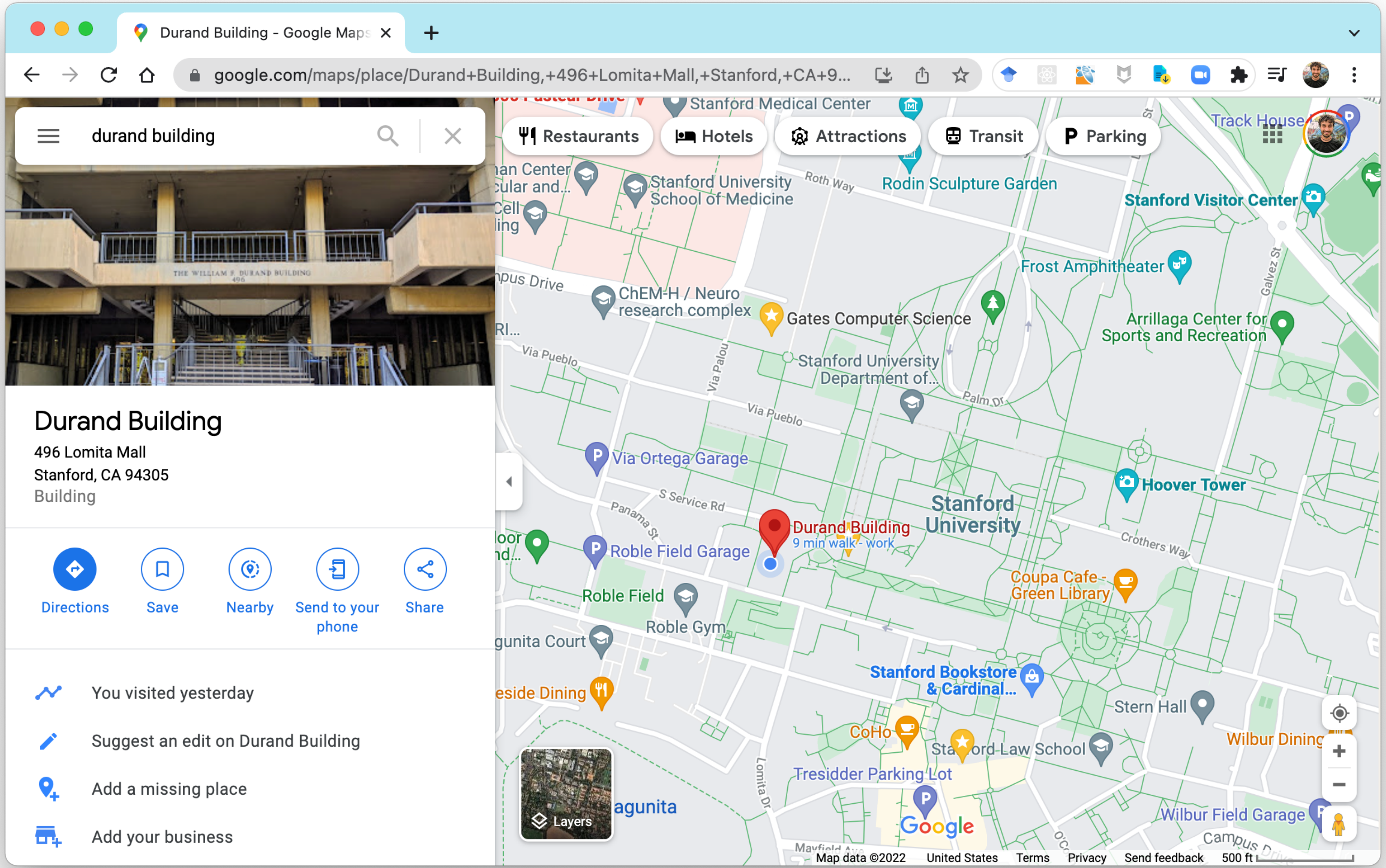
Normal Distribution

Chris Piech
CS109, Stanford University

Some In Person Sections



(Also Durand is a building and a street. If your section is in Durand, go to the building.)

PS3

1
2
3
4
5
6
7
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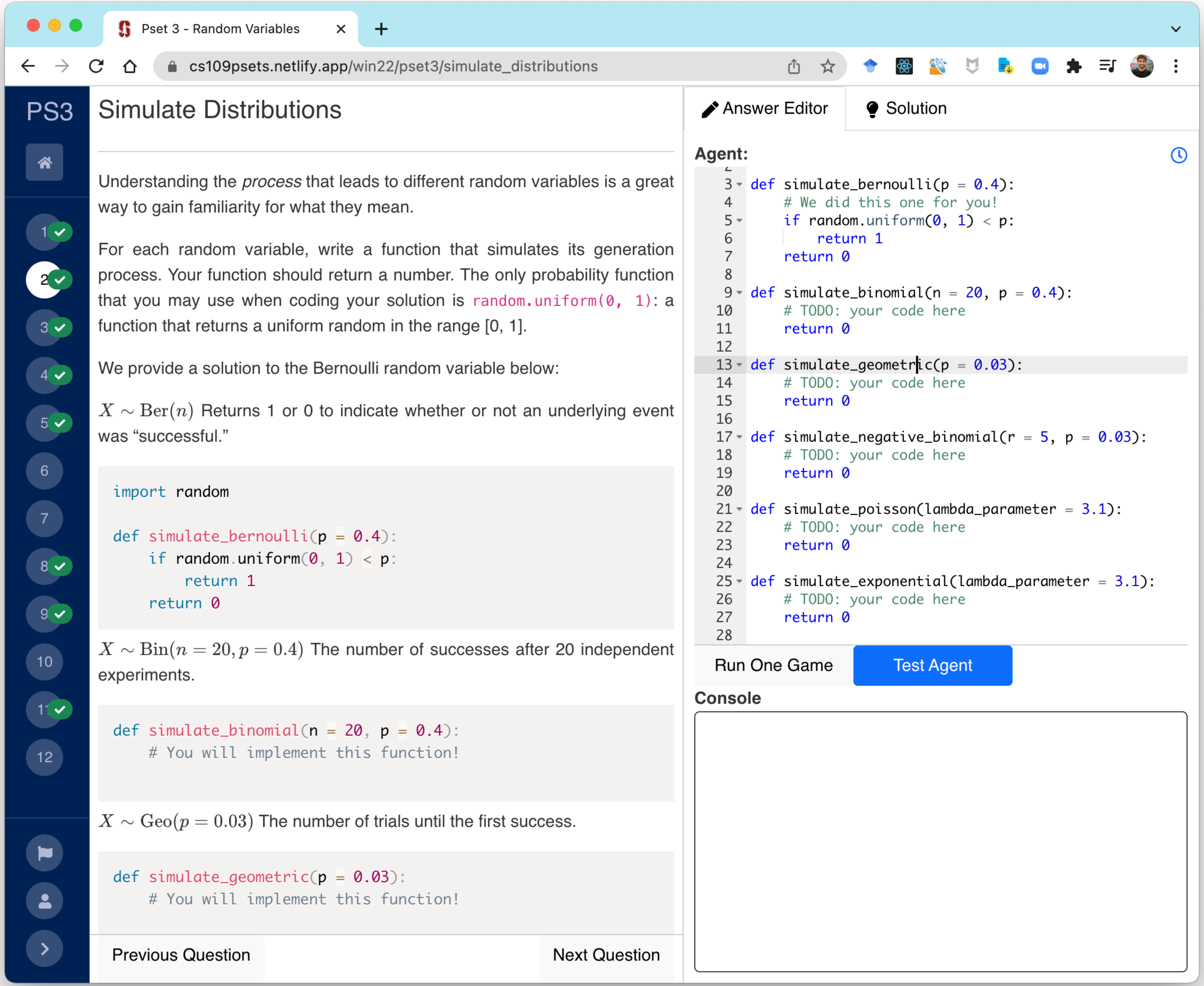
12

PSet 3 is out!

We put a lot of love into it,
and hope you enjoy

Uses discrete random variables

Uses continuous random variables



Simulate!

Great way to make sure you really understand the random variables

Don't use numpy or scipy!

Think about the poisson proof for poisson and exponential

Pset 3 - Random Variables

cs109psets.netlify.app/win22/pset3/measles_test

PS3

Measles Test

To determine whether they have measles, 60 people have their blood tested. However, rather than testing each individual separately, it is decided to first place the people into groups of 6. The blood samples of the 6 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 6 people, whereas if the test is positive, each of the 6 people will also be individually tested and, in all, 7 tests will be made on this group. Note that we assume that the pooled test will be positive if at least one person in the pool has measles. Assume that the probability that a person has measles is 5% for all people, independently of each other, and compute the expected number of tests necessary for each group of 6 people.

Previous Question | Next Question

Answer Editor | Solution

Numeric Answer: 2.589 ✓ Check Answer

Explanation:

Block LaTeX | Image | B | I | U

1. Define a random variable

2. Declare its distribution

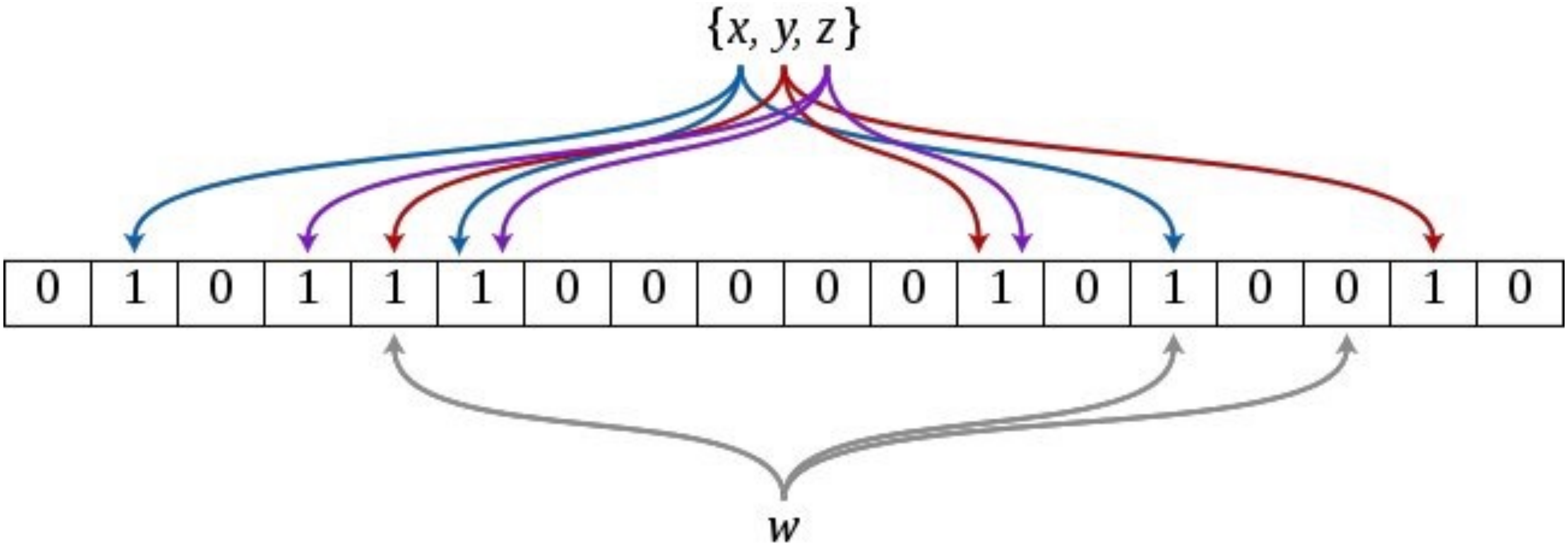
3. Write the question as a probability statement using the random variable

4. Solve

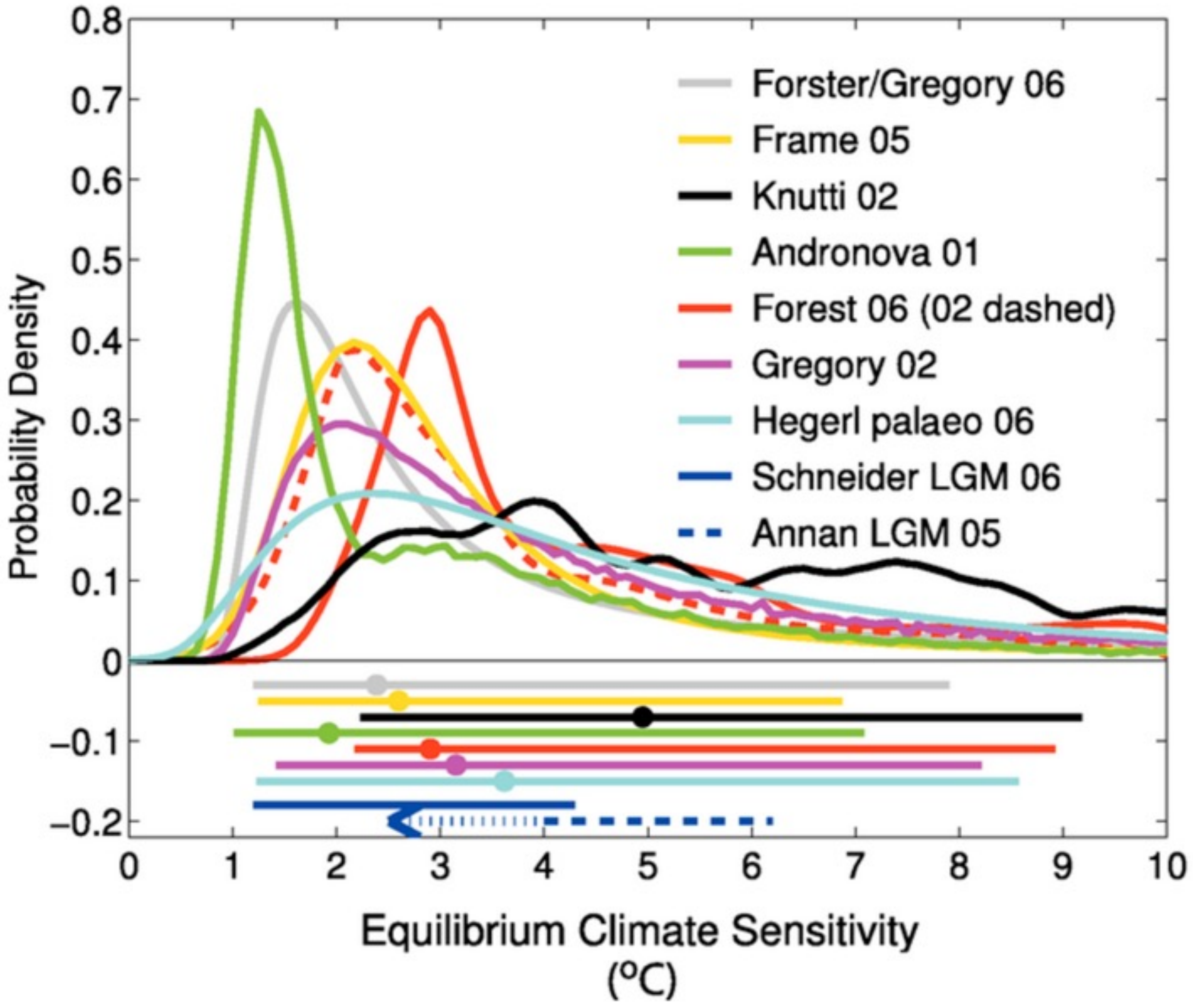
Many prototypical random variable questions.

Batch testing is a thing!

Bloom Filter



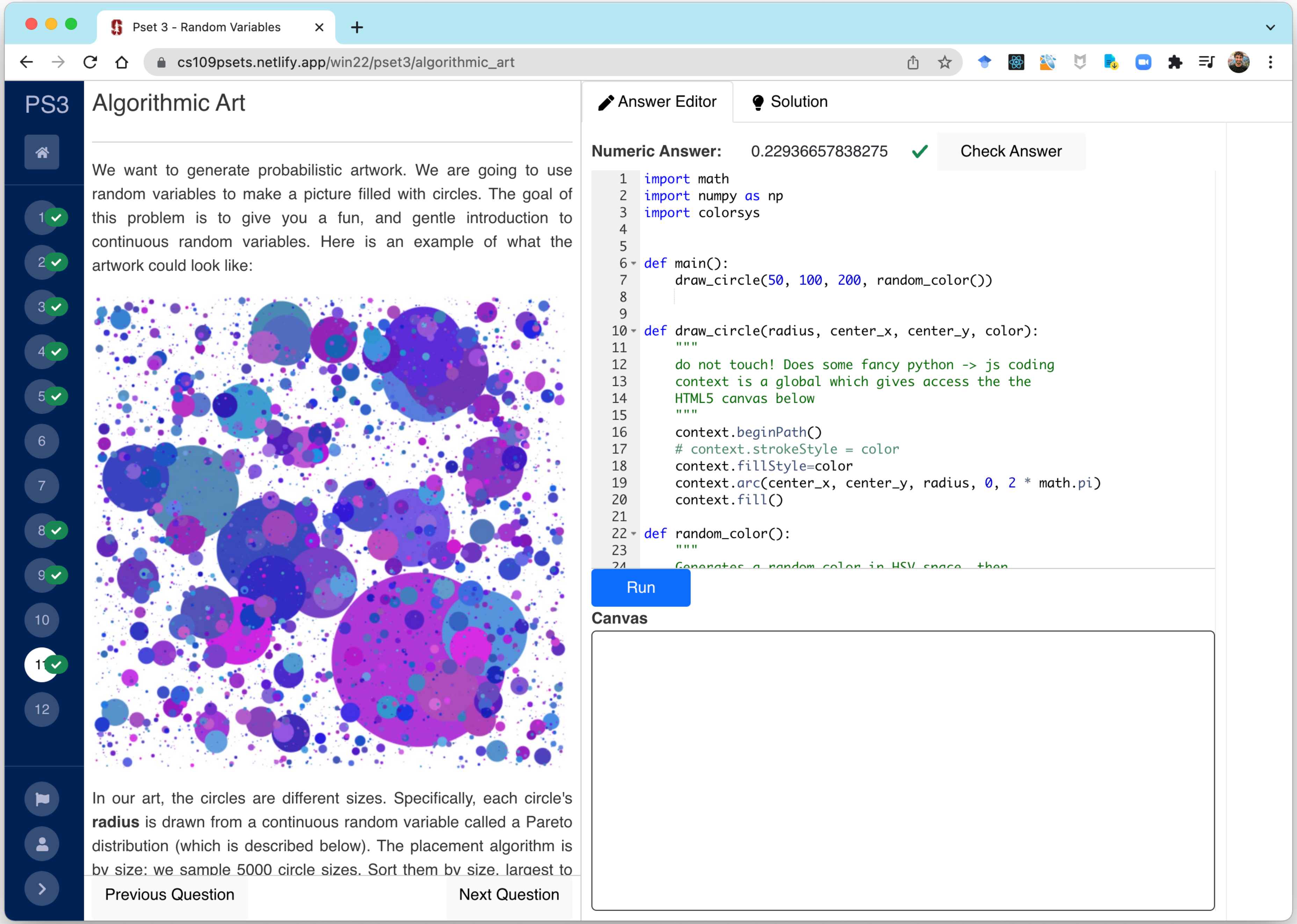
Climate Sensitivity



Which is Random?

Sequence 1:
TTHHTHTTHTTTHTTTHTTTHTTHTHHTHHTHTHHTTTHHTHTHTTHTHHTTHTHHTHTTTH
HTTHHTTHHHTHHTHTTHTHTTHHTHHHTTHTHTTTHHTTHTHTHTHTHTTHTHTHHHTTHT
HTHHTHHHTHTHTTHTTHHTHTHTHTTHHTTHTHTTHHHTHTHTHTTHTTHHTTHTHHTHHH
TTHHTHTTHTHTHTHTHTHTHHHTHTHTHTTHTHHTHTHTTHTTTHHTHTTTHTHHTHHHHT
TTHHTHTHTHTHHHTTHHTHTTTHTHHTHTHTHHTHTTHTTHTHHTHTHTTT

Sequence 2:
HTHHHTHTTHHTTTTTTTTHHHTTTHHTTTTHHTTHHHTTHTHTTTTTTTHTHTTTTHHHHTH
THTTHTTTHTTHTTTTHTHHTHHHHTTTTTHHHHTHHHTTTTHTHTTHHHHTHHHHHHHHTT
HHTHHTHHHHHHHTTHTHTTTHHTTTTHTHHTTHTTHTHTHTTHHHHHTTHTTTHTHTHHTT
TTHTTTTTHHTHTHHHHTTTTHTHHHHHHTHTHTHTHHHTHTTHHHTHHHHHHTHHHTHTTT
HHHTTTHHTHTTHHTHHHTHTTHTTHTTTHHTHTHTTTTHTHTHTTHTHTHT



Final exam question from last year

Meant to be play!

https://chrispiech.github.io/probabilityForComputerScientists/en/examples/algorithmic_art/

Regenerate

In our art, the circles are different sizes. Specifically, each circle's **radius** is drawn from a Pareto distribution (which is described below). The placement algorithm is greedy: we sample 1000 circle sizes. Sort them by size, largest to smallest. Loop over the circle sizes and place circles one by one.

To place a circle on the canvas, we sample the location of the center of the circle. Both the x and y coordinates are uniformly distributed over the dimensions of the canvas. Once we have selected a prospective location we then check if there would be a collision with a circle that has already been placed. If there is a collision we keep trying new locations until you find one that has no collisions.

The Pareto Distribution

Pareto Random Variable

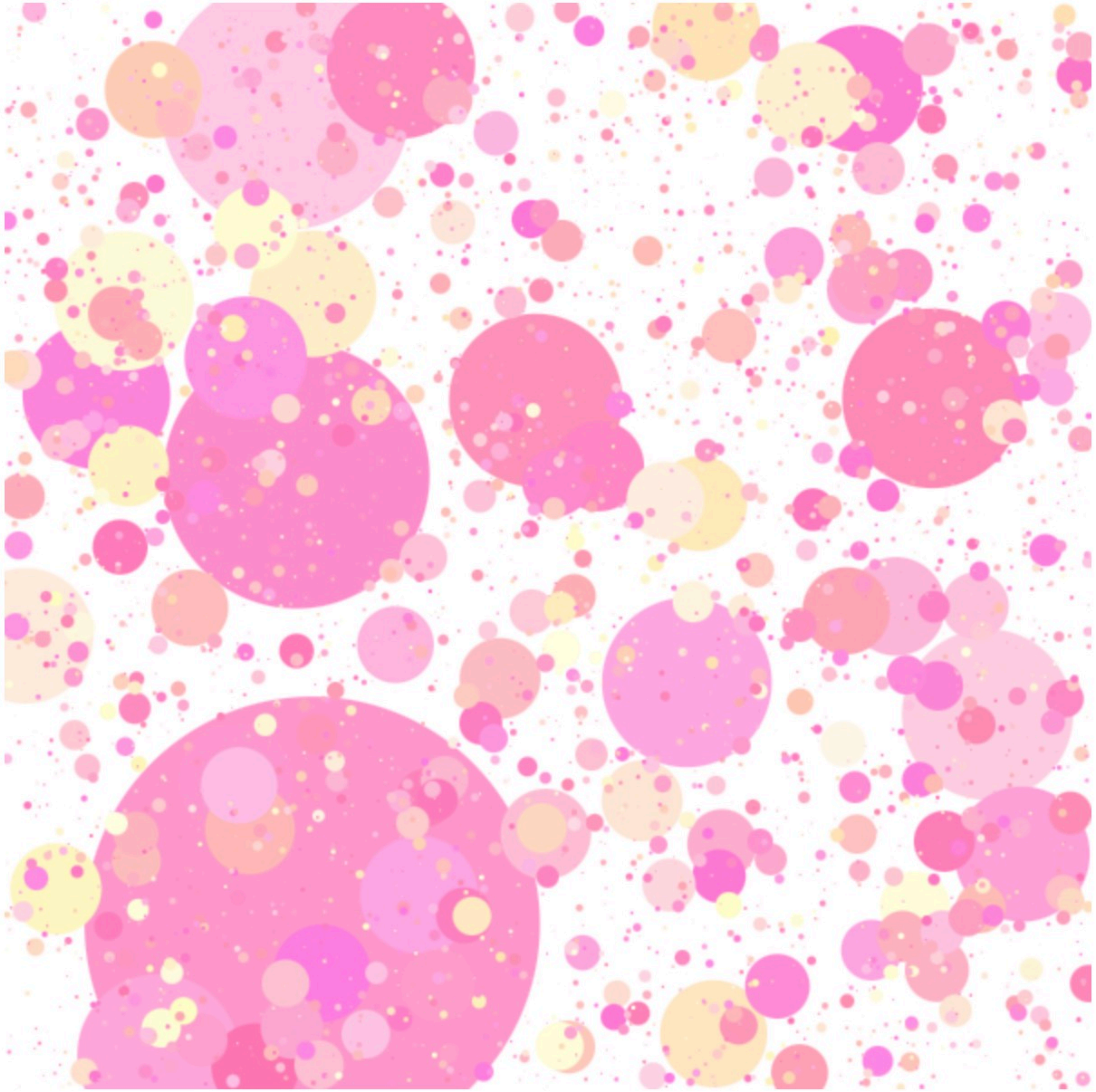
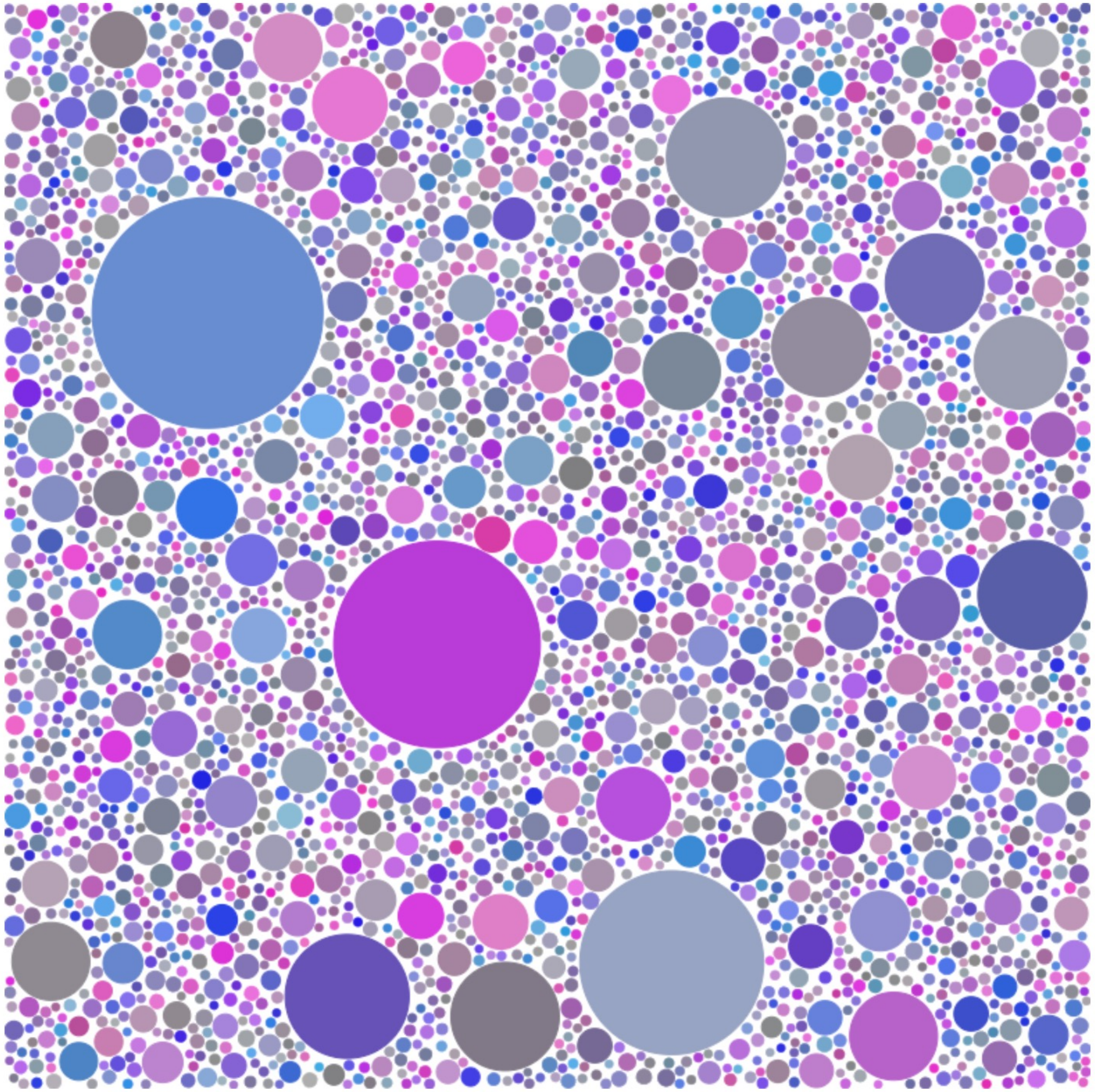
Notation:	$X \sim \text{Pareto}(a)$
Description:	A long tail distribution. Large values are rare and small values are common.
Parameters:	$a \geq 1$, the shape parameter Note there are other optional params. See wikipedia
Support:	$x \in [0, 1]$
PDF equation:	$f(x) = \frac{1}{x^{a+1}}$
CDF equation:	$F(x) = 1 - \frac{1}{x^a}$

Sampling from a Pareto Distribution

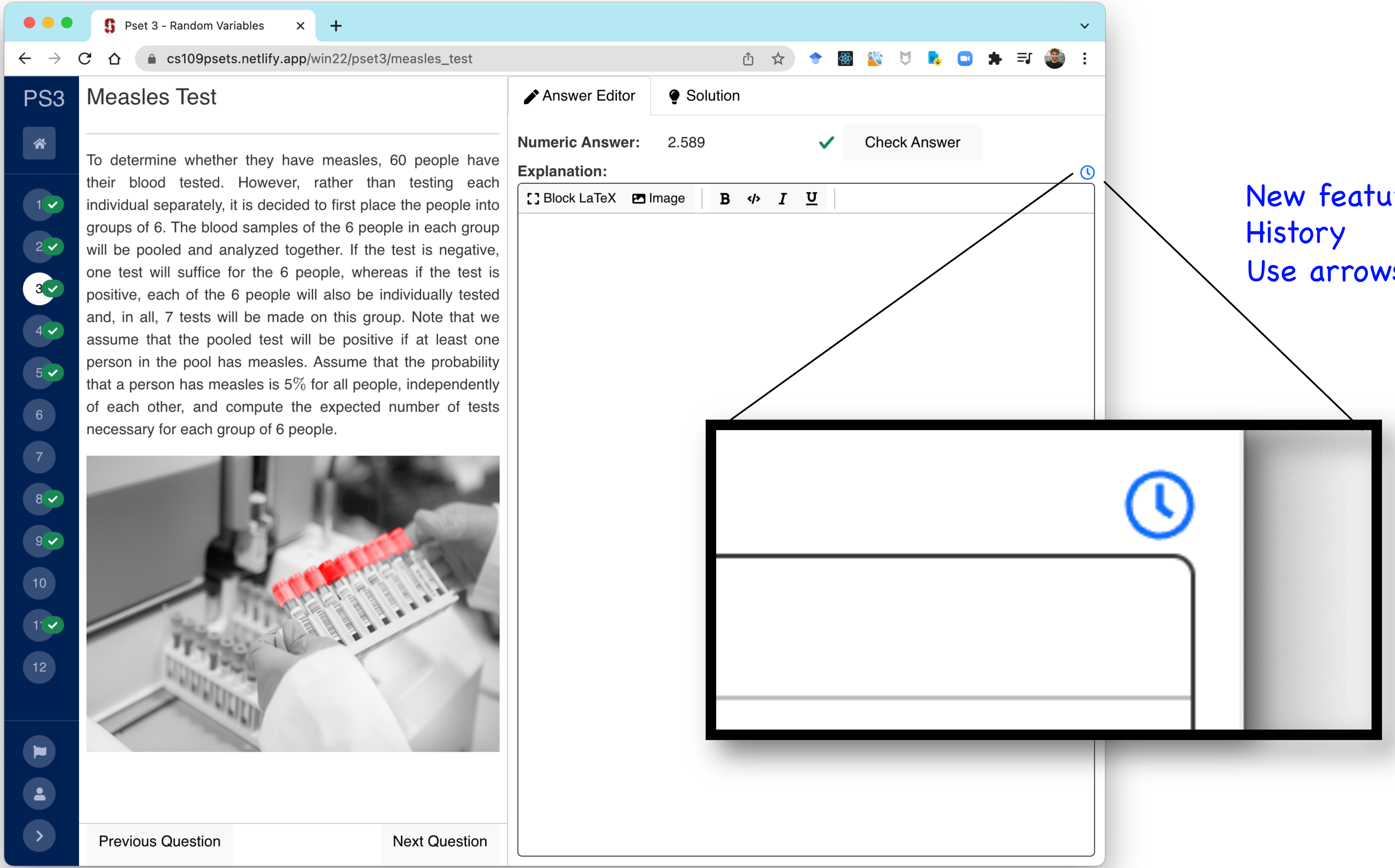
How can we draw samples from a pareto? In python its simple: `stats.pareto.rvs(a)` however in JavaScript, or other languages, it might not be made transparent

$$y = 1 - (\frac{\beta}{x})^\alpha$$

$$(\frac{\beta}{x})^\alpha = 1 - y$$

Please share https://edstem.org/us/courses/16856/discussion/1042885



New features:
History
Use arrows to navigate

Review

Review

	number of **successes**	**time** to get successes	
One trial	$X \sim \text{Ber}(p)$	$X \sim \text{Geo}(p)$	One success
	↑ $n = 1$	↑ $r = 1$	
Several trials	$X \sim \text{Bin}(n, p)$	$X \sim \text{NegBin}(r, p)$	Several successes
Interval of time	$X \sim \text{Poi}(\lambda)$	$X \sim \text{Exp}(\lambda)$	One success

$X \sim \text{Uni}(\alpha, \beta)$ All values are equally likely

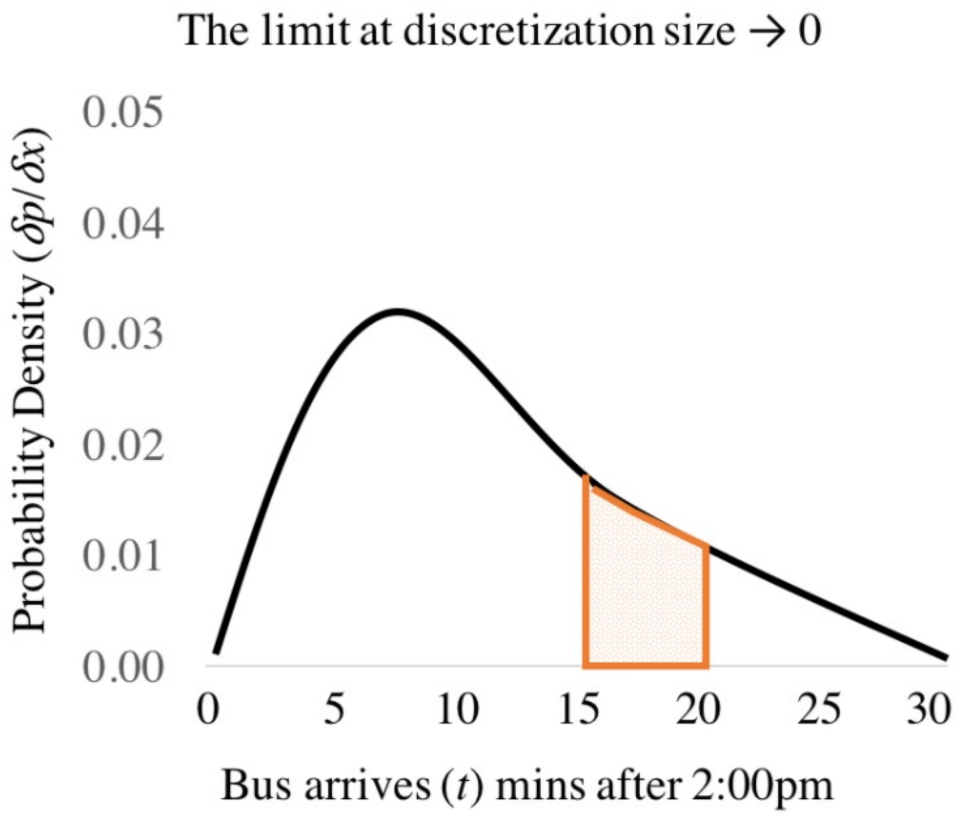
Review: Probability Density Function

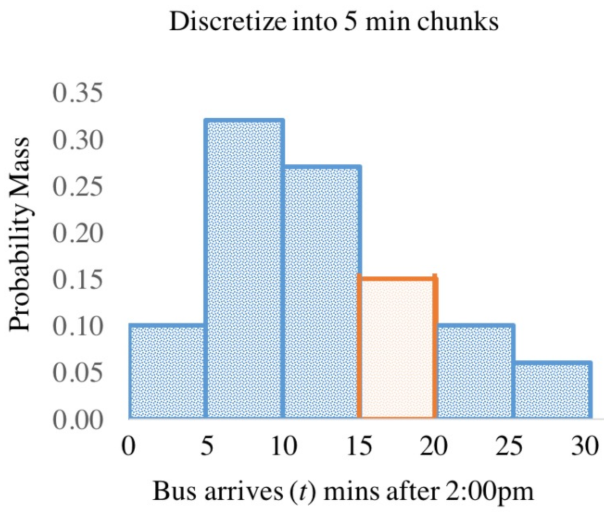
The **probability density function** (PDF) of a continuous random variable represents the **derivative** of probability at a given point.

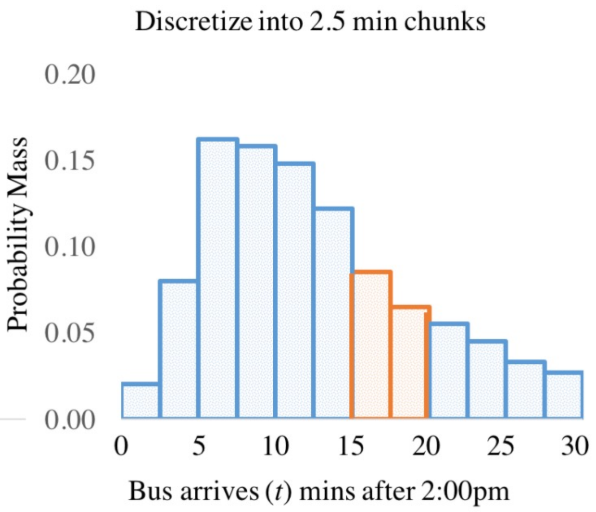
Units of probability *divided by units of X*.
Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^{b} \boxed{f(X = x)}\, dx$$

Review: Probability Density Function







Uniform distribution

Uniform Random Variable

Notation: $X \sim \text{Uni}(\alpha, \beta)$

Description: A continuous random variable that takes on values, with equal likelihood, between α and β

Parameters: $\alpha \in \mathbb{R}$, the minimum value of the variable.
$\beta \in \mathbb{R}$, $\beta > \alpha$, the maximum value of the variable.

Support: $x \in [\alpha, \beta]$

PDF equation:
$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{else} \end{cases}$$

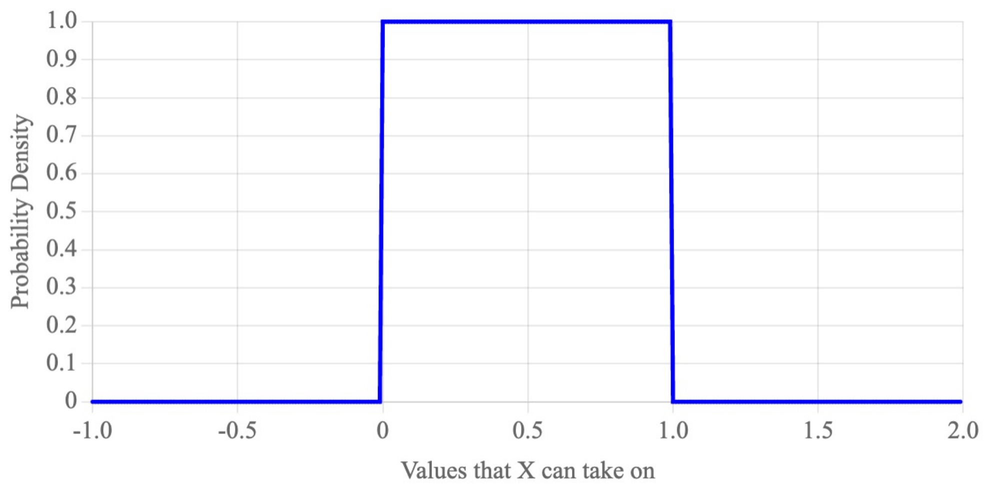
CDF equation:
$$F(x) = \begin{cases} \frac{x-\alpha}{\beta-\alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{for } x < \alpha \\ 1 & \text{for } x > \beta \end{cases}$$

Expectation: $\text{E}[X] = \frac{1}{2}(\alpha + \beta)$

Variance: $\text{Var}(X) = \frac{1}{12}(\beta - \alpha)^2$

PDF graph:

Parameter α: 0 Parameter β: 1



https://chrispiech.github.io/probabilityForComputerScientists/en/part2/uniform/

What do you get if you integrate over a probability *density* function?

A probability!

/Review

Big hole in our knowledge

Say the average rate of earthquakes is 1 every 100 years.

We can talk about the probability distribution of different numbers of earthquakes next year using a Poisson.

We can't talk about the probability distribution of the ***amount of time*** until the next earthquake.

1906 Earthquake
Magnitude 7.8

ILL. No. 65. MEMORIAL ARCH, WITH CHURCH IN BACKGROUND, STANFORD UNIVERSITY, SHOWING TYPES OF CARVED WORK WITH THE SANDSTONE

Exponential Random Variable

Consider an experiment that lasts a duration of time until success occurs.

def An **Exponential** random variable X is the amount of time until success.

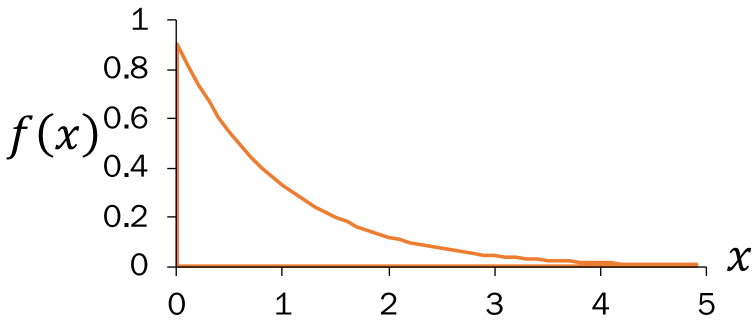
$X \sim \text{Exp}(\lambda)$

Support: $[0, \infty)$

PDF	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
Expectation	$E[X] = \frac{1}{\lambda}$
Variance	$\text{Var}(X) = \frac{1}{\lambda^2}$

Examples:

- Time until next earthquake
- Time for request to reach web server
- Time until end of cell phone contract



How Many Earthquakes

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the probability of **zero major earthquakes magnitude next year?**

X = Number of major earthquakes next year

$$X \sim \text{Poi}(\lambda = 0.002)$$

$$P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = \frac{0.002^0 e^{-0.002}}{0!} \approx 0.998$$

*In California, according to the USGS, 2015

How Long Until the Next Earthquake

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the probability of **a major earthquake in the next 30 years?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002)$$

$$f_Y(y) = \lambda e^{-\lambda y}$$
$$= 0.002^{-0.002y}$$

$$P(Y < 30) = \int_0^{30} 0.002 e^{-0.002y} \, dy$$

*In California, according to the USGS, 2015

Integral Review

$$\int e^{cx}\ dx = \frac{1}{c}e^{cx}$$

How Long Until the Next Earthquake

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the probability of **a major earthquake in the next 30 years?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002)$$

$$f_Y(y) = \lambda e^{-\lambda y}$$
$$= 0.002^{-0.002y}$$

$$P(Y < 30) = \int_0^{30} 0.002 e^{-0.002y} \, dy$$

$$= 0.002 \Big[-500 e^{-0.002y} \Big]_0^{30}$$

$$= \frac{500}{500}(-e^{-0.06} + e^0) \qquad \approx 0.06$$

*In California, according to the USGS, 2015

How Long Until the Next Earthquake

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the **expected number of years until the next earthquake?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002)$$

$$E[Y] = \frac{1}{\lambda} = \frac{1}{0.002} = 500$$

How Long Until the Next Earthquake

Based on historical data, major earthquakes (magnitude 8.0+) happen at a **rate of 0.002** per year*. What is the **standard deviation of years until the next earthquake?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002)$$

$$\text{Var}(Y) = \frac{1}{\lambda^2} = \frac{1}{0.002^2} = 250,000 \text{ years}^2$$

$$\text{Std}(Y) = \sqrt{\text{Var}(X)} = 500 \text{ years}$$

*In California, according to the USGS, 2015

Is there a way to avoid integrals?

Cumulative Density Function

A cumulative density function (CDF) is a “closed form” equation for the probability that a random variable is less than a given value

$$F(x) = P(X < x)$$

If you learn how to use a cumulative density function, you can avoid integrals!

$$F_X(x)$$

This is also shorthand notation for the PMF

Cumulative Density Function

$$F(x) = P(X < x)$$

$$x = 2$$

0.03125

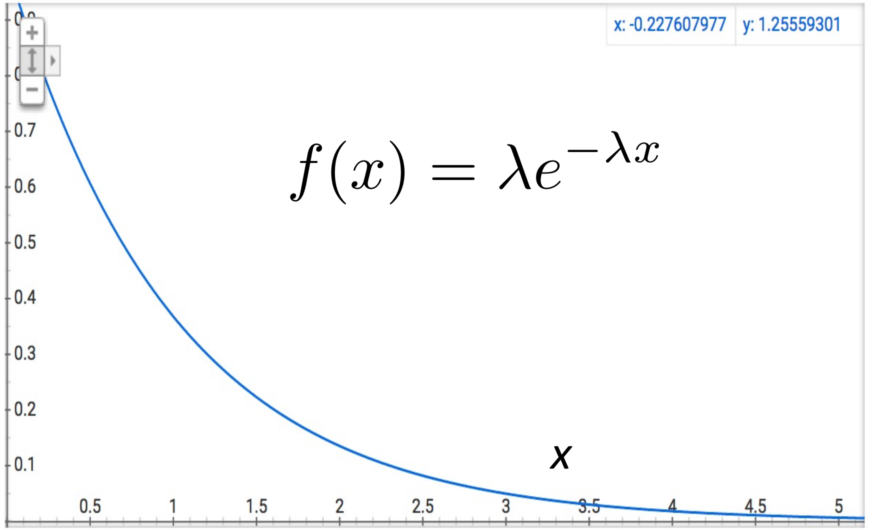
CDF of an Exponential

$$F_X(x) = 1 - e^{-\lambda x}$$

$$\begin{aligned}
P(X < x) &= \int_{y=-\infty}^{x} f(y)\, dy \\
&= \int_{y=0}^{x} \lambda e^{-\lambda y}\, dy \\
&= \frac{\lambda}{\lambda}\Big[-e^{-\lambda y}\Big]_0^x \\
&= [-e^{-\lambda x}] - [-e^{\lambda 0}] \\
&= 1 - e^{-\lambda x}
\end{aligned}$$

Using CDF Example. X is Exp(λ = 1)

Probability density function

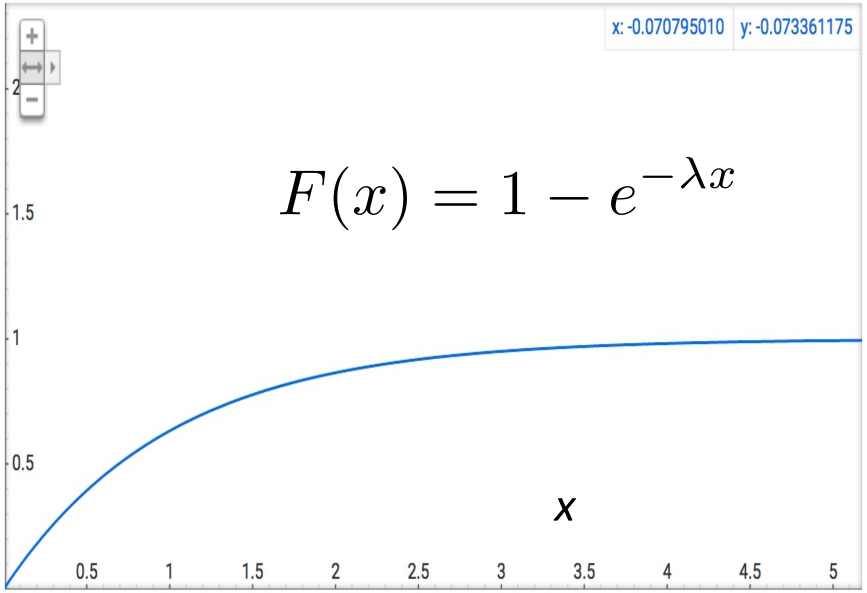


$$f(x) = \lambda e^{-\lambda x}$$

Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



$$F(x) = 1 - e^{-\lambda x}$$

Using CDF Example. X is Exp(λ = 1)

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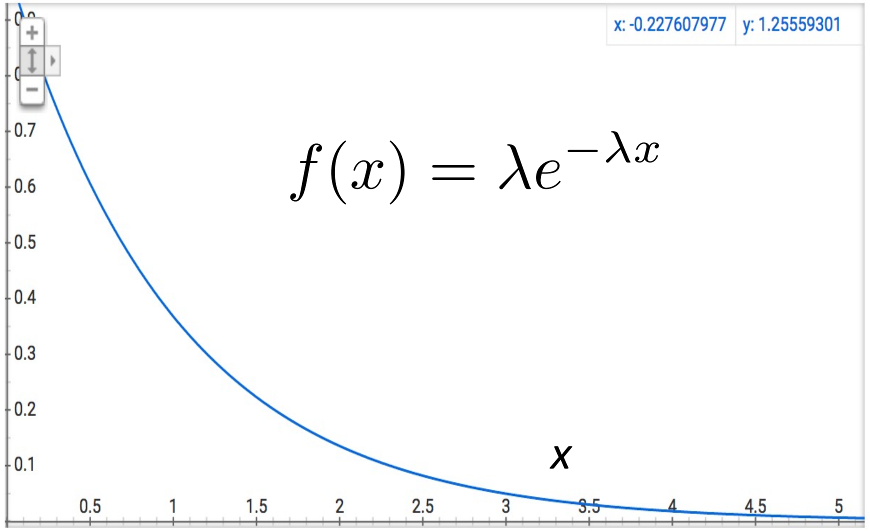
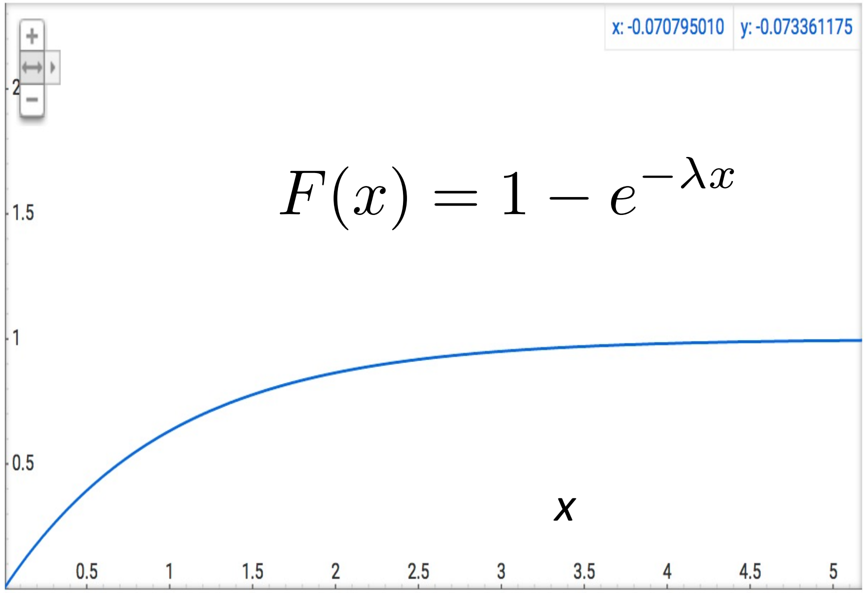


Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\ dy$$

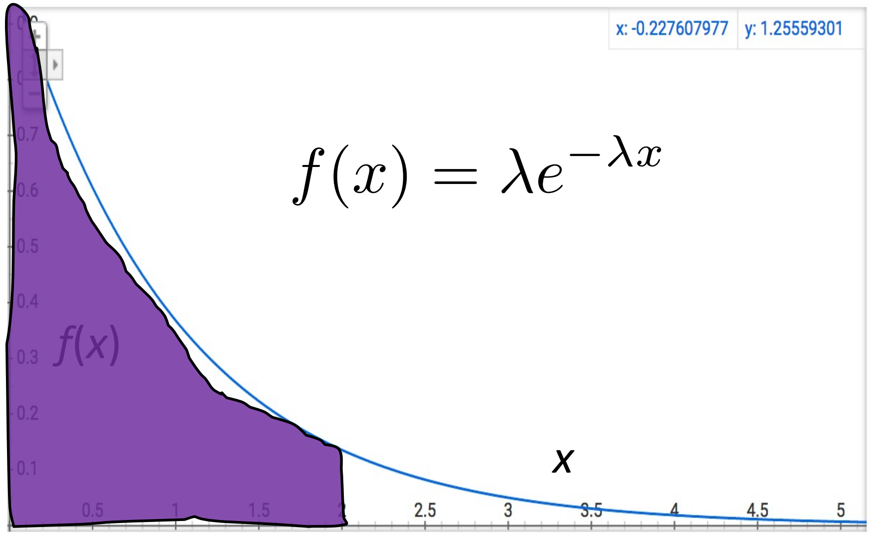


P(X < 2)

____

Using CDF Example. X is Exp(λ = 1)

Probability
density
function



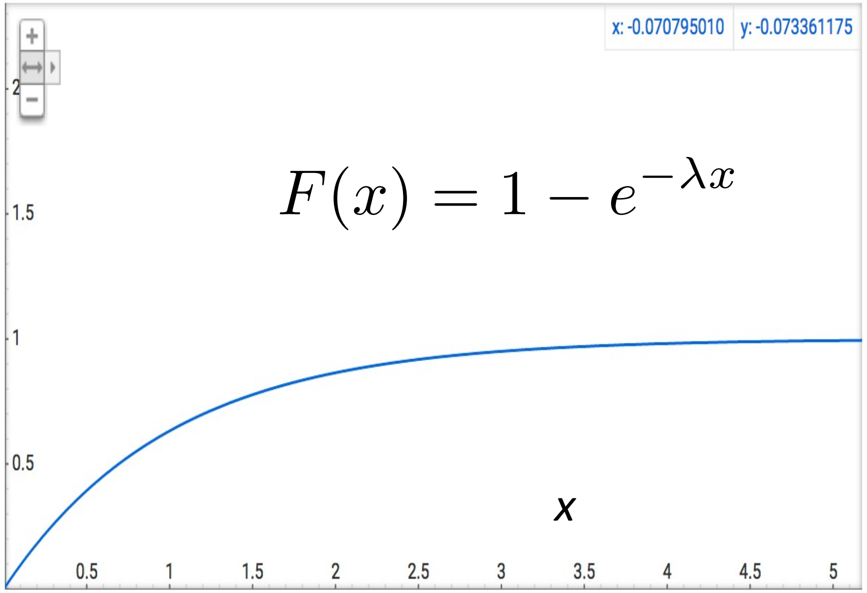
P(X < 2)

$$= \int_{x=-\infty}^{2} f(x)\, dx$$

Cumulative
density function

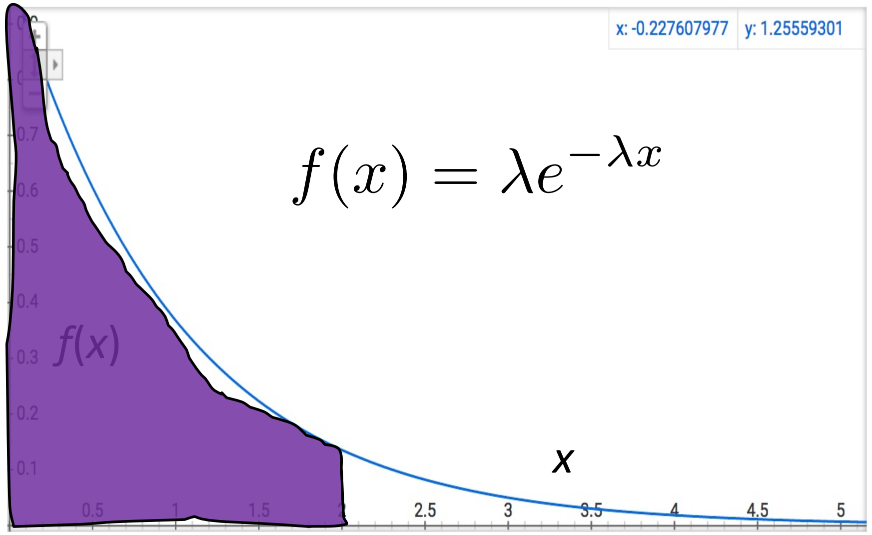
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp($\lambda = 1$)

Probability density function



P(X < 2)

$$= \int_{x=-\infty}^{2} f(x)\,dx$$

or

$$= F(2)$$

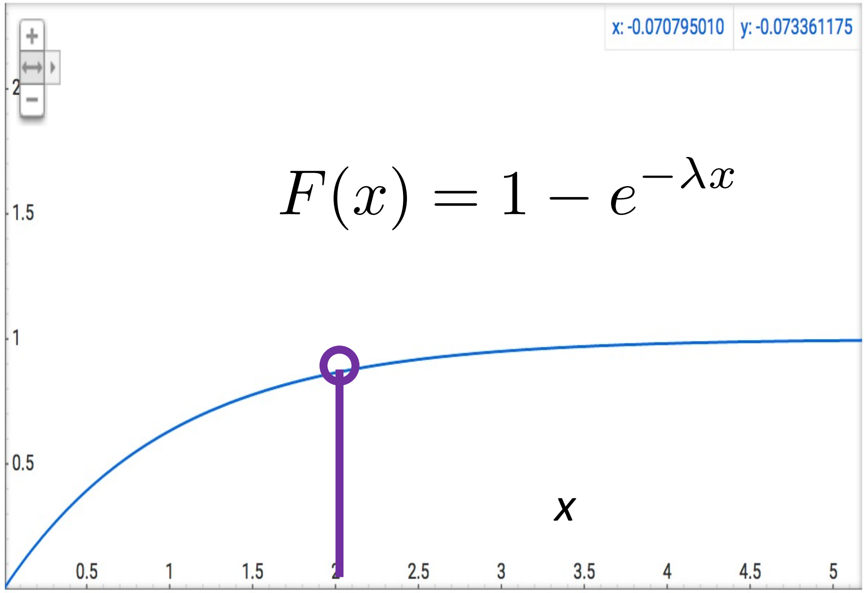
$$= 1 - e^{-2}$$

$$\approx 0.84$$

Cumulative density function

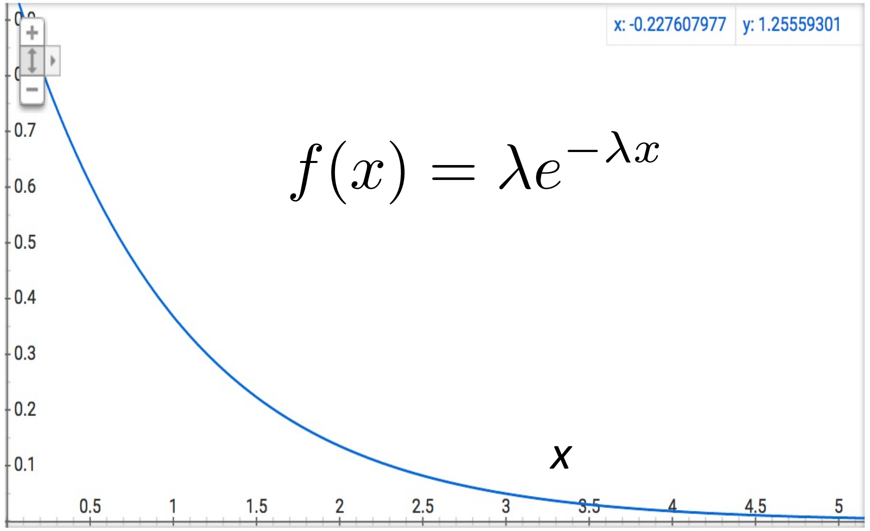
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\,dy$$



Using CDF Example. X is Exp($\lambda = 1$)

Probability density function



Cumulative density function

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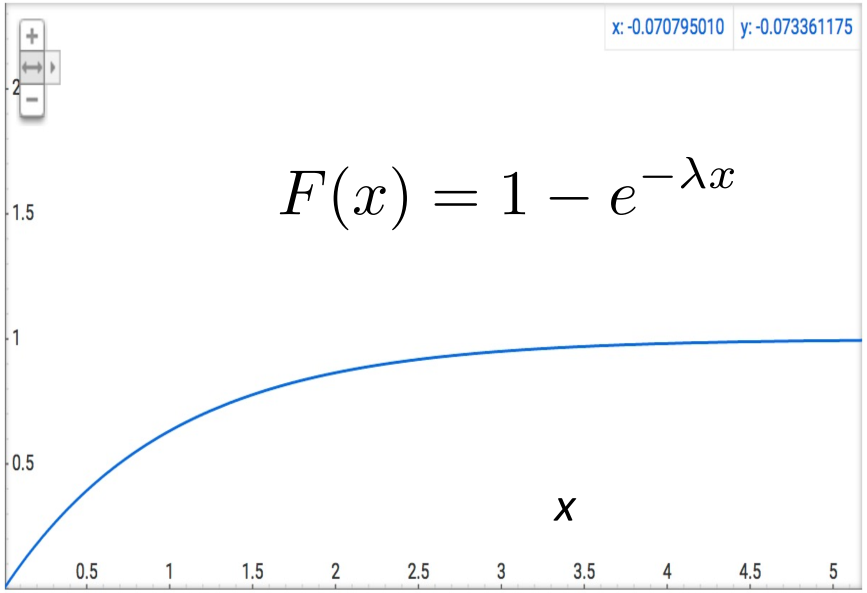
$$= \int_{y=-\infty}^{x} f(y)\, dy$$

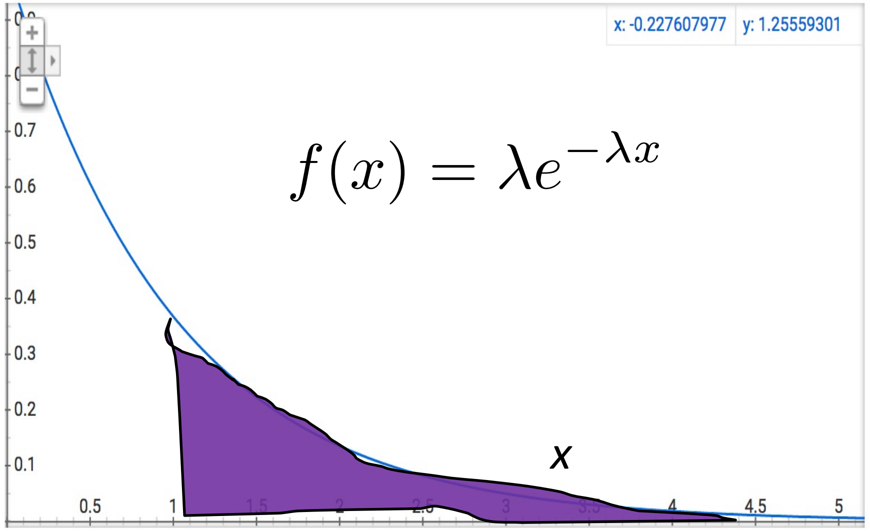


P(X > 1)

____

Using CDF Example. X is Exp(λ = 1)

Probability density function



P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\, dx$$

Cumulative density function

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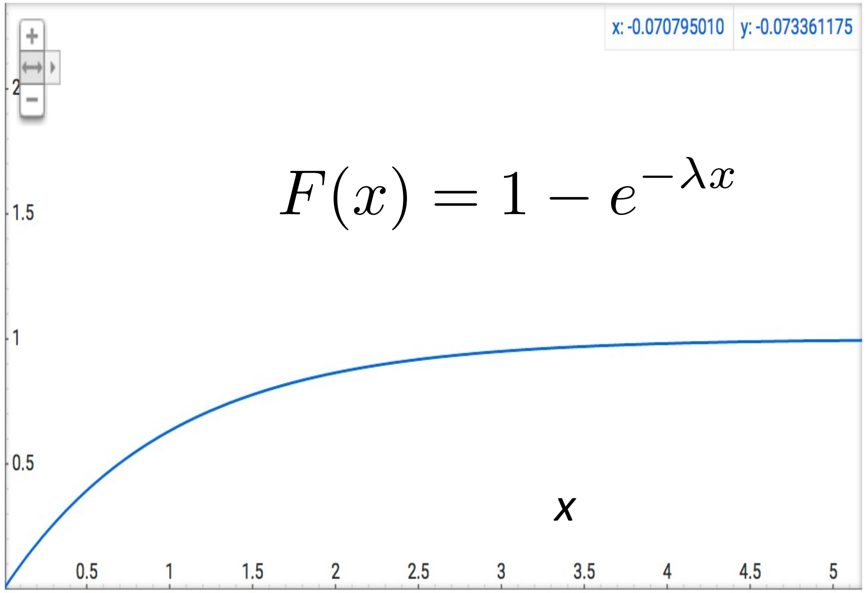
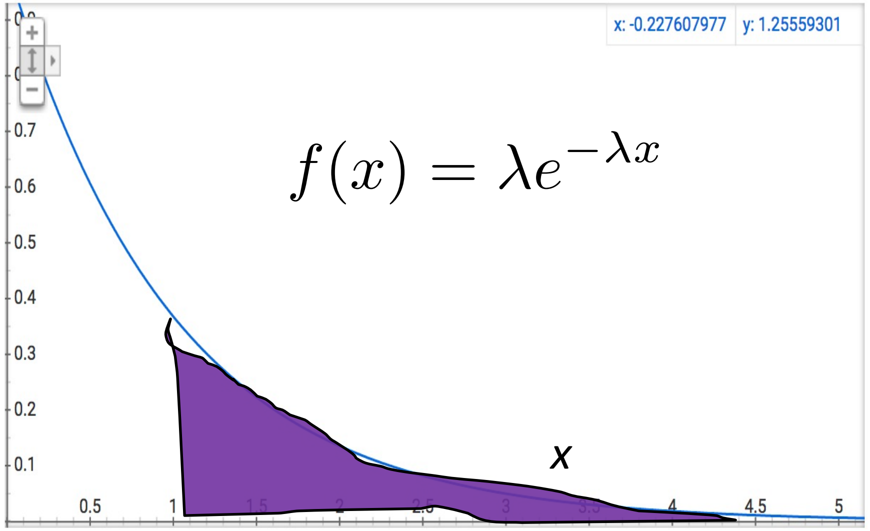


Using CDF Example. X is Exp($\lambda = 1$)

Probability density function



P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\, dx$$

or

$$= 1 - F(1)$$

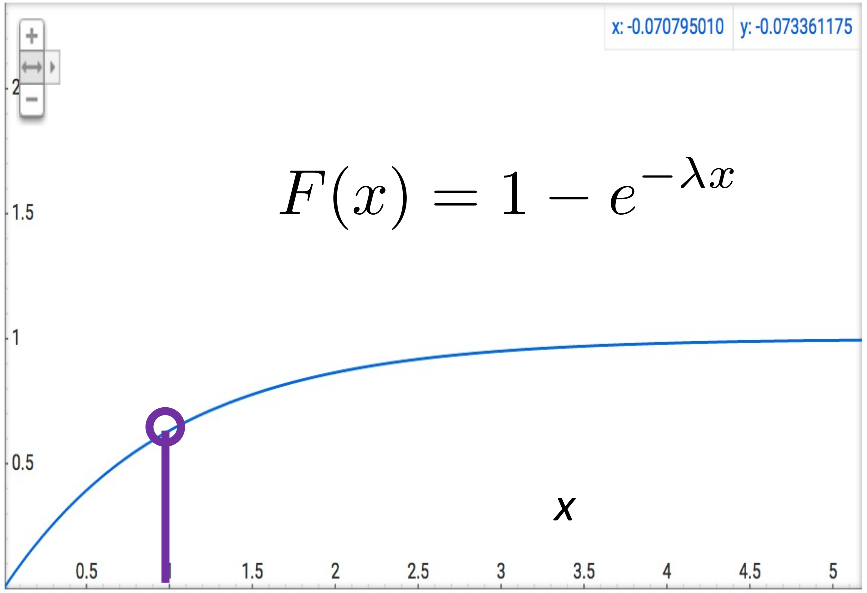
$$= e^{-1}$$

$$\approx 0.37$$

Cumulative density function

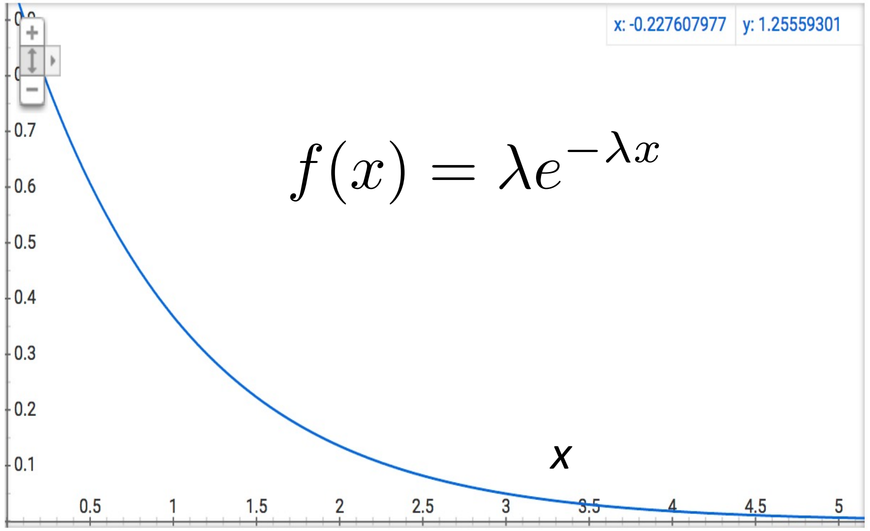
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability
density
function

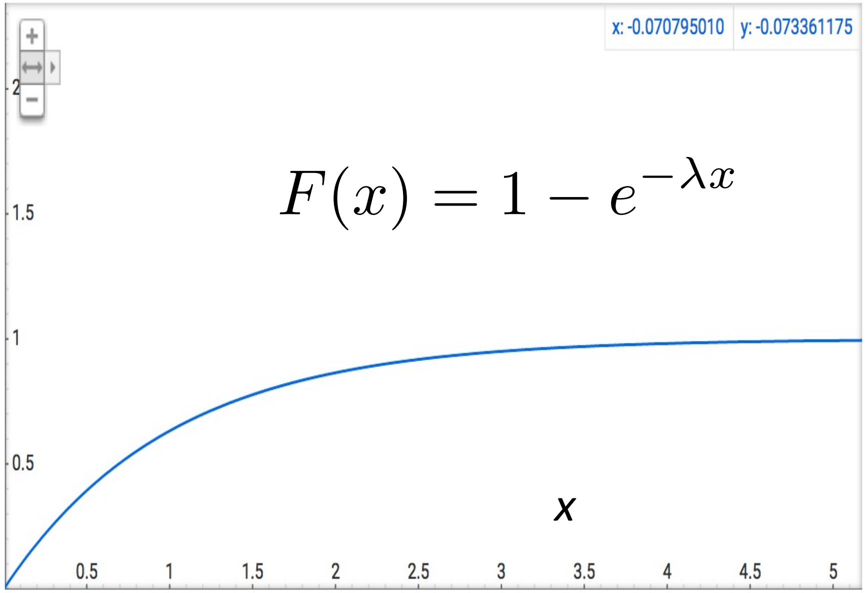


P(1 < X < 2)

Cumulative
density function

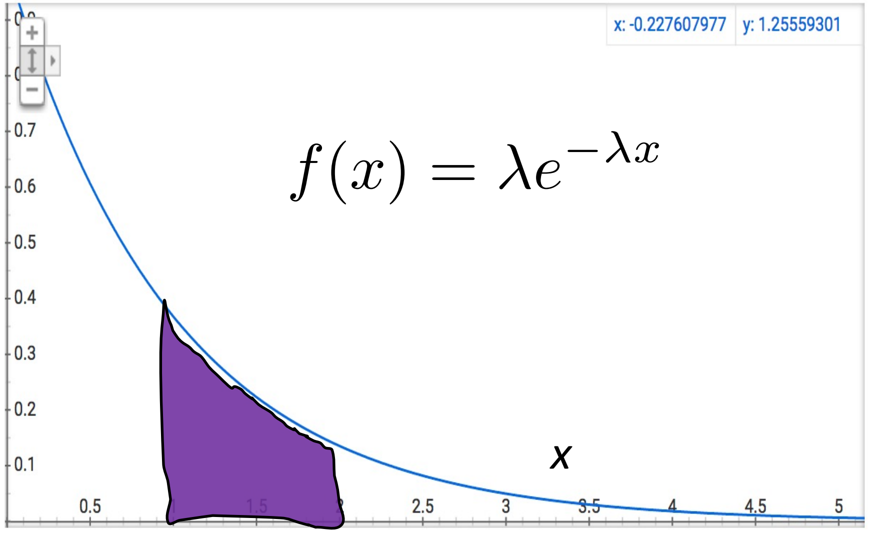
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability density function



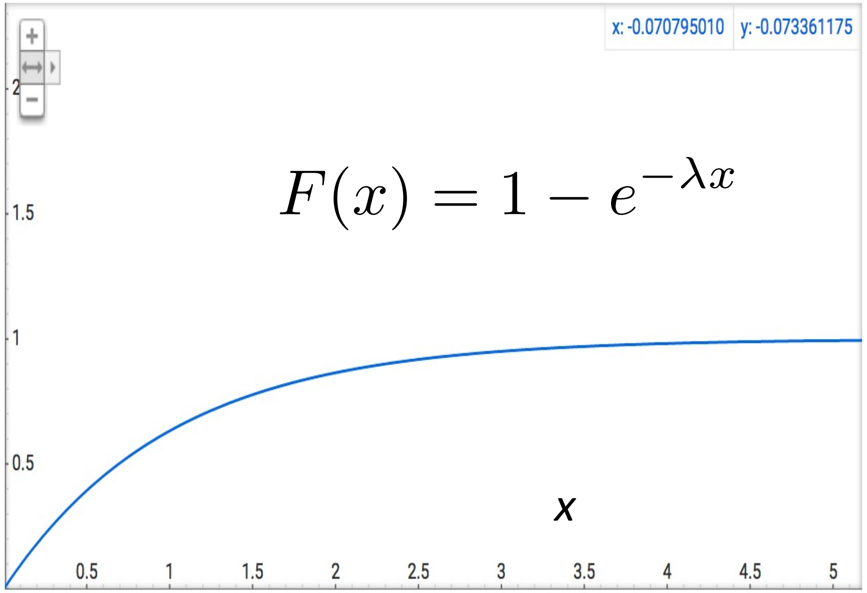
P(1 < X < 2)

$$= \int_{x=1}^{2} f(x)\, dx$$

Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability
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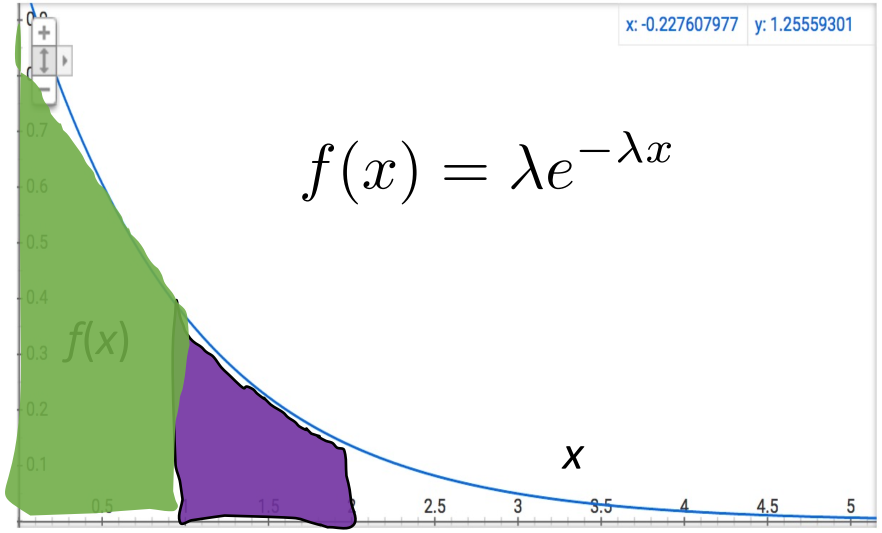


P(1 < X < 2)

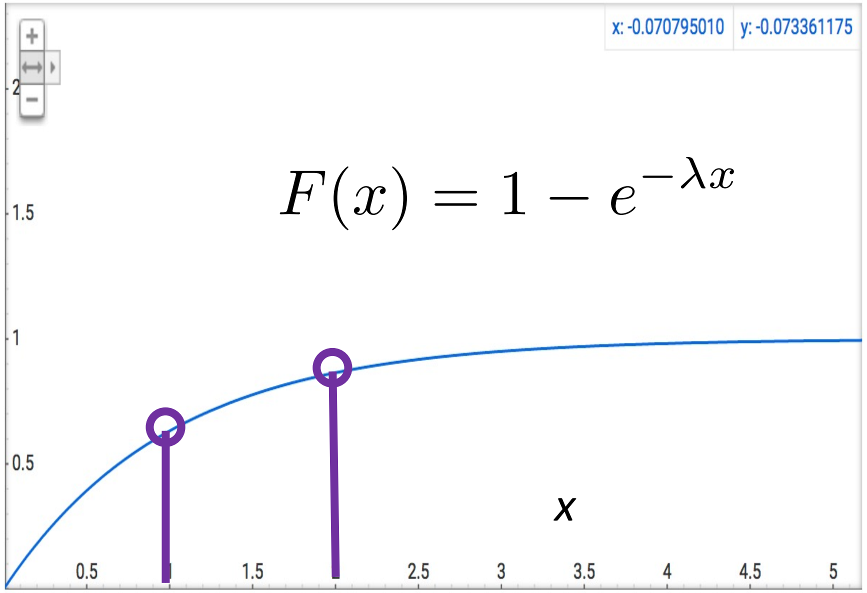
$$= \int_{x=1}^{2} f(x)\, dx$$

or

Cumulative
density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



$$= F(2) - F(1)$$

$$=(1 - e^{-2})$$

$$- (1 - e^{-1})$$

$$\approx 0.23$$

Probability of Earthquake in Next 4 Years?

Based on historical data, earthquakes of magnitude 8.0+ happen at a **rate of 0.002** per year*. What is the probability of **an major earthquake in the next 4 years?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002) \qquad F(y) = 1 - e^{-0.002y}$$

$$\begin{aligned} P(Y < 4) &= F(4) \\ &= 1 - e^{-0.002 \cdot 4} \\ &\approx 0.008 \end{aligned}$$

Feeling lucky?

*According to USGS, 2015

Cumulative Density Function

$$F(x) = P(X < x)$$

Probability Query	Solution	Explanation
$\Pr(X < a)$	$F(a)$	That is the definition of the CDF
$\Pr(X \leq a)$	$F(a)$	Trick question. $\Pr(X = a) = 0$
$\Pr(X > a)$	$1 - F(a)$	$\Pr(X < a) + \Pr(X > a) = 1$
$\Pr(a < X < b)$	$F(b) - F(a)$	$F(a) + \Pr(a < X < b) = F(b)$

Pedagogical pause here

Big Day

Normal Random Variable

<u>def</u> An **Normal** random variable X is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

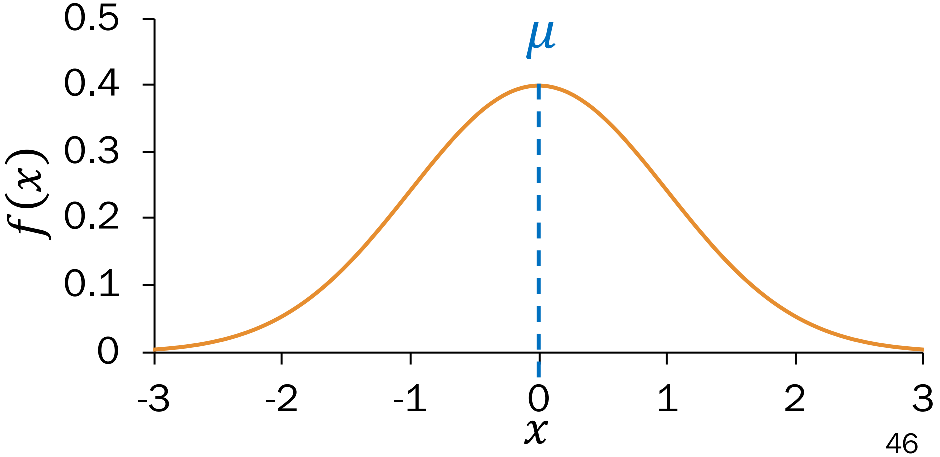
PDF	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Expectation	$E[X] = \mu$
Variance	$\text{Var}(X) = \sigma^2$

Other names: **Gaussian** random variable

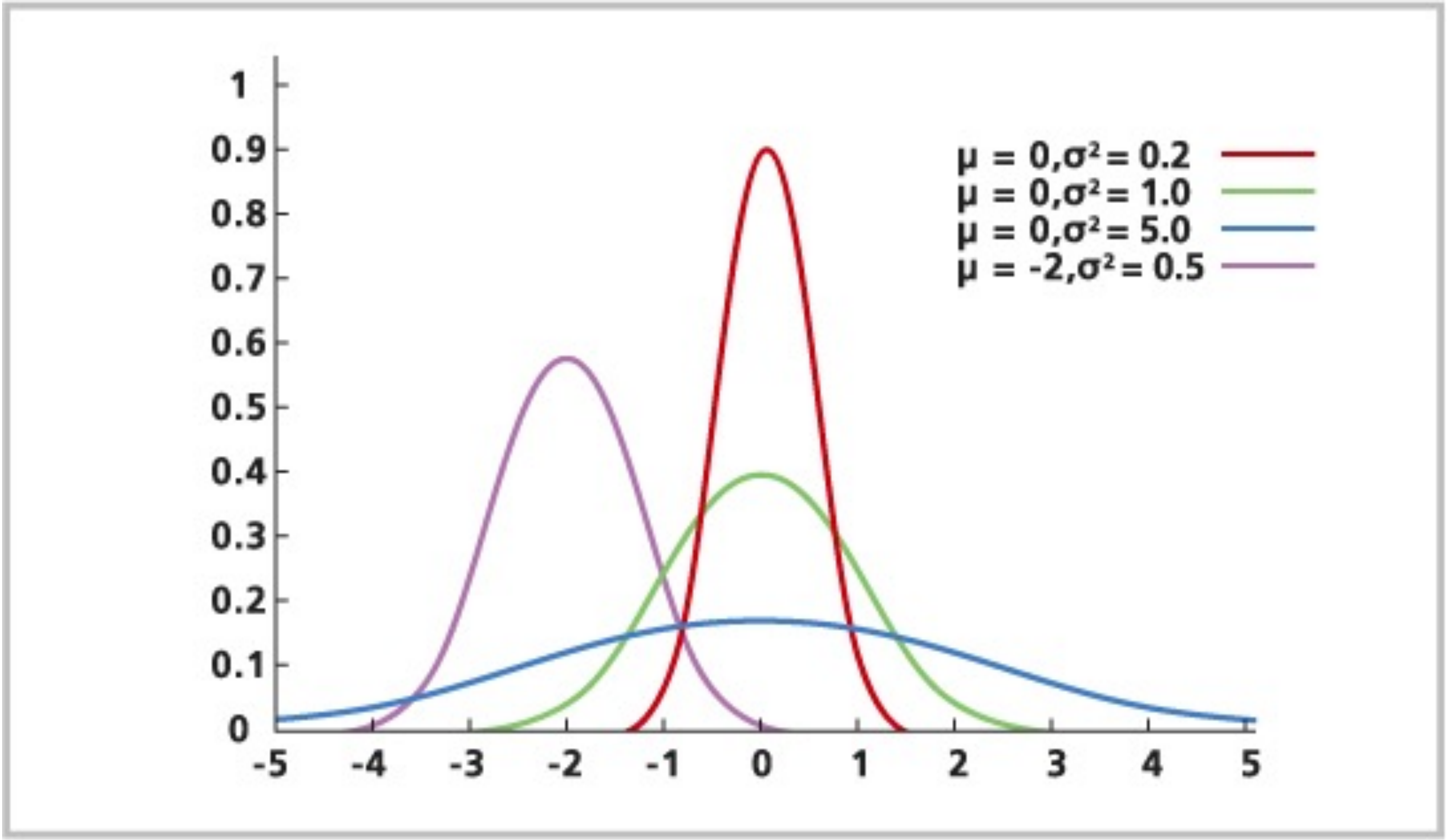
mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

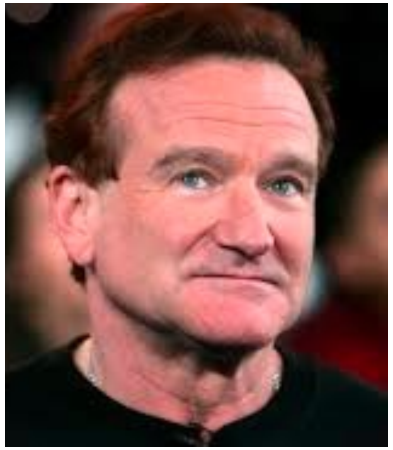


NormCore: A Few Normal Examples



Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician.



Johann Carl Friedrich Gauss (/gaʊs/; German: *Gauß* [gaʊs] (listen); Latin: *Carolus Fridericus Gauss*; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields, including algebra, analysis, astronomy, differential geometry, electrostatics, geodesy, geophysics, magnetic fields, matrix theory, mechanics, number theory, optics and statistics.

Sometimes referred to as the *Princeps mathematicorum*[1] (Latin for "the foremost of mathematicians") and "the greatest mathematician since antiquity", Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.[2]

Did not invent Normal distribution but rather popularized it

Why the Normal?

These are log-normal

- Common for natural phenomena: height, weight, etc.

Most noise is assumed normal

- Most noise in the world is Normal
- Often results from the sum of many random variables

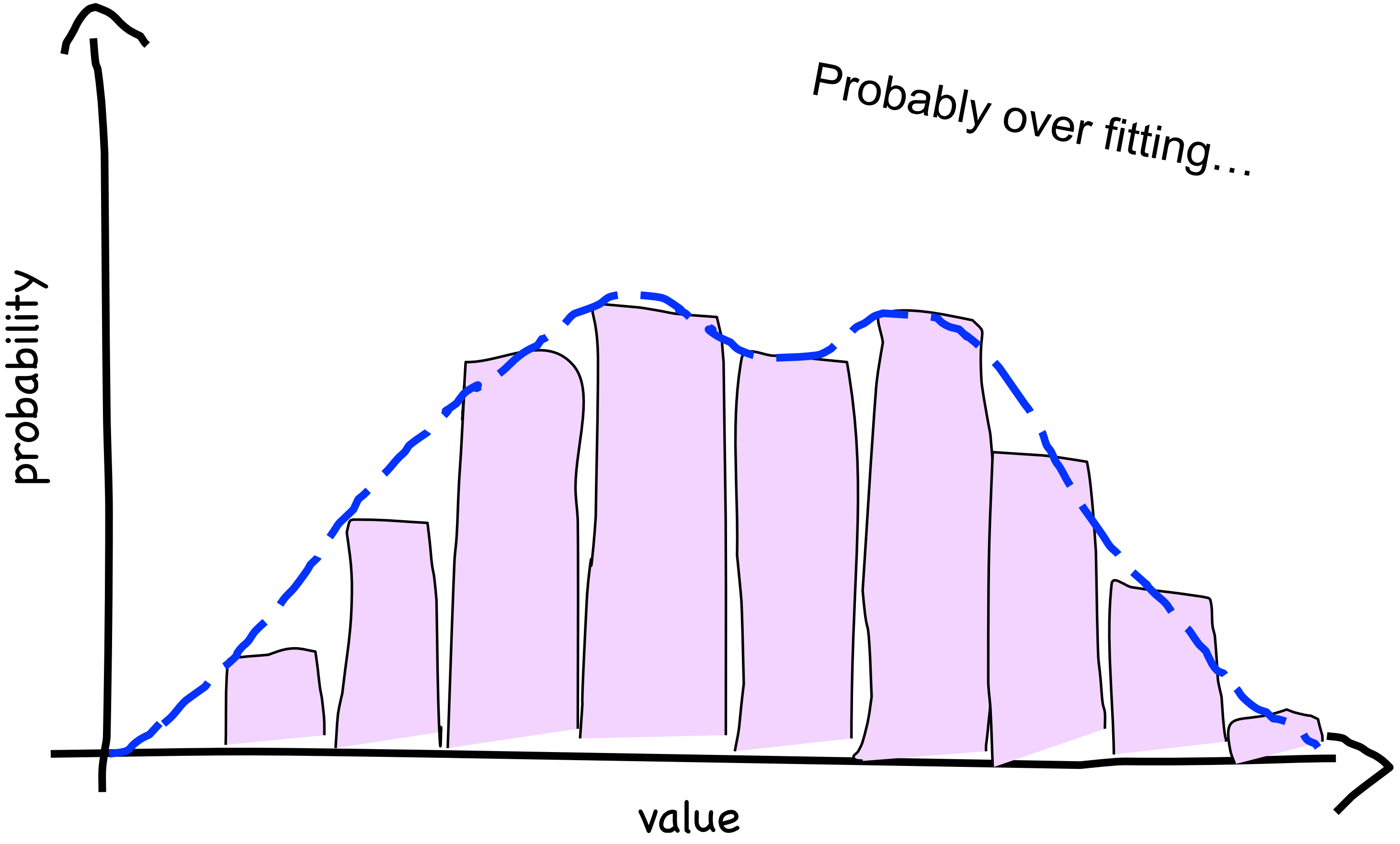
Only if they are equally weighted and independent

- Sample means are distributed normally

That's what they want you to believe…

“The simplest explanation is usually the best one”

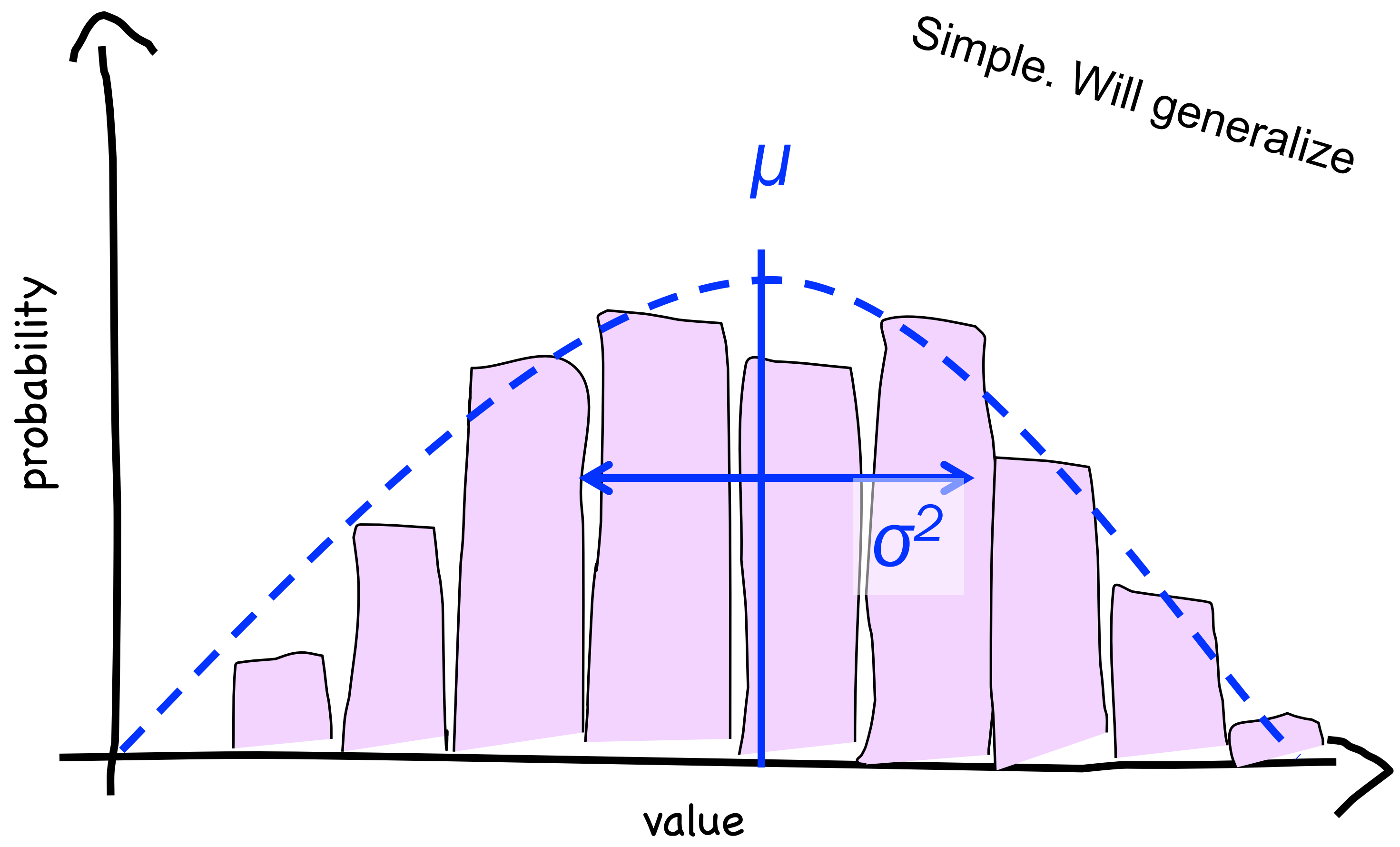
Complexity is Tempting



* That describes the training data, but will it **generalize**?

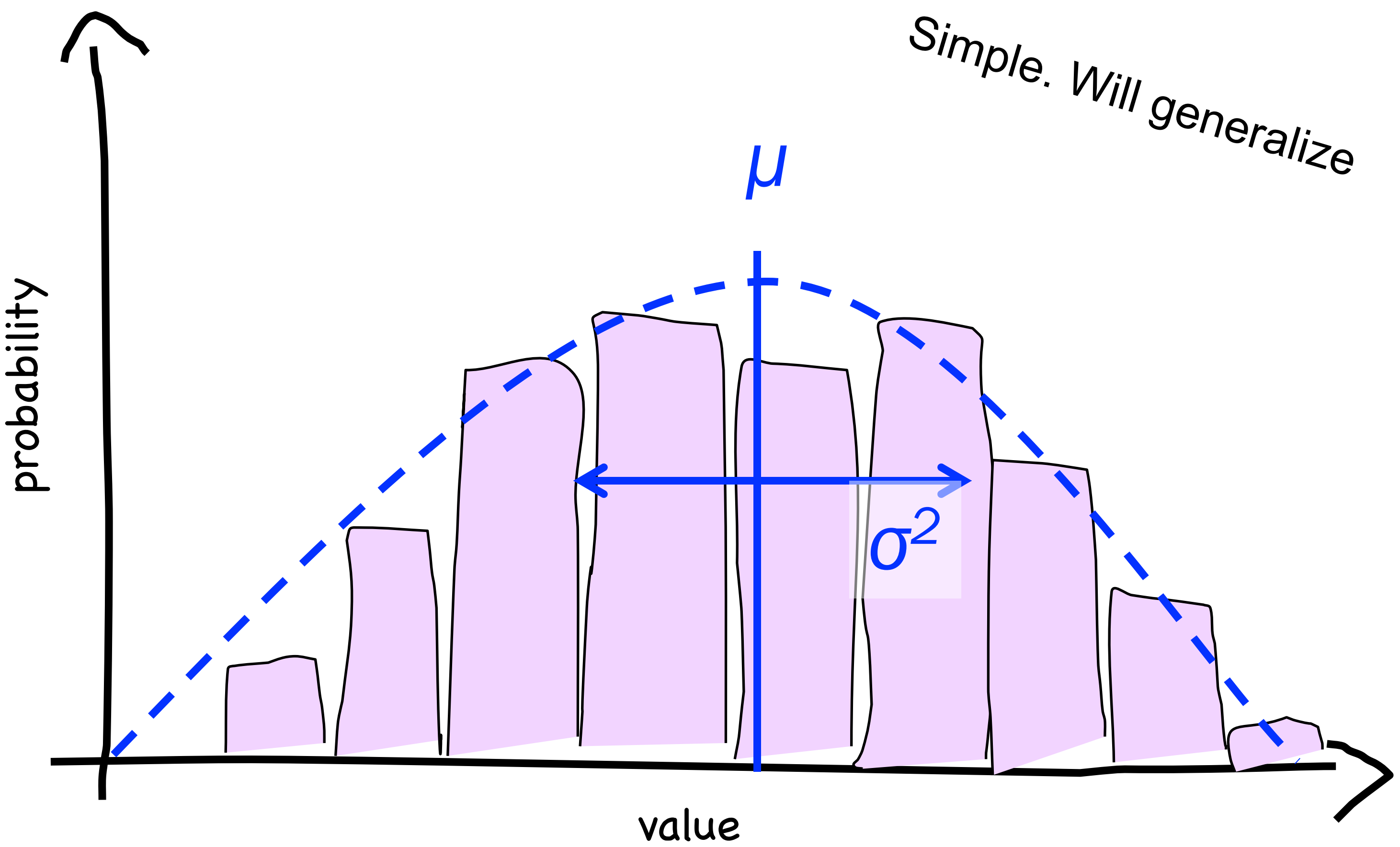
Fewest Assumptions

$$\mathrm{H}(X) = -\sum_{i=1}^{n} \mathrm{P}(x_i) \log \mathrm{P}(x_i)$$



* A Gaussian **maximizes entropy** for a given mean and variance

Fewest Assumptions



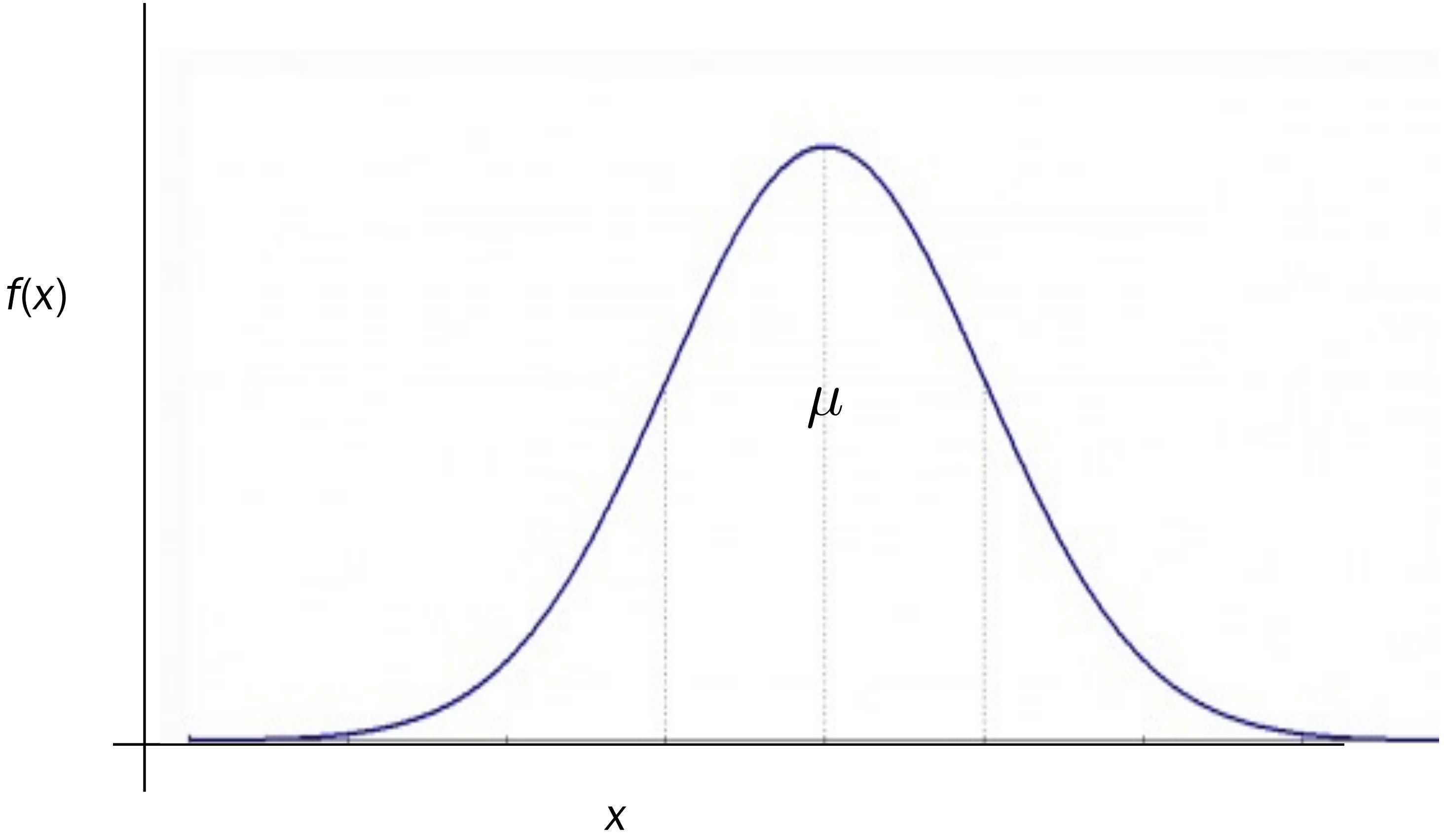
* A Gaussian makes the **fewest assumptions** after matching mean and variance.

But also, it is easy to use

Normal Probability Density Function

$$\mathcal{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



Anatomy of a Beautiful Equation

$$\mathcal{N}(\mu, \sigma^2)$$

"exponential"

the distance to the mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at x

a constant

sigma shows up twice

Campus bikes

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = \int_6^\infty f(x)dx = \int_6^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

(call me if you analytically solve this)

Loving, not scary
...except this time

No closed form for the integral

No closed form for $F(x)$

Spoiler: Numerically Solved CDF

$$\mathcal{N}(\mu, \sigma^2)$$

A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function of any normal

* We are going to spend the next few slides getting here

Linear Transform of Normal is Normal

$$\text{Let } X \sim \mathcal{N}(\mu, \sigma^2)$$

If $Y = aX + b$ then Y is also Normal

$$\begin{aligned} E[Y] &= E[aX + b] \\ &= aE[X] + b \\ &= a\mu + b \end{aligned}$$

$$\begin{aligned} Var(Y) &= Var(aX + b) \\ &= a^2 Var(X) \\ &= a^2\sigma^2 \end{aligned}$$

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Special Linear Transform

If $Y = aX + b$ then *Y* is also Normal

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

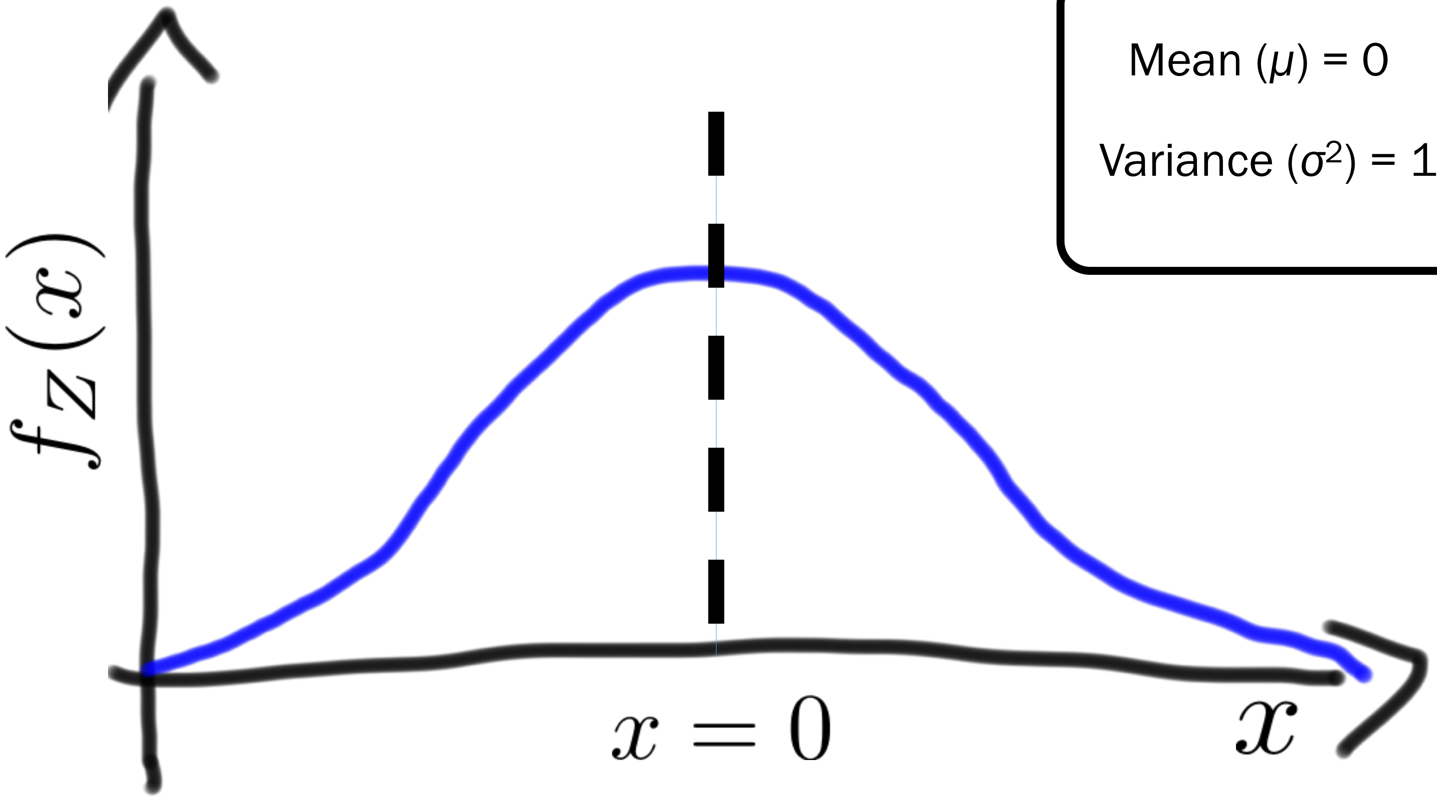
There is a special case of linear transform for any *X*:

$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \qquad a = \frac{1}{\sigma} \quad b = -\frac{\mu}{\sigma}$$

$$\begin{aligned} Z &\sim \mathcal{N}(a\mu + b, a^2\sigma^2) \\ &\sim \mathcal{N}(\frac{\mu}{\sigma} - \frac{\mu}{\sigma}, \frac{\sigma^2}{\sigma^2}) \\ &\sim \mathcal{N}(0, 1) \end{aligned}$$

The Standard Normal

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$



Mean (μ) = 0

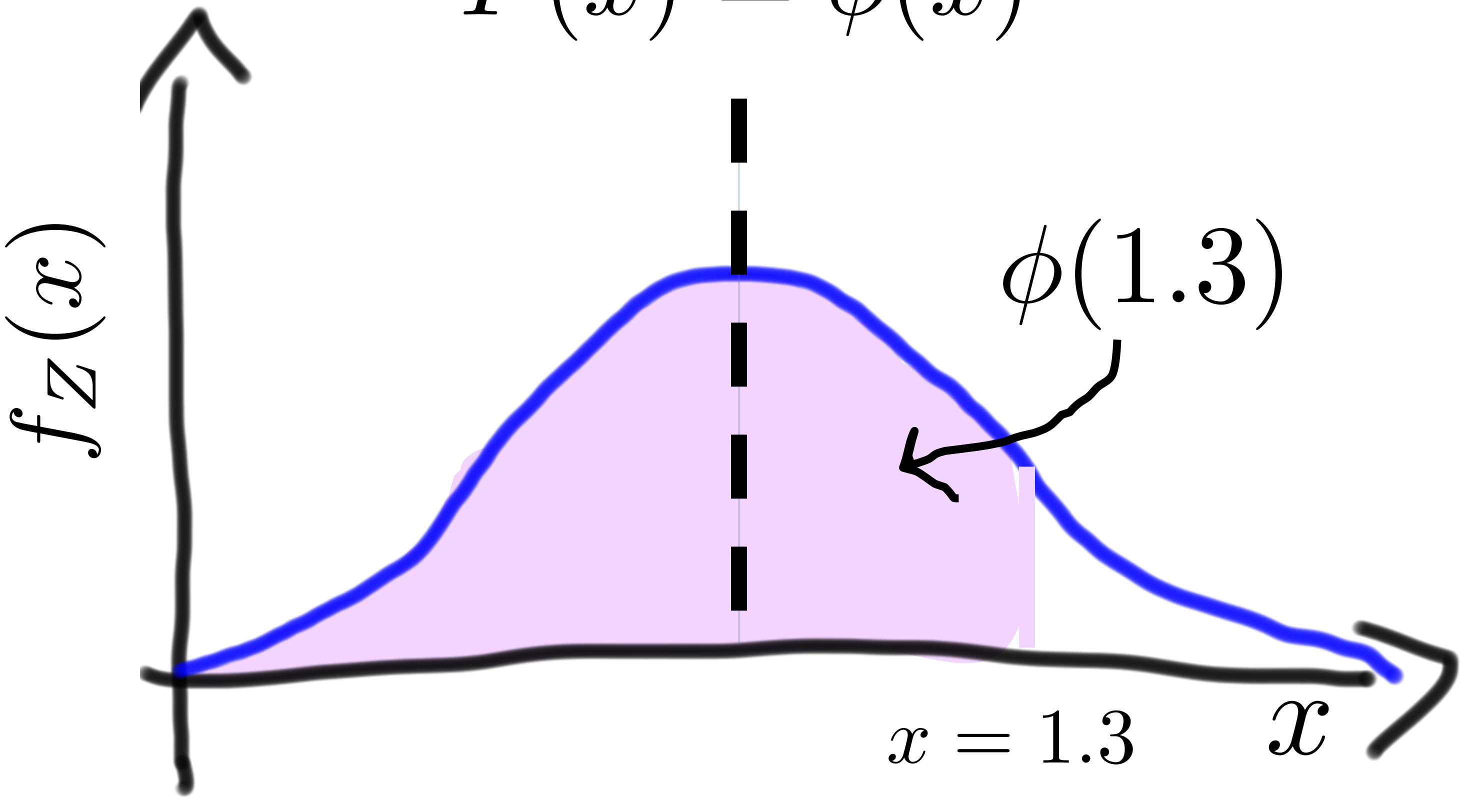
Variance (σ^2) = 1

*This is the probability density function for the standard normal

Phi

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$

$$F(x) = \phi(x)$$

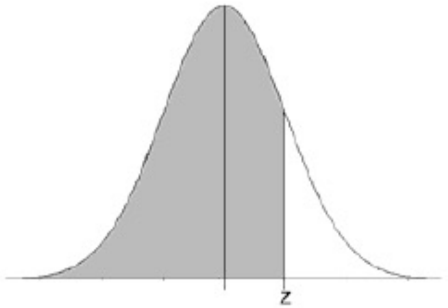


*This is the probability density function for the standard normal

Using Table of ϕ

Standard Normal Cumulative Probability Table

$\Phi(1.31) = 0.7054$

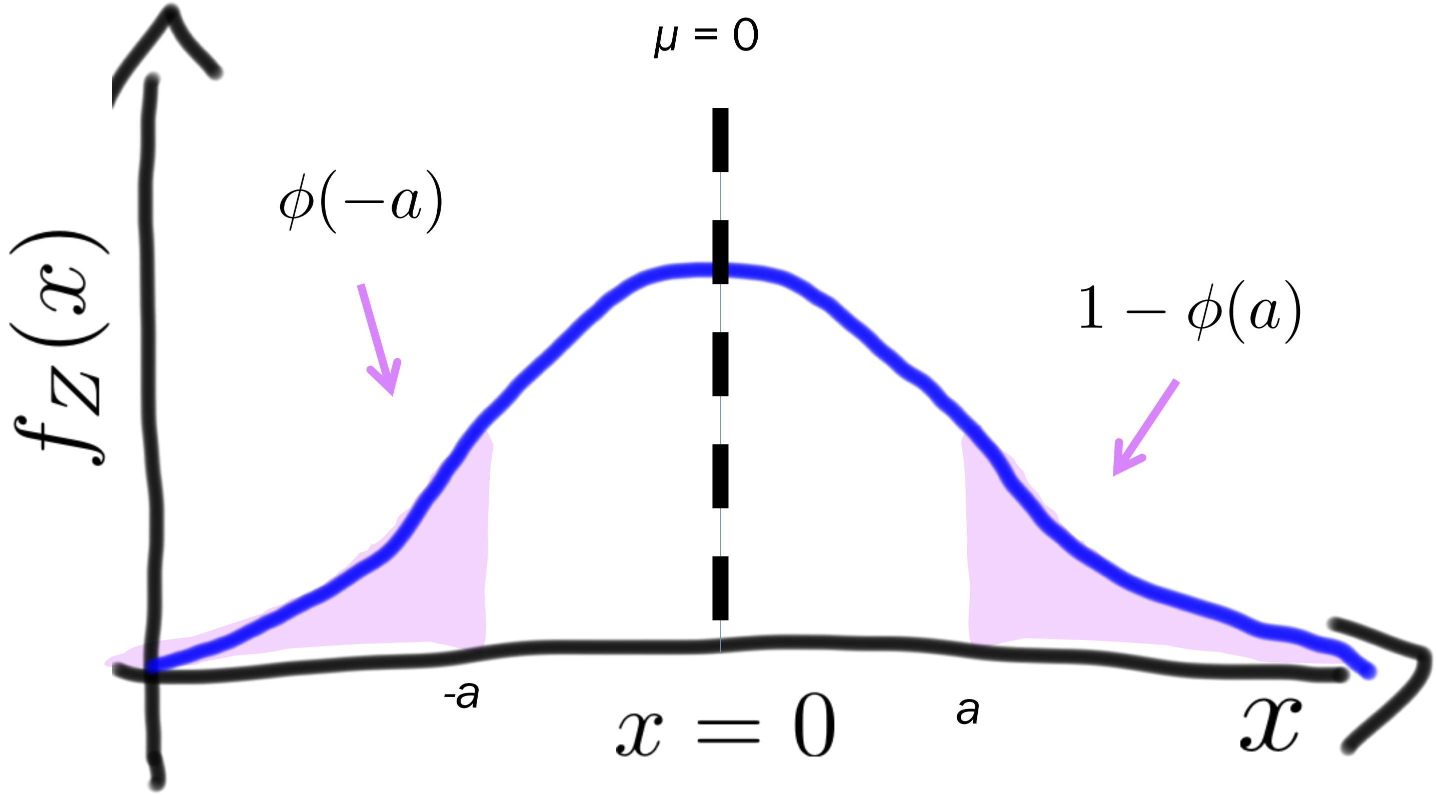


Cumulative probabilities for **POSITIVE** z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Symmetry of Phi

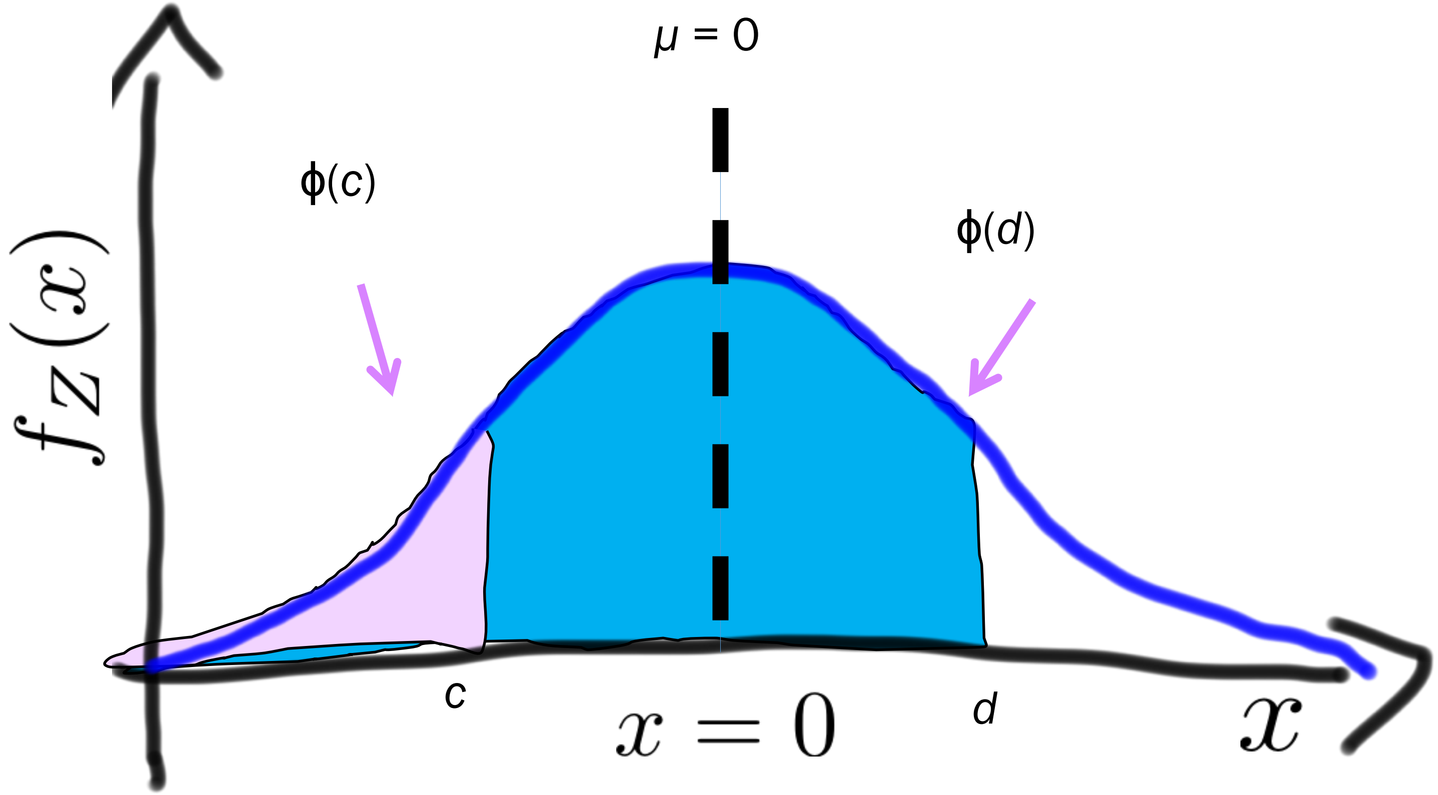
$$\phi(a) = 1 - \phi(a)$$



*This is the probability density function for the standard normal

Interval of Phi

$$P(c < Z < d) = \phi(d) - \phi(c)$$



Compute F(x) via Transform

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ $\qquad Z = \dfrac{X - \mu}{\sigma}$

Use *Z* to compute *F*(*x*)

$$F_X(x) = P(X \leq x)$$

$$= P(X - \mu \leq x - \mu)$$

$$= P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

For normal distribution, F(x) is computed using the phi transform.

And here we are

$\mathcal{N}(\mu, \sigma^2)$

CDF of Standard Normal: A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

Table of $\Phi(Z)$ values in textbook, p. 201 and handout

Using the Phi Table

Standard Normal Cumulative Probability Table

$\Phi(0.54) = 0.7054$

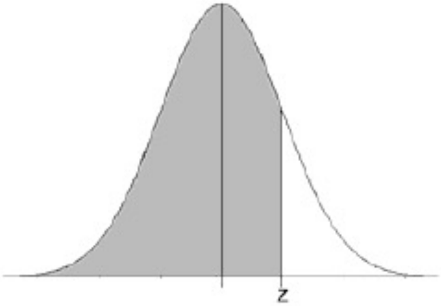


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Table is kinda old school

Using Programming Library

Every modern programming language has a normal library

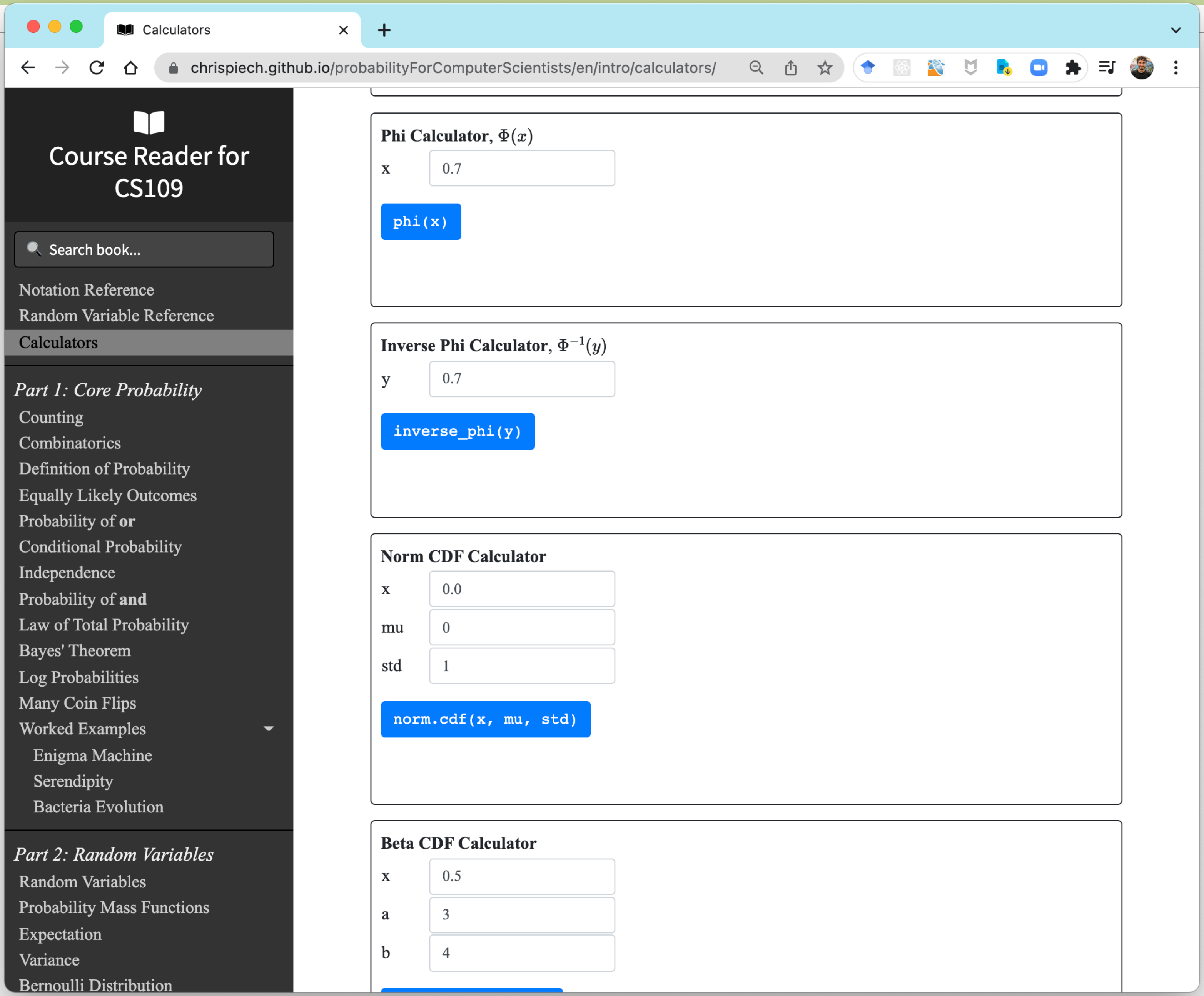
```
stats.norm.cdf(x, mean, std)
```

$$= P(X < x) \text{ where } X \sim \mathcal{N}(\mu, \sigma^2)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

* This is from Python’s scipy library

I made one for you



Campus bikes

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

✕ $P(X \geq 6) = \int_6^\infty f(x)dx$ (no analytic solution)

1. Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X \geq 6) = 1 - F_x(6)$$

$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right)$$

$$\approx 1 - \Phi(1.41)$$

2. Look up $\Phi(z)$ in table

$$1 - \Phi(1.41)$$

$$\approx 1 - 0.9207$$

$$= 0.0793$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$.
Note standard deviation $\sigma = 4$.

How would you write each of the below probabilities as a function of the standard normal CDF, Φ?

1. $P(X > 0)$ (we just did this)
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

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Get your Gaussian On

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- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

Look up $\Phi(z)$ in table

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + \big(1 - F(9)\big)$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + (1 - F(9))$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Look up $\Phi(z)$ in table

$$= \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$= 2\left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$\approx 0.1337$$

Imagine you are taking a quiz…
With no computer!!!

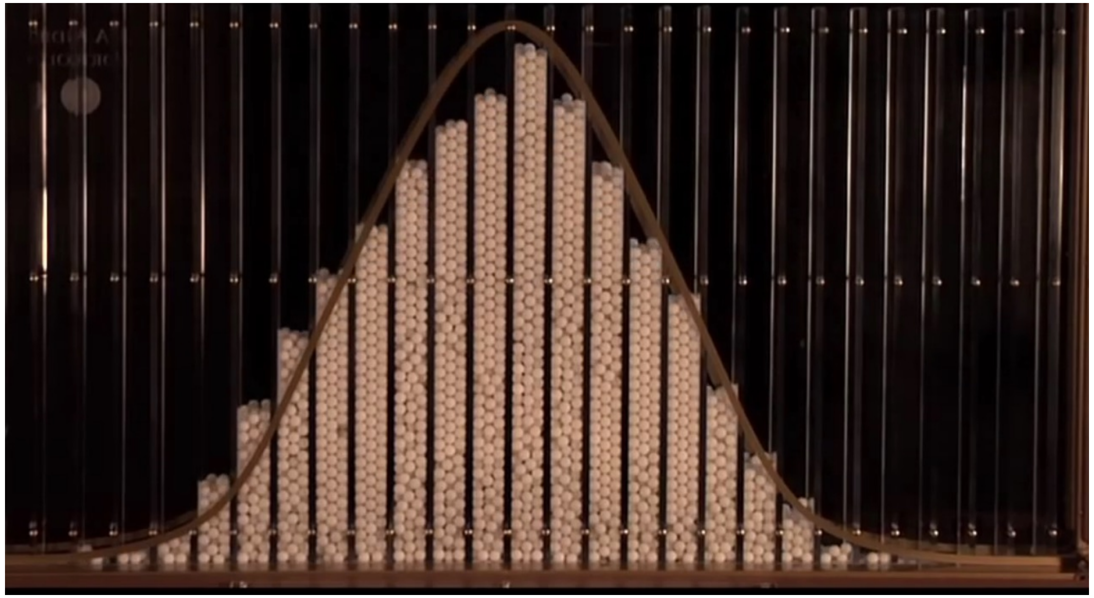
Website Testing

100 people are given a new website design

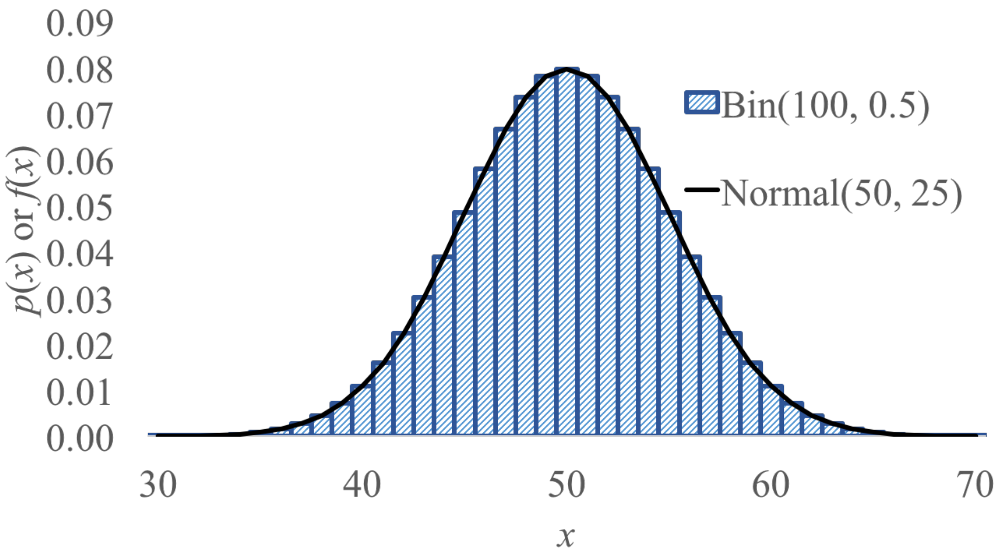
- X = # people whose time on site increases
- CEO will endorse new design if X ≥ 65 What is P(CEO endorses change| it has no effect)?
- X ~ Bin(100, 0.5). Want to calculate P(X ≥ 65)
- Give a numerical answer…

$$P(X \geq 65) = \sum_{i=65}^{100} \binom{100}{i} (0.5)^i (1 - 0.5)^{100-i}$$

Don’t worry, Normal approximates Binomial



Galton Board



(We’ll explain ***why*** in 2 weeks’ time)

Website testing

- 100 people are given a new website design.
- X = # people whose time on site increases
- The design actually has no effect, so P(time on site increases) = 0.5 independently.
- CEO will endorse the new design if $X \geq 65$.

What is $P(\text{CEO endorses change})$? *Give a numerical approximation*.

Approach 1: Binomial

Define

$$X \sim \text{Bin}(n = 100, p = 0.5)$$

Want: $P(X \geq 65)$

Solve

$$P(X \geq 65) \approx 0.0018$$

Approach 2: approximate with Normal

Define

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = np = 50$$
$$\sigma^2 = np(1-p) = 25$$
$$\sigma = \sqrt{25} = 5$$

Solve

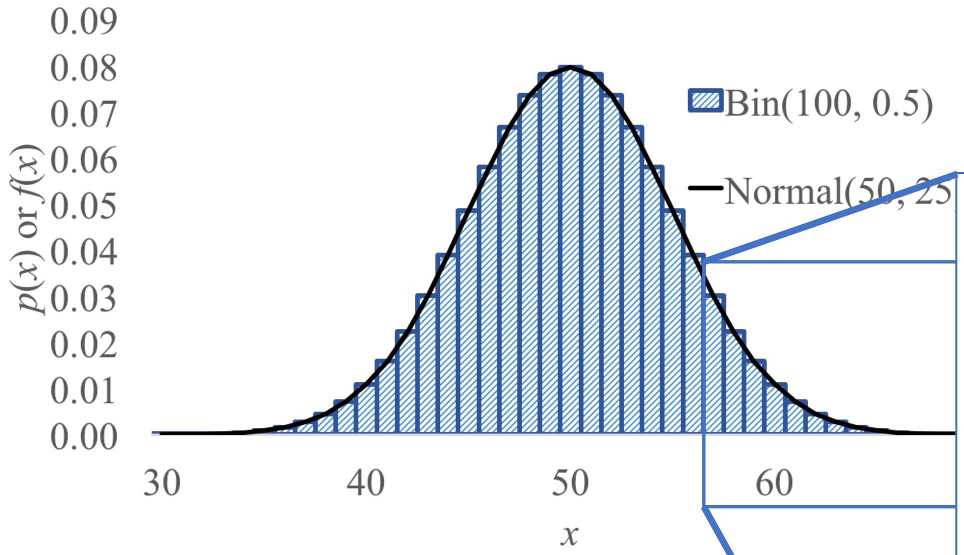
$$P(X \geq 65) \approx P(Y \geq 65) = 1 - F_Y(65)$$

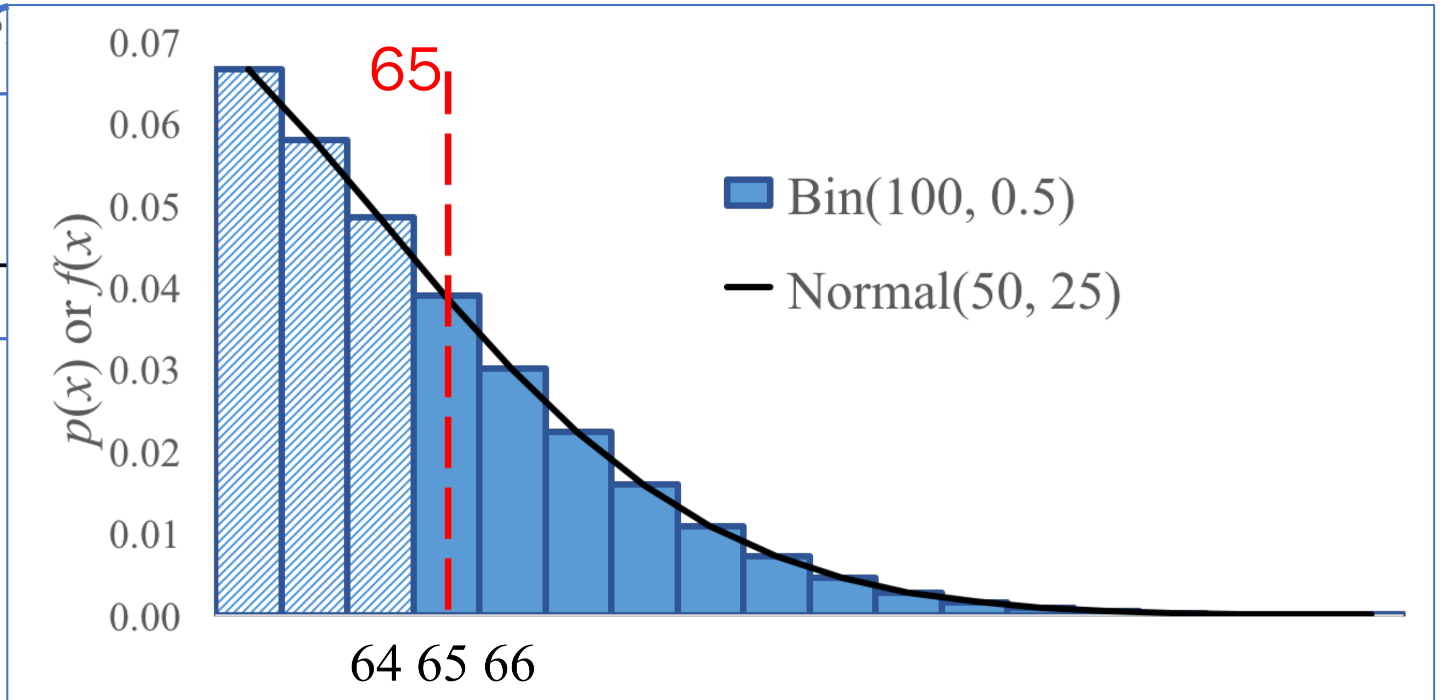
$$= 1 - \Phi\left(\frac{65-50}{5}\right) = 1 - \Phi(3) \approx 0.0013 \text{ ?}$$

⚠️ ⚠️ (this approach is missing something important)

Website testing (with continuity correction)

In our website testing, $Y \sim \mathcal{N}(50, 25)$ approximates $X \sim \text{Bin}(100, 0.5)$.





$P(X \geq 65)$ Binomial

$\approx P(Y \geq 64.5)$ Normal

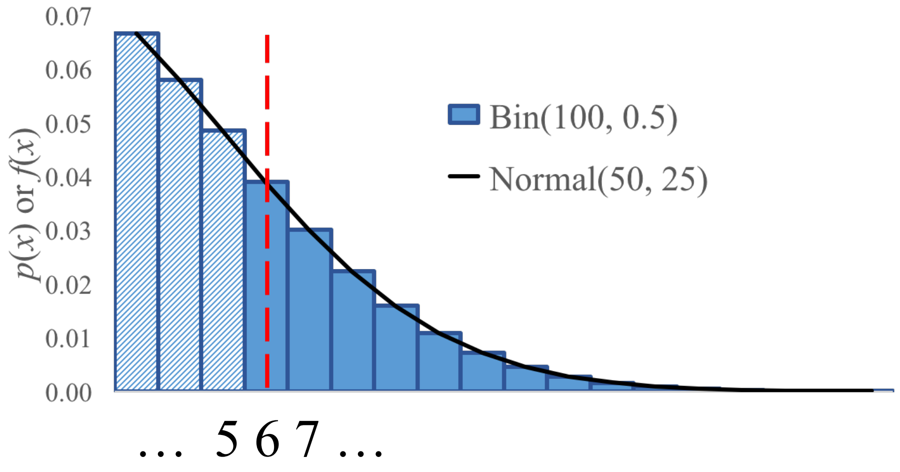
≈ 0.0018 ✅ the better Approach 2

You must perform a **continuity correction** when approximating a Binomial RV with a Normal RV.

Continuity correction

If $Y \sim \mathcal{N}(np, np(1-p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

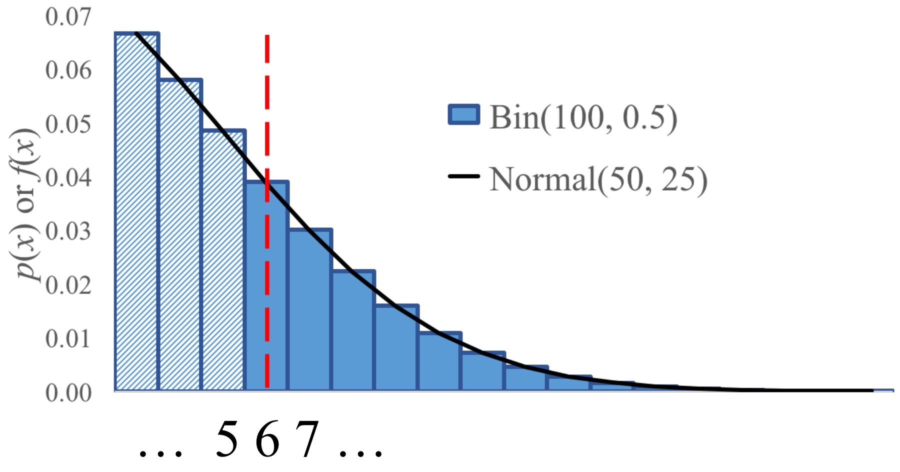
Discrete (e.g., Binomial) probability question	Continuous (Normal) probability question
$P(X = 6)$	
$P(X \geq 6)$	
$P(X > 6)$	
$P(X < 6)$	
$P(X \leq 6)$	



Continuity correction

If $Y \sim \mathcal{N}(np, np(1-p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

Discrete (e.g., Binomial) probability question	Continuous (Normal) probability question
$P(X = 6)$	$P(5.5 \leq Y \leq 6.5)$
$P(X \geq 6)$	$P(Y \geq 5.5)$
$P(X > 6)$	$P(Y \geq 6.5)$
$P(X < 6)$	$P(Y \leq 5.5)$
$P(X \leq 6)$	$P(Y \leq 6.5)$



Normal Approximation

Who gets to approximate?

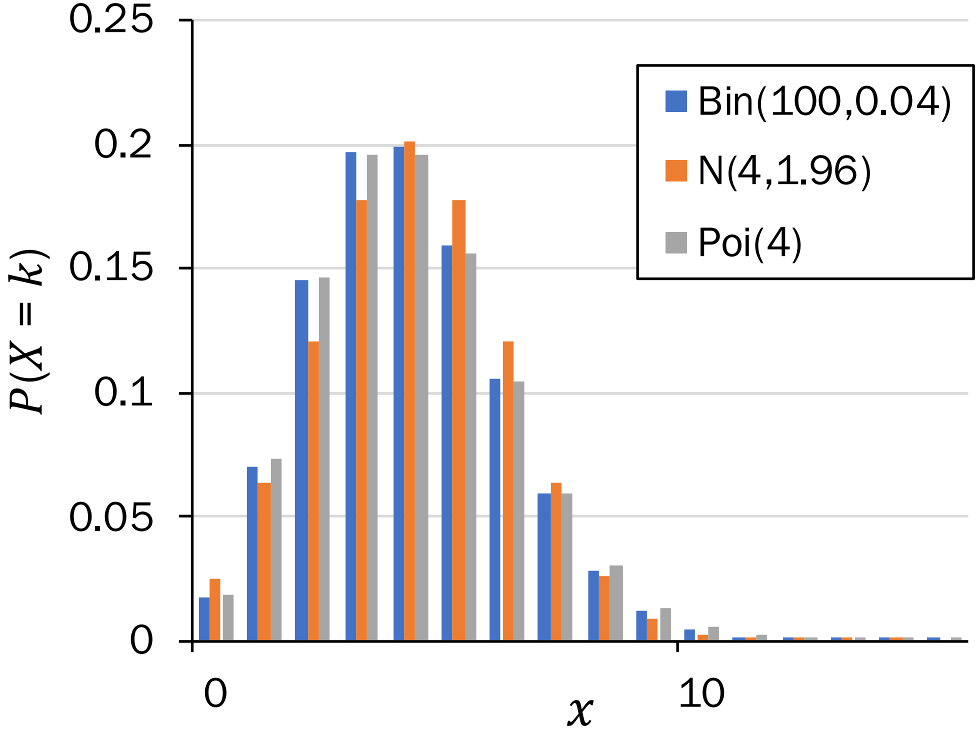
$$X \sim \text{Bin}(n, p)$$
$$E[X] = np$$
$$\text{Var}(X) = np(1-p)$$

$$Y \sim \text{Poi}(\lambda)$$
$$\lambda = np$$

?

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mu = np$$
$$\sigma^2 = np(1-p)$$

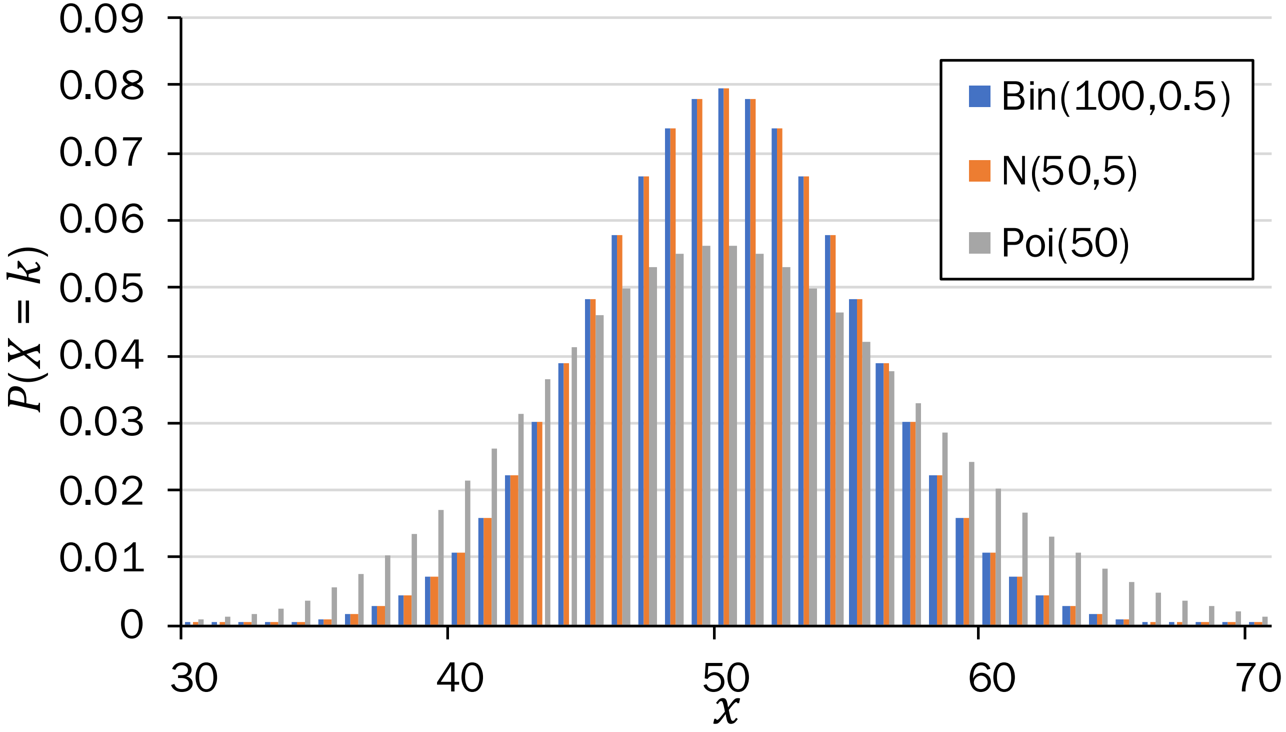
Who gets to approximate?



Poisson approximation

n large (> 20), p small (< 0.05)

slight dependence okay



Normal approximation

n large (> 20), variance large ($np(1-p) > 10$)

independence

1. If there is a choice, either is fine.
2. When using Normal to approximate a discrete RV, use a continuity correction.

Super Question:

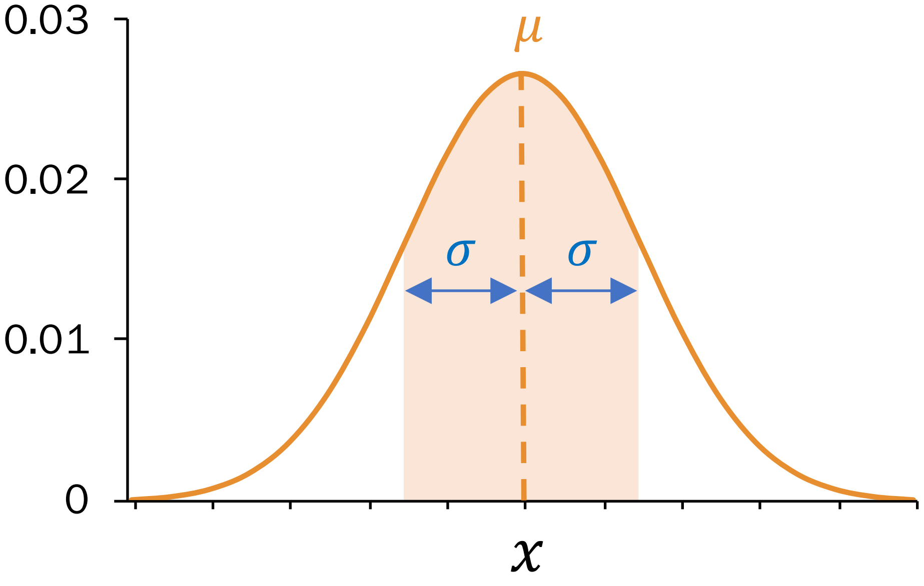
Why Be Normal? 68% rule

You may have heard the statement:

"68% of the class will fall within 1 standard deviation of the exam average."

In general, this is only true of normal distributions:

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF F.



$$P(|X-\mu|<\sigma) = P(\mu-\sigma < X < \mu+\sigma)$$
$$= F(\mu+\sigma) - F(\mu-\sigma)$$
$$= \Phi\left(\frac{(\mu+\sigma)-\mu}{\sigma}\right) - \Phi\left(\frac{(\mu-\sigma)-\mu}{\sigma}\right)$$
$$= \Phi(1) - \Phi(-1) = \Phi(1) - \big(1-\Phi(1)\big)$$
$$= 2\Phi(1) - 1 \quad \approx 2(0.8413) - 1 = 0.6826$$

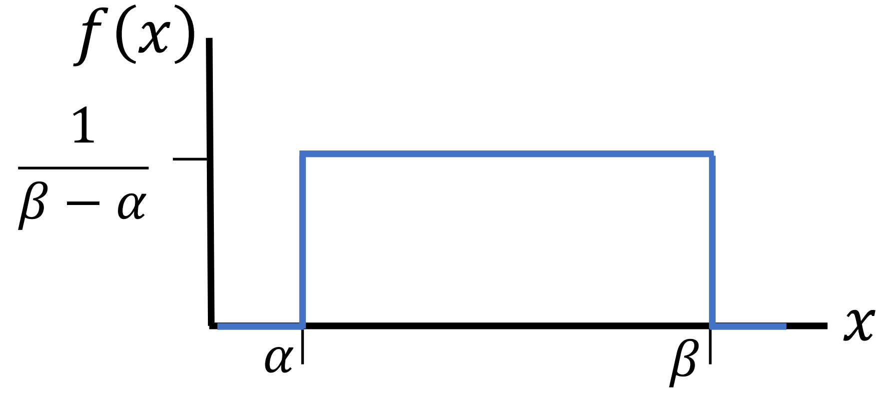
Why Be Normal? 68% rule

You may have heard the statement:

"68% of the class will fall within 1 standard deviation of the exam average."

In general, this is only true of normal distributions:

Counterexample: Let $X \sim \text{Uni}(\alpha, \beta)$.



$$\mu = E[X] = \frac{\alpha + \beta}{2}$$

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12} \Rightarrow \sigma = \text{SD}(X) = \frac{\beta - \alpha}{\sqrt{12}}$$

$$P(|X - \mu| < \sigma) = P(\mu - \sigma < X < \mu + \sigma)$$

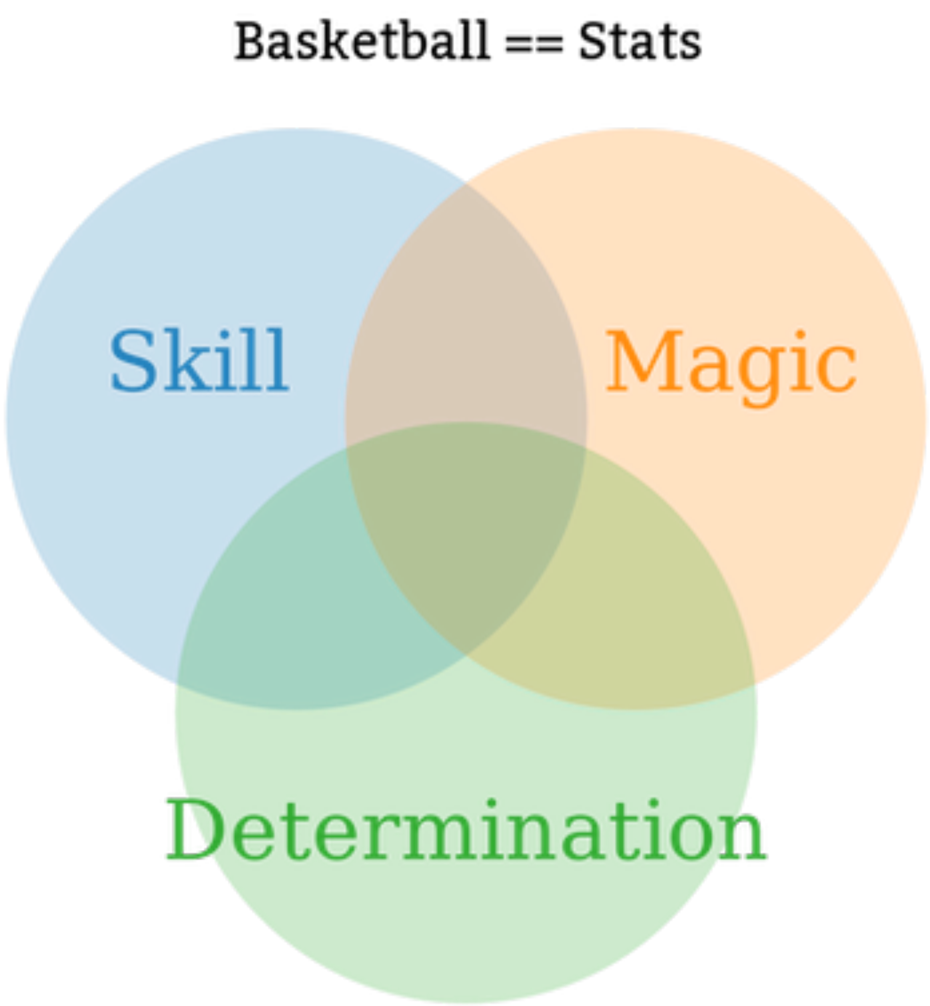
$$= \frac{1}{\beta - \alpha} \cdot [(\mu + \sigma) - (\mu - \sigma)]$$

$$= \frac{1}{\beta - \alpha}[2\sigma] = \frac{1}{\beta - \alpha} \cdot \left[2 \cdot \frac{\beta - \alpha}{\sqrt{12}}\right]$$

$$= 2/\sqrt{12} \approx 0.58$$

If we have time

Gaussian Sampling and ELO ratings



What is the probability that the Warriors win?
How do you model zero-sum games?

Gaussian Sampling and ELO ratings

Each team has an ELO score S, calculated based on its past performance.

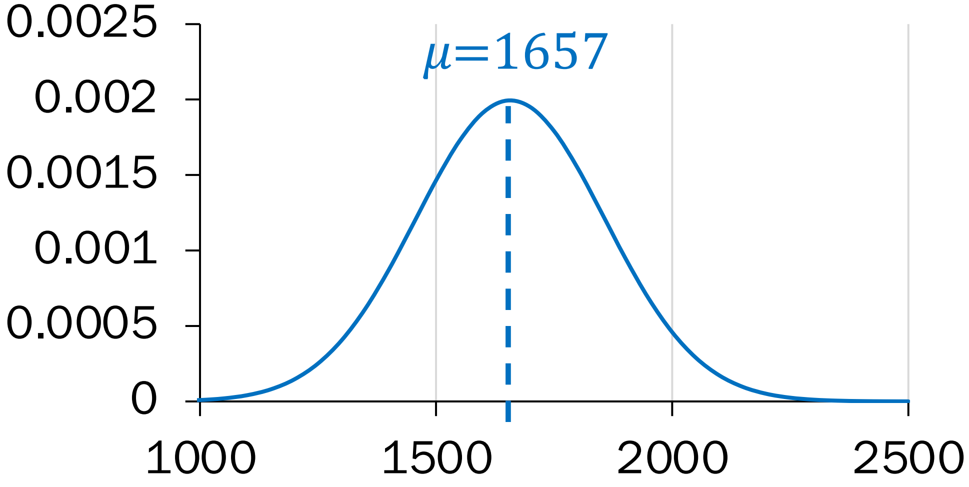
- Each game, a team has ability $A \sim \mathcal{N}(S, 200^2)$.
- The team with the higher sampled ability wins.

Arpad Elo

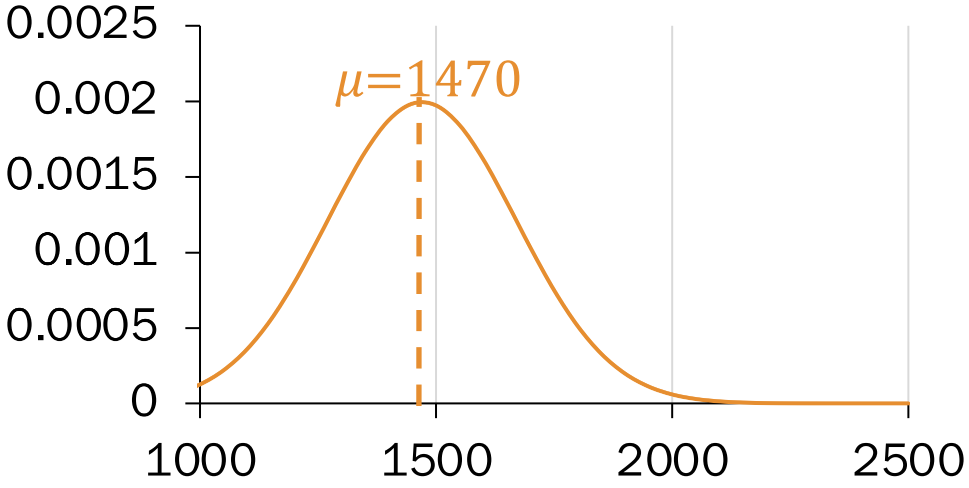
What is the probability that Warriors win this game?

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

Warriors' $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponent's $A_O \sim \mathcal{N}(S = 1470, 200^2)$



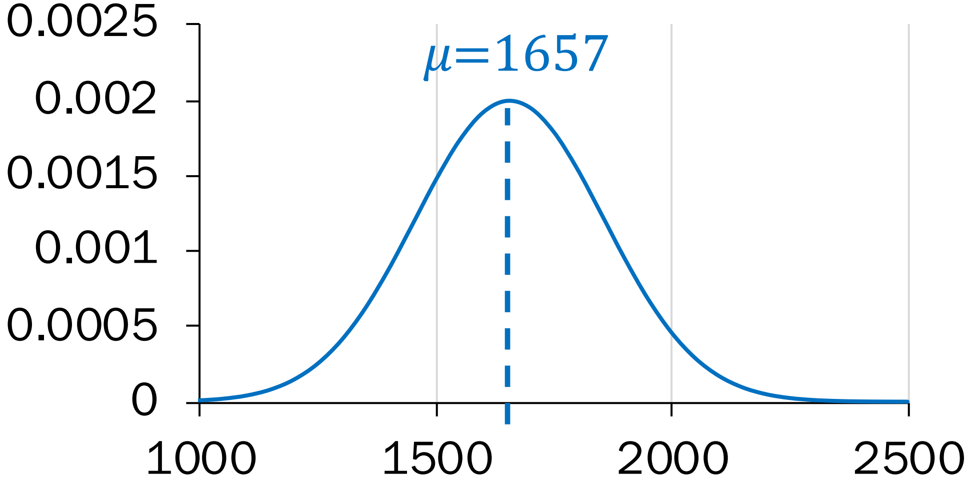
Gaussian Sampling and ELO ratings

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

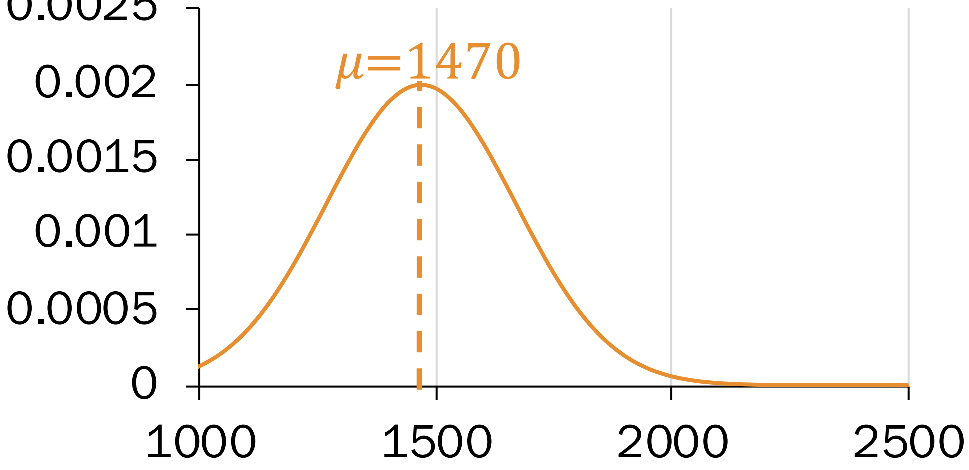
```python
from scipy import stats
WARRIORS_ELO = 1657
OPPONENT_ELO = 1470
STDEV = 200
NTRIALS = 10000

nSuccess = 0
for i in range(NTRIALS):
  w = stats.norm.rvs(WARRIORS_ELO, STDEV)
  o = stats.norm.rvs(OPPONENT_ELO, STDEV)
  if w > o:
    nSuccess += 1
print("Warriors sampled win fraction: ",
      float(nSuccess) / NTRIALS)
```

≈ 0.7488, calculated by sampling

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$



Is there a better way?

$$P(A_W > A_O)$$

- This is a probability of an event
  involving ***two continuous*** random variables!
- We'll solve this problem analytically in two weeks' time.

Big goal for next time: Events involving two ***discrete*** random variables.
Stay tuned!

Anatomy of a Beautiful Equation

$$\mathcal{N}(\mu, \sigma^2)$$

"exponential"

the distance to the mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at *x*

a constant

sigma shows up twice

Does it look less scary like this?

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

This means "e to the power of" and is common function in code math libraries

$$f(x) \propto \frac{1}{\sigma} \cdot \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

This means "proportional to". There is a constant but there are many cases where we don't care what it is!

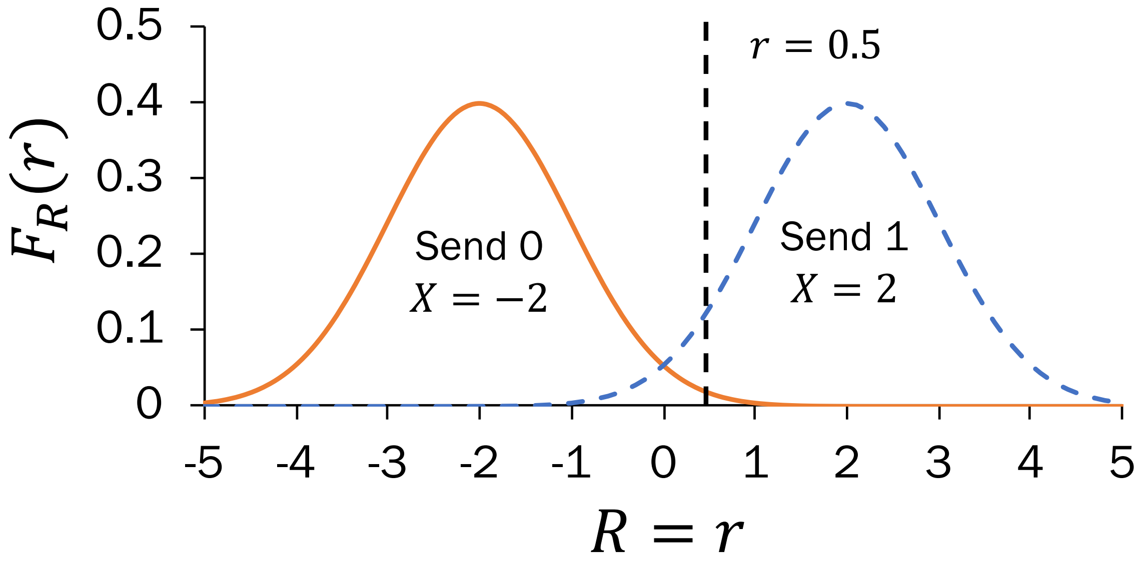
What if you had to take the log of this function?

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: $\begin{cases} 1 & \text{if } R \geq 0.5 \\ 0 & \text{otherwise.} \end{cases}$



1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?
2. What is P(decoding error | original bit is 0)?

These probabilities are unequal. Why might this be useful?

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

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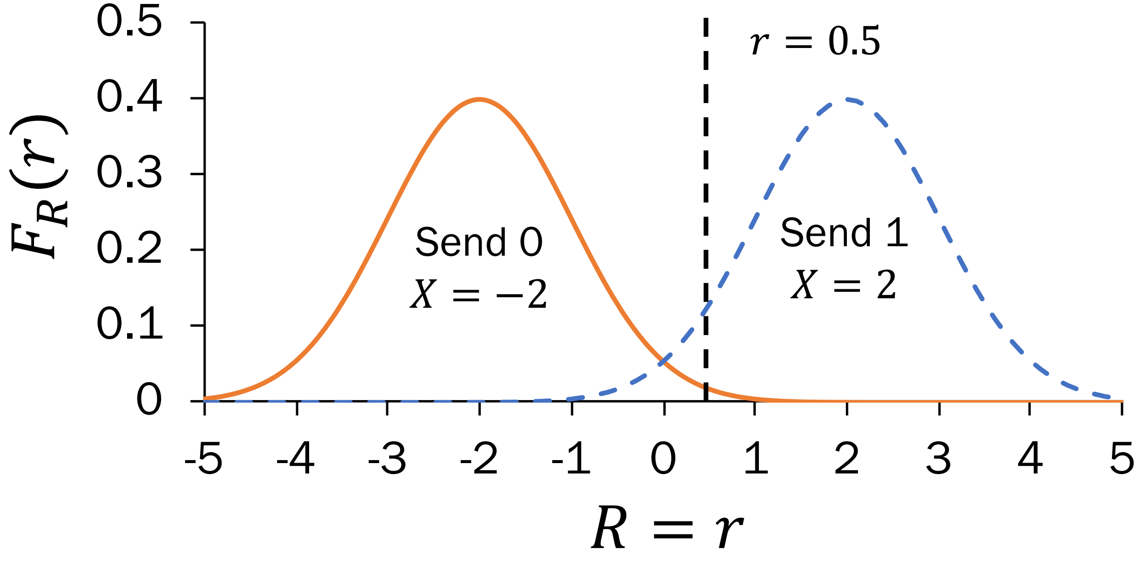

1. What is P(decoding error | original bit is 1)?
   i.e., we sent 1, but we decoded as 0?

$$P(R < 0.5 | X = 2) = P(2 + Y < 0.5) = P(Y < -1.5)$$

Y is Standard Normal

$$= \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

Noisy Wires

Send a voltage of 2 V or -2 V on wire (to denote 1 and 0, respectively).

- X = voltage sent (2 or -2)
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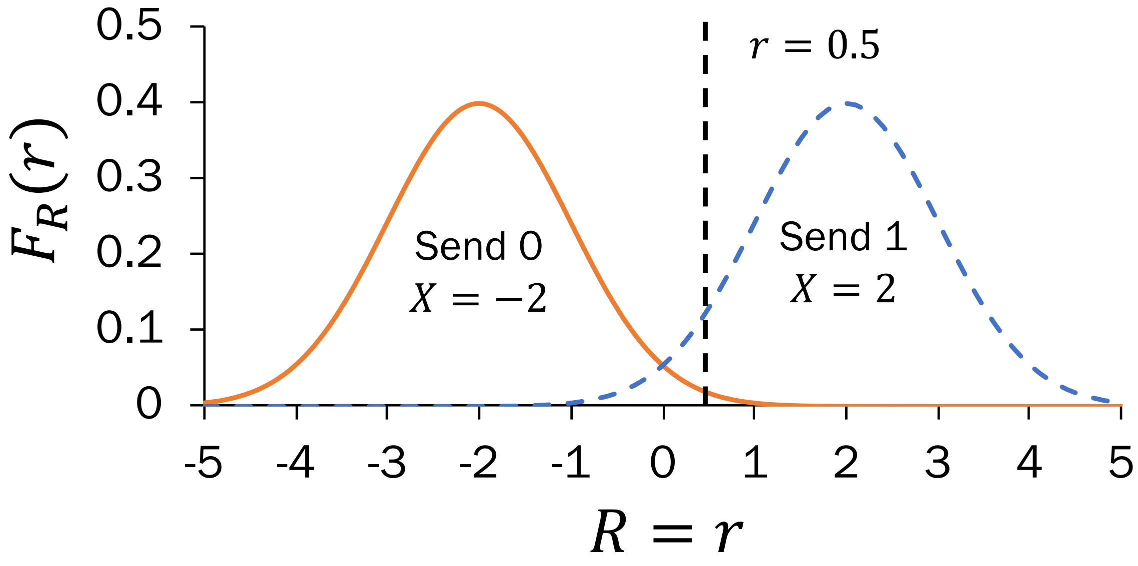

1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?

0.0668

2. What is P(decoding error | original bit is 0)?

$$P(R \geq 0.5 | X = -2) = P(-2 + Y \geq 0.5) = P(Y \geq 2.5) \approx 0.0062$$

Asymmetric decoding probability: We would like to avoid mistaking a 0 for 1. Errors the other way are tolerable.