Naïve Bayes

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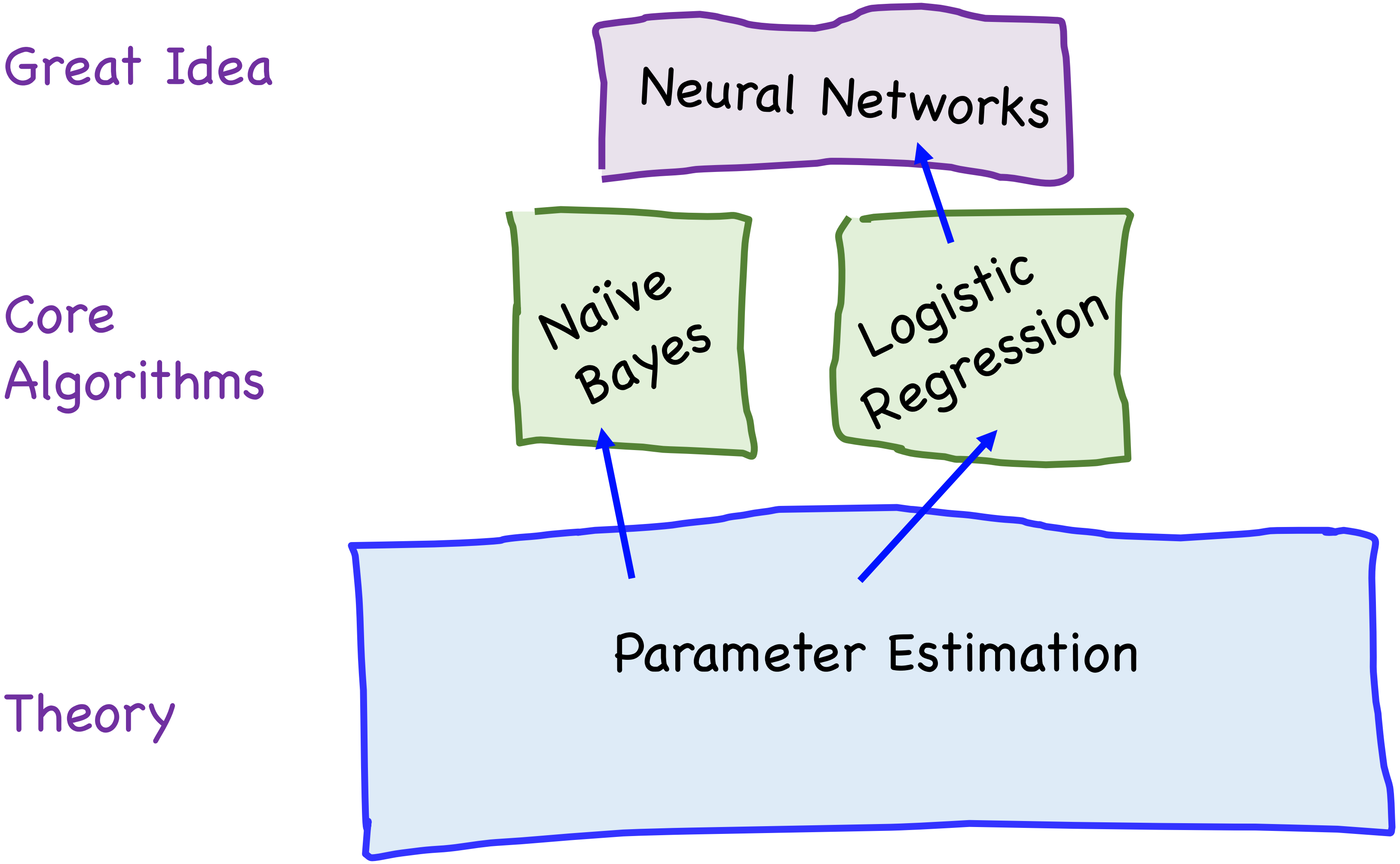
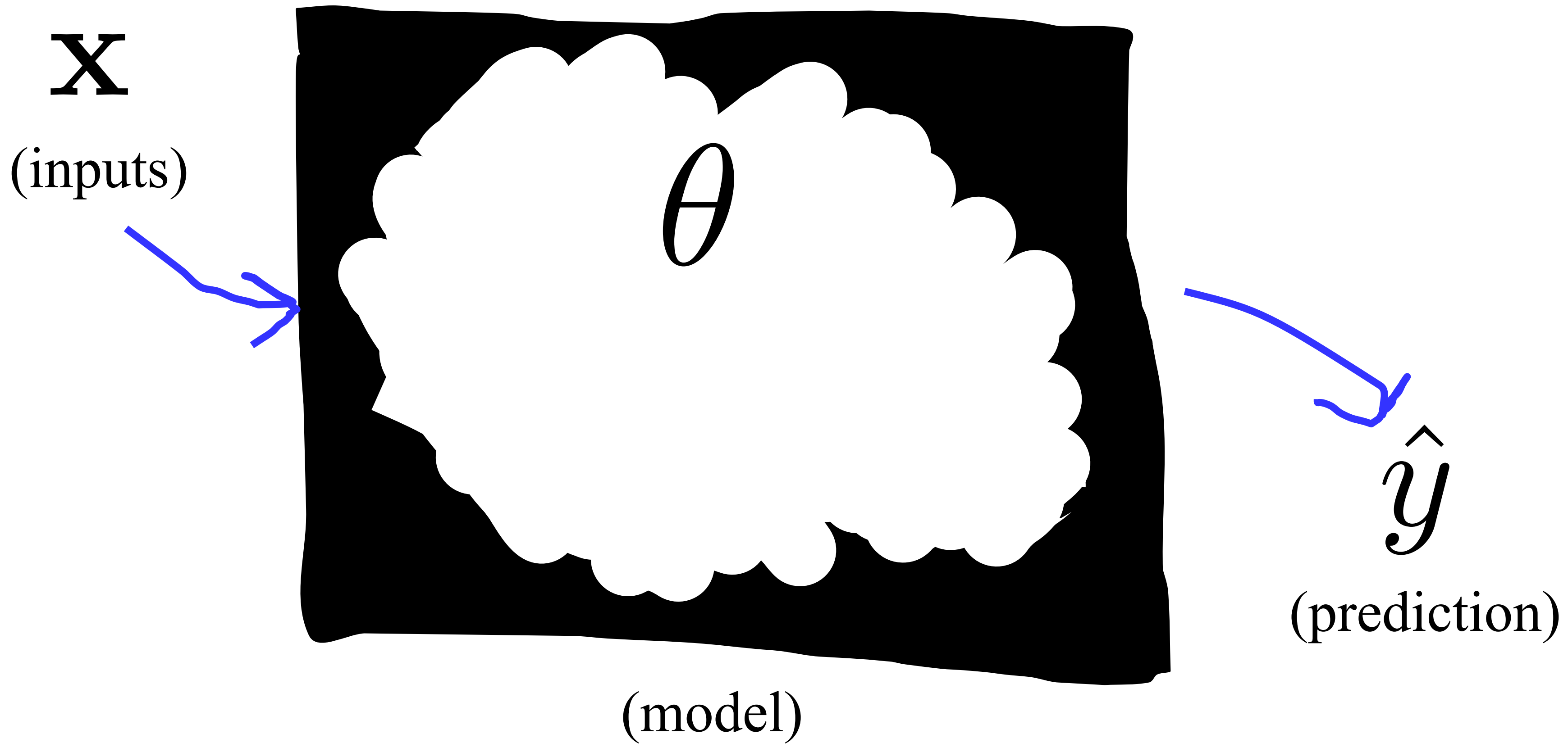
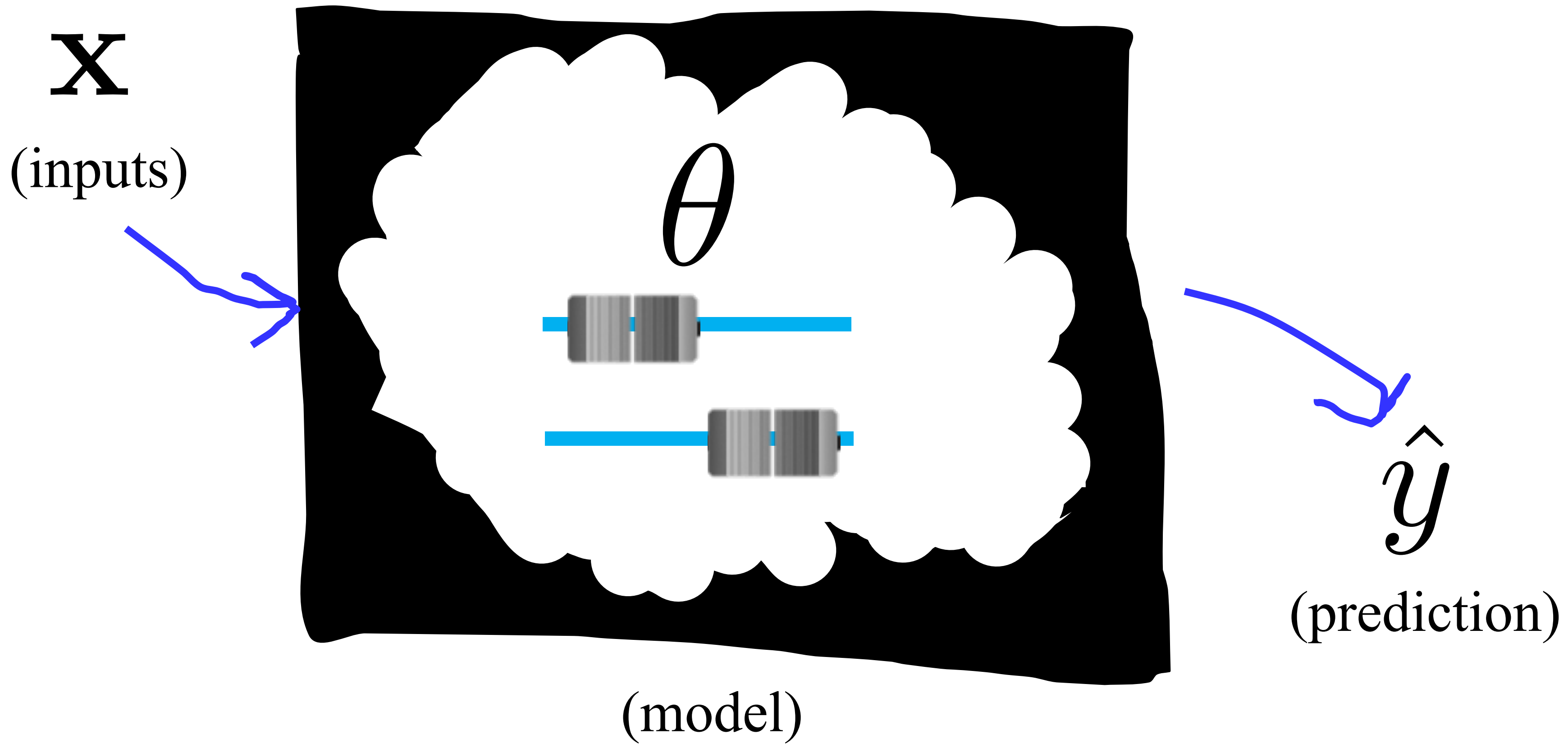
Machine Learning in CS109



Machine Learning



Machine Learning



MLE vs MAP

Data: $x^{(1)}, \ldots, x^{(n)}$

Maximum Likelihood Estimation

$$\hat{\theta}_{MLE} = \underset{\theta}{\text{argmax}}\ f(x^{(1)}, \ldots, x^{(n)} | \theta)$$
$$= \underset{\theta}{\text{argmax}} \left(\sum_i \log f(x^{(i)} | \theta) \right)$$

Maximum A Posteriori

$$\hat{\theta}_{MAP} = \underset{\theta}{\text{argmax}}\ f(\theta | x^{(1)}, \ldots, x^{(n)})$$
$$= \underset{\theta}{\text{argmax}} \left(\log(g(\theta)) + \sum_{i=1}^{n} \log(f(x^{(i)} | \theta)) \right)$$

Event Shorthand

MAP, without shorthand

$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}}\ f(\Theta = \theta | X^{(1)} = x^{(1)}, \ldots, X^{(n)} = x^{(n)})$$

Our shorthand notation

θ is shorthand for the event: $\Theta = \theta$

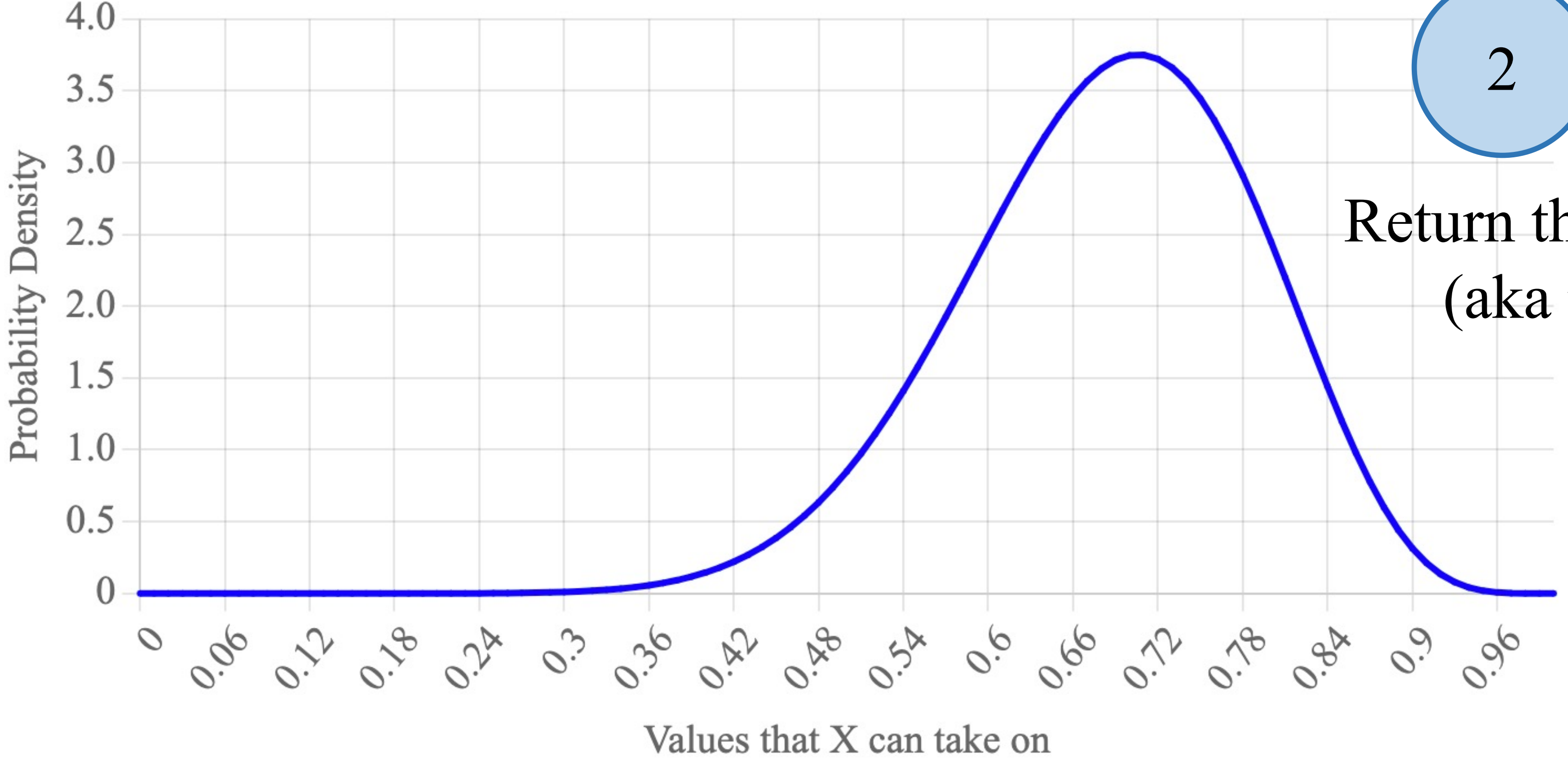
$x^{(i)}$ is shorthand for the event: $X^{(i)} = x^{(i)}$

MAP, now with shorthand

$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}}\ f(\theta | x^{(1)}, \ldots, x^{(n)})$$

MAP For Bernoulli

1 Infer the posterior belief in params



2 Return the argmax (aka mode)

Quick MAP for Bernoulli and Binomial

Review

Beta(a, b) is a conjugate prior for the probability of success in Bernoulli and Binomial distributions.

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$$

Prior Beta(a, b)
Saw $a + b - 2$ imaginary trials: $a - 1$ successes, $b - 1$ failures

Experiment Observe $n + m$ new trials: n successes, m failures

Posterior Beta$(a + n, b + m)$

MAP:

$$p = \frac{a + n - 1}{a + b + n + m - 2}$$

Quick MAP for Bernoulli with Laplace

Beta(a, b) is a conjugate prior for the probability of success in Bernoulli and Binomial distributions.

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$$

Prior Beta$(a = 2, b = 2)$
Saw 2 imaginary trials: 1 successes, 1 failures

Experiment Observe $n + m$ new trials: n successes, m failures

Posterior Beta$(2 + n, 2 + m)$

MAP:

$$p = \frac{n+1}{n+m+2}$$

End Review

Conjugate distributions

MAP estimator:

$$\theta_{MAP} = \arg\max_{\theta} f(\theta | X_1, X_2, \dots, X_n)$$

The mode of the posterior distribution of θ

Distribution parameter	Conjugate distribution
Bernoulli p	Beta
Binomial p	Beta
Multinomial p_i	Dirichlet
Poisson λ	Gamma
Exponential λ	Gamma
Normal μ	Normal
Normal σ^2	Inverse Gamma

Don't need to know Inverse Gamma…
but it will know you ☺

CS109: We'll only focus on MAP for Bernoulli/Binomial p, Multinomial p_i, and Poisson λ.

Multinomial

Each experiment has ***M*** possible outcomes. What is the likelihood of a particular count of each outcome?

multinomial is parameterized by $\boldsymbol{p_i}$*:*
the likelihood of outcome ***i*** *on any one experiment.*

Dice:
$M = 6$
$p_i = 1/6$

Multinomial

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Dice:
$M = 6$
$p_i = 1/6$

MLE for Multinomial

number of observed outcomes of type *i*

MLE estimate of the probability of outcome *i*

$$p_i = \frac{n_i}{n}$$

number of observations

$\theta_{.}$ is $p_{.}$

For a multinomial

MAP for Multinomial, Leplace Prior

number of observed
outcomes of type *i*

MAP estimate of the
probability of outcome *i*

$$p_i = \frac{n_i + 1}{n + m}$$

number of observations

number of outcome
types

$\theta_.$ is $p_.$
For a multinomial

niversity

Your Happy Laplace

Recall example of 6-sides die rolls:

- X ~ Multinomial(p_1, p_2, p_3, p_4, p_5, p_6)
- Roll *n* = 12 times. Result: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes
  - MLE: p_1=3/12, p_2=2/12, p_3=0/12, p_4=3/12, p_5=1/12, p_6=3/12
  - **Laplace estimate**: $$p_i = \frac{n_i + k}{n + k \cdot m}$$
  - Laplace: p_1=4/18, p_2=3/18, p_3=1/18, p_4=4/18, p_5=2/18, p_6=4/18
  - No longer have 0 probability of rolling a three!

More Generally with Dirichlet

Dirichlet$(a_1, a_2, \ldots, a_m)$ is a conjugate for Multinomial.

- Generalizes Beta in the same way Multinomial generalizes Bernoulli/Binomial:

$$f(x_1, x_2, \ldots, x_m) = \frac{1}{B(a_1, a_2, \ldots, a_m)} \prod_{i=1}^{m} x_i^{a_i - 1}$$

Prior Dirichlet$(a_1, a_2, \ldots, a_m)$
Saw $\left(\sum_{i=1}^{m} a_i\right) - m$ imaginary trials, with $a_i - 1$ of outcome i

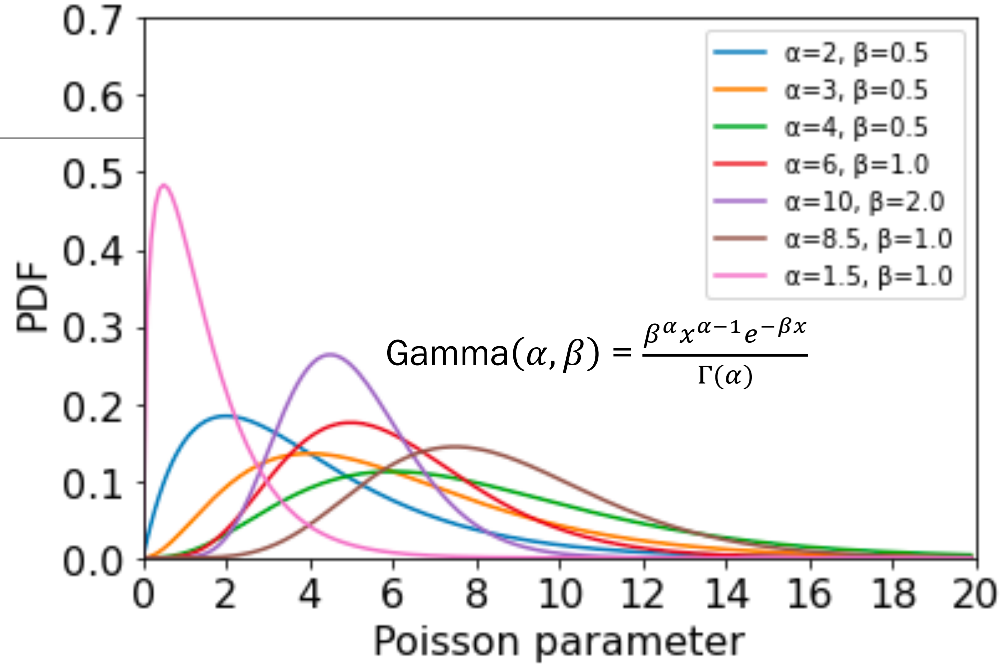
Experiment Observe $n_1 + n_2 + \cdots + n_m$ new trials, with n_i of outcome i

Posterior Dirichlet$(a_1 + n_1, a_2 + n_2, \ldots, a_m + n_m)$

MAP:

$$p_i = \frac{a_i + n_i - 1}{\left(\sum_{i=1}^{m} a_i\right) + \left(\sum_{i=1}^{m} n_i\right) - m}$$

Good times with Gamma



$\text{Gamma}(\alpha, \beta)$ is a conjugate for Poisson.

- Also conjugate for Exponential, but we won't delve into that
- Mode of gamma: $(\alpha - 1)/\beta$

Prior $\theta \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$

Saw $\alpha - 1$ total imaginary events during β prior time periods

Experiment Observe n events during next k time periods

Posterior $(\theta | n \text{ events in } k \text{ periods}) \sim \text{Gamma}(\alpha + n, \beta + k)$

MAP: $$\theta_{MAP} = \frac{a + n - 1}{\beta + k}$$

Future Chris: Write this on the board!!!

MAP for Poisson

Gamma(α, β) is conjugate for Poisson   Mode: $\frac{\alpha-1}{\beta}$

Let λ be the average # of successes in a time period.

1. What does it mean to have a prior of $\theta \sim \text{Gamma}(11,5)$?

Observe 10 imaginary events in 5 time periods,
i.e., observe at Poisson rate = 2

Now perform the experiment and see 11 events in next 2 time periods.

2. Given your prior, what is the posterior distribution?

3. What is θ_{MAP}?

MAP for Poisson

Gamma(α, β) is conjugate for Poisson

Mode: $\frac{\alpha-1}{\beta}$

Let λ be the average # of successes in a time period.

1. What does it mean to have a prior of $\theta \sim \text{Gamma}(11,5)$?

Observe 10 imaginary events in 5 time periods, i.e., observe at Poisson rate = 2

Now perform the experiment and see 11 events in next 2 time periods.

2. Given your prior, what is the posterior distribution?

$(\theta | n$ events in k periods$) \sim \text{Gamma}(22, 7)$

3. What is θ_{MAP}?

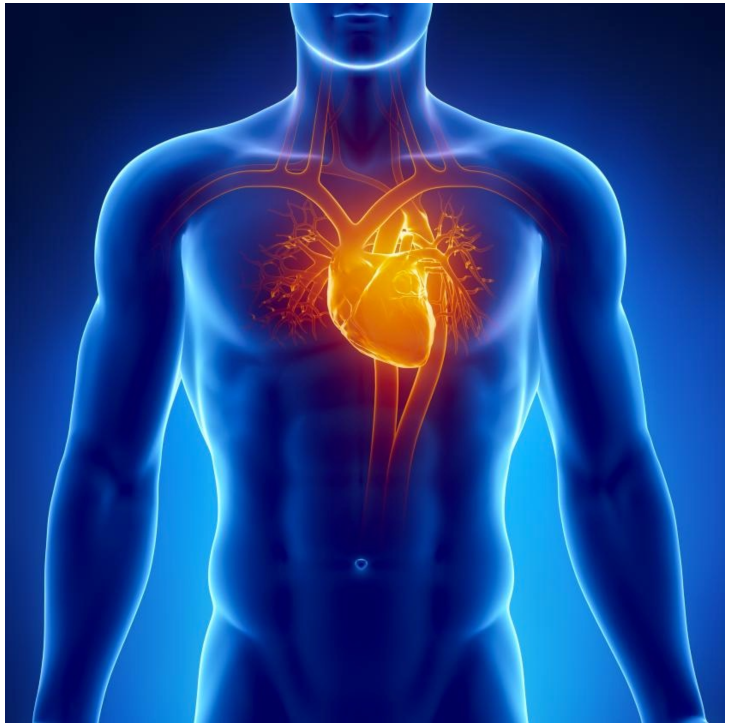
$\theta_{MAP} = 3$, the updated Poisson rate

Pedagogical Pause

The last estimator has risen…

Example Datasets

Heart



Ancestry

Netflix

Training Data

Training Data: assignments all random variables **X** and Y

Assume IID data:

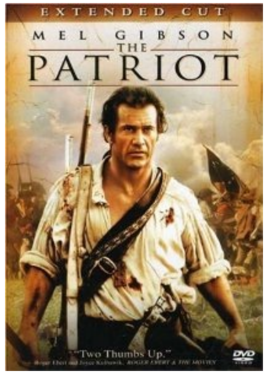
n training datapoints

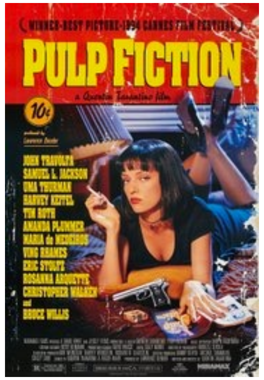
$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots (\mathbf{x}^{(n)}, y^{(n)})$$

$$m = |\mathbf{x}^{(i)}|$$

Each datapoint has m features and a single output

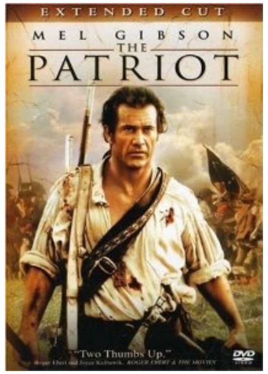
Target Movie "Like" Classification

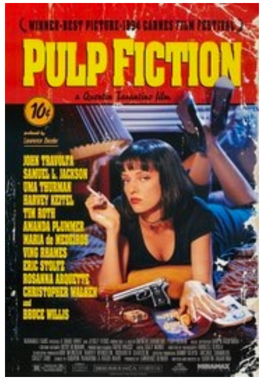




	Movie 1	Movie 2	…	Movie *m*	Output
User 1	1	0		1	1
User 2	1	1		0	0
			⋮		⋮
User *n*	0	0		1	1

Single Instance

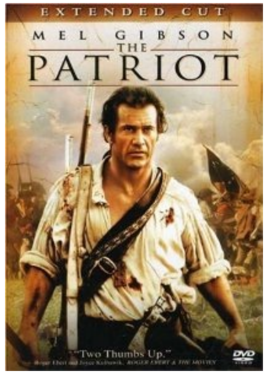
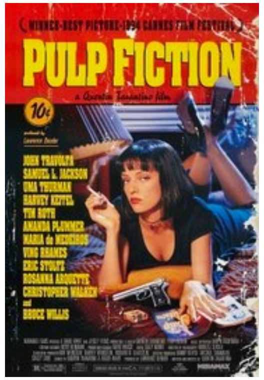




	Movie 1	Movie 2	…	Movie m	Output
User 1	1	0		1	1
User 2	1	1		0	0
⋮		⋮			⋮
User n	0	0		1	1

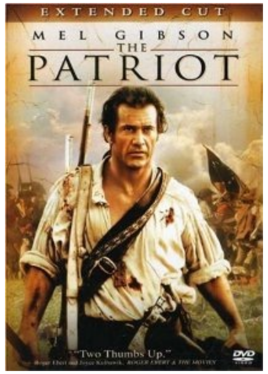
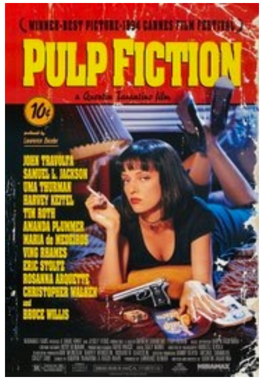
$$(\mathbf{x}^{(i)}, y^{(i)}) \text{ such that } 1 \leq i \leq n$$

Feature Vector

	Movie 1	Movie 2	...	Movie m	Output
			...		
User 1	1	0		1	1
User 2	1	1		0	0
⋮			⋮		⋮
User n	0	0		1	1

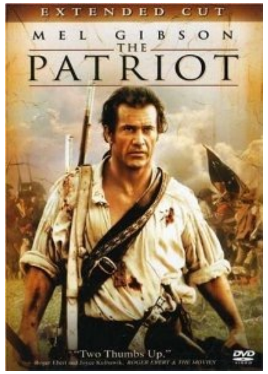
$$(\mathbf{x}^{(i)}, y^{(i)}) \text{ such that } 1 \leq i \leq n$$

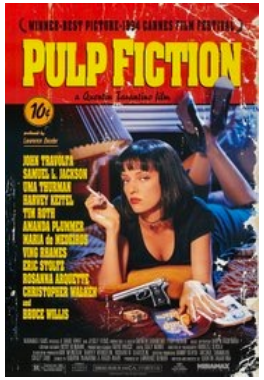
Output Value

	Movie 1	Movie 2		Movie m	Output
			…		
User 1	1	0		1	1
User 2	1	1		0	0
		⋮			⋮
User n	0	0		1	1

$$(\mathbf{x}^{(i)}, y^{(i)}) \text{ such that } 1 \leq i \leq n$$

Single Feature Value

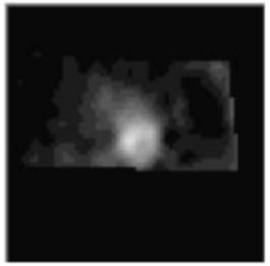
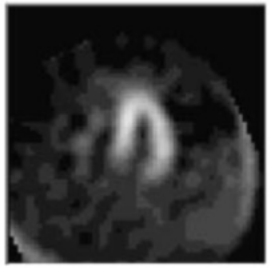




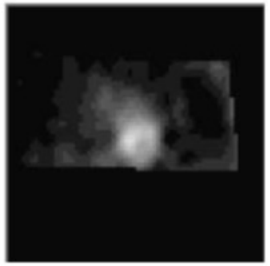
	Movie 1	Movie 2	…	Movie *m*	Output
User 1	1	0		1	1
User 2	1	1		**0**	0
⋮			⋮		⋮
User *n*	0	0		1	1

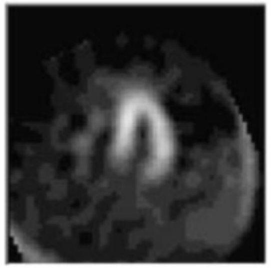
In general: $\mathbf{x}_j^{(i)}$ In this case: $\mathbf{x}_m^{(2)}$

Healthy Heart Classifier

	ROI 1	ROI 2		ROI *m*	Output
			…		
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart *n*	0	0		0	1

Healthy Heart Classifier

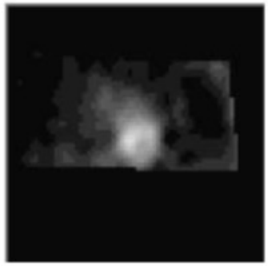
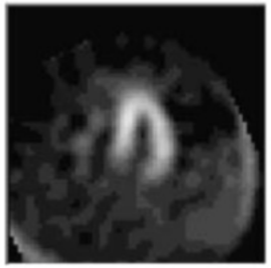




	ROI 1	ROI 2	...	ROI *m*	Output
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart *n*	0	0		0	1

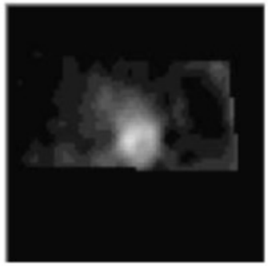
$$x_2^{(1)}$$

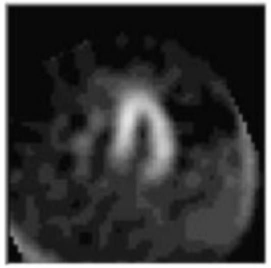
Healthy Heart Classifier

	ROI 1	ROI 2		ROI *m*	Output
			...		
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart *n*	0	0		0	1

$$(\mathbf{x}^{(2)}, y^{(2)})$$

Healthy Heart Classifier



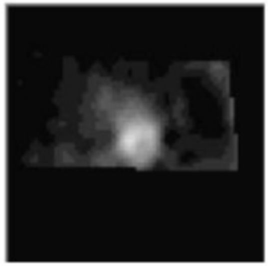
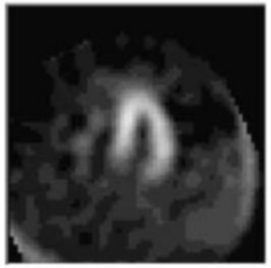


...

	ROI 1	ROI 2		ROI m	Output
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart n	0	0		0	1

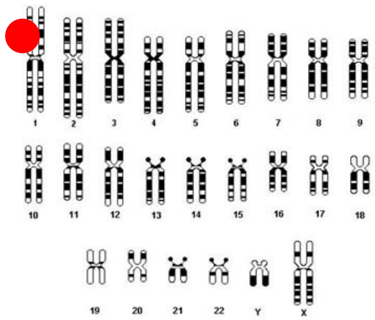
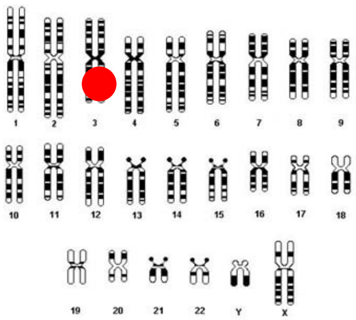
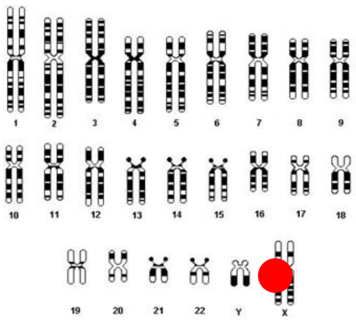
$\mathbf{x}^{(2)}$

Healthy Heart Classifier

	ROI 1	ROI 2		ROI *m*	Output
			…		
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart *n*	0	0		0	1

$$y^{(2)}$$

Ancestry Classifier

	SNP 1	SNP 2	SNP m	Output
		 …		
User 1	1	0	1	0
User 2	0	0	1	1
		⋮		⋮
User n	1	1	0	1

Regression: Predicting Real Numbers

…

	Opposing team ELO	Points in last game	…	At Home?	Output # Points
Game 1	84	105		1	120
Game 2	90	102		0	95
		⋮			⋮
Game *n*	74	120		0	115

Training Data

Training Data: assignments all random variables X and Y

Assume IID data:

n training datapoints

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \dots (\mathbf{x}^{(n)}, y^{(n)})$$

$$m = |\mathbf{x}^{(i)}|$$

Each datapoint has m features and a single output

ML is ubiquitous

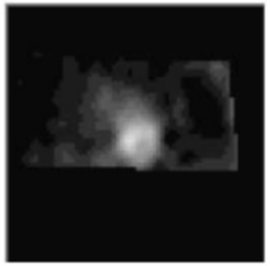
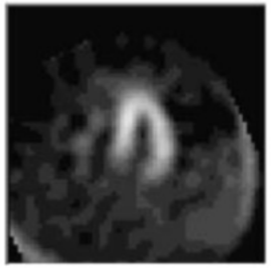
Classification

Classification is Building a Harry Potter Hat

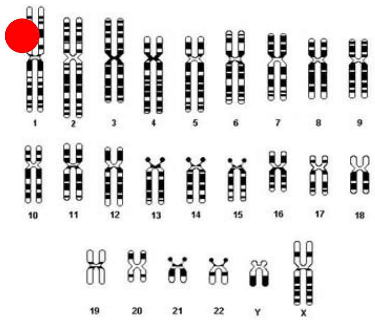
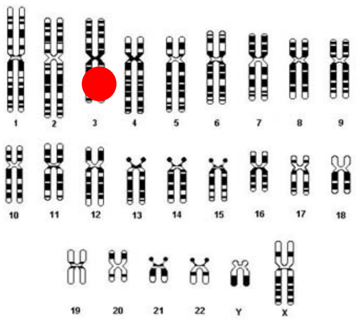
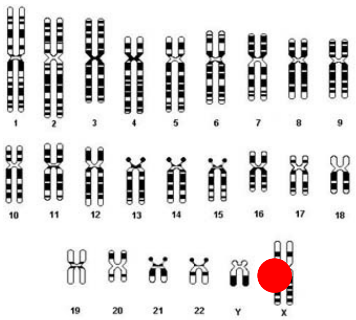
$$\hat{y} = 1$$

$$\mathbf{x} = [0, 1, \ldots, 1]$$

Healthy Heart Classifier

	ROI 1	ROI 2		ROI *m*	Output
			…		
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart *n*	0	0		0	1

Ancestry Classifier



…



	SNP 1	SNP 2	SNP *m*	Output
User 1	1	0	1	0
User 2	0	0	1	1
		⋮		⋮
User *n*	1	1	0	1

NETFLIX

And Learn

Target Movie “Like” Classification

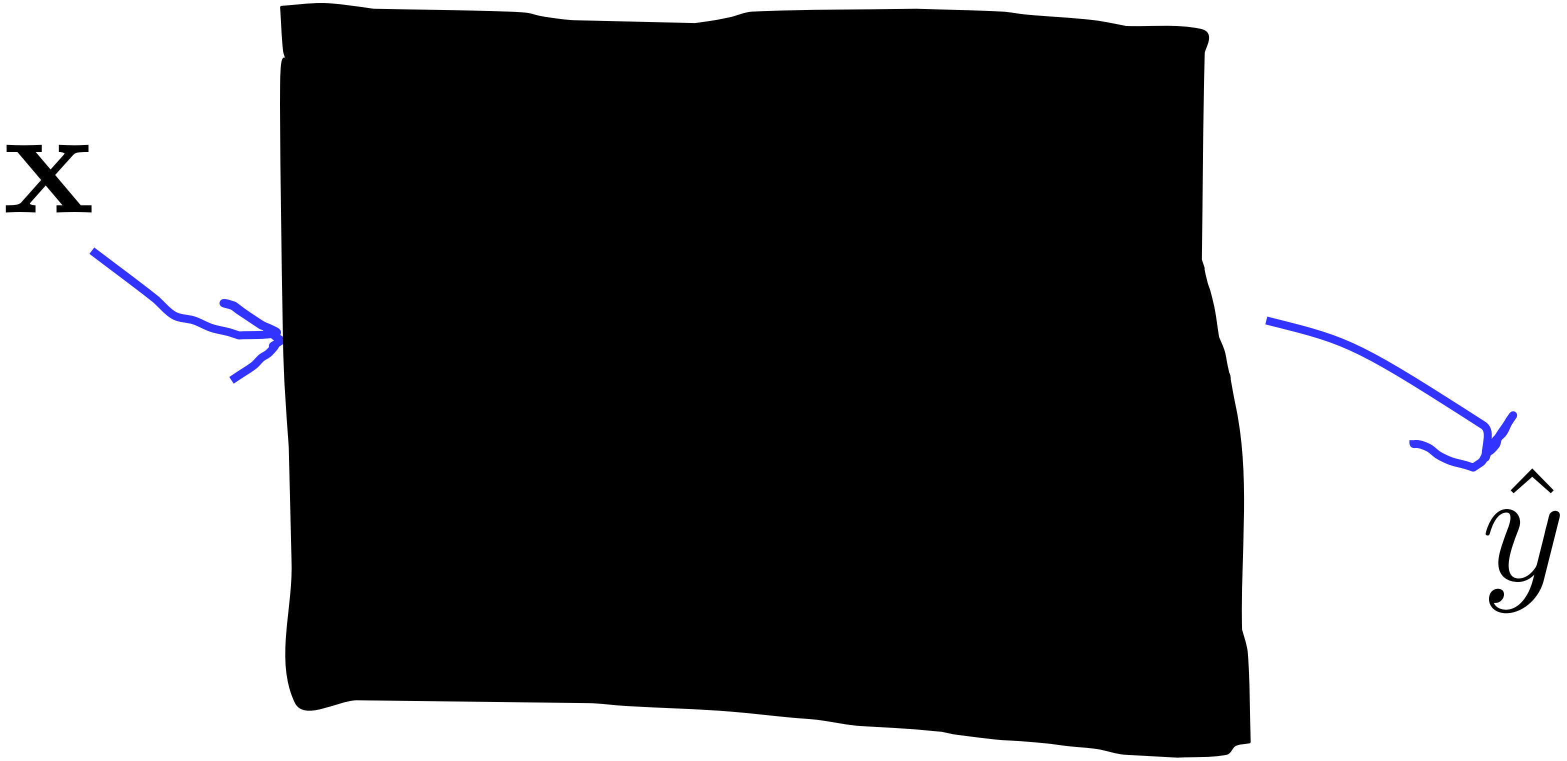
	Feature 1	Output
		
User 1	1	1
User 2	1	0
		⋮
User *n*	0	1

$x_j^{(i)} \in \{0, 1\}$ $y^{(i)} \in \{0, 1\}$

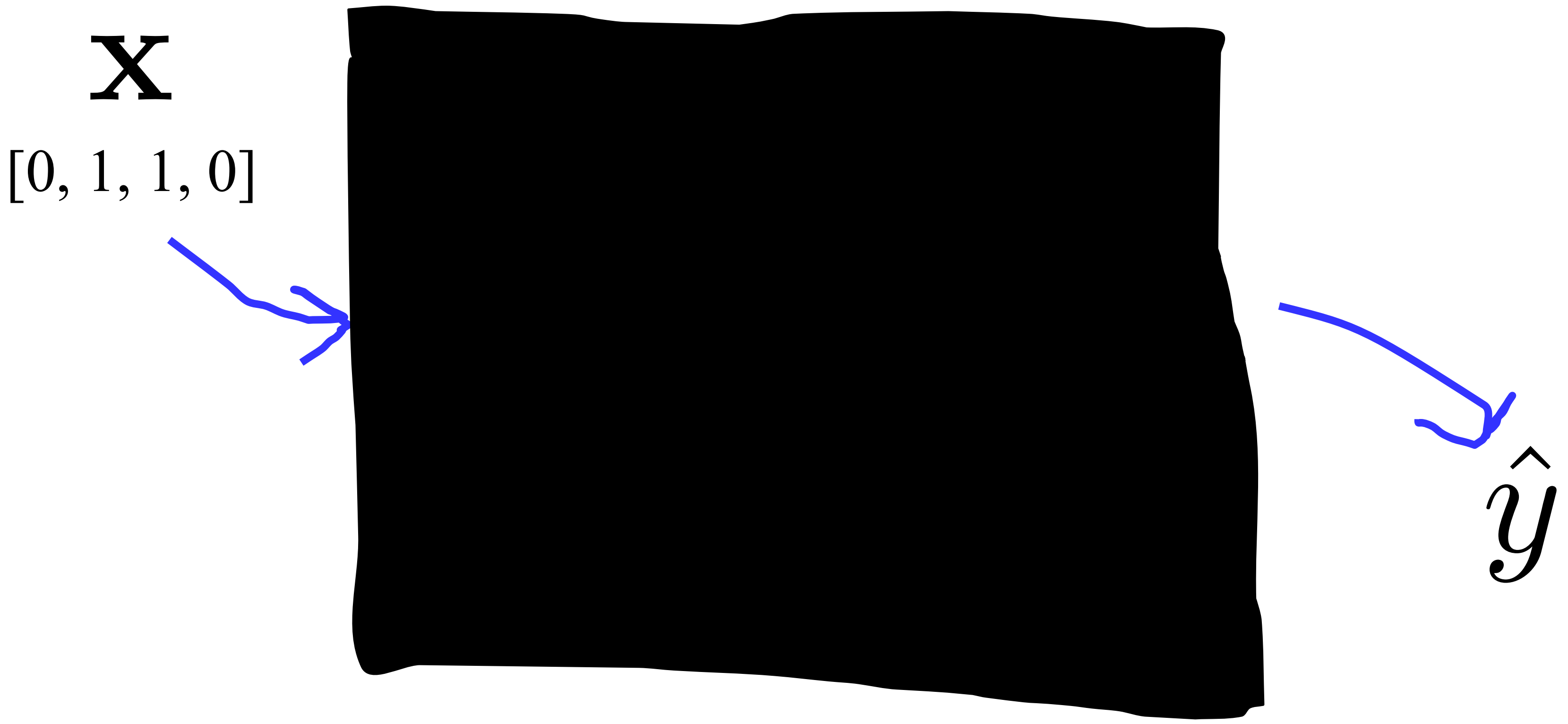
How could we predict the class label:
will the user like life is beautiful?

Fake Algorithm:
Brute Bayes Classifier

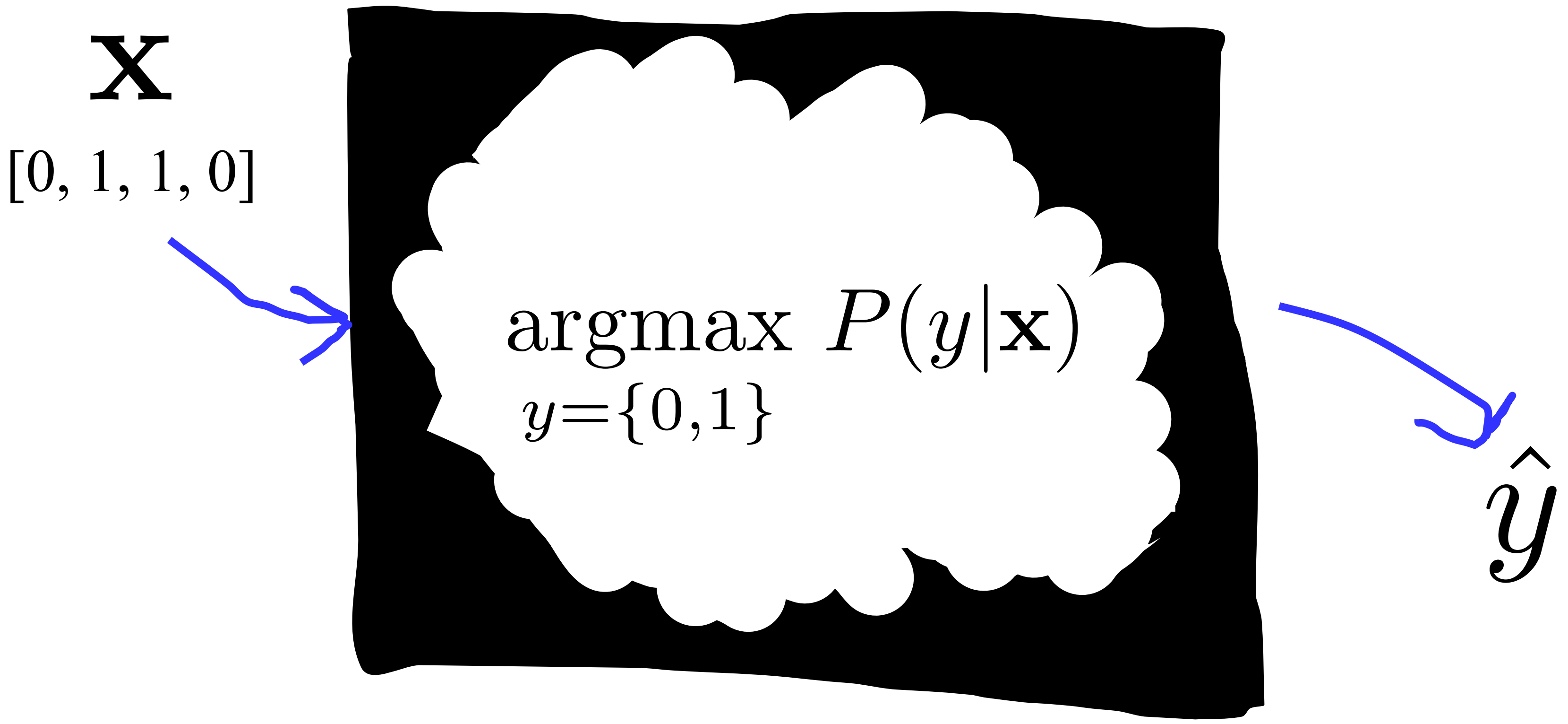
Brute Force Bayes



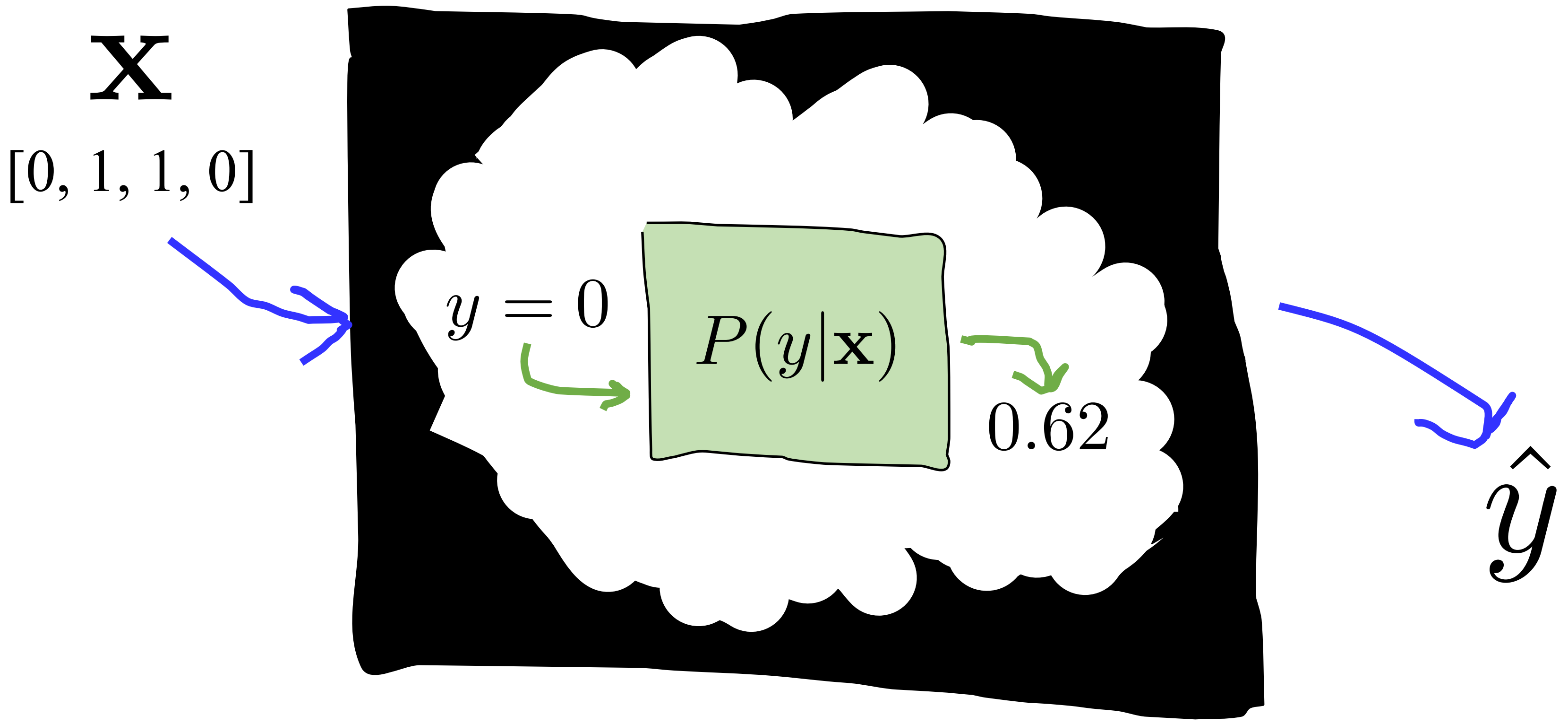
Brute Force Bayes



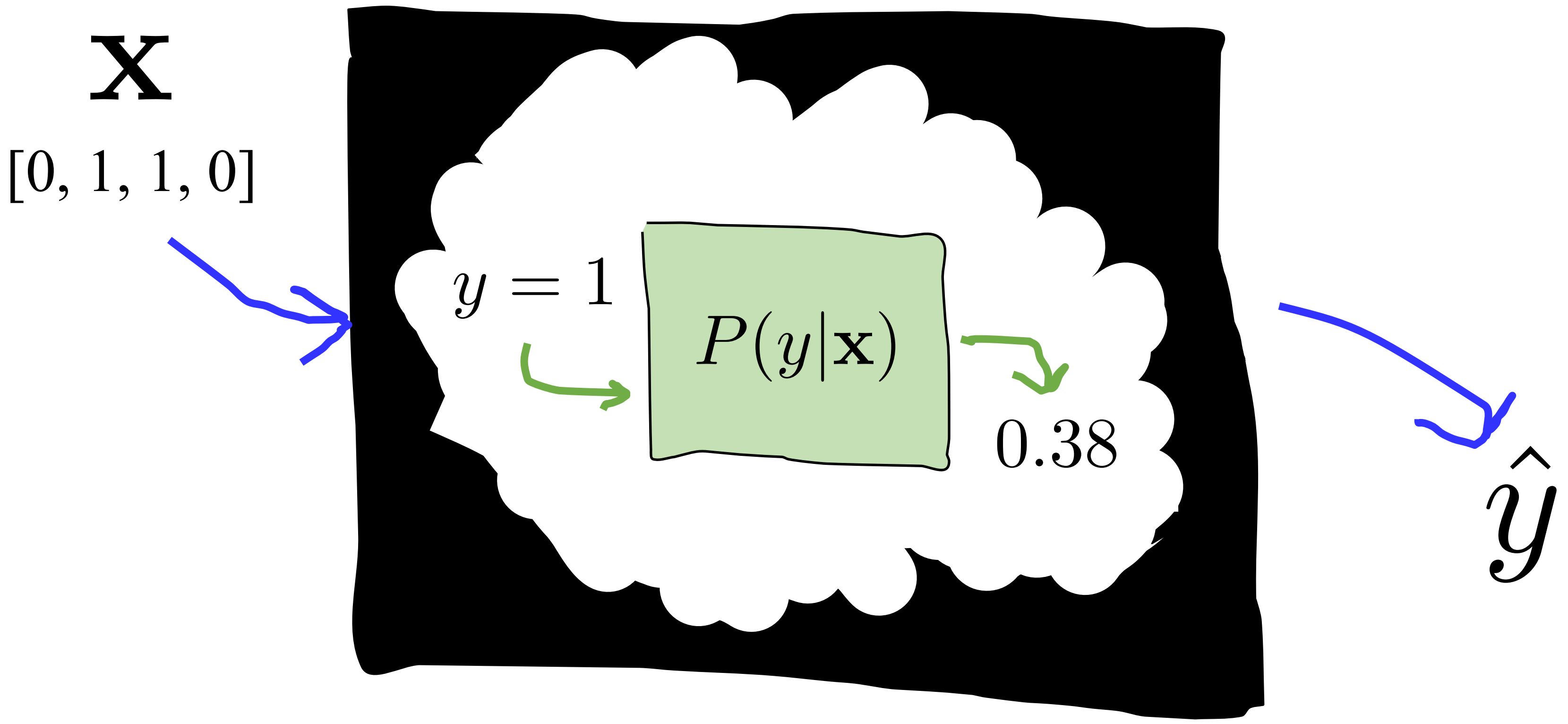
Brute Force Bayes



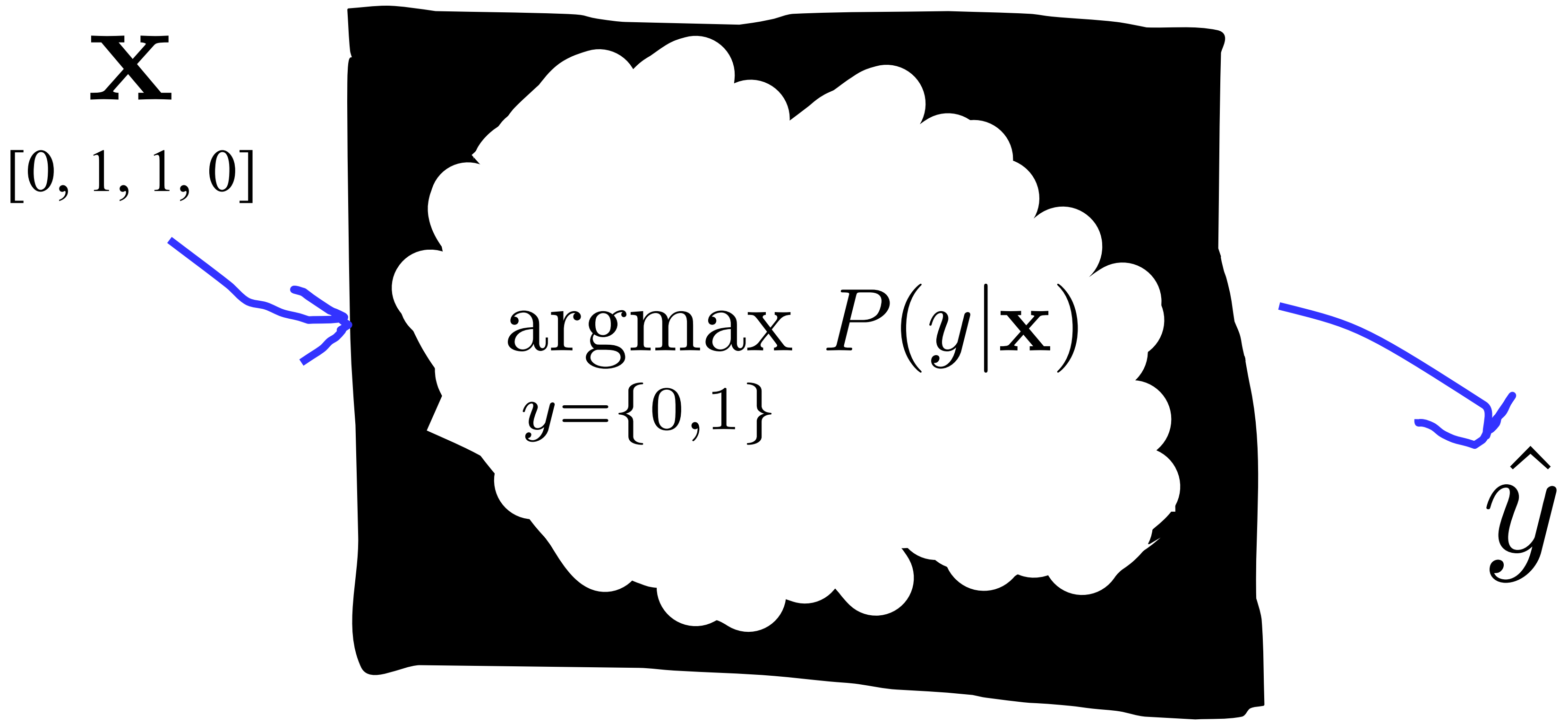
Brute Force Bayes



Brute Force Bayes



Brute Force Bayes



Brute Force Bayes

Prediction: will they like L.I.B.?

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(y|\mathbf{x})$$

Whether or not they liked Independence day

If y = 1, they like L.I.B.?

Simply chose the class label that is the most likely given the data

This is for one user

Brute Force Bayes

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(y|\mathbf{x})$$

Simply chose the class label that is the most likely given the data

This is for one user

Brute Force Bayes

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(y|\mathbf{x})$$

$$= \underset{y=\{0,1\}}{\text{argmax}} \; \frac{P(\mathbf{x}|y)P(y)}{P(\mathbf{x})}$$

$$= \underset{y=\{0,1\}}{\text{argmax}} \; P(\mathbf{x}|y)P(y)$$

Simply chose the class label that is the most likely given the data

This is for one user

* Note how similar this is to Hamilton example ☺

What are the Parameters?

Brute Force Bayes

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(\mathbf{x}|y)P(y)$$

Conditional probability table

Y = 0

$X_1 = 0$	θ_0
$X_1 = 1$	θ_1

Y = 1

$X_1 = 0$	θ_2
$X_1 = 1$	θ_3

Y = 0	θ_4
Y = 1	θ_5

Learn these during training

Brute Force Bayes

$$\hat{y} = \underset{y=\{0,1\}}{\operatorname{argmax}} \; P(\mathbf{x}|y)P(y)$$

Conditional probability table

X_1 \ Y	0	1
0	θ_0	θ_2
1	θ_1	θ_3

Y = 0	θ_4
Y = 1	θ_5

Learn these during training

Training

	x_1	y
		
User 1	1	1
User 2	0	0
		⋮
User *n*	0	1

$P(\mathbf{x}|y)$

X_1 \ Y	0	1
0	θ_0	θ_2
1	θ_1	θ_3

What is $P(X_1 | Y = 0)$?
What is $P(X_1 | Y = 1)$?
Both multinomials with two outcomes

MLE Estimate

	x_1	y
		
User 1	1	1
User 2	0	0
		⋮
User *n*	0	1

$P(\mathbf{x}|y)$

X_1 \ Y	0	1
0	0.0	0.4
1	1.0	0.6

MLE: Just count

MAP Estimate

	x_1	y
		
User 1	1	1
User 2	0	0
		⋮
User *n*	0	1

$P(\mathbf{x}|y)$

X_1 \ Y	0	1
0	0.01	0.42
1	0.99	0.58

MAP: Just count and add imaginary trials

Testing

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(\mathbf{x}|y)P(y)$$

X_1 \ Y	0	1
0	0.01	0.42
1	0.99	0.58

Y = 0	0.21
Y = 1	0.79

Test user: Likes independence day

$$P(x_1 = 1|y = 0)P(y = 0)$$

vs

$$P(x_1 = 1|y = 1)P(y = 1)$$

Testing

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(\mathbf{x}|y)P(y)$$

X_1 \ Y	0	1
0	0.01	0.42
1	0.99	0.58

Y = 0	0.21
Y = 1	0.79

Test user: Likes independence day

$$P(x_1 = 1|y = 0)P(y = 0)$$

0.208

vs

$$P(x_1 = 1|y = 1)P(y = 1)$$

Testing

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(\mathbf{x}|y)P(y)$$

X_1 \ Y	0	1
0	0.01	0.42
1	0.99	0.58

Y = 0	0.21
Y = 1	0.79

Test user: Likes independence day

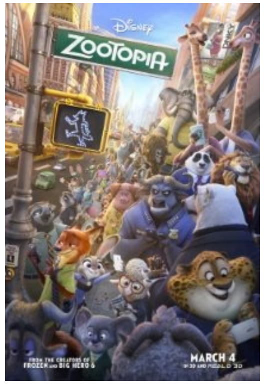
$$P(x_1 = 1|y = 0)P(y = 0)$$ 0.208

vs

$$P(x_1 = 1|y = 1)P(y = 1)$$ 0.458

That was pretty good!

Brute Force Bayes m = 2



	x_1	x_2	y
User 1	1	0	1
User 2	1	0	0
			⋮
User *n*	0	1	1

Brute Force Bayes m = 2

Simply chose the class label that is the most likely given the data

$$\hat{y} = \underset{y=\{0,1\}}{\operatorname{argmax}} \; P(y|\mathbf{x})$$

$$= \underset{y=\{0,1\}}{\operatorname{argmax}} \; \frac{P(\mathbf{x}|y)P(y)}{P(\mathbf{x})}$$

$$= \underset{y=\{0,1\}}{\operatorname{argmax}} \; P(\mathbf{x}|y)P(y)$$

$$P(x_1, x_2|y)$$

Brute Force Bayes

$$\hat{y} = \underset{y=\{0,1\}}{\operatorname{argmax}} \; P(\mathbf{x}|y) P(y)$$

Y = 0

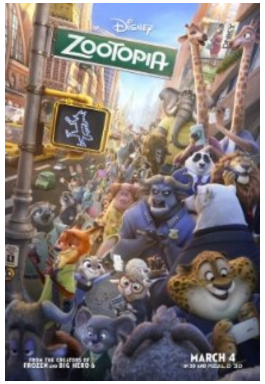
X_2 \ X_1	0	1
0	θ_0	θ_1
1	θ_2	θ_3

Y = 1

X_2 \ X_1	0	1
0	θ_4	θ_5
1	θ_6	θ_7

x_1

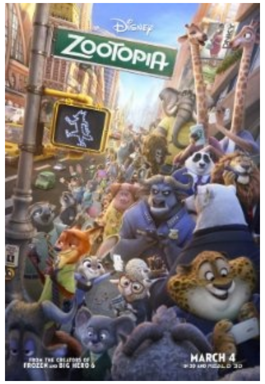
x_2

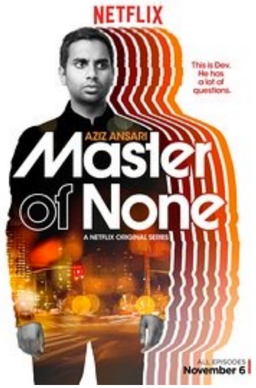


y

Fine

Brute Force Bayes m = 3





	x_1	x_2	x_3	y
User 1	1	0	1	1
User 2	1	0	1	0
				⋮
User *n*	0	1	1	1

Brute Force Bayes m = 3

Simply chose the class label that is the most likely given the data

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(y|\mathbf{x})$$

$$= \underset{y=\{0,1\}}{\text{argmax}} \; \frac{P(\mathbf{x}|y)P(y)}{P(\mathbf{x})}$$

$$= \underset{y=\{0,1\}}{\text{argmax}} \; P(\mathbf{x}|y)P(y)$$

$$P(x_1, x_2, x_3|y)$$

Brute Force Bayes

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}}\ \boxed{P(\mathbf{x}|y)}\,P(y)$$

$X_3 = 0$

Y = 0

X_2 \ X_1	0	1
0	θ_0	θ_1
1	θ_2	θ_3

Y = 1

X_2 \ X_1	0	1
0	θ_8	θ_9
1	θ_{10}	θ_{11}

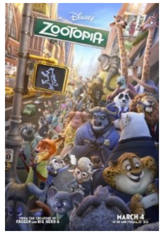
$X_3 = 1$

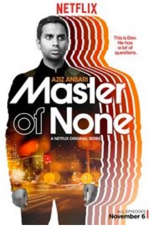
$X_2 = 0$

X_2 \ X_1	0	1
0	θ_4	θ_5
1	θ_6	θ_7

$X_2 = 1$

X_2 \ X_1	0	1
0	θ_{12}	θ_{13}
1	θ_{14}	θ_{15}





And if $m=100$?

Brute Force Bayes m = 100

Simply chose the class label that is the most likely given the data

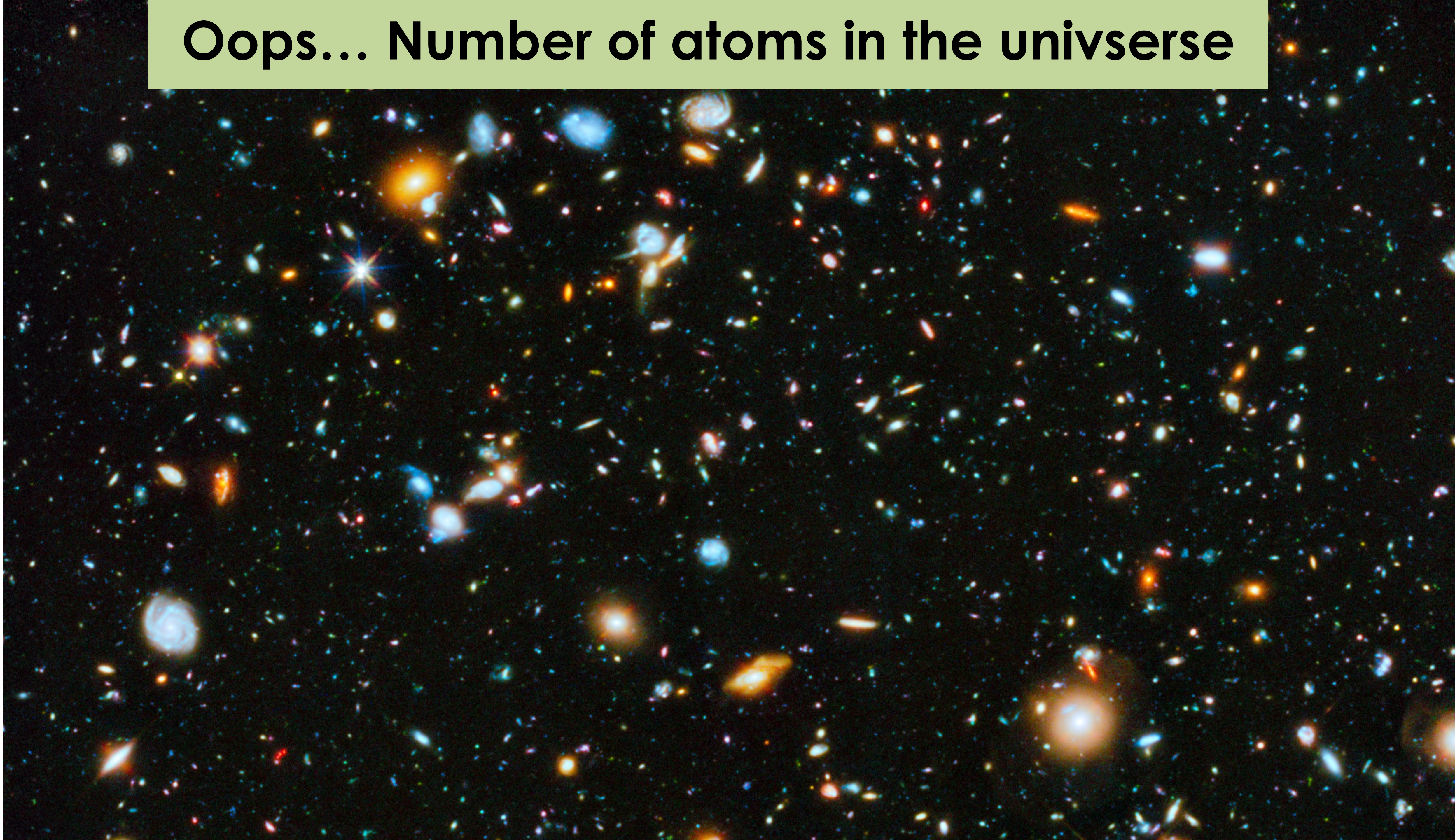
$$\hat{y} = \underset{y=\{0,1\}}{\operatorname{argmax}}\ P(y|\mathbf{x})$$

$$= \underset{y=\{0,1\}}{\operatorname{argmax}}\ \frac{P(\mathbf{x}|y)P(y)}{P(\mathbf{x})}$$

$$= \underset{y=\{0,1\}}{\operatorname{argmax}}\ P(\mathbf{x}|y)P(y)$$

$$P(x_1, x_2, x_3, \ldots, x_{100}|y)$$

Oops… Number of atoms in the univserse



What is the big O for # parameters?
m = # features.

Big O of Brute Force Joint

What is the big O for # parameters?
m = # features.

$$O(2^m)$$

Assuming each feature is binary…

Not going to cut it!

What is the problem here?

$$\hat{y} = \underset{y=\{0,1\}}{\text{argmax}} \; P(y|\mathbf{x})$$

$$= \underset{y=\{0,1\}}{\text{argmax}} \; \frac{P(\mathbf{x}|y)P(y)}{P(\mathbf{x})}$$

$$= \underset{y=\{0,1\}}{\text{argmax}} \; P(\mathbf{x}|y)P(y)$$

$$P(\mathbf{x}|y) = P(x_1, x_2, \ldots, x_m|y)$$

Naïve Bayes Assumption

$$\hat{y} = \underset{y=\{0,1\}}{\operatorname{argmax}}\ P(y|\mathbf{x})$$

$$= \underset{y=\{0,1\}}{\operatorname{argmax}}\ \frac{P(\mathbf{x}|y)P(y)}{P(\mathbf{x})}$$

$$= \underset{y=\{0,1\}}{\operatorname{argmax}}\ P(\mathbf{x}|y)P(y)$$

$$P(\mathbf{x}|y) = P(x_1, x_2, \ldots, x_m|y)$$

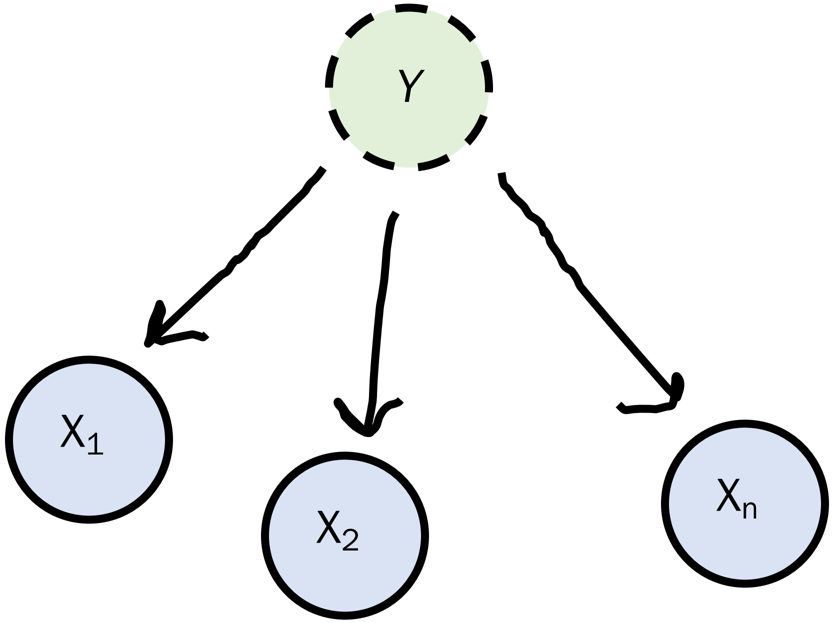
$$= \prod_i P(x_i|y)$$

The Naïve Bayes assumption

Naïve Bayes Assumption:

$$P(\mathbf{x}|y) = \prod_i P(x_i|y)$$

Naïve Bayes as a Model



Class Label

$$Y \sim \text{Bern}$$

Data distribution

$$X_i | Y \sim \text{Bern}$$

$$P(\mathbf{x}, y) = P(y) \prod_i P(x_i | y)$$

Naïve Bayes Classifier

Naïve Bayes

Our prediction for y

Is a function of x

That choses the best value of y given x

$$\hat{y} = g(\mathbf{x}) = \underset{y \in \{0,1\}}{\operatorname{argmax}} \ \hat{P}(y|\mathbf{x})$$

$$= \underset{y \in \{0,1\}}{\operatorname{argmax}} \ \hat{P}(\mathbf{x}|y)\hat{P}(y)$$

Bayes rule!

$$= \underset{y}{\operatorname{argmax}} \ \left(\prod_{i=1}^{n} \hat{P}(x_i|y)\right)\hat{P}(y)$$

Naïve Bayes Assumption

$$= \underset{y}{\operatorname{argmax}} \ \log \hat{P}(y) + \sum_{i=1}^{m} \log \hat{P}(x_i|y)$$

This log version is useful for numerical stability

Computing Probabilities from Data

Various probabilities you will need to compute for Naive Bayesian Classifier (using MLE here):

$$\hat{p}(X_i = 1 | Y = 0) = \frac{(\#\text{ training examples where } X_i = 1 \text{ and } Y = 0)}{(\#\text{ training examples where } Y = 0)}$$

$$\hat{p}(Y = 1) = \frac{(\#\text{ training examples where } Y = 1)}{(\#\text{ training examples})}$$

Computing Probabilities from Data With Laplace

Various probabilities you will need to compute for Naive Bayesian Classifier (using MAP with Laplace):

$$\hat{p}(X_i = 1 | Y = 0) = \frac{(\#\text{ training examples where } X_i = 1 \text{ and } Y = 0) + 1}{(\#\text{ training examples where } Y = 0) + 2}$$

$$\hat{p}(Y = 1) = \frac{(\#\text{ training examples where } Y = 1) + 1}{(\#\text{ training examples}) + 2}$$

Naïve Bayes Example

Predict Y based on observing variables X_1 and X_2

- X_1 and X_2 are both indicator variables
  - X_1 denotes "likes Star Wars", X_2 denotes "likes Harry Potter"
- Y is indicator variable: "likes Lord of the Rings"
  - Use training data to estimate params: $\hat{P}(x_i|y)$ $\hat{P}(y)$

Y \ X_1	0	1	MLE estimates	
0	3	10	0.23	0.77
1	4	13	0.24	0.76

Y \ X_2	0	1	MLE estimates	
0	5	8	0.38	0.62
1	7	10	0.41	0.59

Y	#	MLE est.
0	13	0.43
1	17	0.57

- Say someone likes **Star Wars (X_1 = 1),** but not **Harry Potter (X_2 = 0)**
- Will they like "Lord of the Rings"? Need to predict Y:

$$\hat{y} = \underset{y \in \{0,1\}}{\text{argmax}}\ \hat{P}(\mathbf{x}|y)\hat{P}(y) = \underset{y \in \{0,1\}}{\text{argmax}}\ \hat{P}(x_1|y)\hat{P}(x_2|y)\hat{P}(y)$$

Naïve Bayes Example

Predict Y based on observing variables X_1 and X_2

- X_1 and X_2 are both indicator variables
  - X_1 denotes "likes Star Wars", X_2 denotes "likes Harry Potter"
- Y is indicator variable: "likes Lord of the Rings"
  - Use training data to estimate params: $\hat{P}(x_i|y)$ $\hat{P}(y)$

Y \ X_1	0	1	MLE estimates	
0	3	10	0.23	0.77
1	4	13	0.24	0.76

Y \ X_2	0	1	MLE estimates	
0	5	8	0.38	0.62
1	7	10	0.41	0.59

Y	#	MLE est.
0	13	0.43
1	17	0.57

- Say someone likes **Star Wars (X_1 = 1),** but not **Harry Potter (X_2 = 0)**
- Will they like "Lord of the Rings"? Need to predict Y:

$$\hat{y} = \underset{y \in \{0,1\}}{\operatorname{argmax}} \ \hat{P}(X_1 = x_1|Y = y)\hat{P}(X_2 = x_2|Y = y)\hat{P}(Y = y)$$

Naïve Bayes Example

Predict Y based on observing variables X_1 and X_2

- X_1 and X_2 are both indicator variables
  - X_1 denotes "likes Star Wars", X_2 denotes "likes Harry Potter"
- Y is indicator variable: "likes Lord of the Rings"
  - Use training data to estimate params: $\hat{P}(x_i|y)$ $\hat{P}(y)$

Y \ X_1	0	1	MLE estimates	
0	3	10	0.23	0.77
1	4	13	0.24	0.76

Y \ X_2	0	1	MLE estimates	
0	5	8	0.38	0.62
1	7	10	0.41	0.59

Y	#	MLE est.
0	13	0.43
1	17	0.57

- Say someone likes **Star Wars ($X_1 = 1$)**, but not **Harry Potter ($X_2 = 0$)**
- Will they like "Lord of the Rings"? Need to predict Y:

$$\hat{y} = \underset{y \in \{0,1\}}{\text{argmax}}\ \hat{P}(X_1 = 1|Y = y)\hat{P}(X_2 = 0|Y = y)\hat{P}(Y = y)$$

One Slide to Rule them All

Y \ X_1	0	1	MLE estimates	
0	3	10	0.23	0.77
1	4	13	0.24	0.76

Y \ X_2	0	1	MLE estimates	
0	5	8	0.38	0.62
1	7	10	0.41	0.59

Y	#	MLE est.
0	13	0.43
1	17	0.57

$$\hat{y} = \underset{y \in \{0,1\}}{\text{argmax}}\ \hat{P}(X_1 = 1|Y = y)\hat{P}(X_2 = 0|Y = y)\hat{P}(Y = y)$$

- Let Y = 0

$$\hat{P}(X_1 = 1|Y = 0)\hat{P}(X_2 = 0|Y = 0)\hat{P}(Y = 0)$$
$$= (0.77)(0.38)(0.43) = 0.126$$

- Let Y = 1

$$\hat{P}(X_1 = 1|Y = 1)\hat{P}(X_2 = 0|Y = 1)\hat{P}(Y = 1)$$
$$= (0.76)(0.41)(0.57) = 0.178$$

Since term is greatest when Y = 1, we predict Ŷ = 1

$$P(Y = 1) = K \cdot 0.178 \qquad P(Y = 0) = K \cdot 0.126 \qquad K = \frac{1}{0.126 + 0.178}$$

MAP Naïve Bayes

Predict Y based on observing variables X_1 and X_2

- X_1 and X_2 are both indicator variables
  - X_1 denotes "likes Star Wars", X_2 denotes "likes Harry Potter"
- Y is indicator variable: "likes Lord of the Rings"
  - Use training data to estimate PMFs: $\hat{P}(x_i|y)$ $\hat{P}(y)$

Y \ X_1	0	1	MAP estimates
0	3	10	
1	4	13	

Y \ X_2	0	1	MAP estimates
0	5	8	
1	7	10	

Y	#	MAP est.
0	13	
1	17	

What prior?

MAP Naïve Bayes

Predict Y based on observing variables X_1 and X_2

- X_1 and X_2 are both indicator variables
  - X_1 denotes "likes Star Wars", X_2 denotes "likes Harry Potter"
- Y is indicator variable: "likes Lord of the Rings"
  - Use training data to estimate PMFs: $\hat{P}(x_i|y)$ $\hat{P}(y)$

Y \ X_1	0	1	MAP estimates	
0	3	10	0.27	0.73
1	4	13		

Y \ X_2	0	1	MAP estimates
0	5	8	
1	7	10	

Y	#	MAP est.
0	13	
1	17	

Laplace! $p_i = \frac{n_i + 1}{n + m}$ $p_i = \frac{n_i + 1}{n + 2}$

MAP Naïve Bayes

Predict Y based on observing variables X_1 and X_2

- X_1 and X_2 are both indicator variables
  - X_1 denotes "likes Star Wars", X_2 denotes "likes Harry Potter"
- Y is indicator variable: "likes Lord of the Rings"
  - Use training data to estimate PMFs: $\hat{P}(x_i|y)$ $\hat{P}(y)$

Y \ X_1	0	1	MAP estimates	
0	3	10	0.27	0.73
1	4	13	0.26	0.74

Y \ X_2	0	1	MAP estimates	
0	5	8	0.4	0.6
1	7	10	0.42	0.58

Y	#	MAP est.
0	13	0.45
1	17	0.55

Laplace!

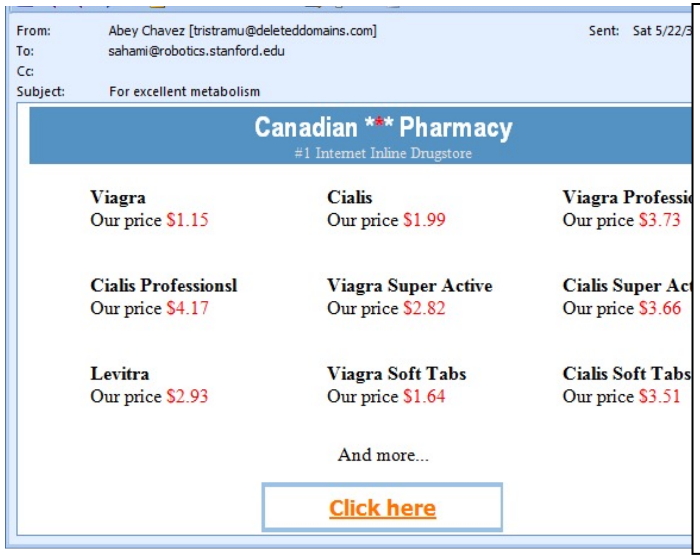
$$p_i = \frac{n_i + 1}{n + m} \qquad p_i = \frac{n_i + 1}{n + 2}$$

Training Naïve Bayes, is estimating parameters for a multinomial (or bernoulli).

Thus training is just counting.

What is Bayes Doing in my Mail Server

This is spam:



Let's get Bayesian on your spam:

```
Content analysis details:   (49.5 hits, 7.0 required)
 0.9 RCVD_IN_PBL            RBL: Received via a relay in Spamhaus PBL
                            [93.40.189.29 listed in zen.spamhaus.org]
 1.5 URIBL_WS_SURBL         Contains an URL listed in the WS SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_JP_SURBL         Contains an URL listed in the JP SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_OB_SURBL         Contains an URL listed in the OB SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_SC_SURBL         Contains an URL listed in the SC SURBL blocklist
                            [URIs: recragas.cn]
 2.0 URIBL_BLACK            Contains an URL listed in the URIBL blacklist
                            [URIs: recragas.cn]
 8.0 BAYES_99               BODY: Bayesian spam probability is 99 to 100%
                            [score: 1.0000]
```

A Bayesian Approach to Filtering Junk E-Mail

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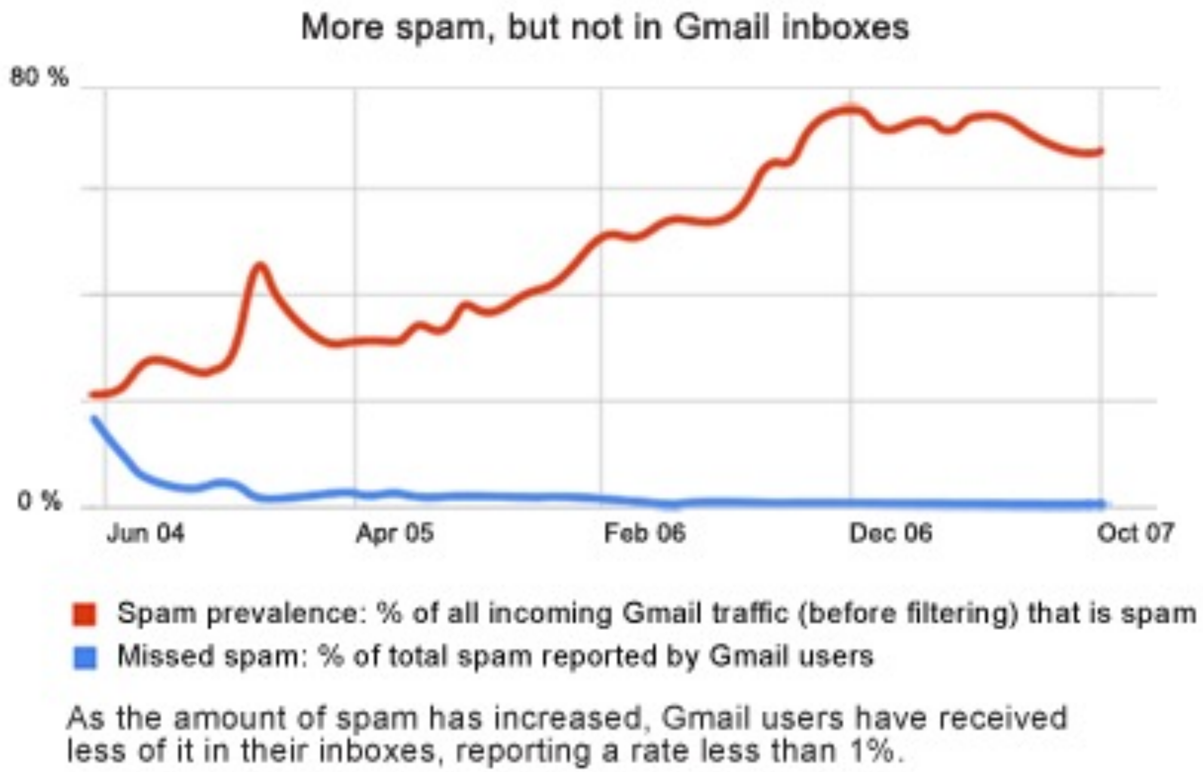
Abstract

In addressing the growing problem of junk E-mail on the Internet, we examine methods for the automated

contain offensive material (such as graphic pornography), there is often a higher cost to users of actually viewing this mail than simply the time to sort out the

Spam, Spam… Go Away!

The constant battle with spam



"And machine-learning algorithms developed to merge and rank large sets of Google search results allow us to combine hundreds of factors to classify spam."

Source: `http://www.google.com/mail/help/fightspam/spamexplained.html`

Email Classification

Want to predict if an email is spam or not

- Start with the input data
  - Consider a lexicon of *m* words (Note: in English $m \approx 100{,}000$)
  - Define *m* indicator variables $\mathbf{X} = \langle X_1, X_2, \ldots, X_m \rangle$
  - Each variable X_i denotes if word *i* appeared in a document or not
  - Note: *m* is huge, so make "Naive Bayes" assumption
- Define output classes Y to be: {spam, non-spam}
- Given training set of *N* previous emails
  - For each email message, we have a training instance: $\mathbf{X} = \langle X_1, X_2, \ldots, X_m \rangle$ noting for each word, if it appeared in email
  - Each email message is also marked as spam or not (value of Y)

Training the Classifier

Given N training pairs:

$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots, (\mathbf{x}^{(n)}, y^{(n)})$

Learning

- Estimate probabilities $P(y)$ and $P(x_i \mid y)$ for all i
  - Many words are likely to not appear at all in given set of email
- Laplace estimate: $\hat{p}(X_i = 1 \mid Y = spam)_{Laplace} = \dfrac{(\#\text{spam emails with word } i) + 1}{\text{total } \#\text{ spam emails} + 2}$

Classification

- For a new email, generate $\mathbf{X} = \langle X_1, X_2, \ldots, X_m \rangle$
- Classify as spam or not using: $\hat{y} = \underset{y \in \{0,1\}}{\text{argmax}}\ \hat{P}(\mathbf{x}|y)\hat{P}(y)$
- Employ Naive Bayes assumption: $P(\mathbf{x}|y) = \prod_i P(x_i|y)$

Training Naïve Bayes, is estimating parameters for Bernoullis.

Thus it is just counting.

How Does This Do?

After training, can test with another set of data

- "Testing" set also has known values for Y, so we can see how often we were right/wrong in predictions for Y
- Spam data
  - Email data set: 1789 emails (1578 spam, 211 non-spam)
  - First, 1538 email messages (by time) used for training
  - Next 251 messages used to test learned classifier
- Criteria:
  - Precision = # *correctly* predicted class Y/ # predicted class Y
  - Recall = # *correctly* predicted class Y / # real class Y messages

	Spam		Non-spam	
	Precision	**Recall**	**Precision**	**Recall**
Words only	**97.1%**	**94.3%**	**87.7%**	**93.4%**
Words + add'l features	**100%**	**98.3%**	**96.2%**	**100%**

Naïve Bayes Classification

Computing Probabilities from Data

Various probabilities you will need to compute for Naive Bayesian Classifier (using MLE here):

$$\hat{p}(X_i = 1 | Y = 0) = \frac{(\#\text{ training examples where } X_i = 1 \text{ and } Y = 0)}{(\#\text{ training examples where } Y = 0)}$$

$$\hat{p}(Y = 1) = \frac{(\#\text{ training examples where } Y = 1)}{(\#\text{ training examples})}$$