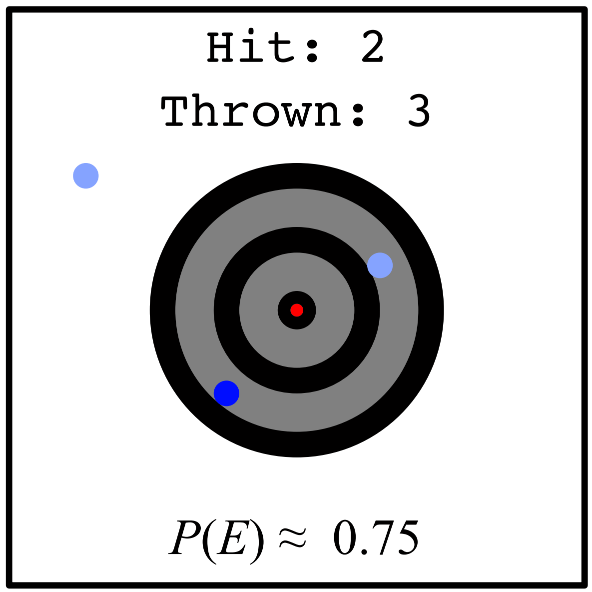

The "event" *E* is that you hit the target

What is a Probability?

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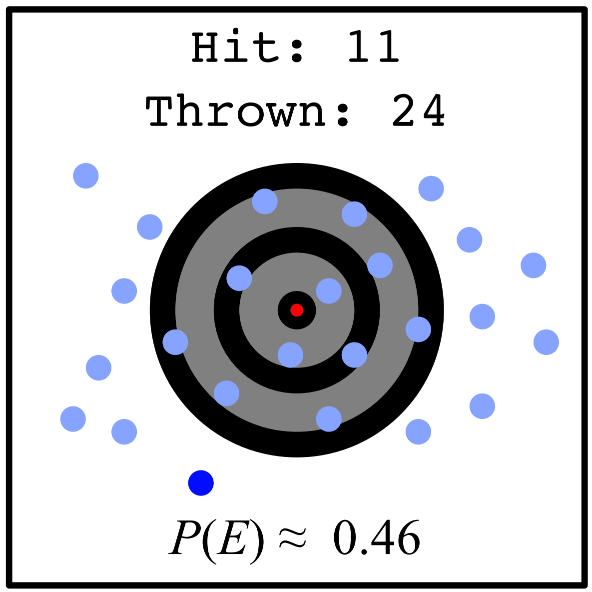
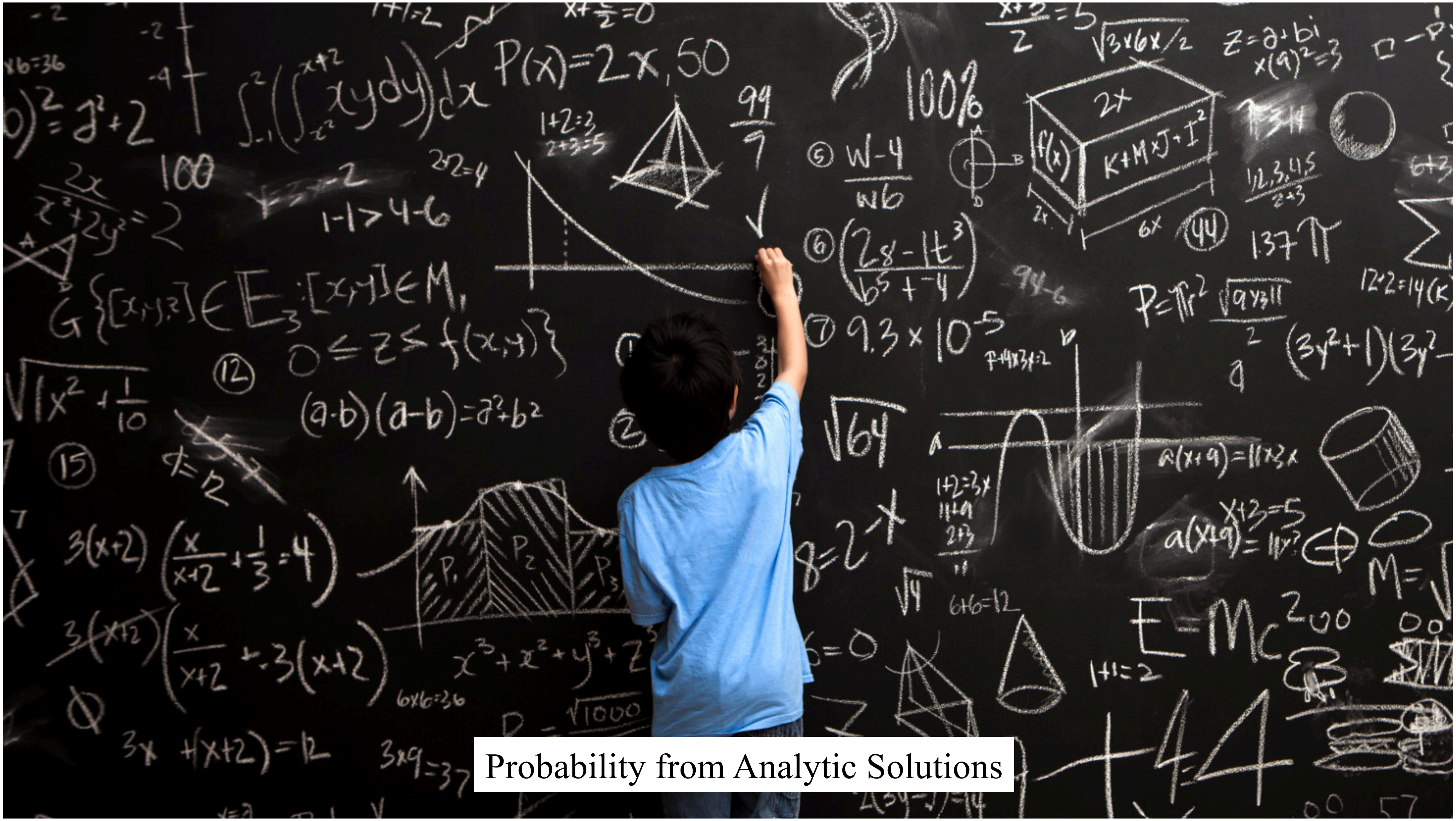
n is the number of trails



The “event” *E* is that you hit the target

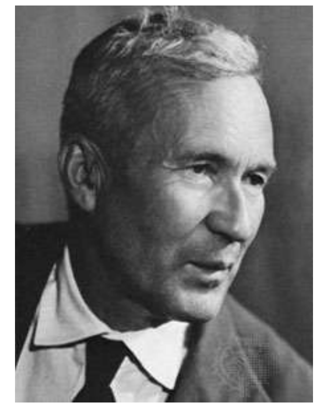

90
MLB 040
VOLVO



Probability from Analytic Solutions

Axioms of Probability

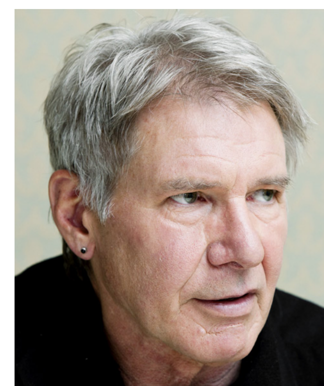
Kolmogorov



Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$



Harrison Ford

Learning Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Identity 3: $P(E^c) = 1 - P(E)$

Technically Identity 3 can be proved from the 3 axioms.

Special Case of Analytic Probability

03c_elo

Equally Likely Outcomes

Equally Likely Outcomes

Some sample spaces have **equally likely outcomes**.

- Coin flip: S = {Head, Tails}
- Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
- Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

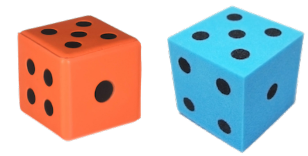
If we have equally likely outcomes, then P(Each outcome) $= \frac{1}{|S|}$

Therefore $$P(E) = \frac{\text{\# outcomes in E}}{\text{\# outcomes in S}} = \frac{|E|}{|S|}$$ (by Axiom 3)

Roll two dice

$P(E) = \frac{|E|}{|S|}$ Equally likely outcomes

Roll two 6-sided fair dice. What is P(sum = 7)?



S = { (1, 1) , (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1) , (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1) , (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1) , (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1) , (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1) , (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

$E =$

Not Everything is Equally Likely

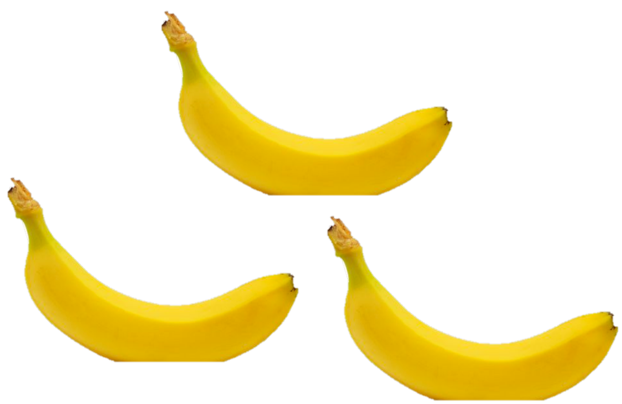
- Play lottery.
  - What is P(Win)?

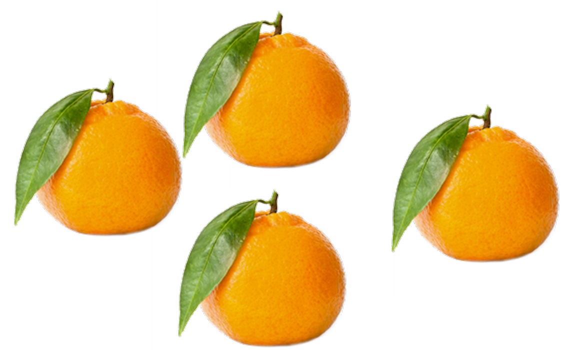
- S = {Lose, Win}
- E = {Win}
- P(Win) = |E|/|S| = 1/2 = 50%

Mandarins and Bananas

- 4 Mandarins and 3 Bananas in a Bag. 3 drawn.
  - What is P(1 Mandarin and 2 Bananas drawn)?

Equally likely sample space? Thought experiment





The Choice of Sample Space is Yours!

	Distinct	Indistinct
Unordered	$\{M_1, B_2, B_3\}$ $\{B_1, B_2, B_3\}$	{3 Bananas} {3 Mandarins}
Ordered	$[M_1, B_2, B_3]$ $[B_1, B_2, B_3]$	[Banana, Banana, Banana] [Mandarin, Mandarin, Mandarin]

Which choice will lead to equally likely outcomes?

Mandarins and Bananas

- 4 Mandarins and 3 Bananas in a Bag. 3 drawn.
  - What is P(1 Mandarin and 2 Bananas drawn)?
- Ordered:
  - Pick 3 ordered items: |S| = 7 * 6 * 5 = 210
  - Pick Mandarin as either 1st, 2nd, or 3rd item:
    |E| = (4 * 3 * 2) + (3 * 4 * 2) + (3 * 2 * 4) = 72
  - P(1 Mandarin, 2 Banana) = 72/210 = 12/35
- Unordered:
  - $|S| = \binom{7}{3} = 35$
  - $|E| = \binom{4}{1}\binom{3}{2} = 12$
  - P(1 Mandarin, 2 Banana) = 12/35

Make indistinct items **distinct** to get equally likely sample space outcomes

*You will need to use this “trick” with high probability

Straight Poker Hand

- Consider 5 card poker hands.
  - "straight" is 5 consecutive rank cards of any suit
  - What is P(straight)?

$$|S| = \binom{52}{5}$$

$$|E| = 10 \cdot \binom{4}{1}^5$$

What is an example of one outcome?

Is each outcome equally likely?

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10 \cdot \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$$

Straight Poker Hand

- Consider 5 card poker hands.
  - "straight" is 5 consecutive rank cards of any suit
  - "straight flush" is 5 consecutive rank cards of same suit
  - What is P(straight, but not straight flush)?

$$|S| = \binom{52}{5}$$

$$|E| = 10\binom{4}{1}^5 - 10\binom{4}{1}$$

$$P(\text{straight}) = \frac{|E|}{|S|} = \frac{10\binom{4}{1}^5 - 10\binom{4}{1}}{\binom{52}{5}} \approx 0.00392$$

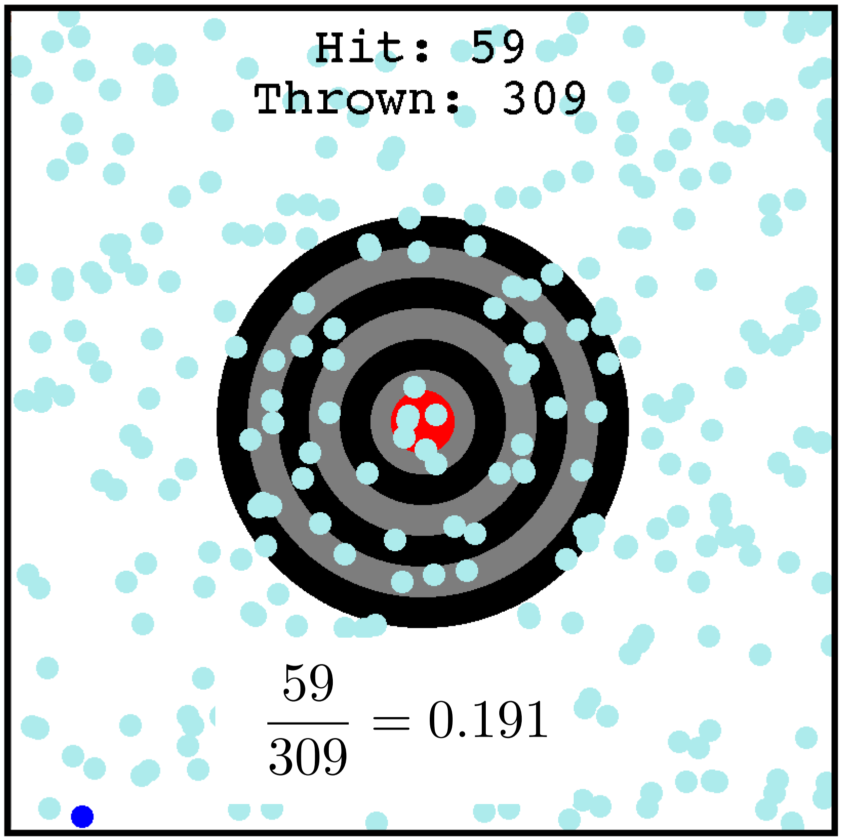
When approaching an "**equally likely probability**" problem, start by defining **sample spaces** and **event spaces**.

Chip Defect Detection

- n chips manufactured, 1 of which is defective.
- k chips randomly selected from n for testing.
  - What is P(defective chip is in k selected chips)?
- $|S| = \binom{n}{k}$
- $|E| = \binom{1}{1}\binom{n-1}{k-1}$
- P(defective chip is in k selected chips)

$$= \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{k}{n}$$

Target Revisited



$$\frac{59}{309} = 0.191$$

Screen size = 800x800
Radius of target = 200

The dart is equally likely to land anywhere on the screen.

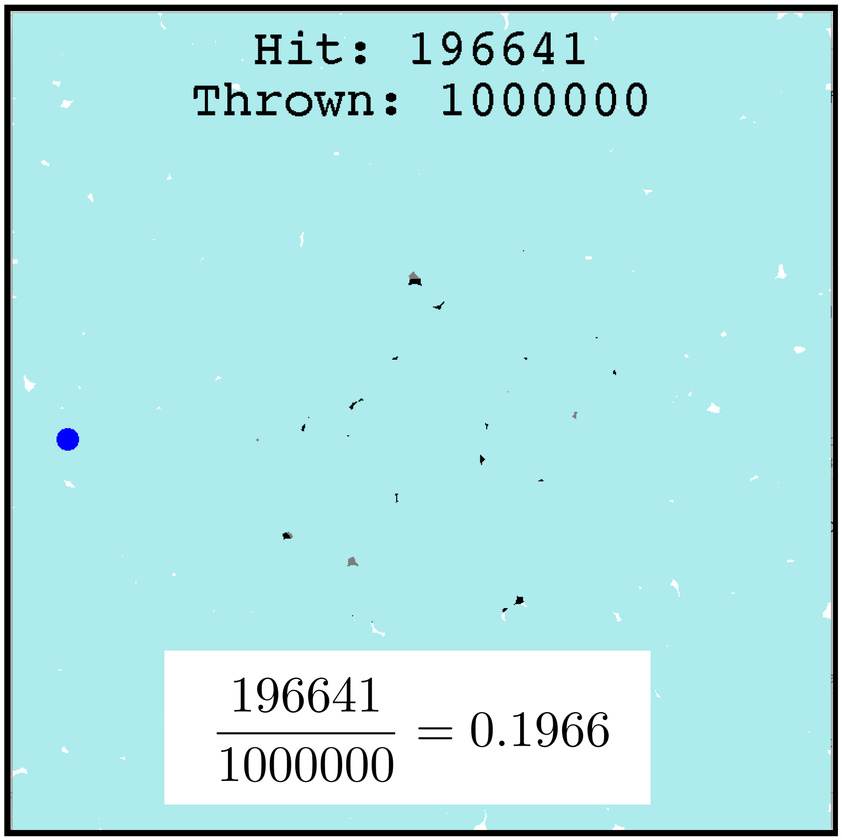
What is the probability of hitting the target?

$$|S| = 800^2$$

$$|E| = \pi 200^2$$

$$p(E) = \frac{\pi \cdot 200^2}{800^2} \approx 0.1963$$

Target Revisited



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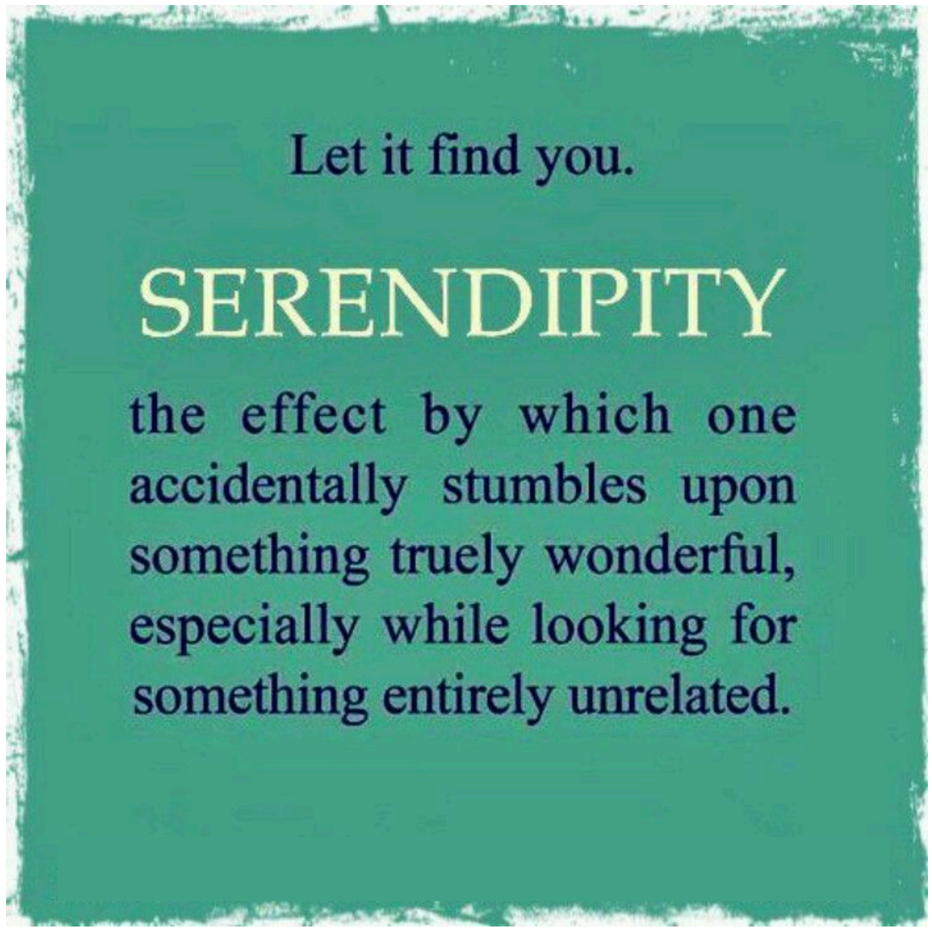
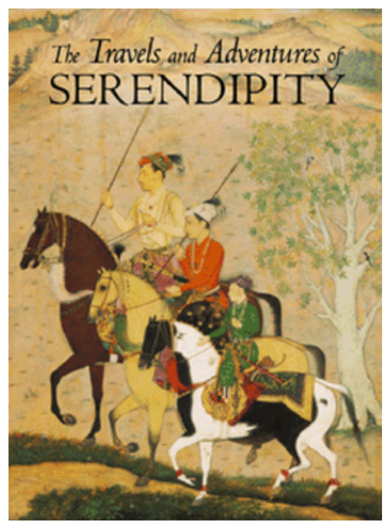

Let it find you.
SERENDIPITY
the effect by which one accidentally stumbles upon something truely wonderful, especially while looking for something entirely unrelated.

WHEN YOU MEET YOUR BEST FRIEND

Somewhere you didn't expect to.

Serendipity

- Say the population of Stanford is 17,000 people
  - You are friends with ?
  - Walk into a room, see 268 random people.
  - What is the probability that you see someone you know?
  - Assume you are equally likely to see each person at Stanford



Many times it is easier to calculate $P(E^C)$.

Back to Axiom 3

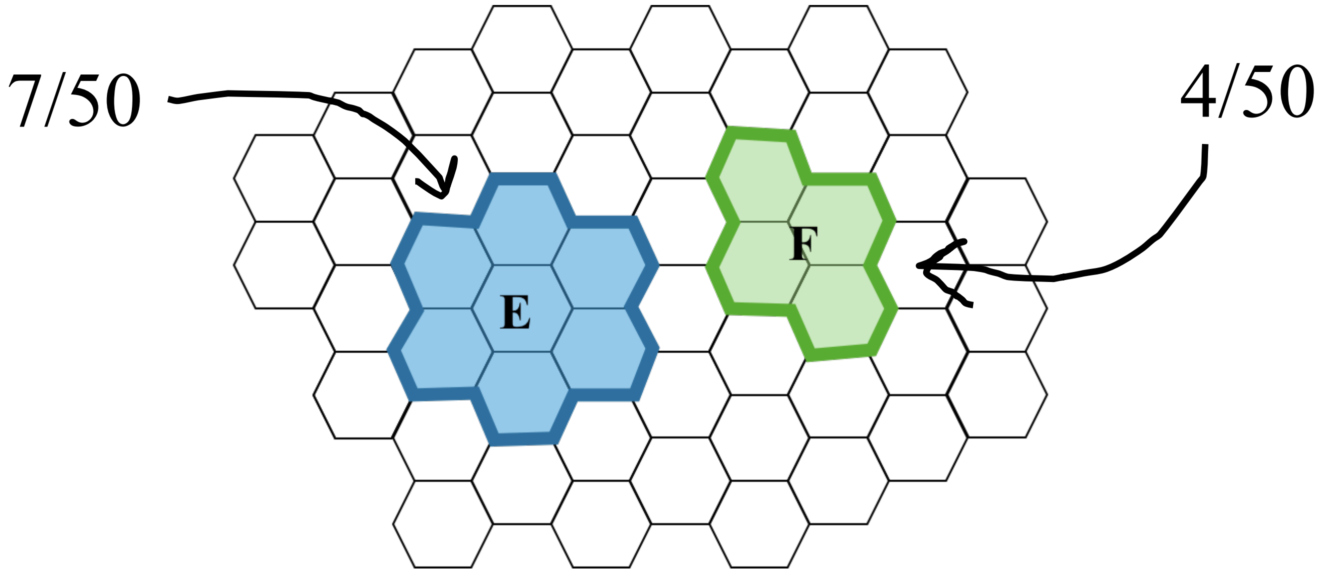
Axioms of Probability

Recall: S = all possible outcomes. E = the event.

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Axiom 3: If events E and F are mutually exclusive:

$$P(E \cup F) = P(E) + P(F)$$

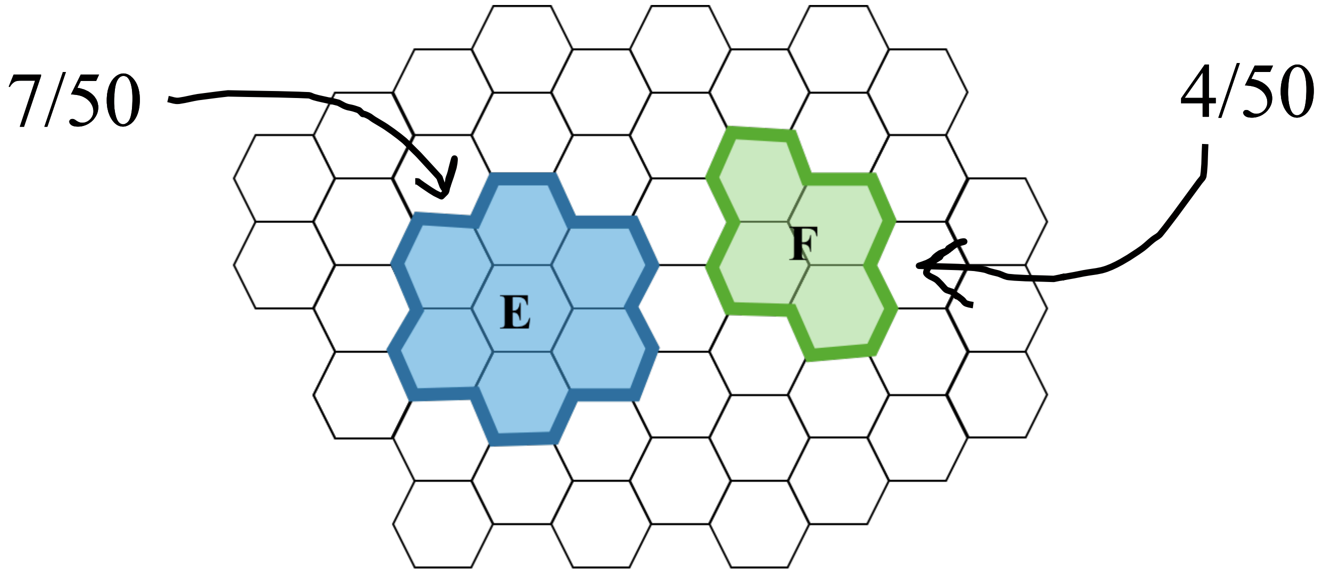
Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

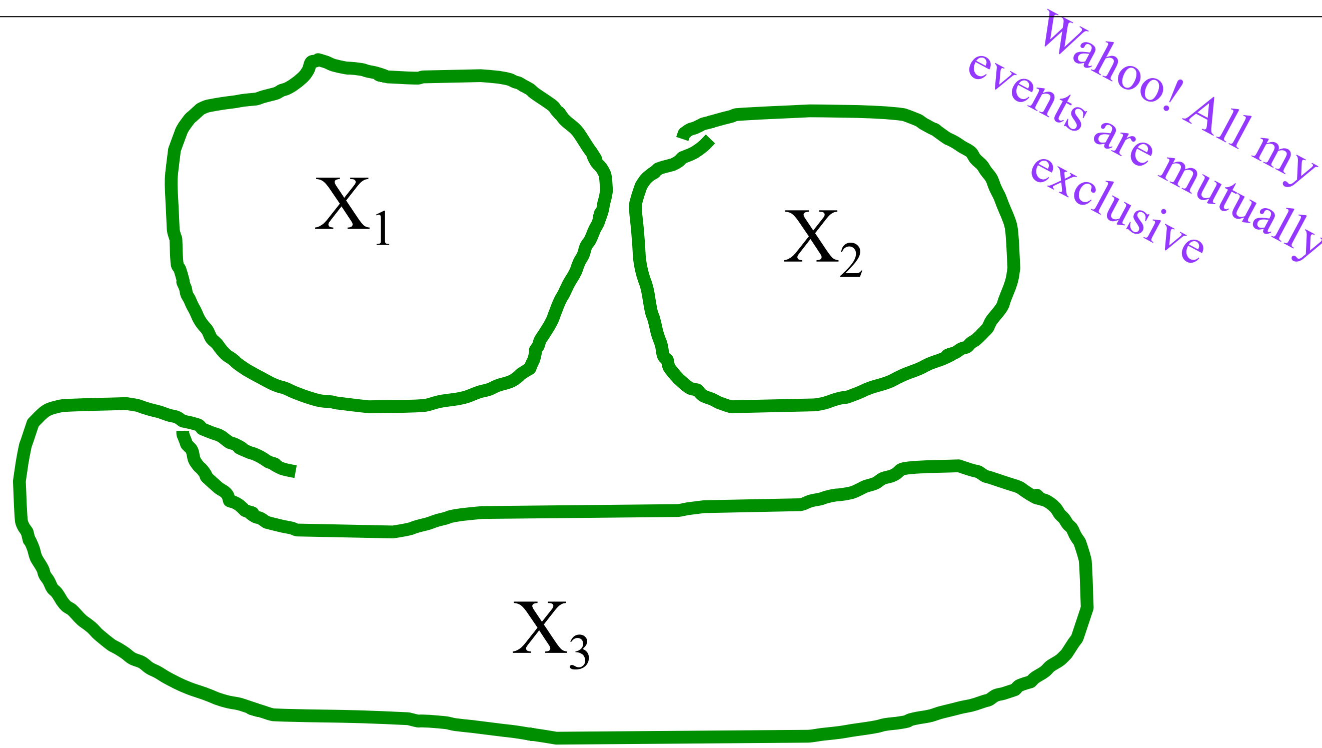
Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{5} = \frac{11}{50}$$

Probability of "or"



$$P(X_1 \cup X_2 \cup \cdots \cup X_n) = \sum_{i=1}^{n} P(X_i)$$

If events are *mutually exclusive* probability of OR is easy!

$$P(E^c) = 1 - P(E)?$$

$$P(E \cup E^c) = P(E) + P(E^c)$$

Since E and E^c are mutually exclusive

$$P(S) = P(E) + P(E^c)$$

Since everything must either be in E or E^c

$$1 = P(E) + P(E^c)$$

Axiom 2

$$P(E^c) = 1 - P(E)$$

Rearrange


Trailing the dovetail shuffle to it’s lair – Persi Diaconosis

Trailing the Dovetail Shuffle to Its Lair

The Annals of Applied Probability
1992, Vol. 2, No. 2, 294–313

TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR

BY DAVE BAYER[1] AND PERSI DIACONIS[2]

Columbia University and Harvard University

We analyze the most commonly used method for shuffling cards. The main result is a simple expression for the chance of any arrangement after any number of shuffles. This is used to give sharp bounds on the approach to randomness: $\frac{3}{2}\log_2 n + \theta$ shuffles are necessary and sufficient to mix up n cards.

Key ingredients are the analysis of a card trick and the determination of the idempotents of a natural commutative subalgebra in the symmetric group algebra.

1. Introduction. The dovetail, or riffle shuffle is the most commonly used method of shuffling cards. Roughly, a deck of cards is cut about in half and then the two halves are riffled together. Figure 1 gives an example of a riffle shuffle for a deck of 13 cards.

A mathematically precise model of shuffling was introduced by Gilbert and Shannon [see Gilbert (1955)] and independently by Reeds (1981). A deck of n cards is cut into two portions according to a binomial distribution; thus, the chance that k cards are cut off is $\binom{n}{k}/2^n$ for $0 \le k \le n$. The two packets are then riffled together in such a way that cards drop from the left or right heaps with probability proportional to the number of cards in each heap. Thus, if there are A and B cards remaining in the left and right heaps, then the chance that the next card will drop from the left heap is $A/(A+B)$. Such shuffles are easily described backwards: Each card has an equal and independent chance of being pulled back into the left or right heap. An inverse riffle shuffle is illustrated in Figure 2.

Experiments reported in Diaconis (1988) show that the Gilbert–Shannon–Reeds (GSR) model is a good description of the way real people shuffle real cards. It is natural to ask how many times a deck must be shuffled to mix it up. In Section 3 we prove:

THEOREM 1. *If n cards are shuffled m times, then the chance that the deck is in arrangement π is $\binom{2^m+n-r}{n}/2^{mn}$, where r is the number of rising sequences in π.*

Rising sequences are defined and illustrated in Section 2 through the analysis of a card trick. Section 3 develops several equivalent interpretations of

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[2]Partially supported by NSF Grant DMS-89-05874.
AMS 1980 *subject classifications*. 20B30, 60B15, 60C05, 60F99.
Key words and phrases. Card shuffling, symmetric group algebra, total variation distance.

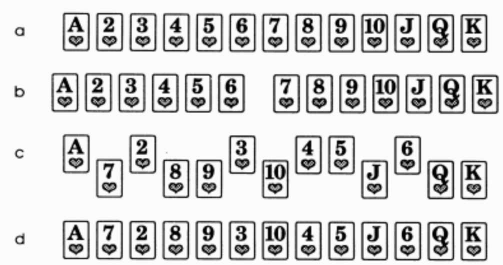

FIG. 1. *A riffle shuffle.* (a) *We begin with an ordered deck.* (b) *The deck is divided into two packets of similar size.* (c) *The two packets are riffled together.* (d) *The two packets can still be identified in the shuffled deck as two distinct "rising sequences" of face values.*

the GSR distribution for riffle shuffles, including a geometric description as the motion of n points dropped at random into the unit interval under the baker's transformation $x \to 2x \pmod 1$. This leads to a proof of Theorem 1.

Section 3 also relates shuffling to some developments in algebra. A permutation π has a descent at i if $\pi(i) > \pi(i+1)$. A permutation π has r rising sequences if and only if π^{-1} has $r-1$ descents. Let

$$A_k = \sum_{\pi \text{ has } k \text{ descents}} \pi$$

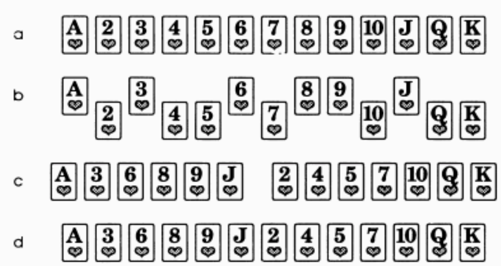

FIG. 2. *An inverse riffle shuffle.* (a) *We begin with a sorted deck.* (b) *Each card is moved one way or the other uniformly at random, to "pull apart" a riffle shuffle and retrieve two packets.* (c) *The two packets are placed in sequence.* (d) *The two packets can still be identified in the shuffled deck; they are separated by a "descent" in the face values. This shuffle is inverse to the shuffle diagrammed in Figure 1.*

Probability of "or"

- What is the probability that in the *n* shuffles seen since the start of time, yours is unique?
  - $|S| = (52!)^n$
  - $|E| = (52! - 1)^n$
  - P(no deck matching yours) = $(52!-1)^n/(52!)^n$

- For $n = 10^{20}$,
  - P(deck matching yours) < 0.000000001

* Assume 7 billion people have been shuffling cards once a second since cards were invented