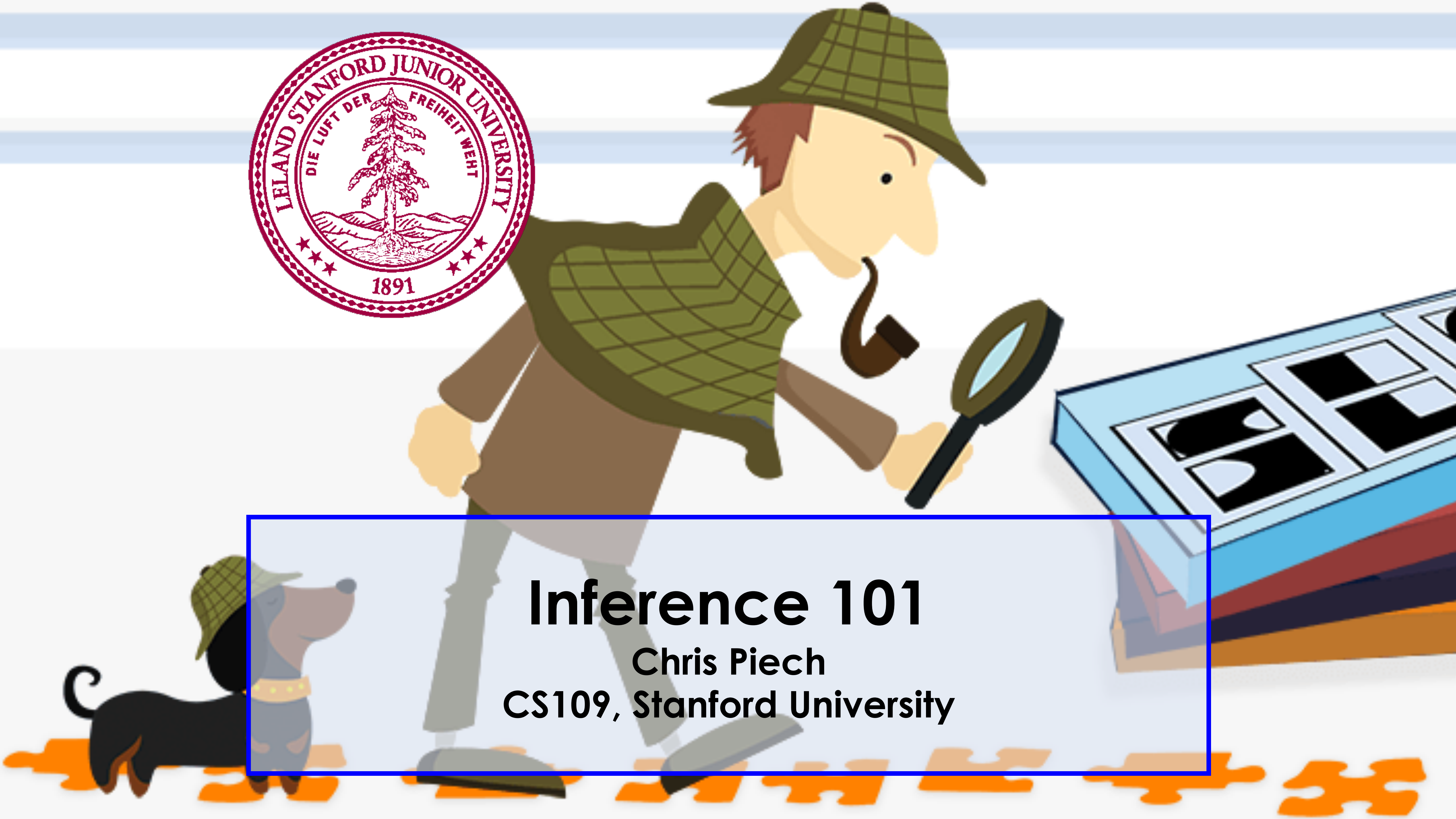


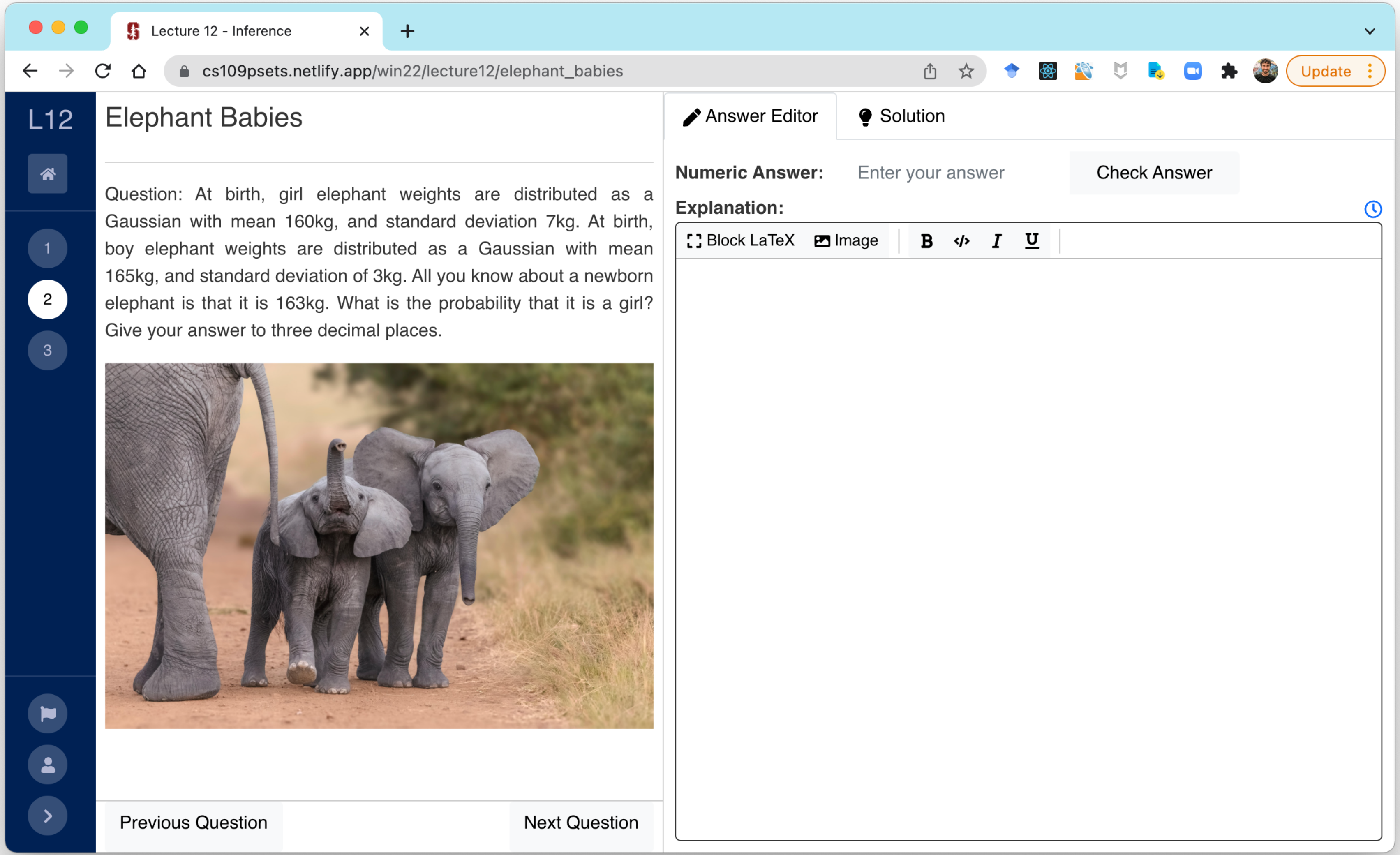
Inference 101

Chris Piech
CS109, Stanford University

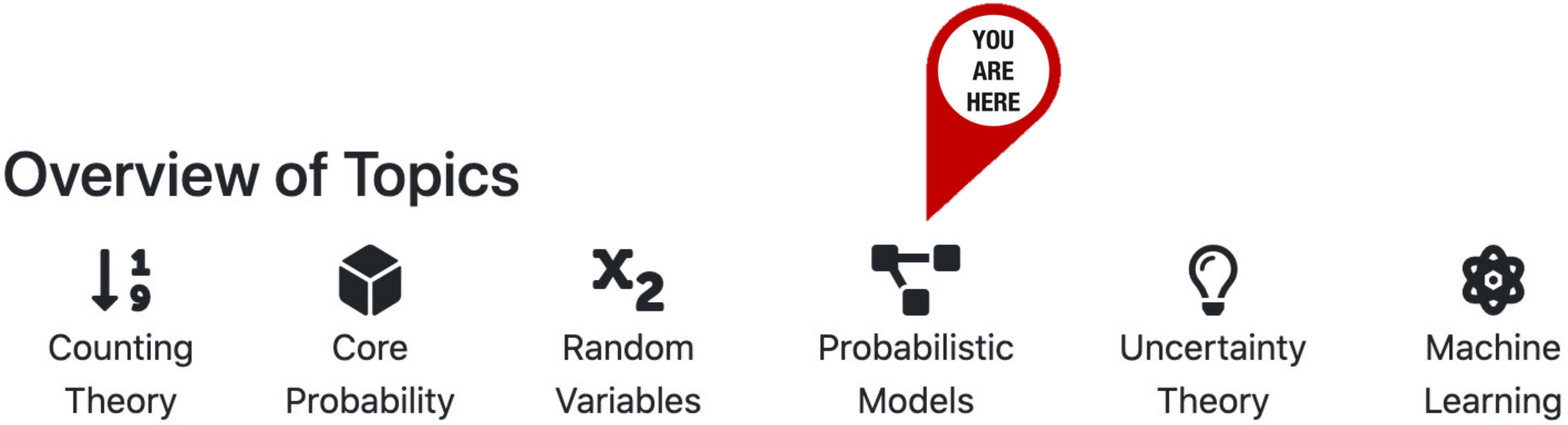
Announcements

- Midterm in 1 week and 1 day.
  - 7-9p in three rooms Hewlett 200, Hewlett 201, CEMEX. 33% the density of a normal exam.
  - Feel uncomfortable taking an exam in these rooms? Let us know by next Wednesday 9a. Get sick? Don't come! Let us know.
  - Need accommodations? Let us know before next Thursday 9a
  - Open notes, closed book, closed computer
  - Cumulative up until this Friday. Most emphasis will be on the topics in the first 3 psets, sections.

https://cs109psets.netlify.app/win22/lecture12



Where are we in CS109?



Where are we locally?

Discrete Models:
General Case, Multinomial

Inference
Conclusions from Observations

Modelling:
Make your own!

General Inference:
Use computers to infer

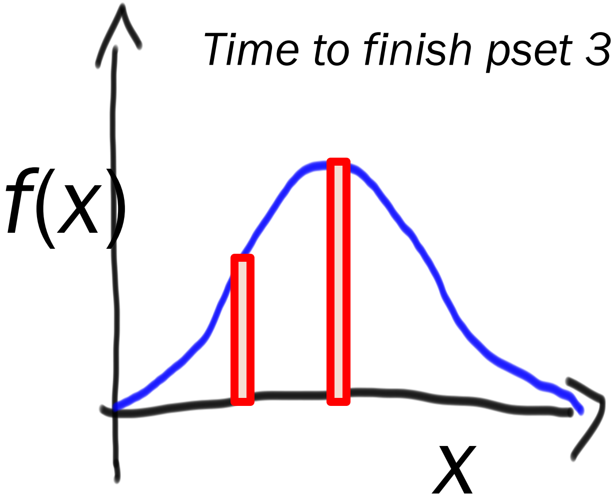
Learning Goals

1. Know how to use a multinomial
2. Be able to calculate large bayes problems using a computer
3. Combine Bayes Theorem and Random Variables

Relative Probability of Continuous Variables

X = time to finish pset 3

$X \sim N(\mu = 10, \sigma^2 = 2)$



How much more likely are you to complete in 10 hours than in 5?

$$\frac{P(X=10)}{P(X=5)} = \frac{\varepsilon f(X=10)}{\varepsilon f(X=5)}$$

$$= \frac{f(X=10)}{f(X=5)}$$

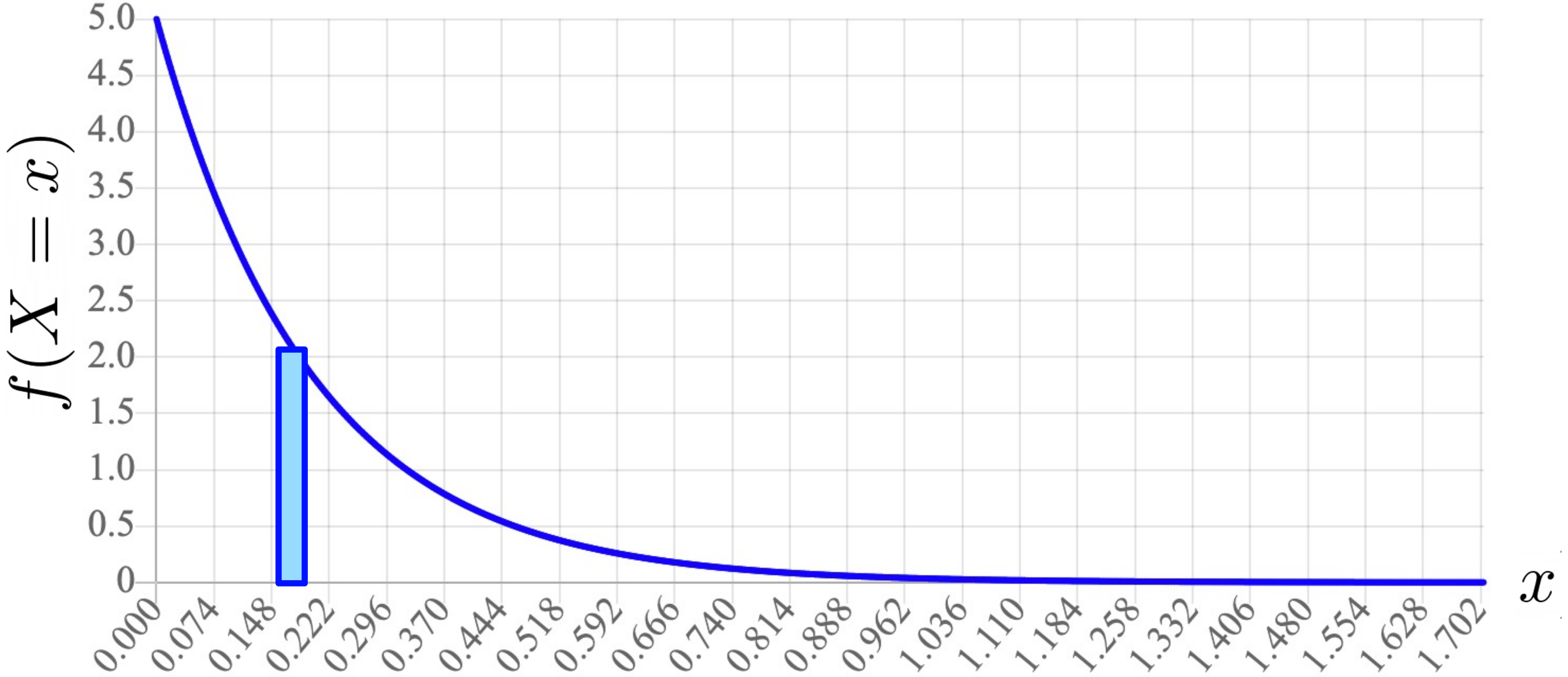
$$= \frac{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}}$$

$$= \frac{\frac{1}{\sqrt{4\pi}} e^{-\frac{(10-10)^2}{4}}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{(5-10)^2}{4}}}$$

$$= \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

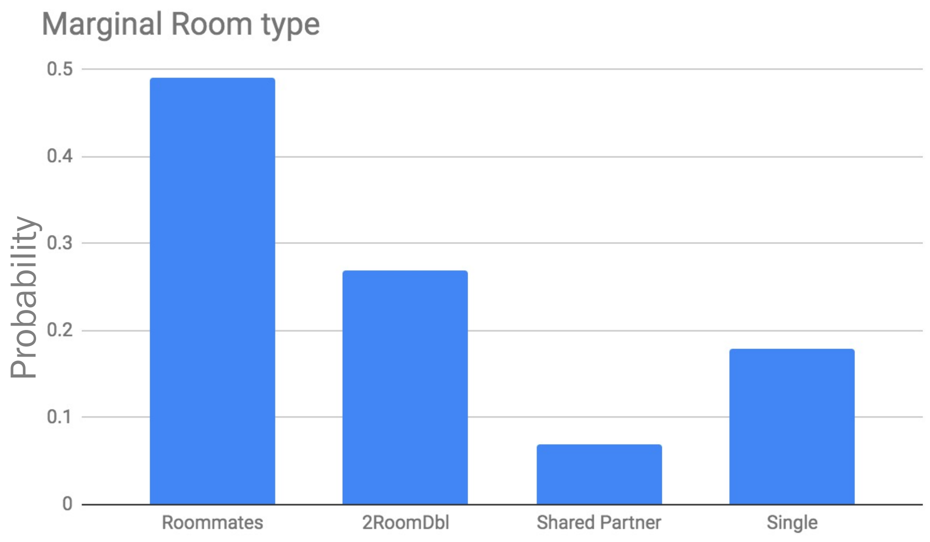
Calculation is based on this claim

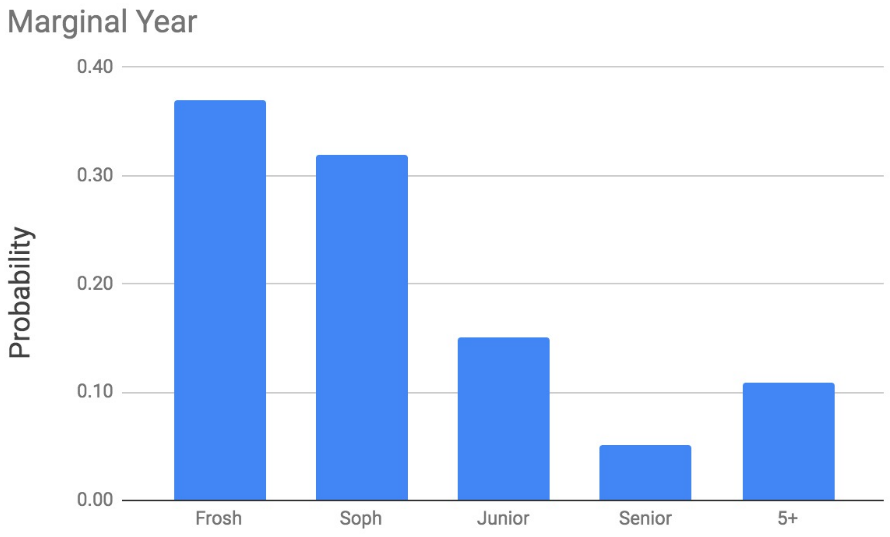
$$P(X = x) = f(X = x) \cdot \epsilon_x$$



Joint Probability Table

	Roommates	2RoomDbl	Shared Partner	Single	
Frosh	0.30	0.07	0.00	0.00	0.37
Soph	0.12	0.18	0.00	0.03	0.32
Junior	0.04	0.01	0.00	0.10	0.15
Senior	0.01	0.02	0.02	0.01	0.05
5+	0.02	0.00	0.05	0.04	0.11
	0.49	0.27	0.07	0.18	1.00





Last Week

Joint Distribution *noun*

The probability of a simultaneous assignment to ***all*** the random variables in a probabilistic model.

Eg:

$$P(X = x, Y = y)$$

$$f(X = x, Y = y)$$

$$P(X = x, Y = y, \cdots, Z = z)$$

Notation: These are all the same

$$P(X = x, Y = y)$$

$$P_{X,Y}(x, y)$$

$$P(x, y)$$

The Multinomial

Multinomial distribution

- *n* independent trials of experiment performed
- Each trial results in one of *m* outcomes, with respective probabilities: $p_1, p_2, \ldots, p_m$ where $\sum_{i=1}^{m} p_i = 1$
- X_i = number of trials with outcome *i*

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \ldots p_m^{c_m}$$

Joint distribution

Multinomial # ways of ordering the successes

Probabilities of each ordering are equal and mutually exclusive

where $\sum_{i=1}^{m} c_i = n$ and $\binom{n}{c_1, c_2, \ldots, c_m} = \frac{n!}{c_1! c_2! \cdots c_m!}$

Probabilistic Text Analysis

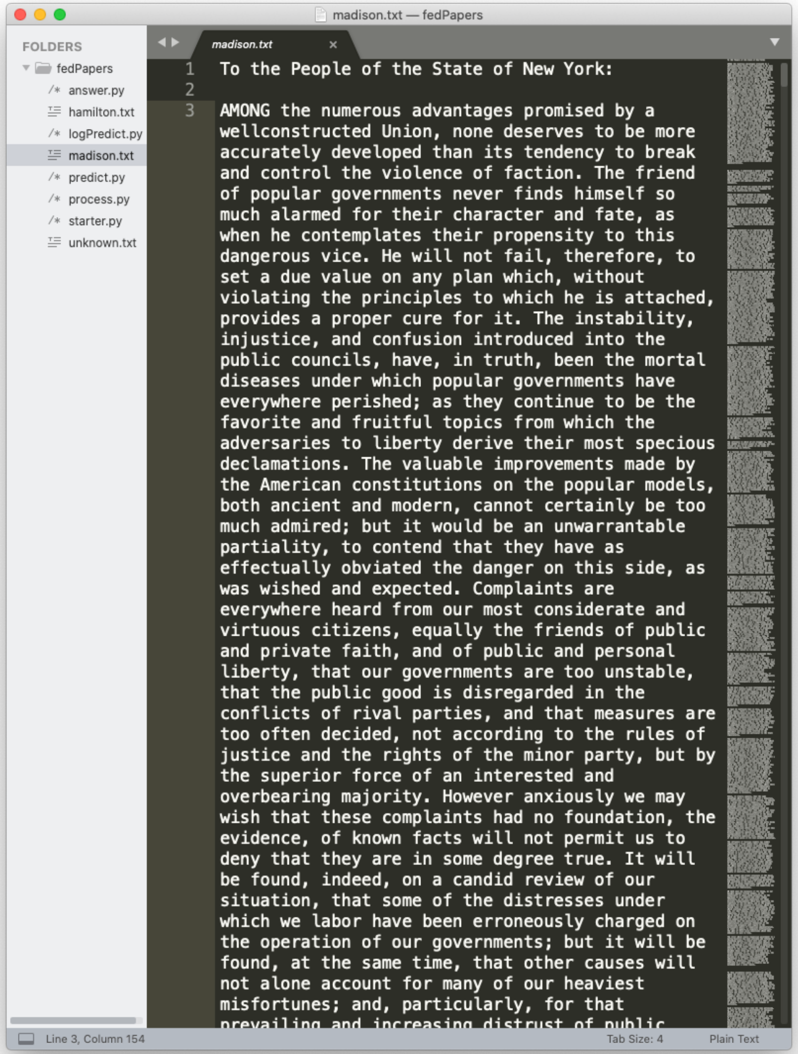
According to the Global Language Monitor there are 988,968 words in the english language used on the internet.

End Review

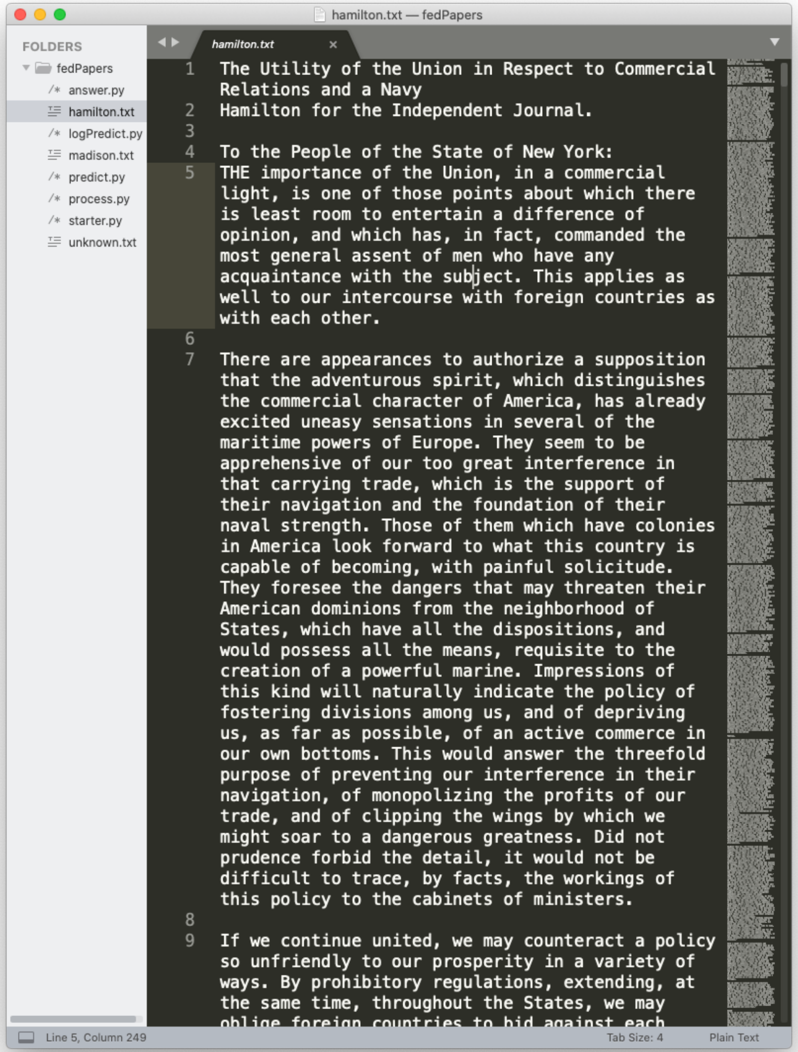
Who wrote the federalist papers?

Who wrote Federalist Paper 53?

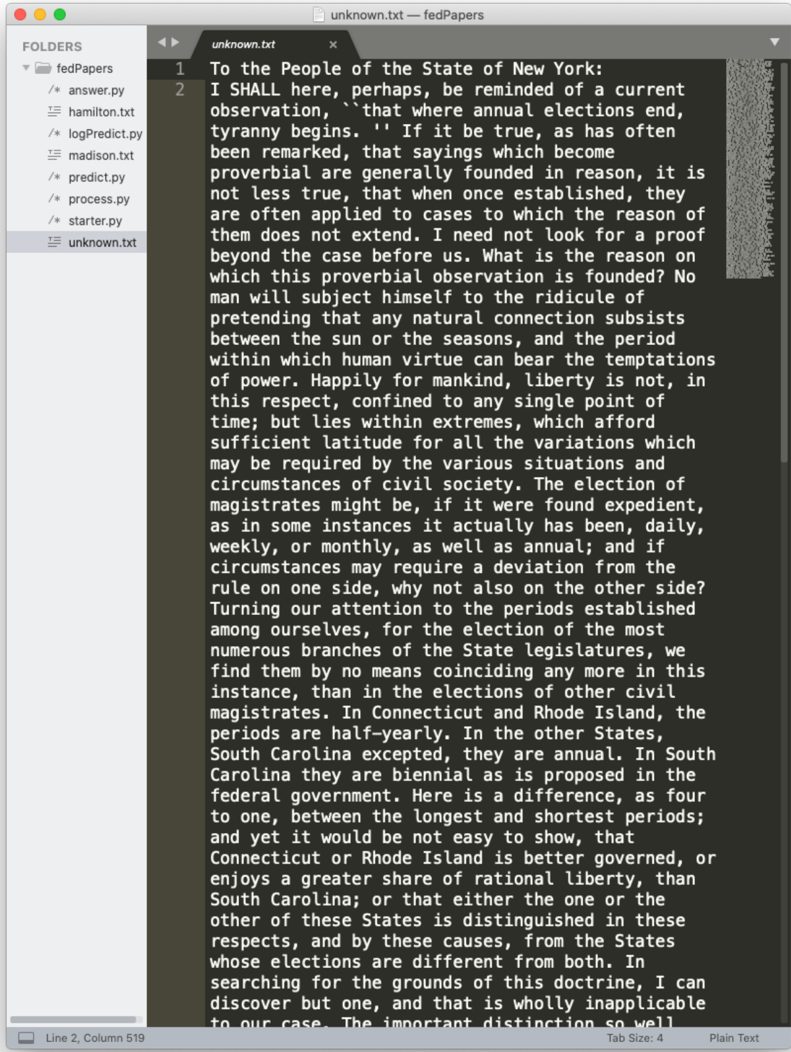
madison.txt



hamilton.txt



unknown.txt



Where to start?

We have words, we want to know probability of authorship. We also know probability of words given author…

Well hello again…

Who wrote Federalist Paper 53?

Prob Document given Hamilton

Prior belief it was Hamilton

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

Prob Hamilton given Document

Prob of the document???

Who wrote Federalist Paper 53?

Model document as a multinomial where we care about count of words

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

Who wrote Federalist Paper 53?

$$P(H|D) = \frac{\binom{n}{c_1 \ldots c_k} \cdot \prod_i h_i^{c_i} \cdot P(H)}{P(D)}$$

Number of times word i is in the doc

Loop over unique words

Prob hamilton would write word i

Prior belief it was Hamilton

Prob Hamilton given Document

Prob of the document???

Who wrote Federalist Paper 53?

Prob that Hamilton wrote it

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

$$= \frac{P(H) \cdot \binom{n}{c_1 \ldots c_m} \cdot \prod_i h_i^{c_i}}{P(D)}$$

Prob that Madison wrote it

$$P(M|D) = \frac{P(D|M)P(M)}{P(D)}$$

$$= \frac{P(M) \cdot \binom{n}{c_1 \ldots c_m} \cdot \prod_i m_i^{c_i}}{P(D)}$$

$$\frac{P(H|D)}{P(M|D)} = \frac{P(M) \cdot \binom{n}{c_1 \ldots c_k} \cdot \prod_i h_i^{c_i}}{P(D)} \Big/ \frac{P(H) \cdot \binom{n}{c_1 \ldots c_k} \cdot \prod_i m_i^{c_i}}{P(D)}$$

$$= \frac{\prod_i h_i^{c_i}}{\prod_i m_i^{c_i}}$$

To the code

What happened?

All our probabilities are zero…

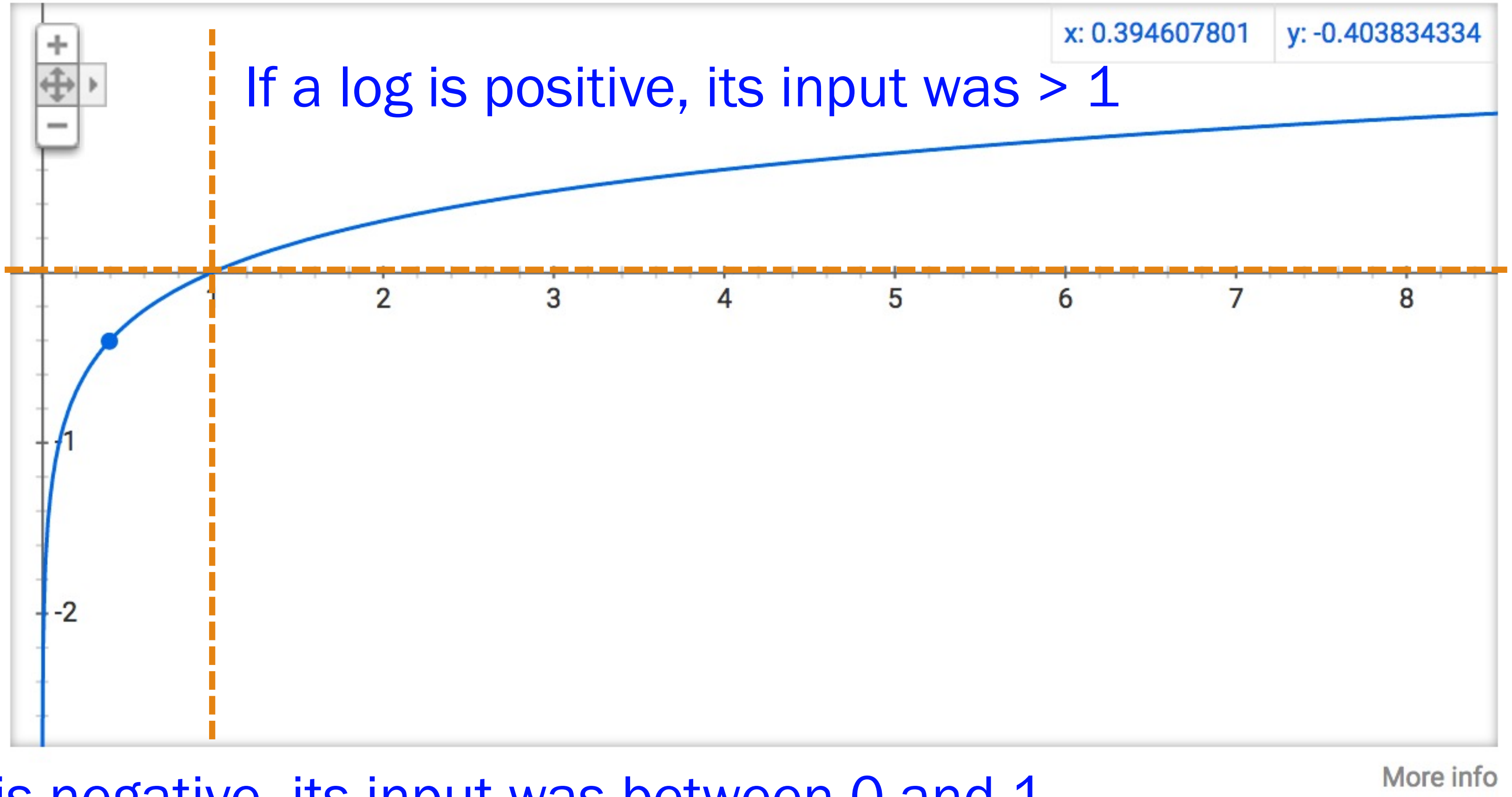
Use logs when probabilities become too small!

$$\frac{P(H|D)}{P(M|D)} = \frac{\prod_i m_i^{c_i}}{\prod_i h_i^{c_i}}$$

$$\begin{aligned}
\log \frac{P(H|D)}{P(M|D)} &= \log \frac{\prod_i h_i^{c_i}}{\prod_i m_i^{c_i}} \\
&= \sum_i \log h_i^{c_i} - \sum_i \log m_i^{c_i} \\
&= \sum_i c_i \cdot \log h_i - \sum_i c_i \log m_i
\end{aligned}$$

What does it mean if a log value is positive / negative

Graph for log(x)



If a log is negative, its input was between 0 and 1 woot

Joint Random Variables

- [ ] Use a joint table, or joint function to solve probability question
- [ ] Think about **conditional** probabilities with joint variables (which might be continuous)
- [ ] Use and find **independence** of random variables
- [ ] Use and find **expectation** of random variables

Joint Probability Table

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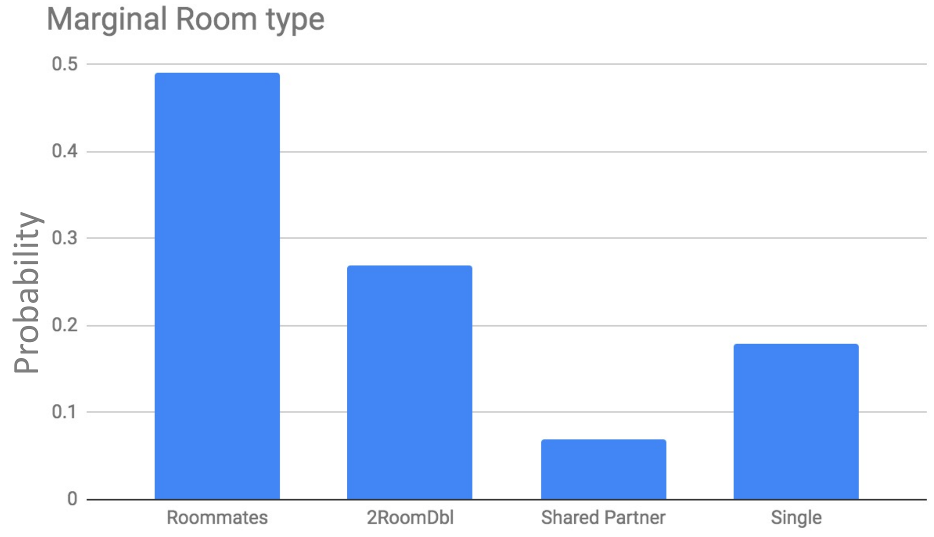
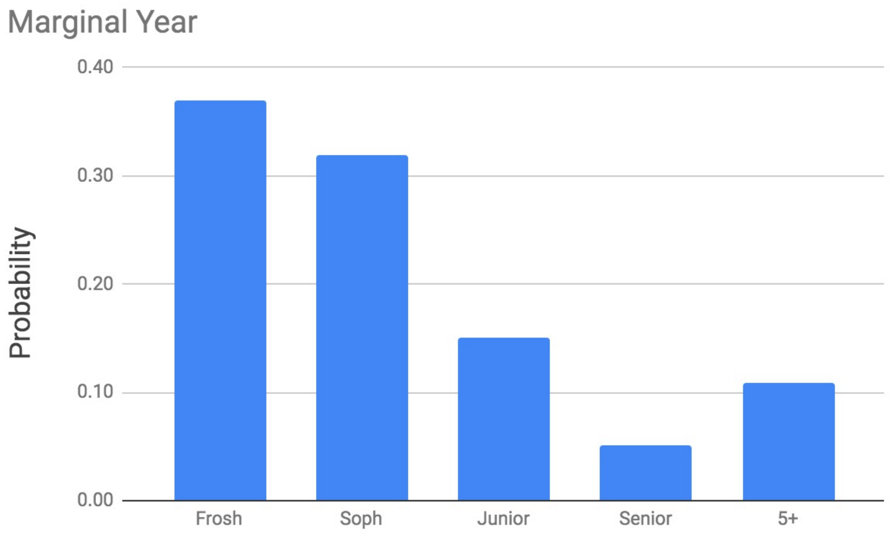
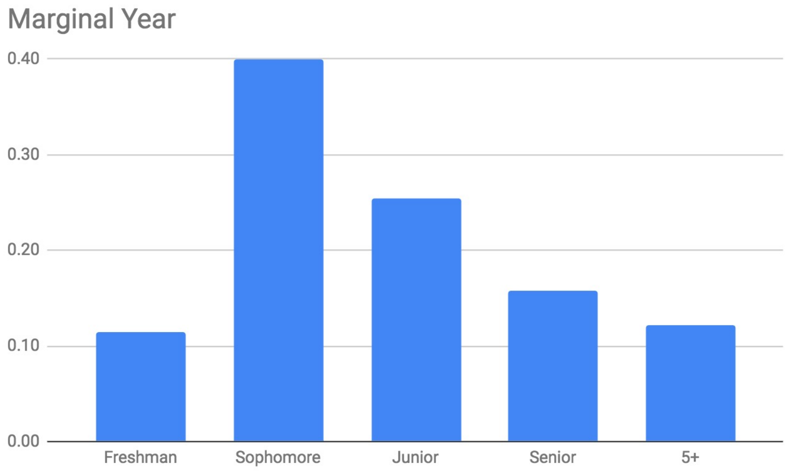
Senior	0.01	0.02	0.02	0.01	0.05
5+	0.02	0.00	0.05	0.04	0.11
	0.49	0.27	0.07	0.18	1.00

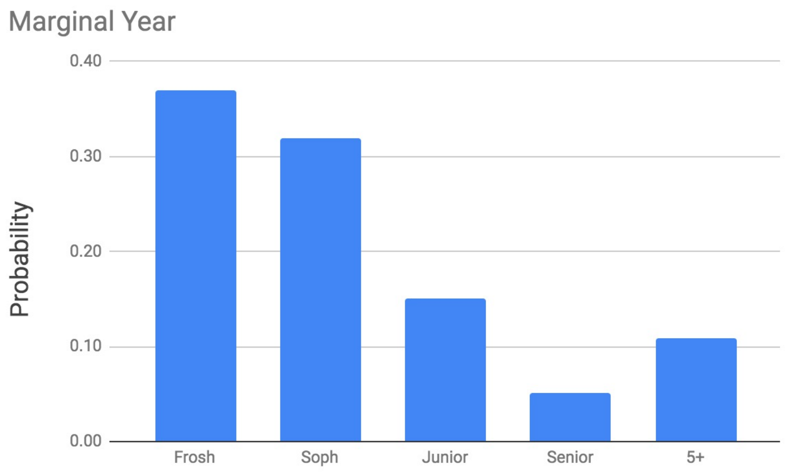




Change in Marginal Year



Fall quarter ‘18



Spr quarter ‘19

Joint Random Variables

- [x] Use a joint table, or joint function to solve probability question
- [ ] Think about **conditional** probabilities with joint variables (which might be continuous)
- [ ] Use and find **independence** of random variables
- [ ] Use and find **expectation** of random variables

Today: Inference

Inference *noun*

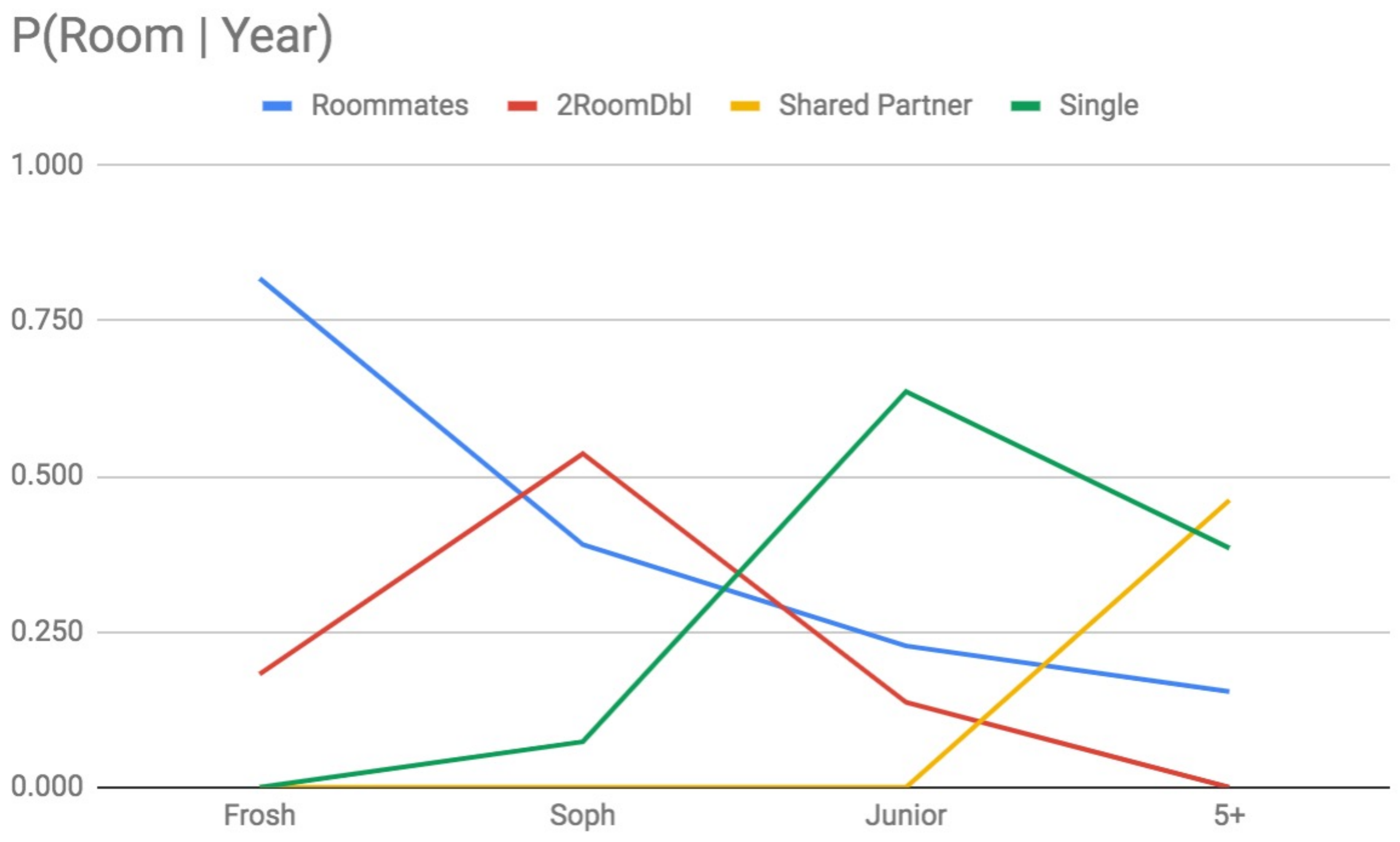
An updated belief about a random variable (or multiple) based on conditional knowledge regarding another random variable (or multiple) in a probabilistic model.

TLDR: conditional probability with random variables.

Warmup Inference

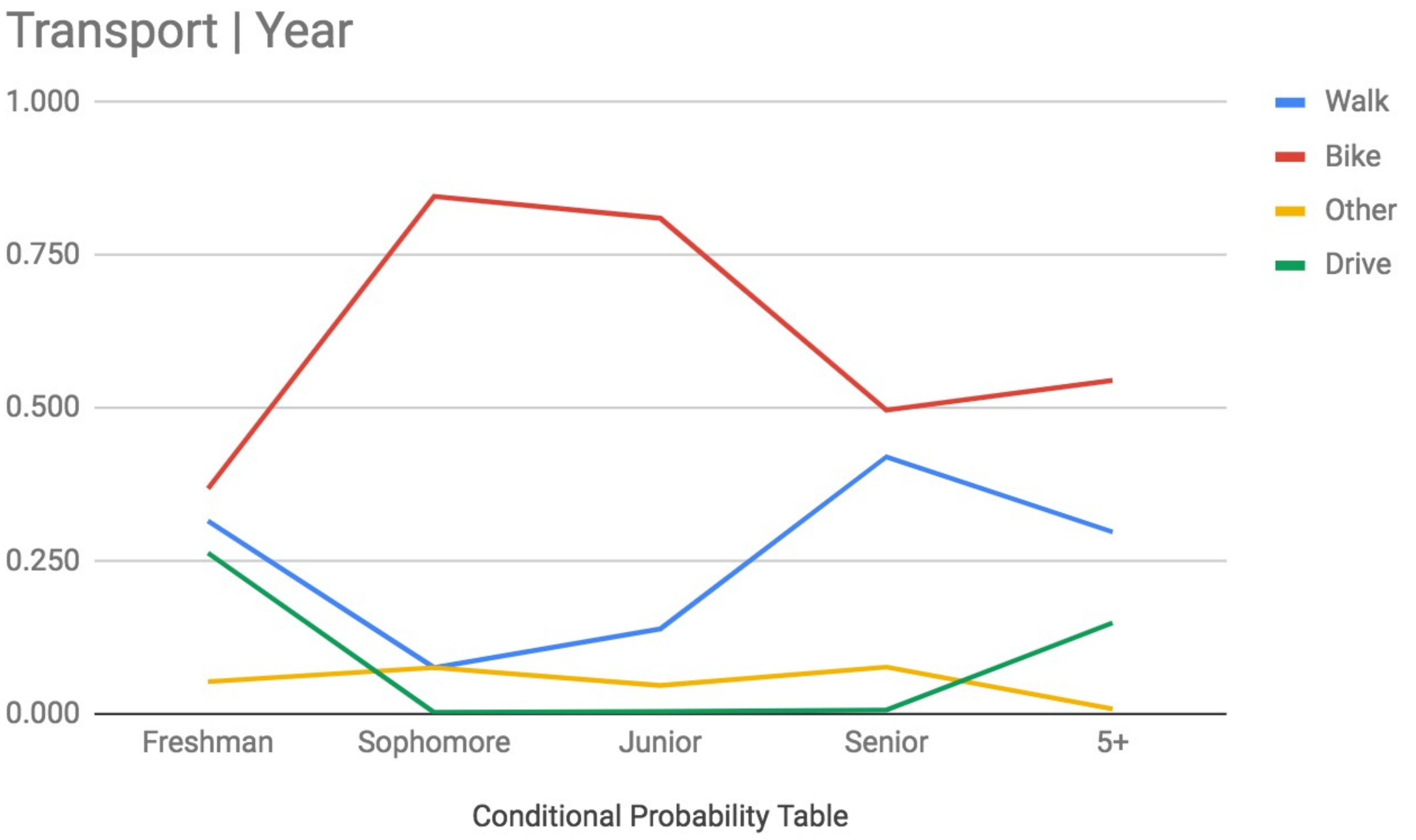
Q: What is the probability that someone has a single, given that they are a senior?

Room | Year

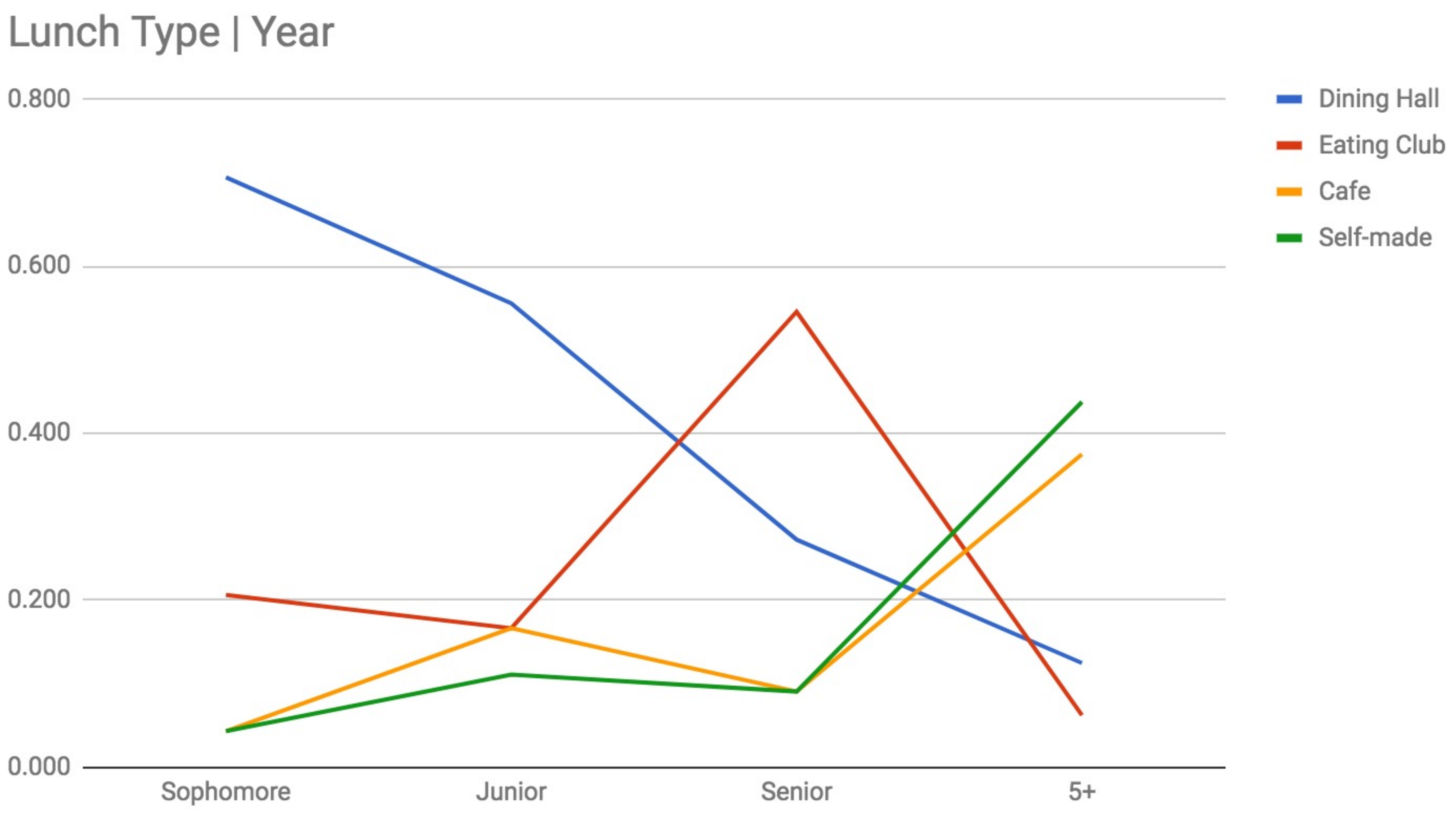


P(Room | Year)
Roommates
2RoomDbl
Shared Partner
Single
1.000
0.750
0.500
0.250
0.000
Frosh
Soph
Junior
5+

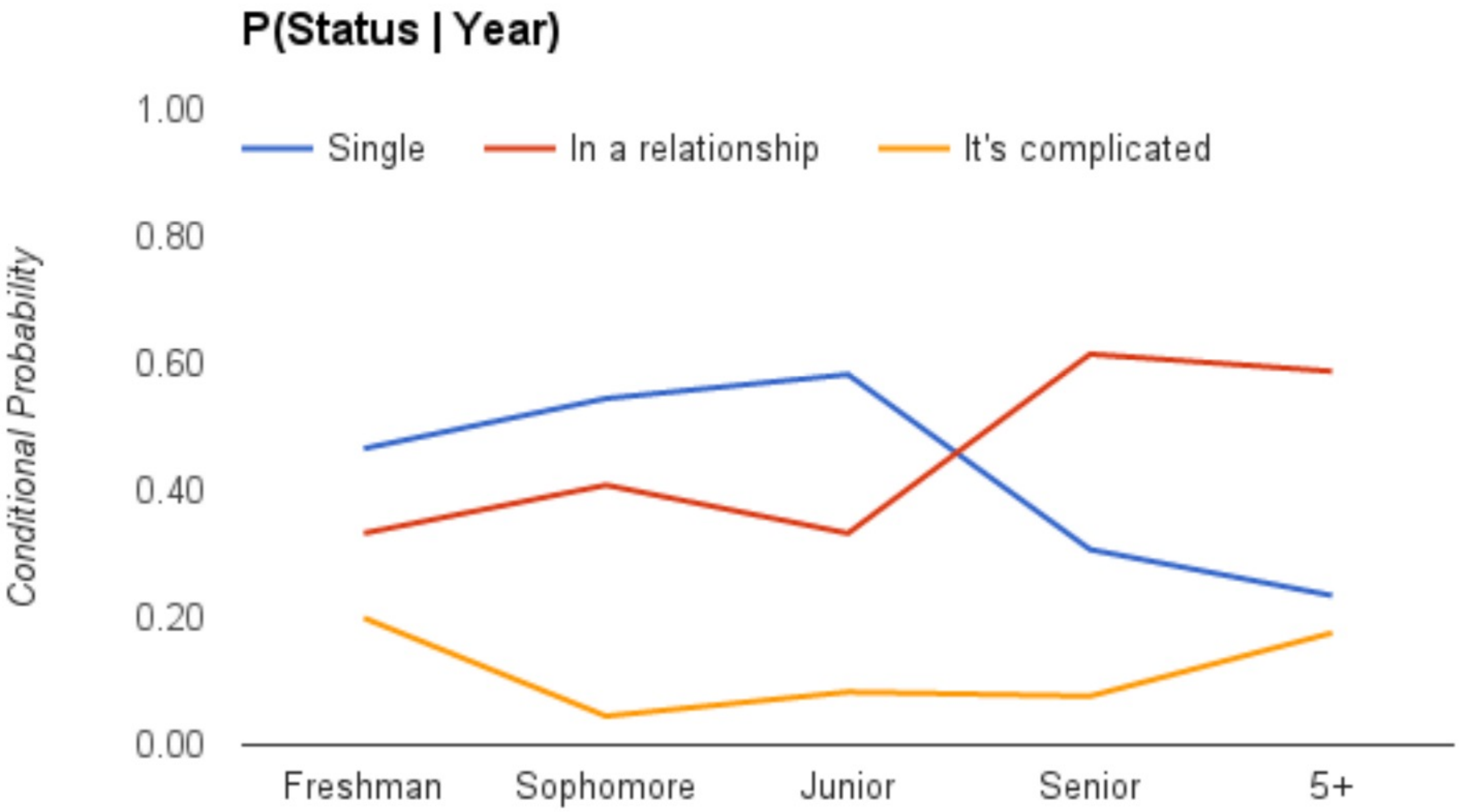
Transport | Year



Lunch | Year


Lunch Type | Year
0.800
0.600
0.400
0.200
0.000
Sophomore
Junior
Senior
5+
Dining Hall
Eating Club
Cafe
Self-made

Relationship Status | Year



Number or Function?

$$P(X = 2|Y = 5)$$

Number

Number or Function?

$$P(X = x | Y = 2)$$

Random Variable

(also a function or 1D table)

Number or Function?

$$P(X = x | Y = y)$$

2D Function

(or 2D table)

Pedagogical Pause

Warmup: Bayes Revisited

Posterior belief

Likelihood of evidence

Prior belief

$$P(B|E) = \frac{P(E|B) \quad P(B)}{P(E)}$$

Normalization constant

Bayes Theorm with Discrete

Let M be a **discrete** random variable

Let N be a **discrete** random variable

$$P(M=2|N=3) = \frac{P(N=3|M=2)P(M=2)}{P(N=3)}$$

More generally

$$P(M=m|N=n) = \frac{P(N=n|M=m)P(M=m)}{P(N=n)}$$

Shorthand notation

$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

I Heard That

Let X be the **change in gaze** (measured in degrees) over 3 seconds after a sound is played

Value of X	PMF of X given Baby can hear the sound	PMF of X given Baby can **not** hear the sound
0 to 5	0.08	0.40
5 to 10	0.15	0.30
10 to 15	0.35	0.12
15 to 20	0.20	0.08
20 to 25	0.12	0.05
Above 25	0.10	0.05

$$\text{P(can hear the sound)} = \frac{3}{4}$$

You observe $X = 0$. What is the probability the baby **can** hear the sound?

I Heard That

Value of X	PMF of X given Baby can hear the sound	PMF of X given Baby can **not** hear the sound
0 to 5	0.08	0.40
5 to 10	0.15	0.30
10 to 15	0.35	0.12
15 to 20	0.20	0.08
20 to 25	0.12	0.05
Above 25	0.10	0.05

$$\text{P(can hear the sound)} = \frac{3}{4}$$

You observe $X = 0$. What is the probability the baby **can** hear the sound?

$$P(Y=1|X=0) = \frac{P(X=0|Y)P(Y)}{P(X=0|Y)P(Y) + P(X=0|Y^C)P(Y^C)}$$

$$P(Y=1|X=0) = \frac{0.08 * 0.75}{0.08 * 0.75 + 0.40 * 0.25} = \frac{3}{8}$$

Inference with Continuous

Q: At birth, girl elephant weights are distributed as a Gaussian with mean = 160kg, std = 7kg. At birth, boy elephant weights are distributed as a Gaussian with mean = 165kg, std = 3kg. All you know about a newborn elephant is that it is 163kg. What is the probability that it is a girl?

Inference with Continuous

Q: At birth, girl elephant weights are distributed as a Gaussian with mean = 160kg, std = 7kg. At birth, boy elephant weights are distributed as a Gaussian with mean = 165kg, std = 3kg. All you know about a newborn elephant is that it is 163kg. What is the probability that it is a girl?

Model:

Let G be an indicator that the elephant is a girl. G is Bern(p = 0.5)

Let X be the distribution of weight of the elephant.

$X \mid G = 1$ is $N(\mu = 160, \sigma^2 = 7^2)$

$X \mid G = 0$ is $N(\mu = 165, \sigma^2 = 3^2)$

Inference with Continuous

Q: What is $P(G = 1 \mid X = 163)$

Let G be an indicator that the elephant is a girl. G is $\text{Bern}(p = 0.5)$

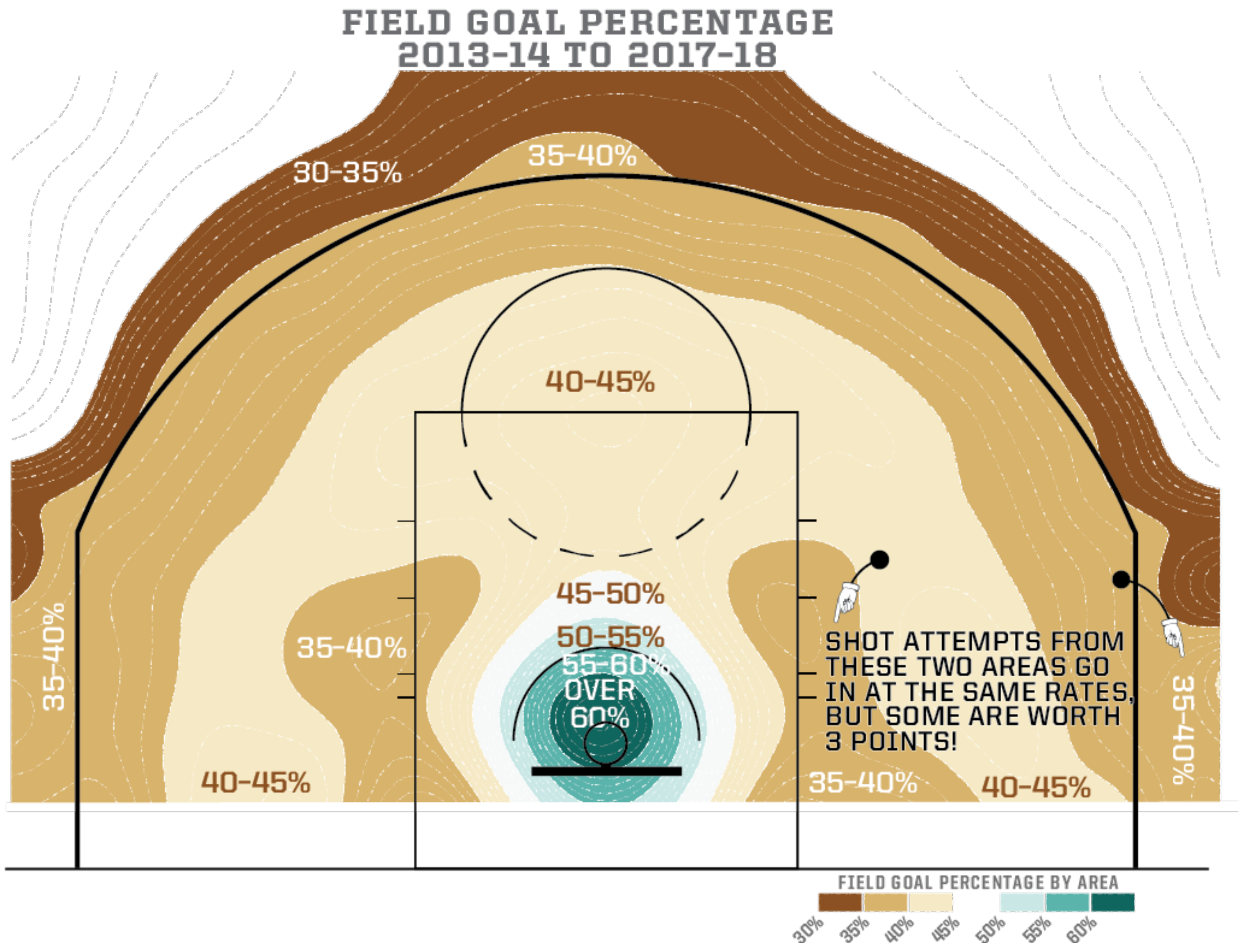
Let X be the distribution of weight of the elephant.

$X \mid G = 1$ is $N(\mu = 160, \sigma^2 = 7^2)$

$X \mid G = 0$ is $N(\mu = 165, \sigma^2 = 3^2)$

Aside: Models with continuous RVs

Probabilistic Models can have Continuous Random Vars



X location of the shot

Y location of the shot

$$f(X = x, Y = y)$$

Joint: If any random variable is continuous, then we consider the joint a density

Joint is Complete Information!

A joint distribution is complete information. It can be used to answer any probability question.

Still true when some variables are continuous

Mixing Discrete and Continuous

Let X be a **continuous** random variable

Let N be a **discrete** random variable

$$P(N = n | X = x) = \frac{P(X = x | N = n)P(N = n)}{P(X = x)}$$

$$P(N = n | X = x) = \frac{f(X = x | N = n) \cdot \epsilon \cdot P(N = n)}{f(X = x) \cdot \epsilon}$$

$$P(N = n | X = x) = \frac{f(X = x | N = n) \cdot P(N = n)}{f(X = x)}$$

Mixing Discrete and Continuous

Let X be a **continuous** random variable

Let N be a **discrete** random variable

$$P(X = x|N = n) = \frac{P(N = n|X = x)P(X = x)}{P(N = n)}$$

Change notation

$$P(x|n) = \frac{P(n|x)P(x)}{P(n)}$$

$$f(x|n) \cdot \epsilon_x = \frac{P(n|x)f(x) \cdot \epsilon_x}{P(n)}$$

$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

All the Bayes Belong to Us

M,N are discrete. X, Y are continuous

OG Bayes

$$P(m|n) = \frac{P(n|m)P(m)}{P(n)}$$

Mix Bayes #1

$$f(x|n) = \frac{P(n|x)f(x)}{P(n)}$$

Mix Bayes #2

$$P(n|x) = \frac{f(x|n)P(n)}{f(x)}$$

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)}$$

LOTP? Chain Rule? You can play too!

N is discrete. X is continuous

$$f(N = n, X = x) = f(X = x|N = n)P(N = n)$$

$$f(X = x) = \sum_{n} f(X = x|N = n)P(N = n)$$

End Aside

Inference with Continuous

Q: What is $P(G = 1 \mid X = 163)$

Let G be an indicator that the elephant is a girl. G is $\text{Bern}(p = 0.5)$

Let X be the distribution of weight of the elephant.

$X \mid G = 1$ is $N(\mu = 160, \sigma^2 = 7^2)$

$X \mid G = 0$ is $N(\mu = 165, \sigma^2 = 3^2)$

Inference with Continuous

Q: What is P(G = 1 | X = 163)

Let G be an indicator that the elephant is a girl. G is Bern(p = 0.5)

Let X be the distribution of weight of the elephant.

X | G = 1 is N(μ = 160, $\sigma^2 = 7^2$)

X | G = 0 is N(μ = 165, $\sigma^2 = 3^2$)

Joint Distribution is Implied:

$$f(G = 1, X = 72.3) = f(X = 72.3 | G = 1)P(G = 1)$$

More generally

$$f(G = g, X = x) = f(X = x | G = g)P(G = g)$$

Inference with Continuous

Q: What is $P(G = 1 \mid X = 163)$

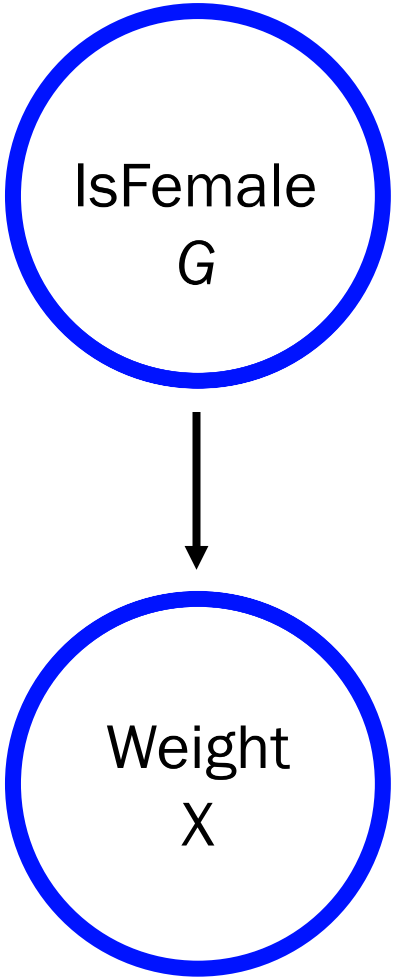
Let G be an indicator that the elephant is a girl. G is $\text{Bern}(p = 0.5)$

Let X be the distribution of weight of the elephant.

$X \mid G = 1$ is $N(\mu = 160, \sigma^2 = 7^2)$

$X \mid G = 0$ is $N(\mu = 165, \sigma^2 = 3^2)$

Model Shown Graphically



G = 1 is Bern(p = 0.5)

X | G = 1 is N(μ = 160, $\sigma^2 = 7^2$)

X | G = 0 is N(μ = 165, $\sigma^2 = 3^2$)

Does this define the joint?

$$f(G = g, X = x)$$

$$= f(X = x | G = g) P(G = g)$$

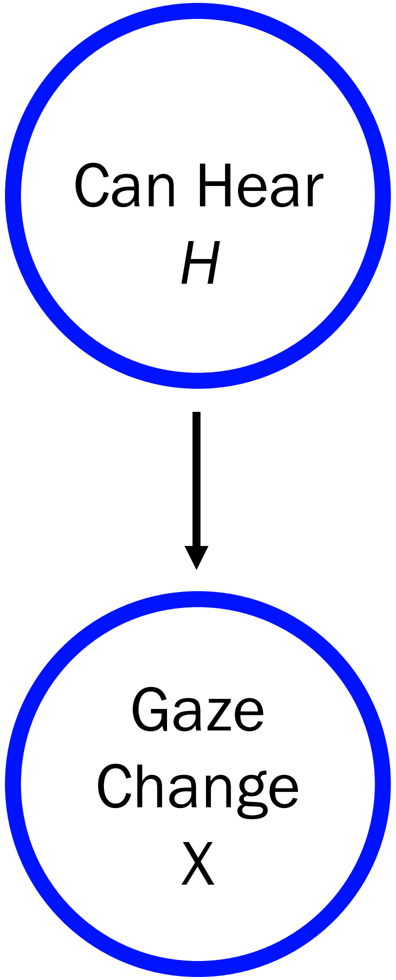
Q: What is P(G = 1 | X = 163)

I Heard That Redux

Normal Assumption: We choose to approximate eye movements with normal distributions. For babies who can hear sounds, we approximate their gaze movement after the sound is played as: $N(\mu = 15, \sigma^2 = 50)$. For babies who can **not** hear sounds, we approximate gaze movement as $N(\mu = 8, \sigma^2 = 50)$.

For a new baby we observe a 14 degree movement after the sound is played.
What is your belief that a baby can hear, under The **Normal Assumption**?

I Heard That Redux



H = 1 is Bern(p = 0.75)

X | H = 1 is $N(\mu = 15, \sigma^2 = 50)$
X | H = 0 is $N(\mu = 8, \sigma^2 = 50)$

Q: What is P(H = 1 | X = 14)

Joint Random Variables

- [x] Use a joint table, or joint function to solve probability question
- [x] Think about **conditional** probabilities with joint variables (which might be continuous)
- [ ] Use and find **independence** of random variables
- [ ] Use and find **expectation** of random variables