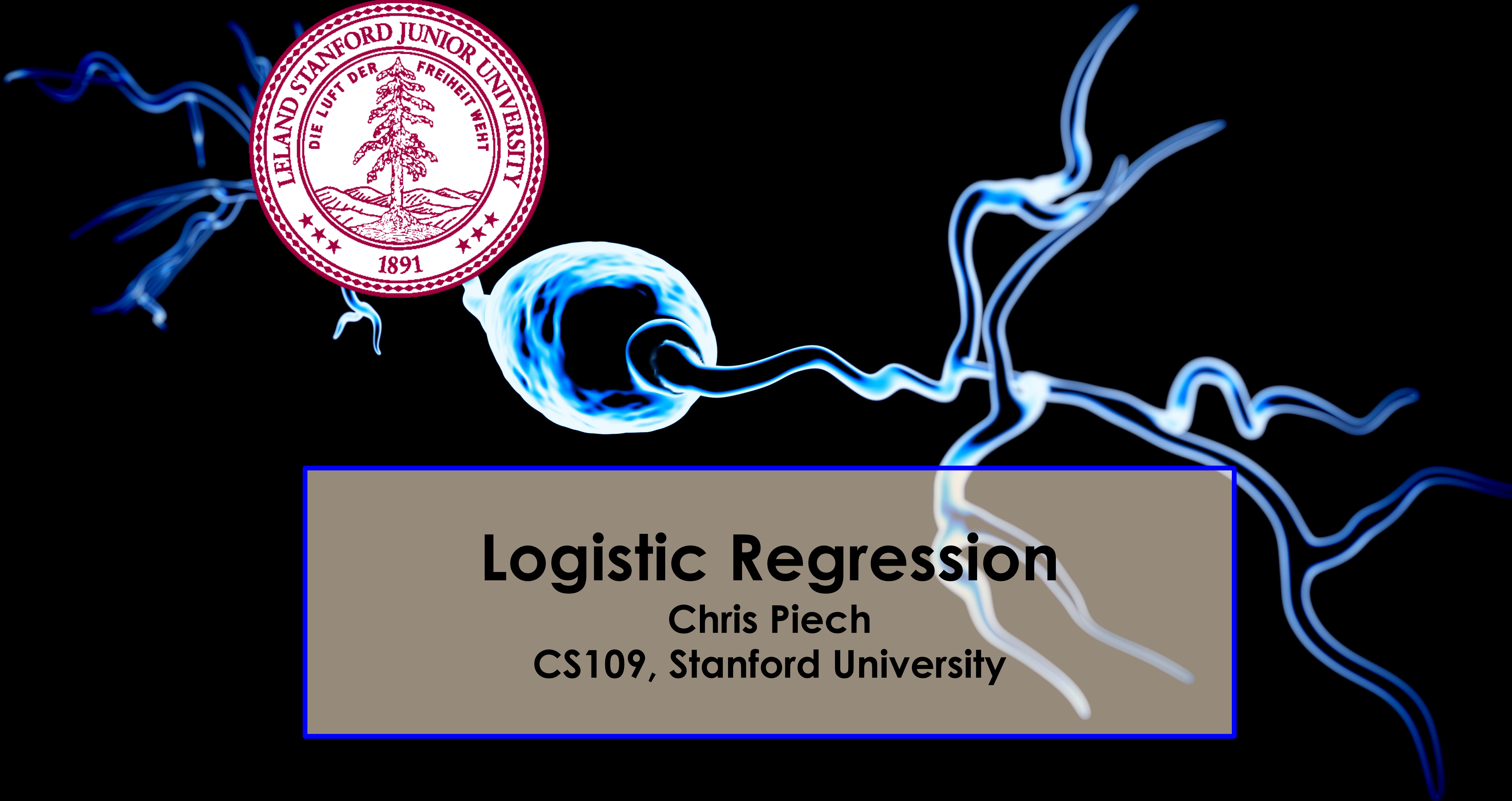
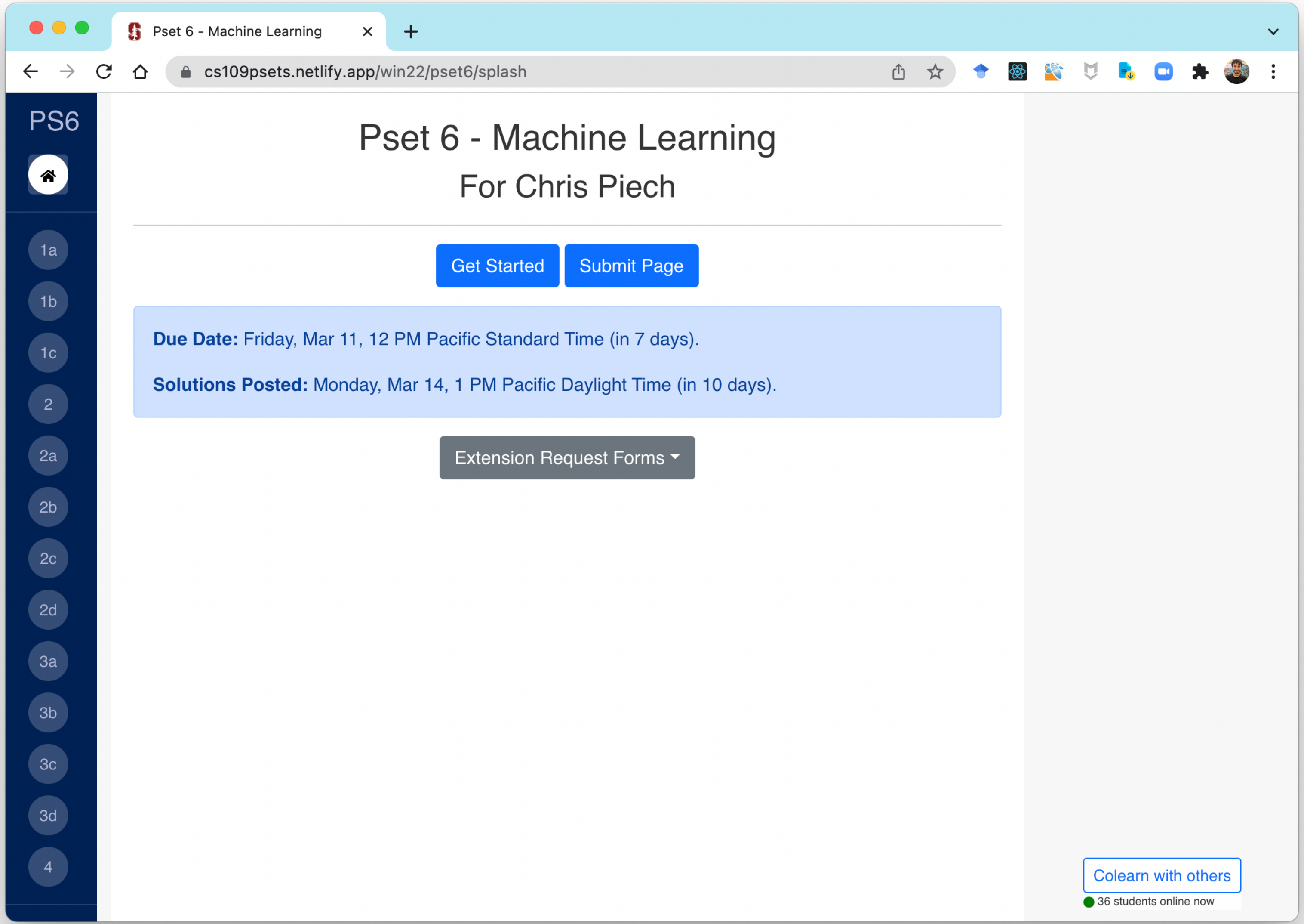


Logistic Regression

Chris Piech
CS109, Stanford University

Problem Set #6

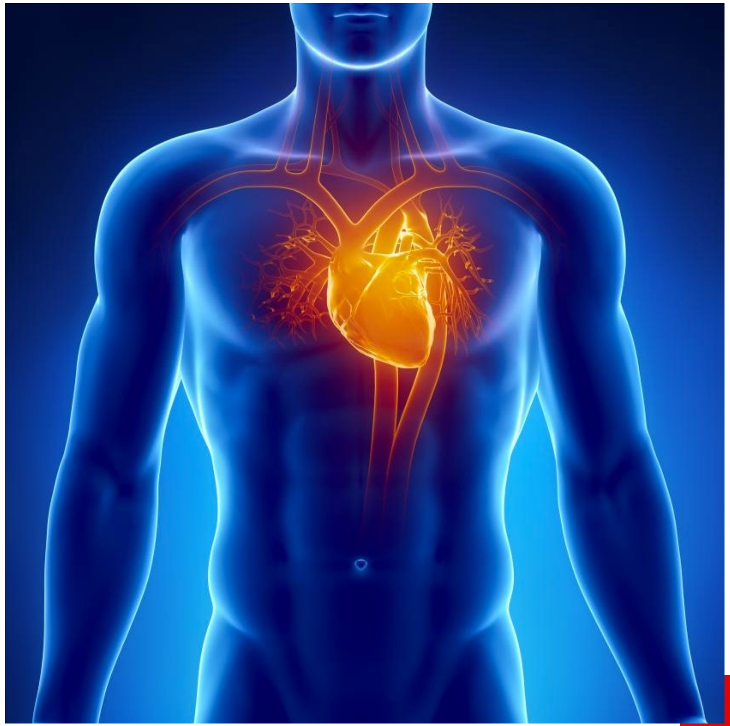


Some Fun Times

$$f_X(x) = \begin{cases} \frac{x}{\theta} e^{-x^2/2\theta} & x \geq 0 \\ 0 & else \end{cases}$$

We wish to model the wind speed on a wind farm. To this end we collect N independent measurements of wind speeds $w_1, w_2, \cdots, w_N$. Find a maximum likelihood estimate of θ if we are modeling the wind speed as coming from a Rayleigh distribution.

Heart



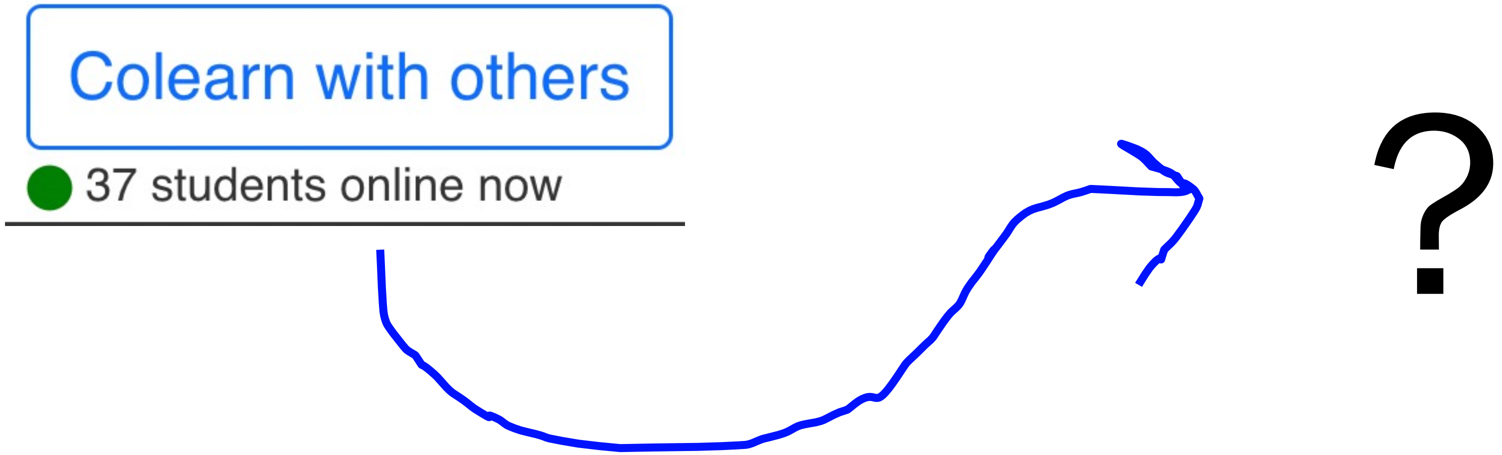
Ancestry

23andMe

Netflix

NETFLIX

Colearn with Others



Colearn with Others

Colearn with others

37 students online now

Opt In

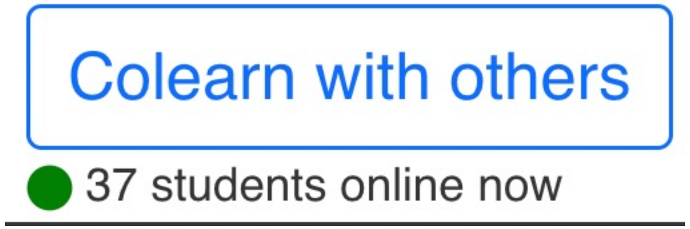
Be so kind

Be a team

Enjoy Win win

Enjoy community of learning

Warning: It is a pilot.



There will be some rough edges ☺

Come co-design with me!!!
This time next week

Creative Challenge

Extension until Saturday noon.

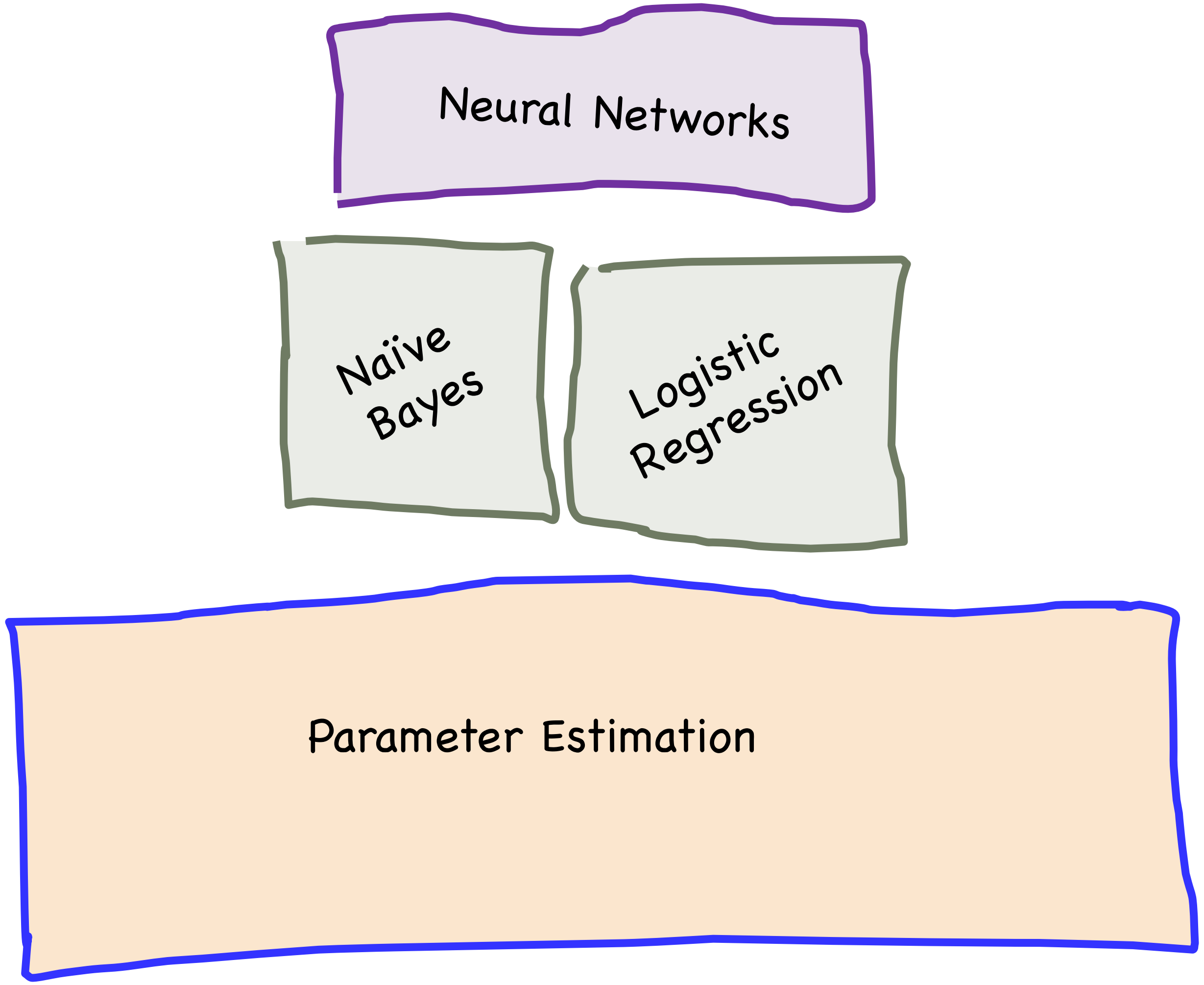
Review

Machine Learning

Great Idea

Classification Algorithms

Theory

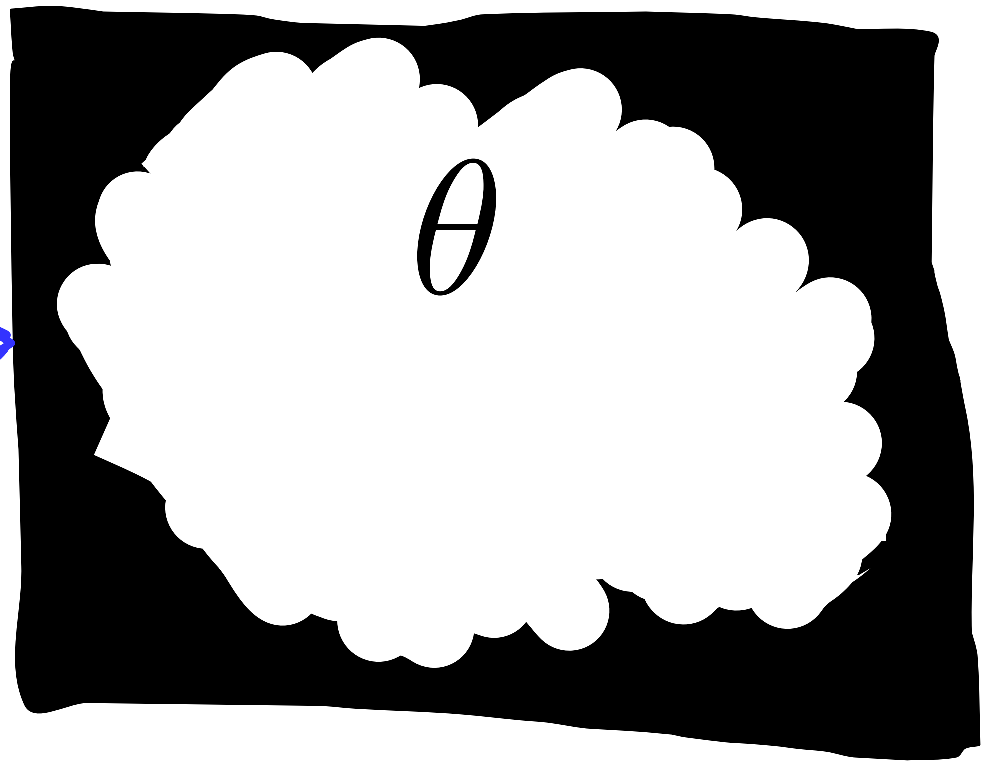


Machine Learning
(aka Applied Probability)

Machine Learning

$\mathbf{x}$

(inputs)

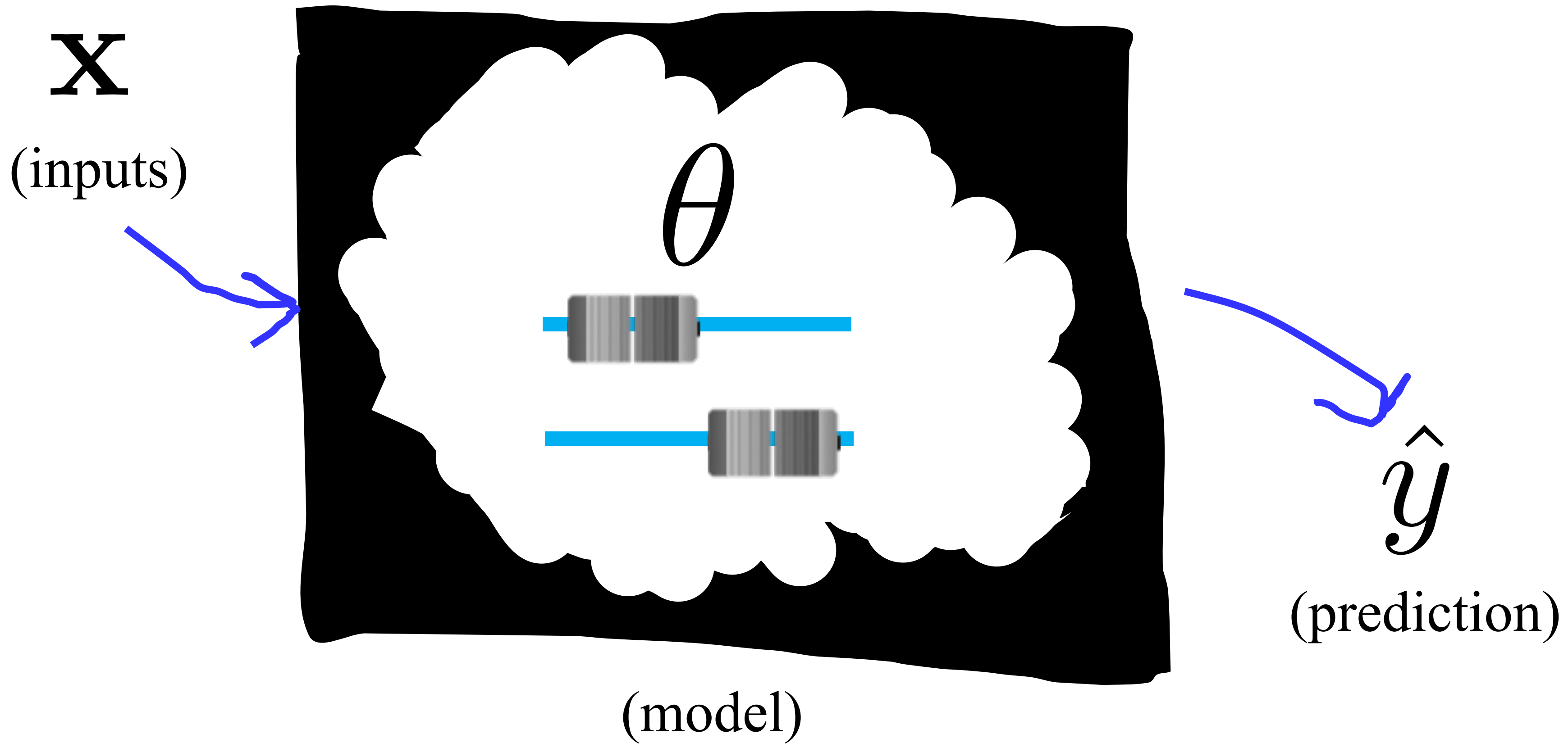


(model)

$\hat{y}$

(prediction)

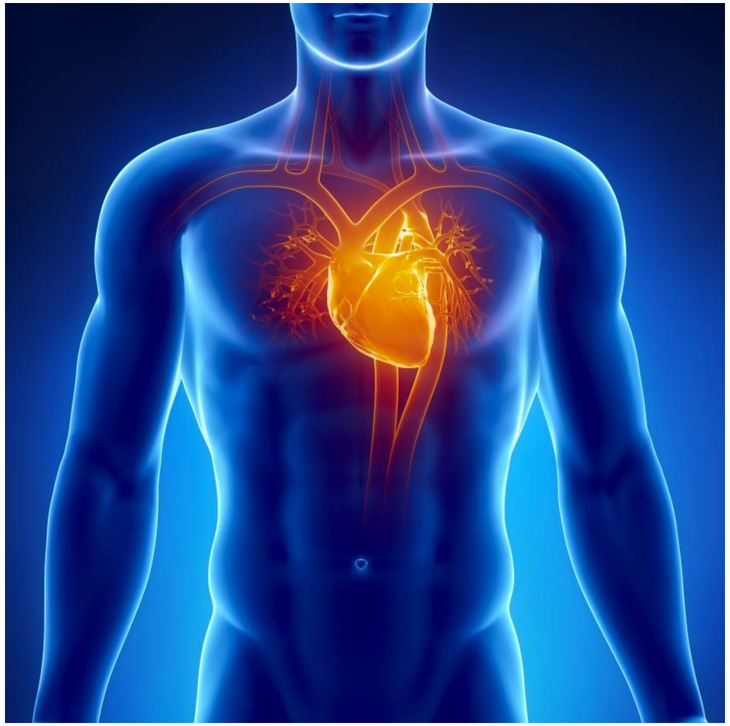
Machine Learning



Classification

Classification Task

Heart



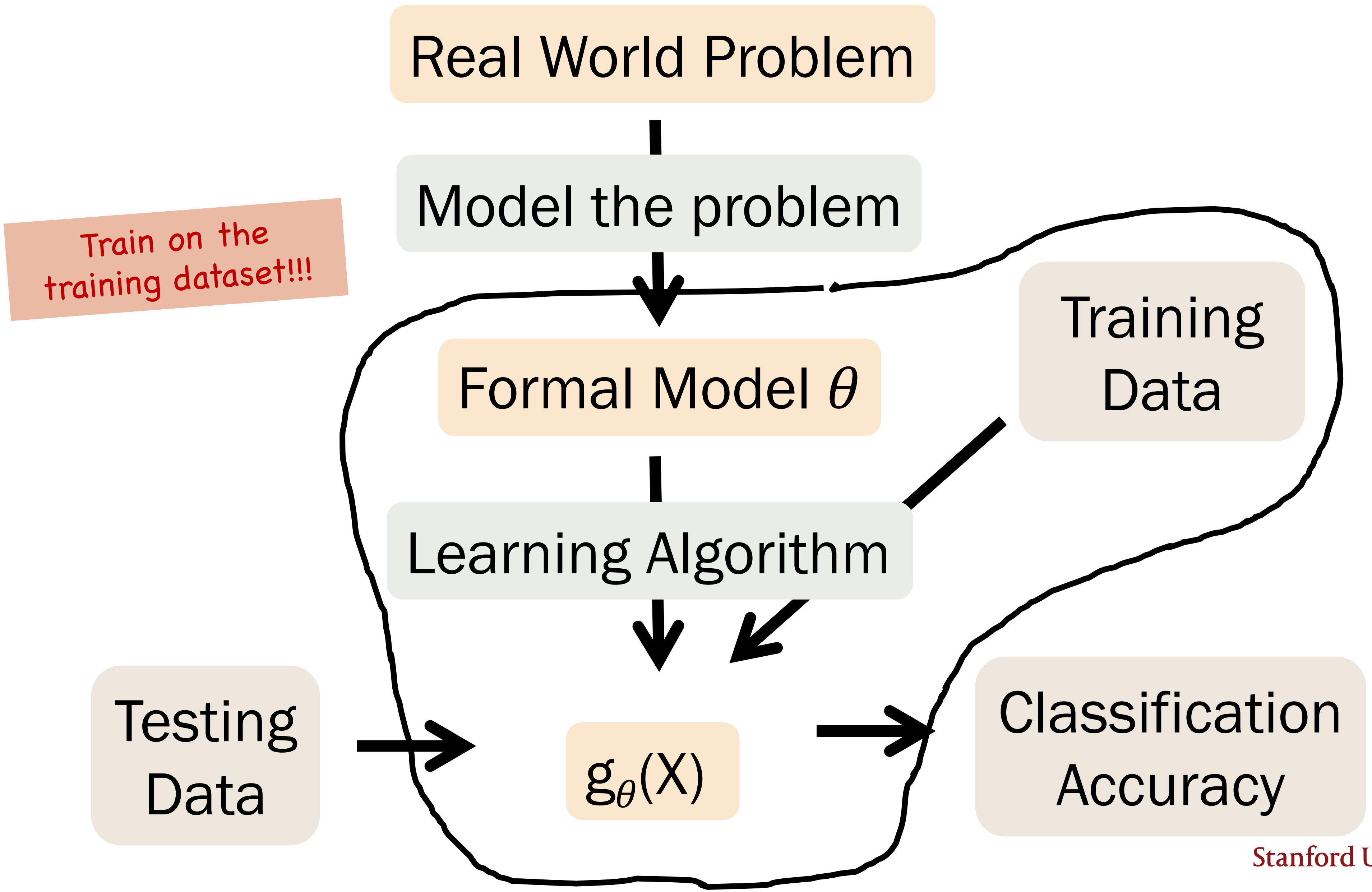
Ancestry

23andMe

Netflix

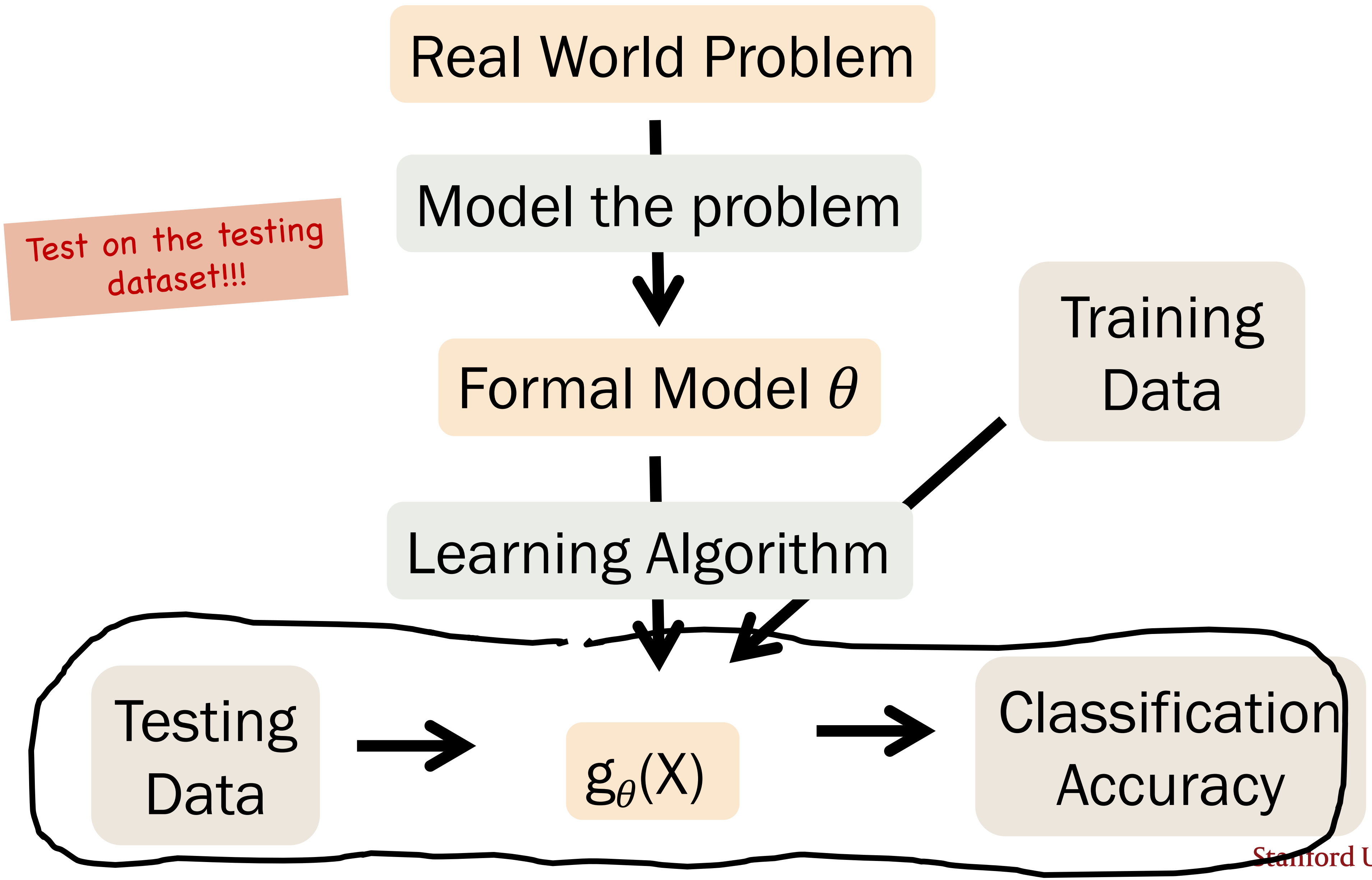
NETFLIX

Training



Real World Problem
Model the problem
Train on the training dataset!!!
Formal Model θ
Training Data
Learning Algorithm
Testing Data
$g_\theta(X)$
Classification Accuracy

Testing


Real World Problem
Model the problem
Formal Model θ
Training Data
Learning Algorithm
Test on the testing dataset!!!
Testing Data
$g_\theta(X)$
Classification Accuracy

Training Data

Assume IID data:

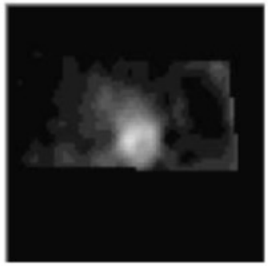
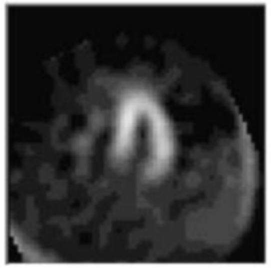
n training datapoints

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots (\mathbf{x}^{(n)}, y^{(n)})$$

$$m = |\mathbf{x}^{(i)}|$$

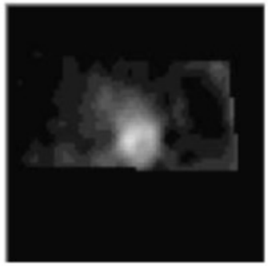
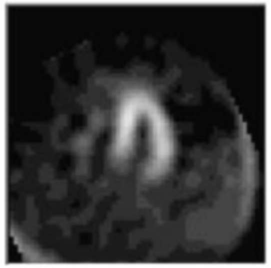
Each datapoint has m features and a single output

Training: Heart Disease Classifier

	ROI 1	ROI 2		ROI m	Output
			…		
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart n	0	0		0	1

$g_{\theta}(X)$

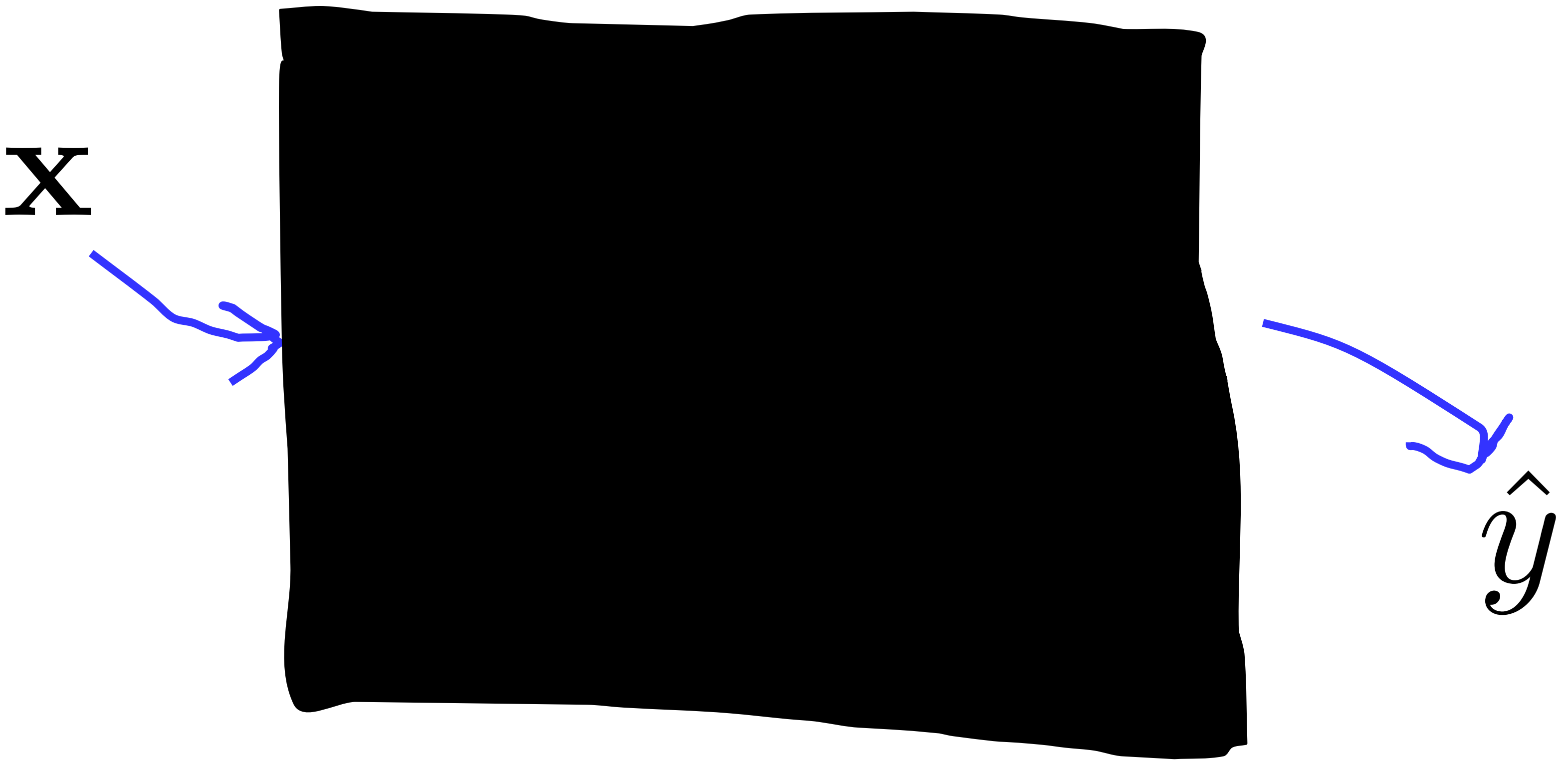
Testing: Heart Disease Classifier

	ROI 1	ROI 2		ROI *m*	Output
			...		
New Heart	1	0		1	$\boxed{1}$

$g_\theta(X)$

Naïve Bayes Classification

$g_\theta(\mathbf{x})$?

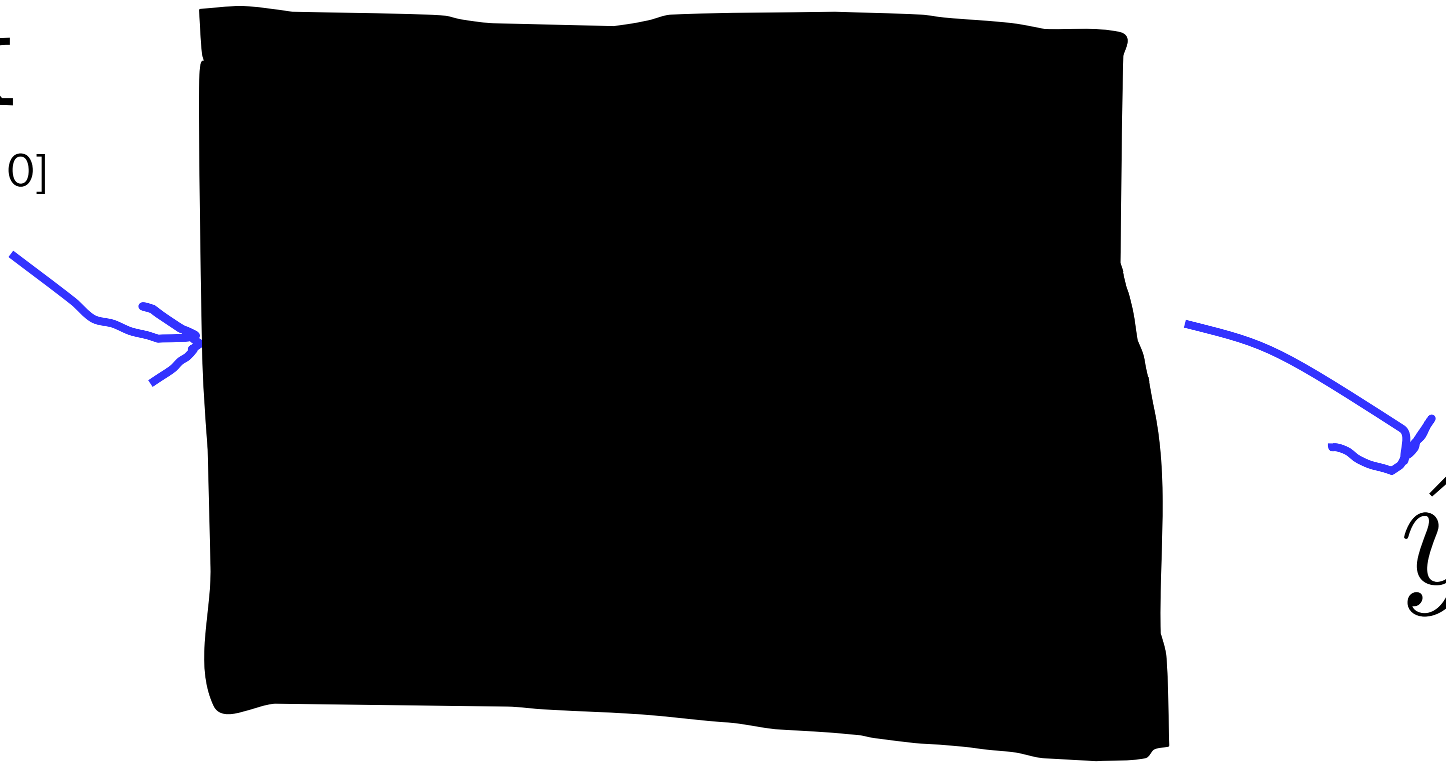


Making a prediction…

$g_\theta(\mathbf{x})$?

$\mathbf{x}$

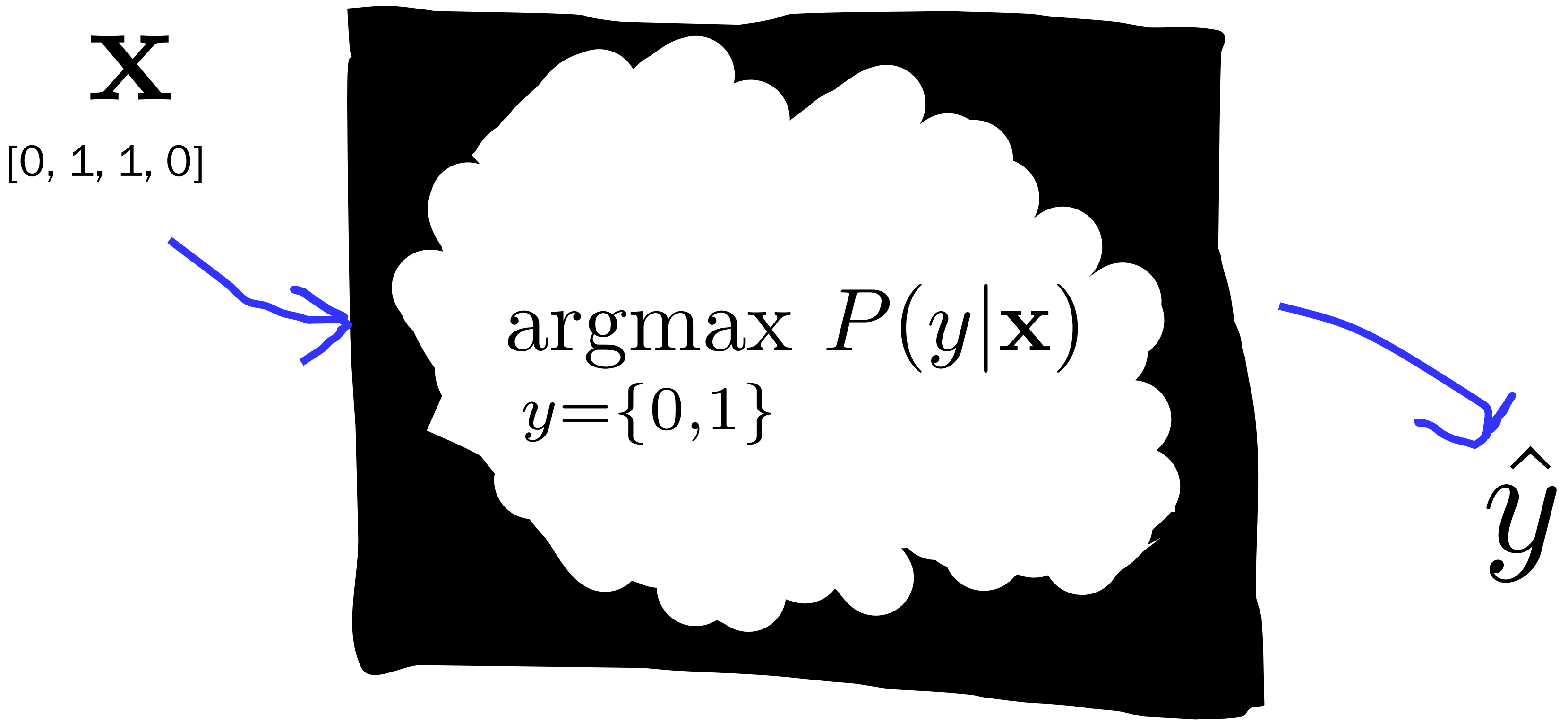
[0, 1, 1, 0]



$\hat{y}$

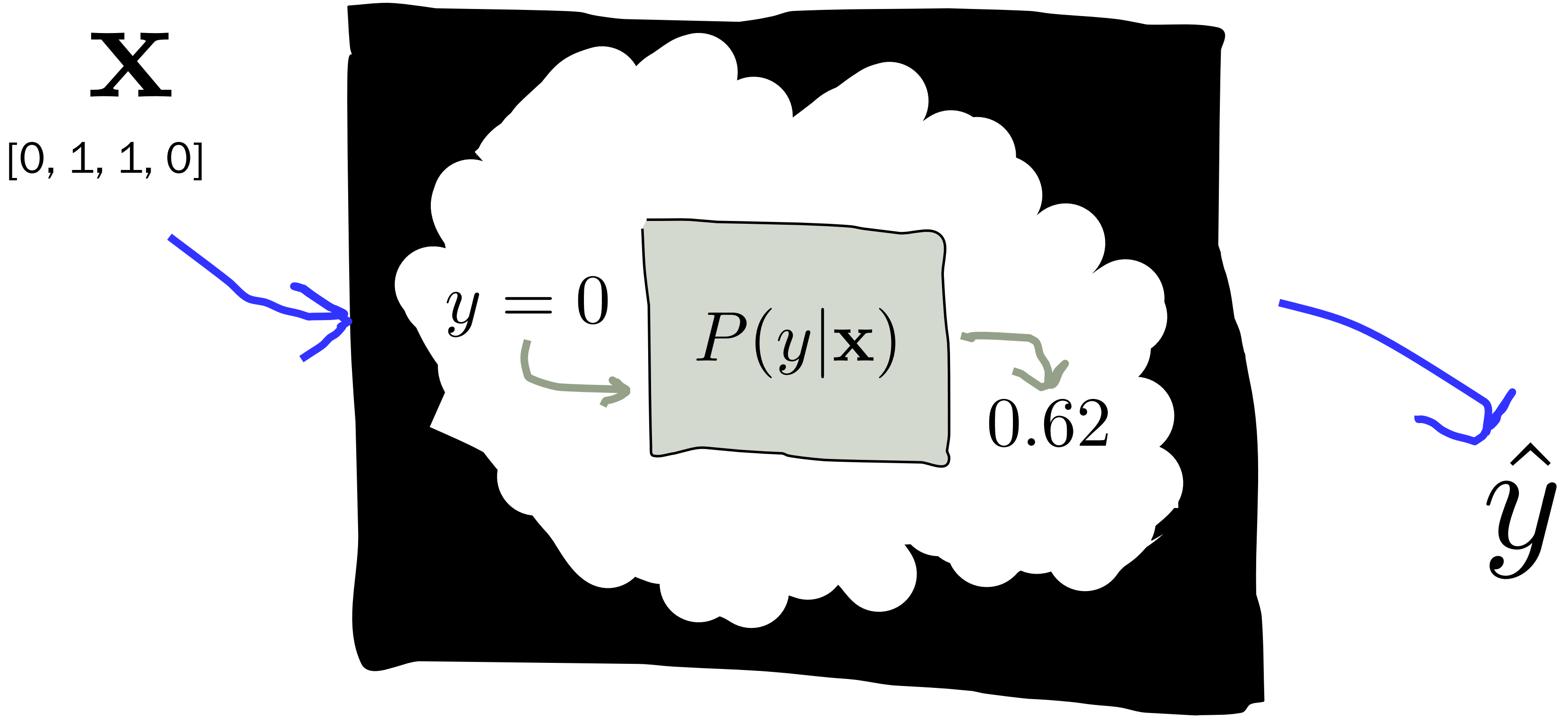
Making a prediction…

$g_\theta(\mathbf{x})$?



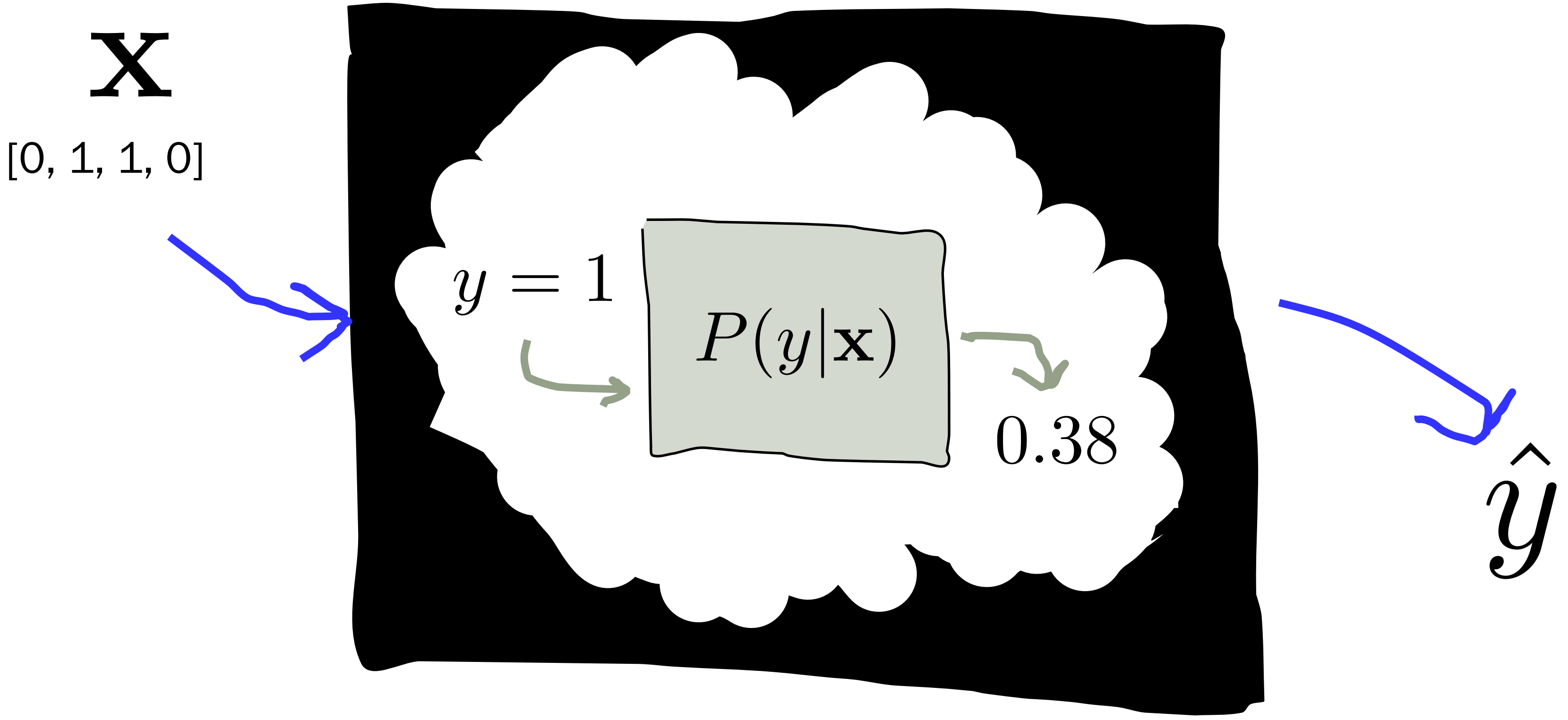
Making a prediction…

$g_\theta(\mathbf{x})$?



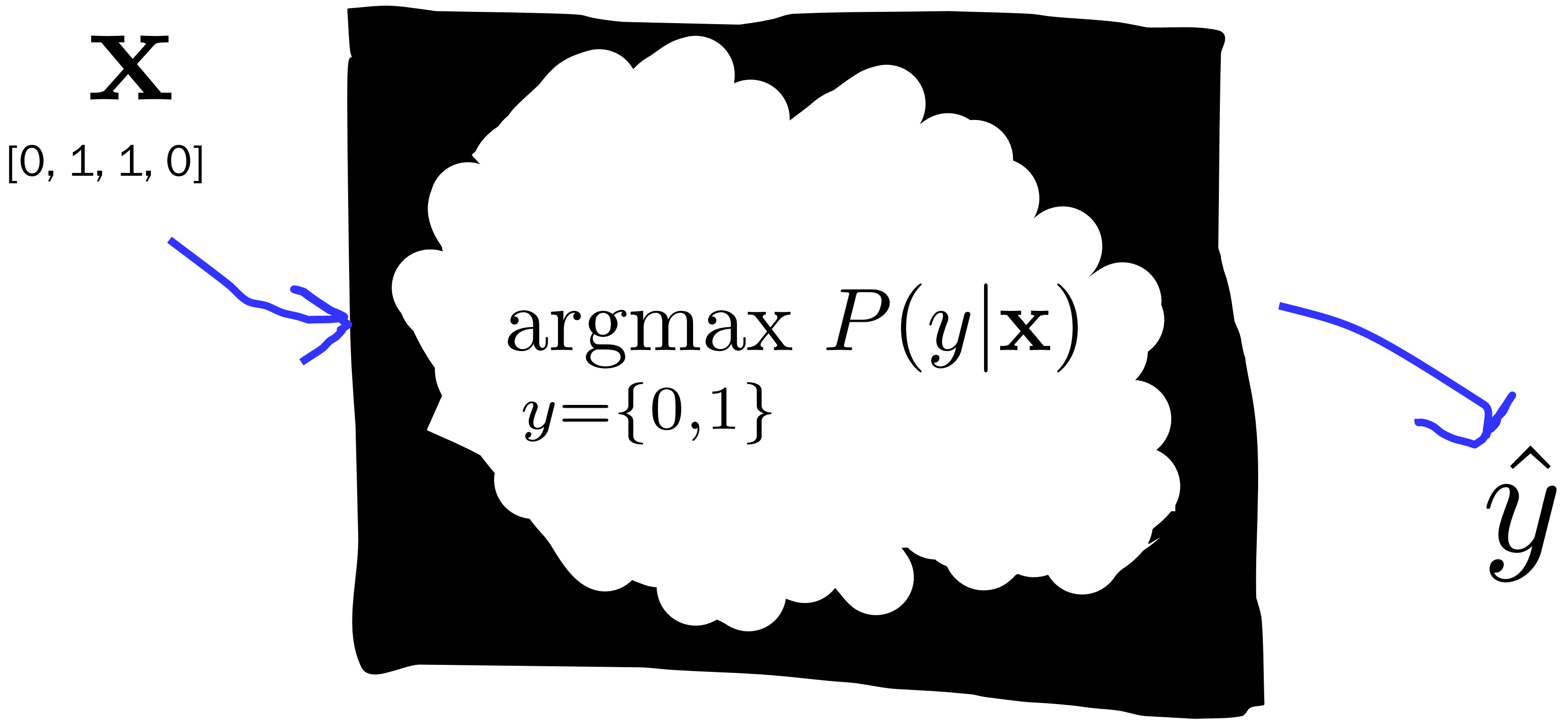
Making a prediction…

$\mathrm{g}_\theta(\mathbf{x})$?



Making a prediction…

$g_\theta(\mathbf{x})$?

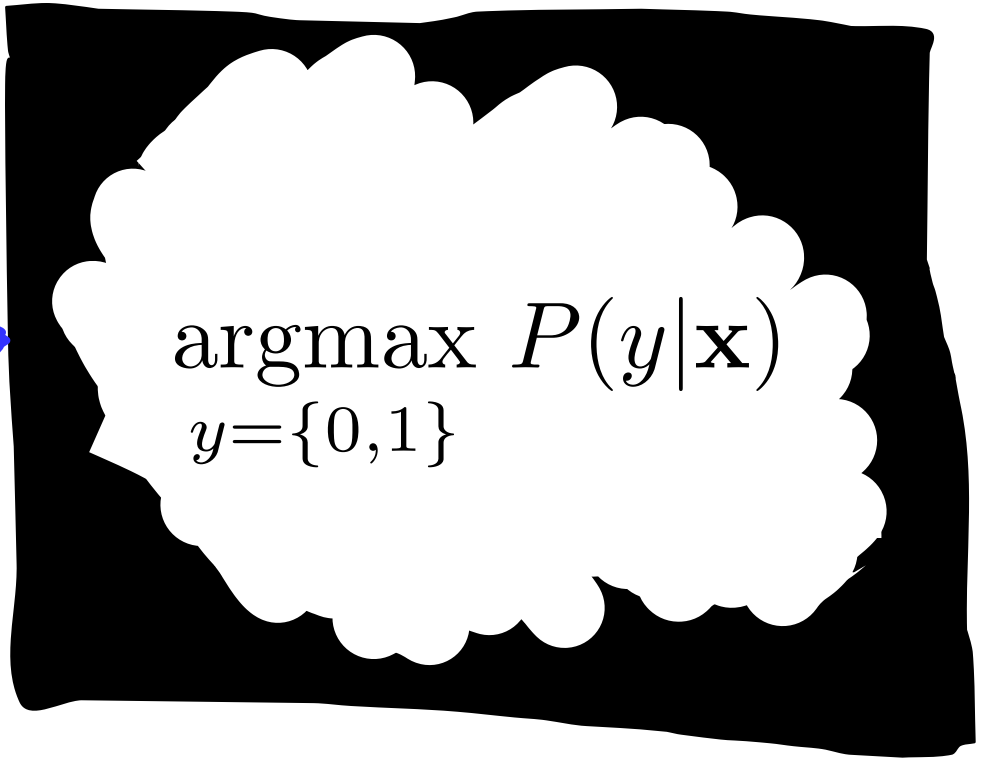


Making a prediction…

$\mathrm{g}_\theta(\mathbf{x})$?

$\mathbf{x}$

[0, 1, 1, 0]



$\hat{y} = 0$

Making a prediction…

Big Assumption

Naïve Bayes Assumption:

$$P(\mathbf{x}|y) = \prod_i P(x_i|y)$$

Prediction

Simply chose the class label that is the most likely given the data. Make Naïve Bayes assumption

$$\hat{y} = \arg\max_{y=\{0,1\}} P(Y = y|\mathbf{X} = \mathbf{x})$$

$$= \arg\max_{y=\{0,1\}} \frac{P(Y = y)P(\mathbf{X} = \mathbf{x}|Y = y)}{P(\mathbf{X} = \mathbf{x})}$$

$$= \arg\max_{y=\{0,1\}} P(Y = y)P(\mathbf{X} = \mathbf{x}|Y = y)$$

$$= \arg\max_{y=\{0,1\}} P(Y = y)\prod_i P(X_i = x_i|Y = y)$$

$$= \arg\max_{y=\{0,1\}} \log P(Y = y) + \sum_i \log P(X_i = x_i|Y = y)$$

Must learn params for this

Must learn params for this

Learning Probabilities from Data

Various probabilities you will need to compute for Naive Bayesian Classifier (using **MLE** here):

$$\hat{p}(X_i = 1 | Y = 0) = \frac{(\#\text{ training examples where } X_i = 1 \text{ and } Y = 0)}{(\#\text{ training examples where } Y = 0)}$$

$$\hat{p}(Y = 1) = \frac{(\#\text{ training examples where } Y = 1)}{(\#\text{ training examples})}$$

Learning Probabilities from Data

Various probabilities you will need to compute for Naive Bayesian Classifier (using **MAP with Laplace prior** here):

$$\hat{p}(X_i = 1 | Y = 0) = \frac{(\#\text{ training examples where } X_i = 1 \text{ and } Y = 0) + 1}{(\#\text{ training examples where } Y = 0) + 2}$$

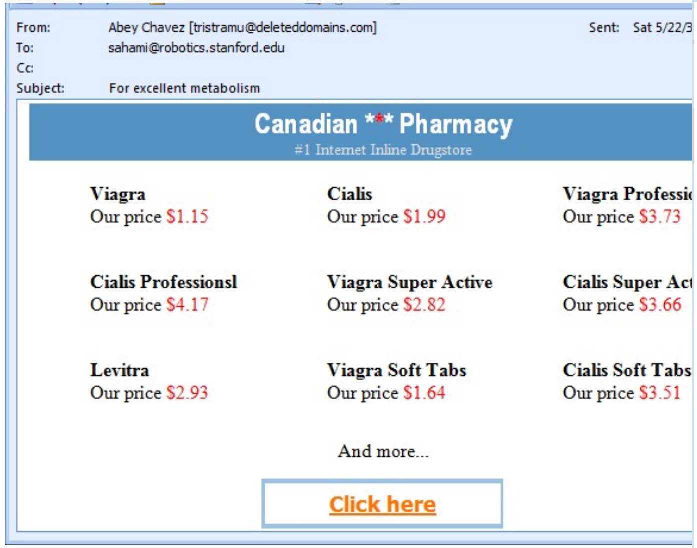
$$\hat{p}(Y = 1) = \frac{(\#\text{ training examples where } Y = 1) + 1}{(\#\text{ training examples}) + 2}$$

Training Naïve Bayes, is estimating parameters for a multinomial (or bernoulli).

Thus training is just counting.

What is Bayes Doing in my Mail Server

This is spam:



Let's get Bayesian on your spam:

```
Content analysis details:   (49.5 hits, 7.0 required)
 0.9 RCVD_IN_PBL            RBL: Received via a relay in Spamhaus PBL
                            [93.40.189.29 listed in zen.spamhaus.org]
 1.5 URIBL_WS_SURBL         Contains an URL listed in the WS SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_JP_SURBL         Contains an URL listed in the JP SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_OB_SURBL         Contains an URL listed in the OB SURBL blocklist
                            [URIs: recragas.cn]
 5.0 URIBL_SC_SURBL         Contains an URL listed in the SC SURBL blocklist
                            [URIs: recragas.cn]
 2.0 URIBL_BLACK            Contains an URL listed in the URIBL blacklist
                            [URIs: recragas.cn]
 8.0 BAYES_99               BODY: Bayesian spam probability is 99 to 100%
                            [score: 1.0000]
```

A Bayesian Approach to Filtering Junk E-Mail

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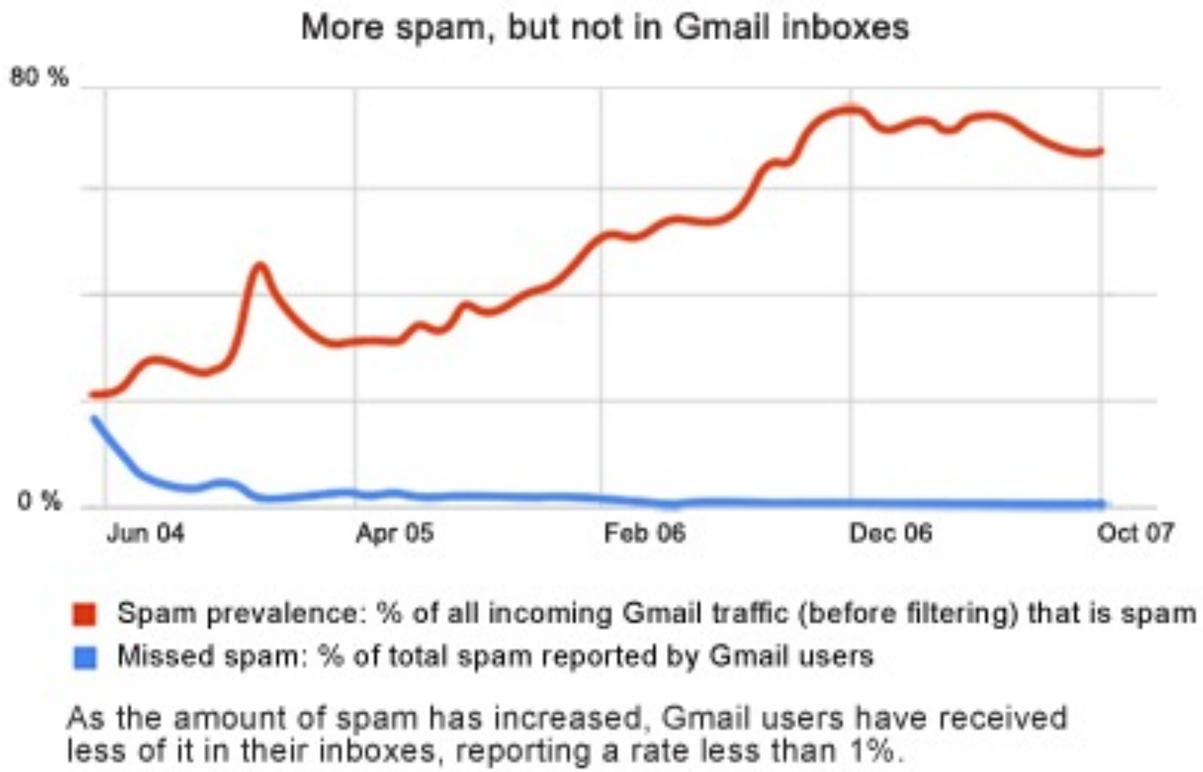
Abstract

In addressing the growing problem of junk E-mail on the Internet, we examine methods for the automated

contain offensive material (such as graphic pornography), there is often a higher cost to users of actually viewing this mail than simply the time to sort out the junk. Lastly, junk mail not only wastes user time, but

Spam, Spam... Go Away!

The constant battle with spam



“And machine-learning algorithms developed to merge and rank large sets of Google search results allow us to combine hundreds of factors to classify spam.”

Source: `http://www.google.com/mail/help/fightspam/spamexplained.html`

Email Classification

Want to predict if an email is spam or not

- Start with the input data
  - Consider a lexicon of m words (Note: in English $m \approx 100{,}000$)
  - Define m indicator variables $\mathbf{X} = \langle X_1, X_2, \ldots, X_m \rangle$
  - Each variable X_i denotes if word i appeared in a document or not
  - Note: m is huge, so make "Naive Bayes" assumption
- Define output classes Y to be: {spam, non-spam}
- Given training set of N previous emails
  - For each email message, we have a training instance: $\mathbf{X} = \langle X_1, X_2, \ldots, X_m \rangle$ noting for each word, if it appeared in email
  - Each email message is also marked as spam or not (value of Y)

Training the Classifier

Given N training pairs:

$(\mathbf{x}^{(1)},y^{(1)}), (\mathbf{x}^{(2)},y^{(2)}), \ldots, (\mathbf{x}^{(n)}, y^{(n)})$

Learning

- Estimate probabilities $P(y)$ and $P(x_i \mid y)$ for all i
  - Many words are likely to not appear at all in given set of email
- Laplace estimate: $\hat{p}(X_i = 1 \mid Y = spam)_{Laplace} = \dfrac{(\#\,\text{spam emails with word } i) + 1}{\text{total}\,\#\,\text{spam emails} + 2}$

Classification

- For a new email, generate $\mathbf{X} = <X_1, X_2, \ldots, X_m>$
- Classify as spam or not using: $\hat{y} = \underset{y \in \{0,1\}}{\operatorname{argmax}}\ \hat{P}(\mathbf{x}|y)\hat{P}(y)$
- Employ Naive Bayes assumption: $P(\mathbf{x}|y) = \prod_i P(x_i|y)$

How Does This Do?

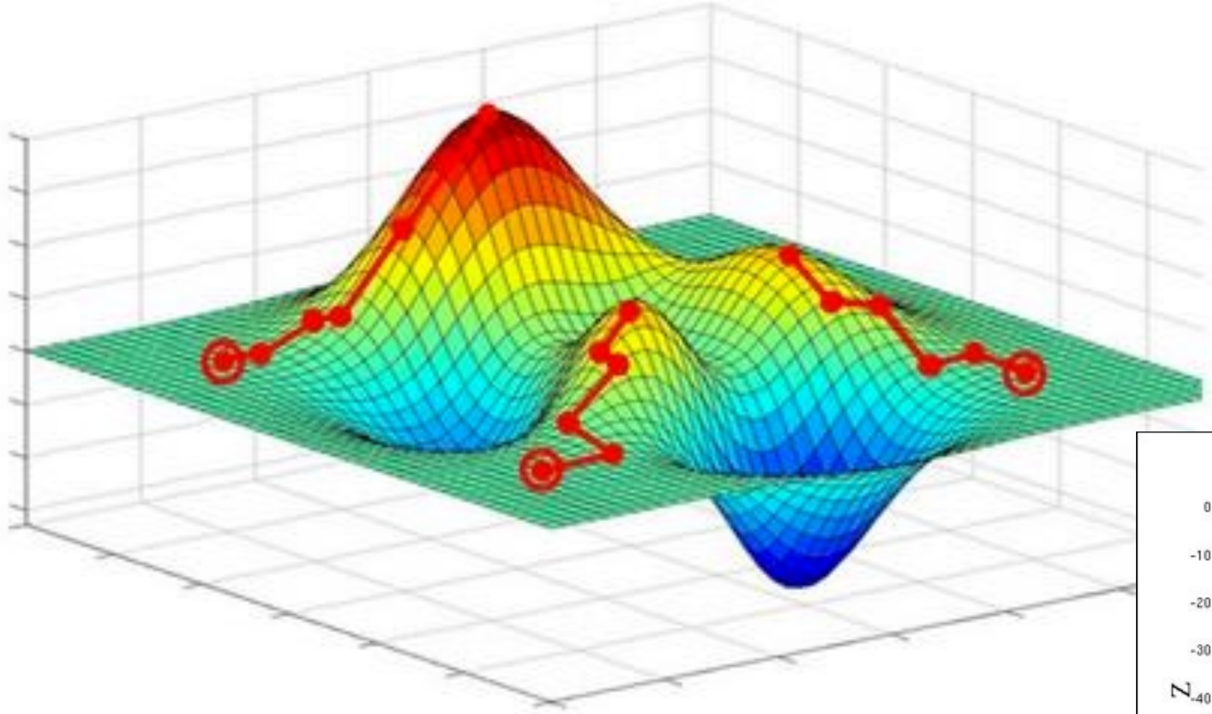
After training, can test with another set of data

- "Testing" set also has known values for Y, so we can see how often we were right/wrong in predictions for Y
- Spam data
  - Email data set: 1789 emails (1578 spam, 211 non-spam)
  - First, 1538 email messages (by time) used for training
  - Next 251 messages used to test learned classifier
- Criteria:
  - Precision = # *correctly* predicted class Y/ # predicted class Y
  - Recall = # *correctly* predicted class Y / # real class Y messages

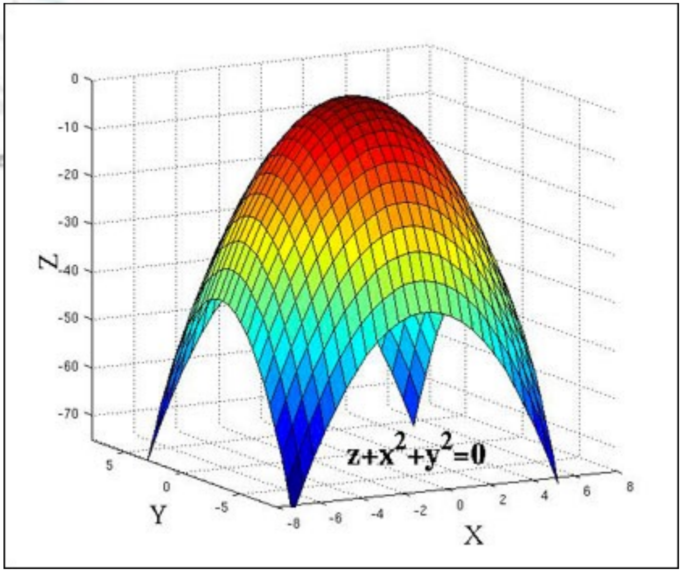
	Spam		Non-spam	
	Precision	**Recall**	**Precision**	**Recall**
Words only	**97.1%**	**94.3%**	**87.7%**	**93.4%**
Words + add'l features	**100%**	**98.3%**	**96.2%**	**100%**

Optimization

Gradient Ascent



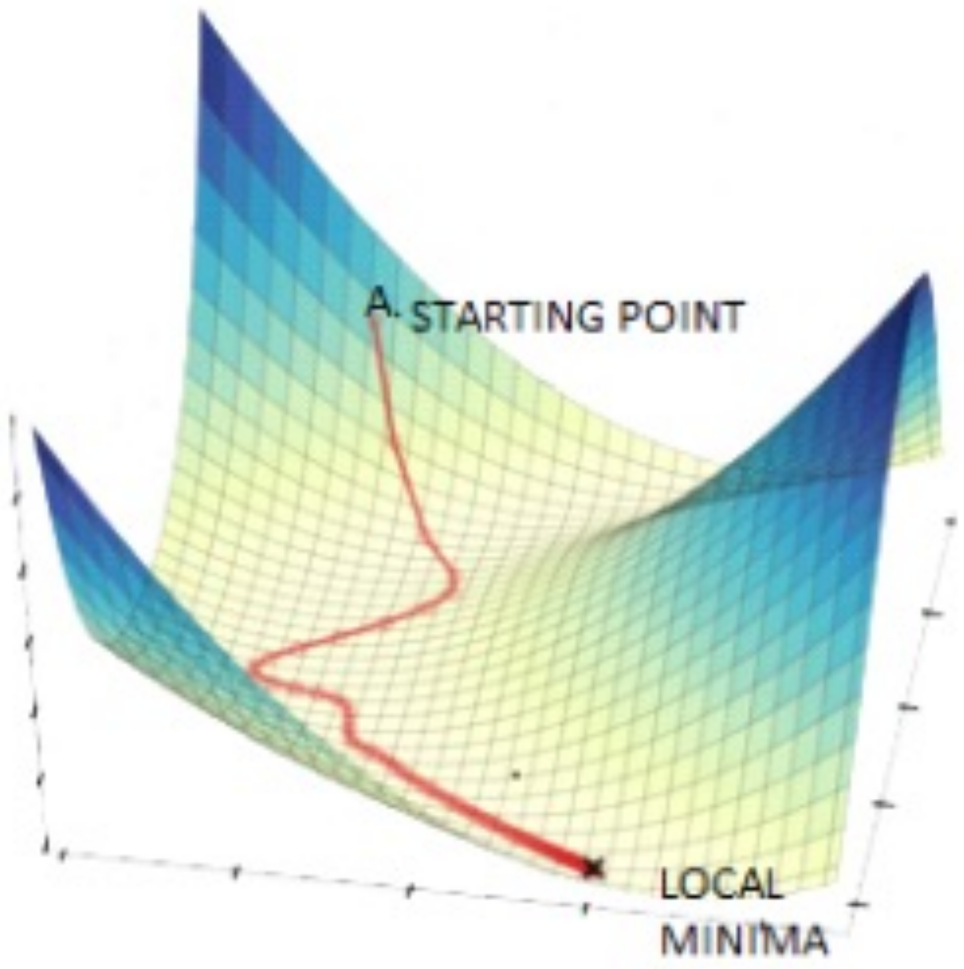
Logistic regression LL function is convex



Walk uphill and you will find a local maxima
(if your step size is small enough)

Gradient descent is your bread and butter algorithm for optimization (eg argmax)

Gradient Decent



Walk downhill and you will find a local maxima
(if your step size is small enough)

If someone gives you a gradient descent package, you should minimize **negative** log likelihood.

If you are writing optimization yourself, feel free to gradient **ascent** on log likelihood :-)

End Review

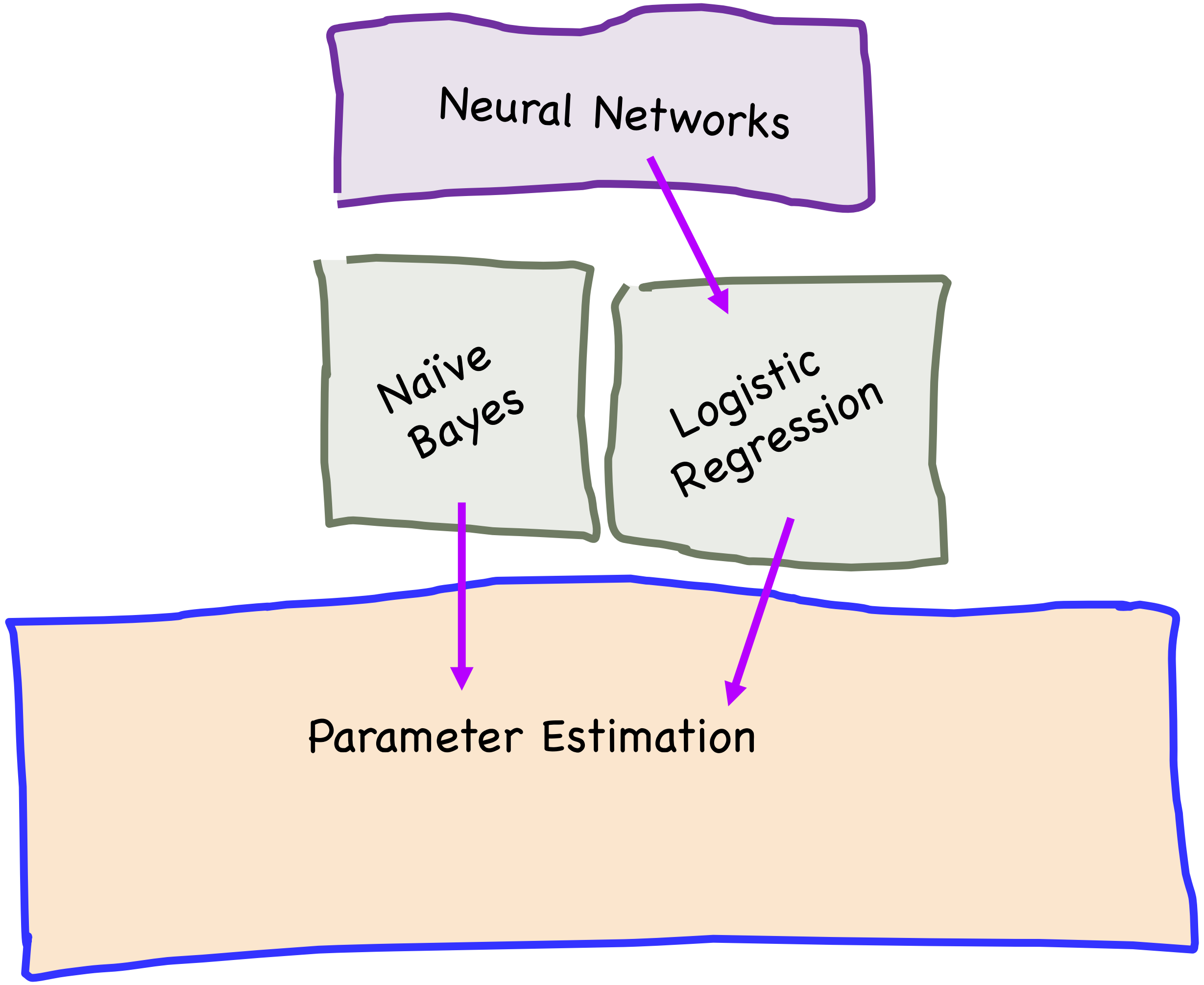
Logistic Regression

Machine Learning *Dependencies*

Great Idea

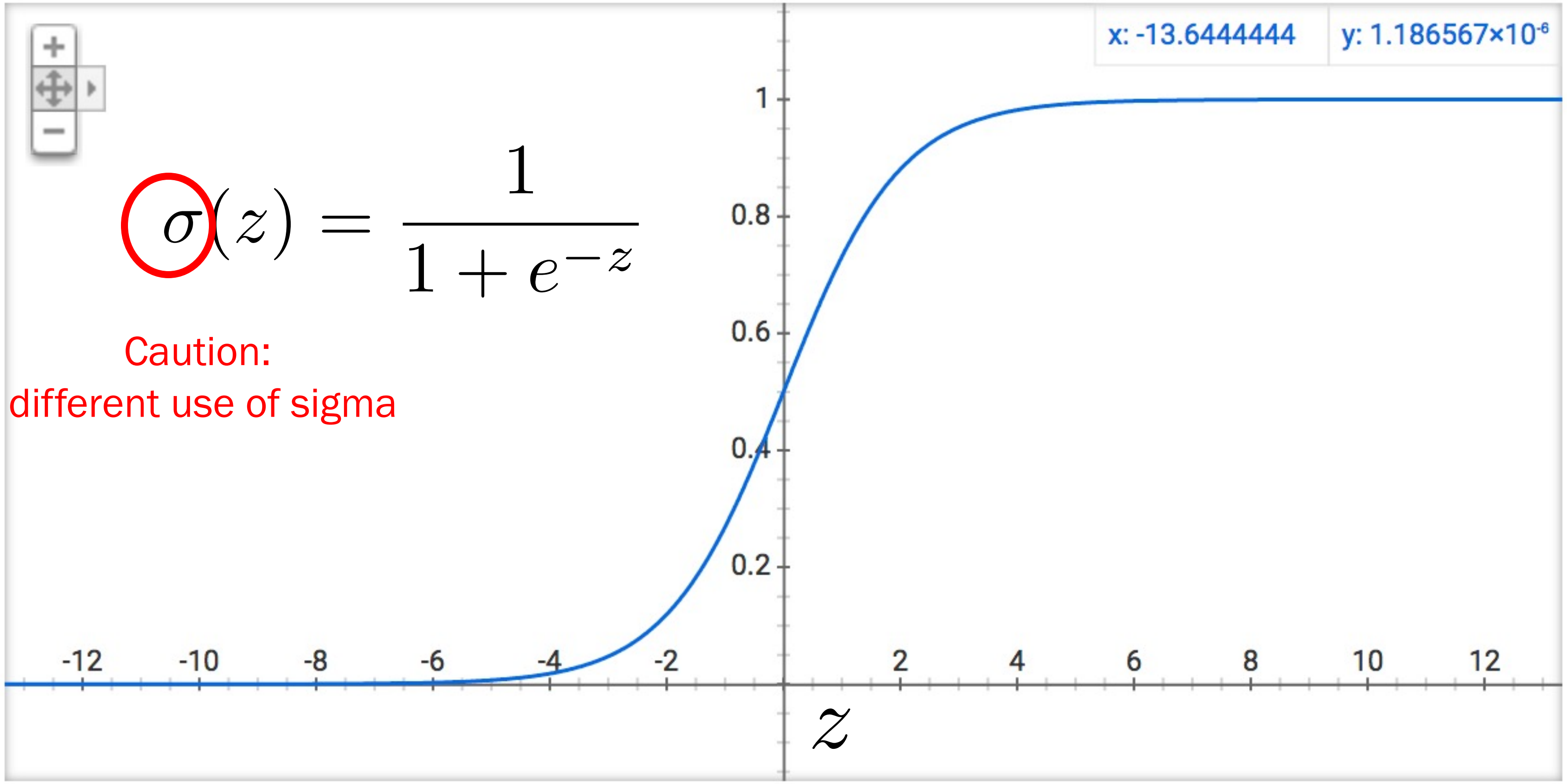
Core Algorithms

Theory



Chapter 0: Background

Background: Sigmoid Function



$$\sigma(z) = \frac{1}{1+e^{-z}}$$

Caution:
different use of sigma

The sigmoid function squashes z to be a number between 0 and 1

Background: Key Notation

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Sigmoid function

$$\theta^T \mathbf{x} = \sum_{i=1}^{n} \theta_i x_i$$

$$= \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$$

Weighted sum
(aka dot product)

$$\sigma(\theta^T \mathbf{x}) = \frac{1}{1 + e^{-\theta^T \mathbf{x}}}$$

Sigmoid function of
weighted sum

Background: Chain Rule

Who knew calculus would be so useful?

$$\frac{\partial f(x)}{\partial x} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x}$$

Aka decomposition of composed functions

$$f(x) = f(z(x))$$

Chapter 1: Big Picture

From Naïve Bayes to Logistic Regression

In classification we care about P(Y | **X**)

Recall the Naive Bayes Classifier

- Predict P(Y | **X**)
- Use assumption that $P(\mathbf{X} \mid Y) = P(X_1, X_2, \ldots X_m \mid Y) = \prod_{i=1}^{m} P(X_i \mid Y)$
- That is a pretty big assumption…

Could we model P(Y | **X**) directly?

- Welcome our friend: logistic regression!

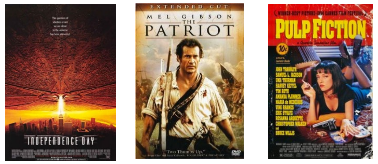
Logistic Regression Assumption

Could we model P(Y | **X**) directly?

- Welcome our friend: logistic regression!

$\mathbf{x}, \theta$

[1, 1, 0]



$P(Y = 1|\mathbf{x})$

0.81

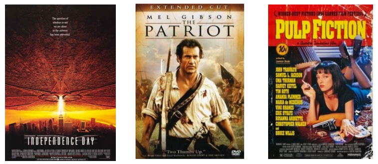
Logistic Regression Assumption

Could we model P(Y | **X**) directly?

- Welcome our friend: logistic regression!

$\mathbf{x}, \theta$

[1, 1, 0]



$P(Y = 1|\mathbf{x})$

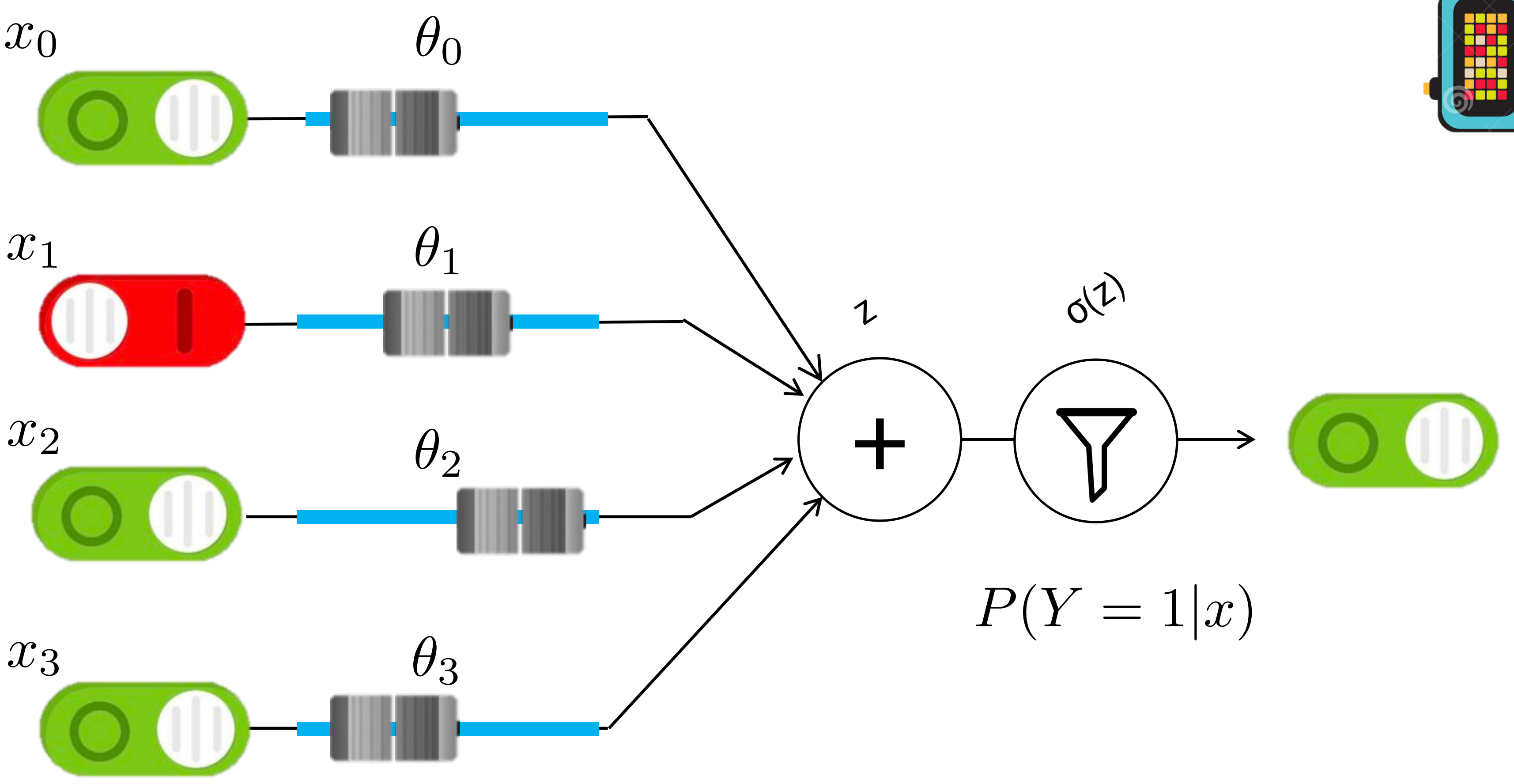
0.81

$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression

$$P(Y = 1 | \mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression

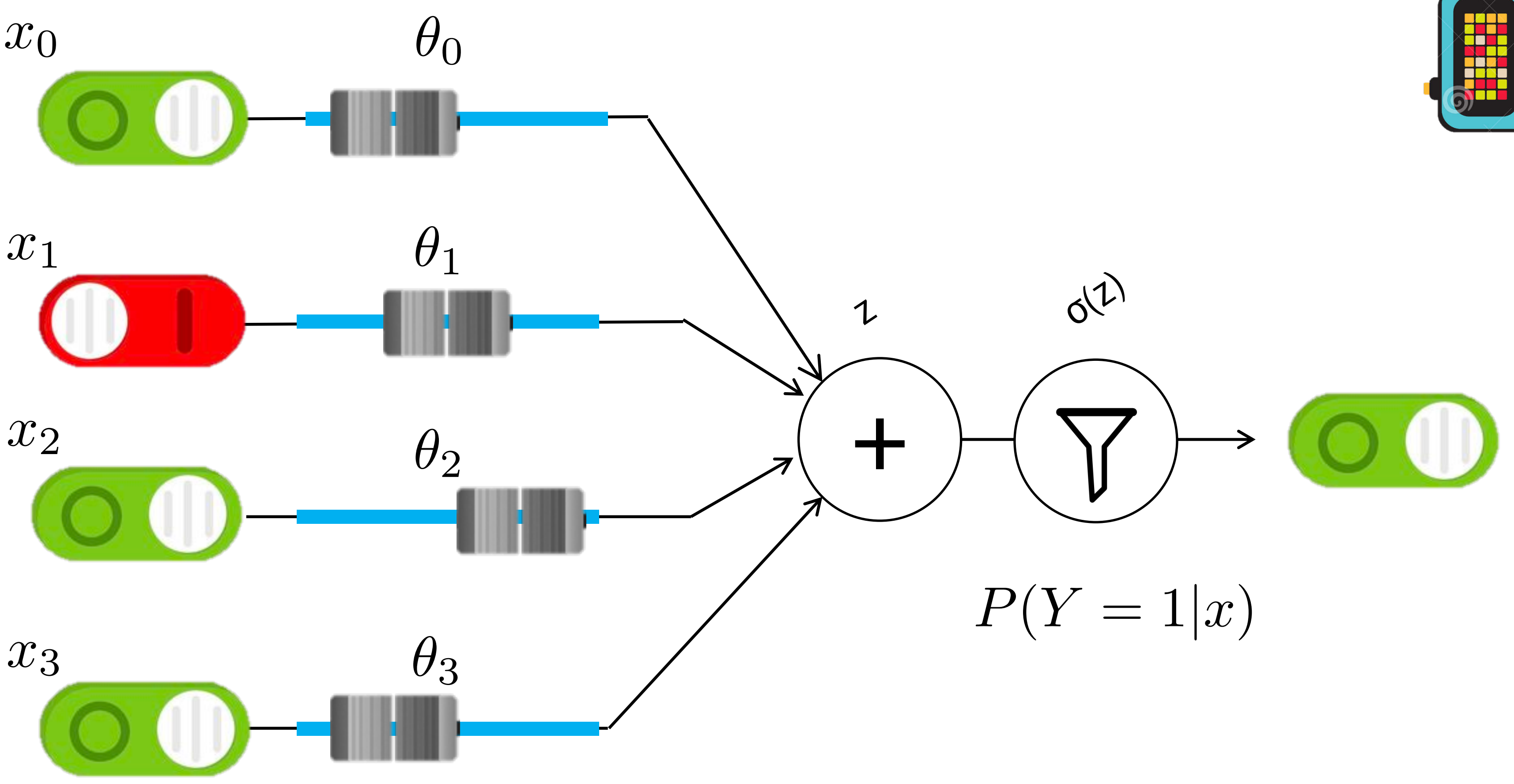


$$P(Y=1|\mathbf{X}=\mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression

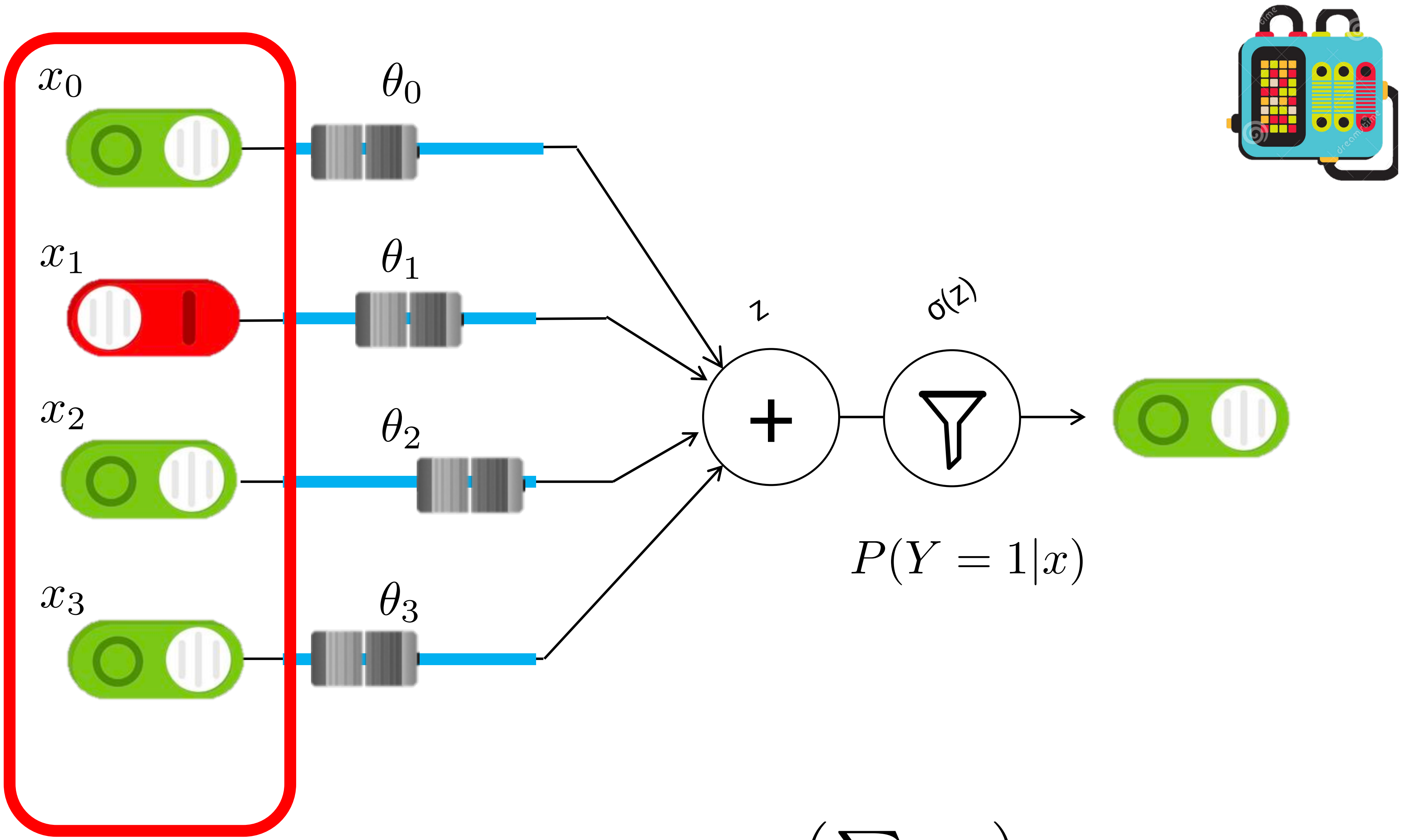
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression Cartoon



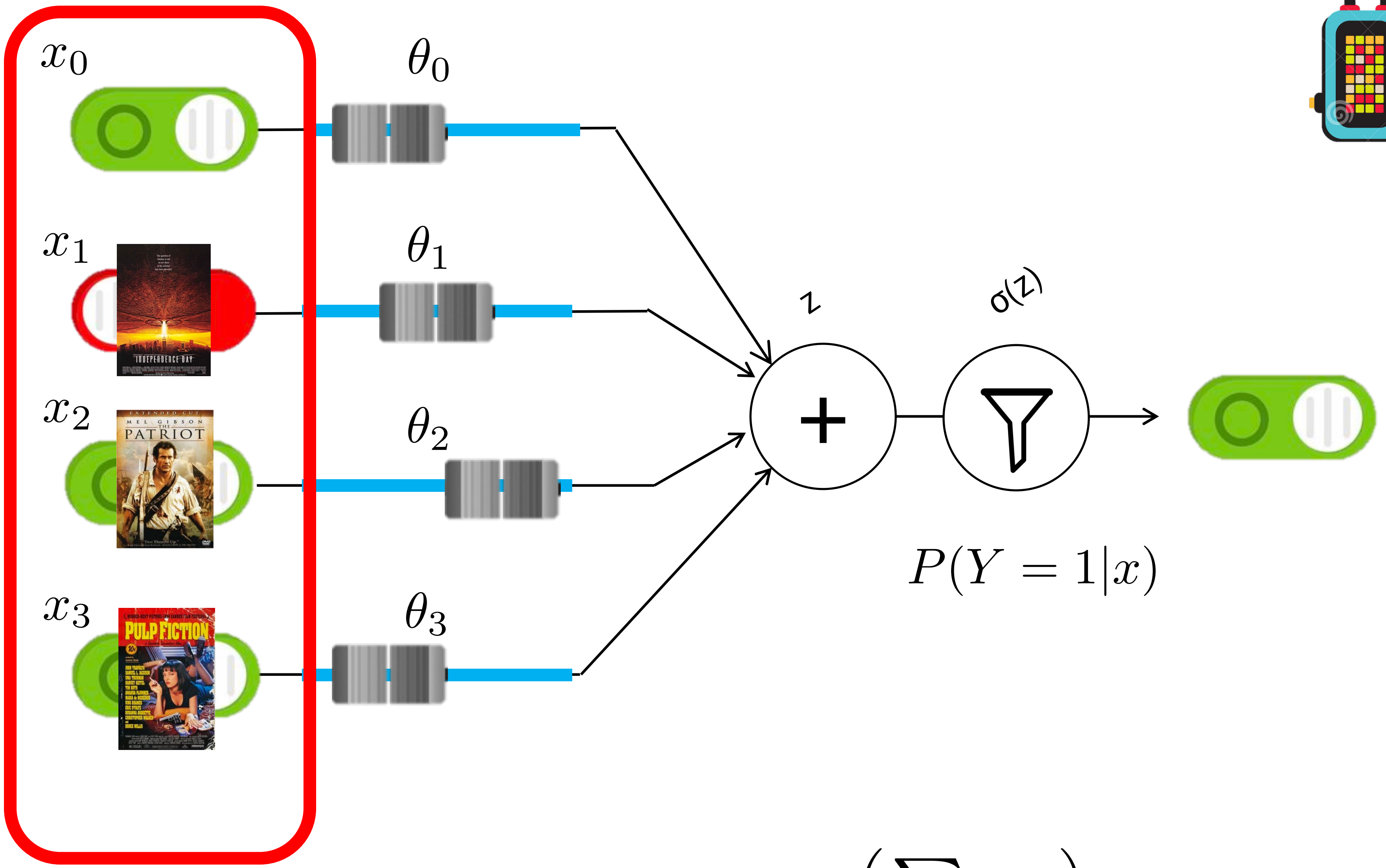
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs $x = [0, 1, 1]$



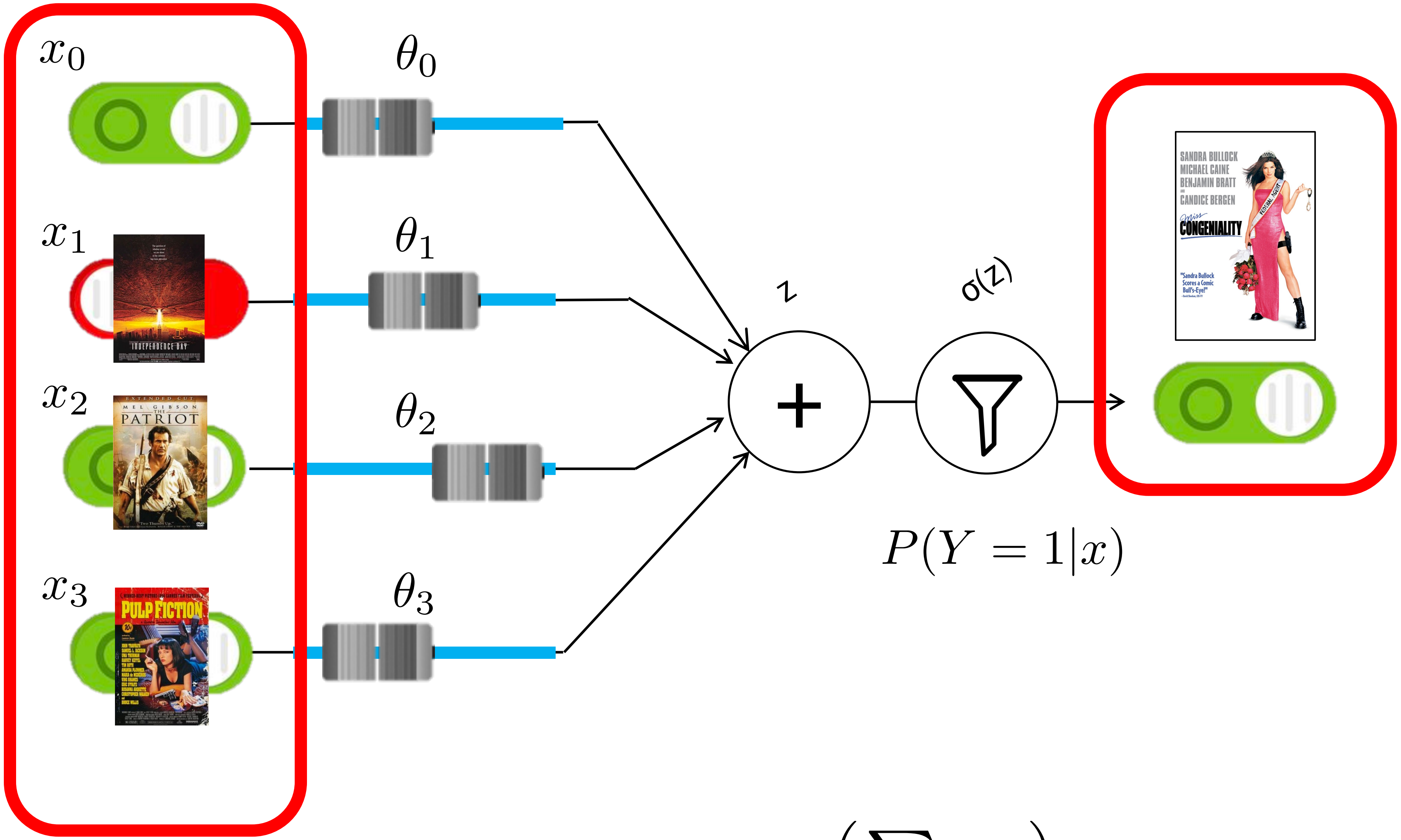
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs



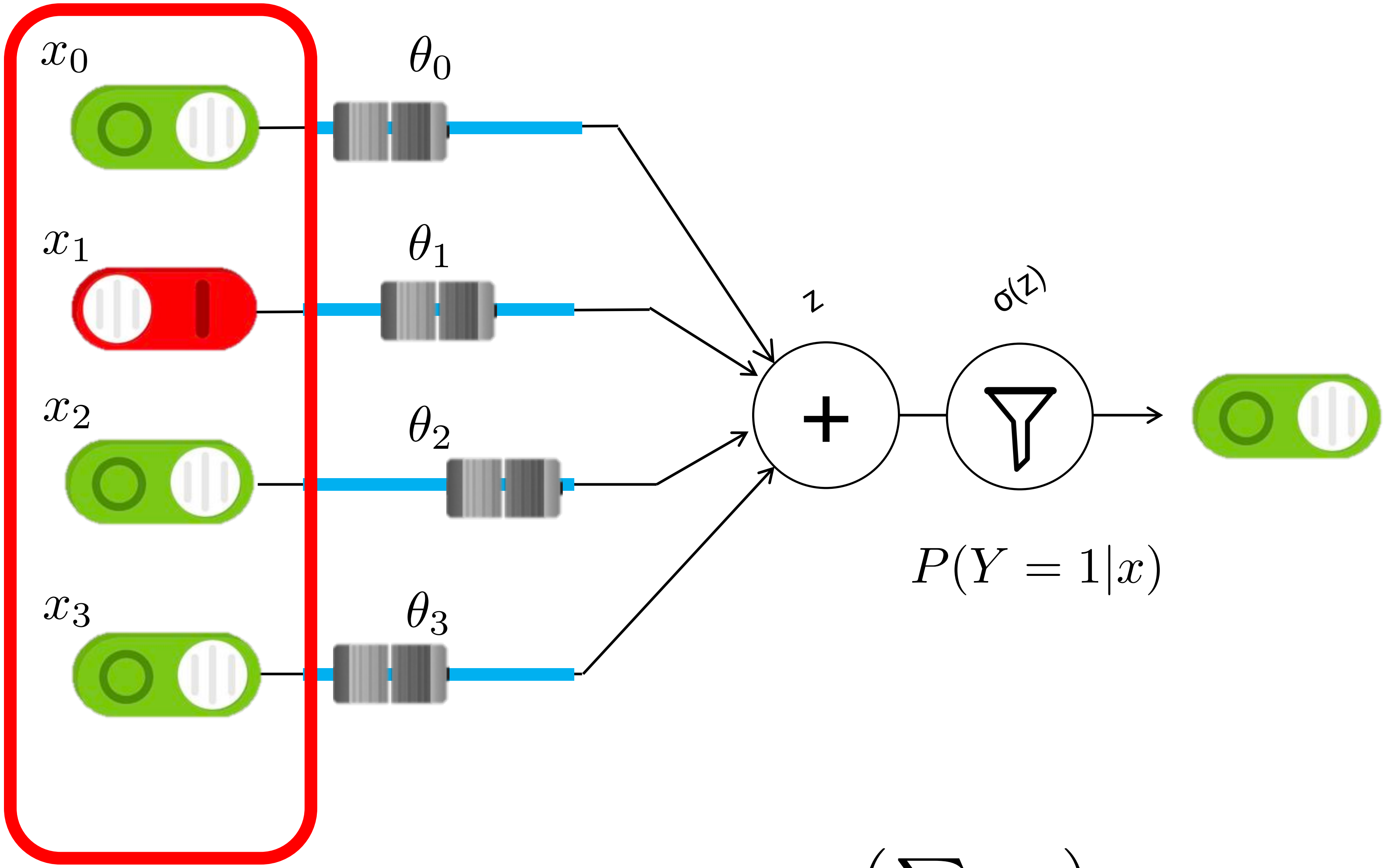
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs + Output



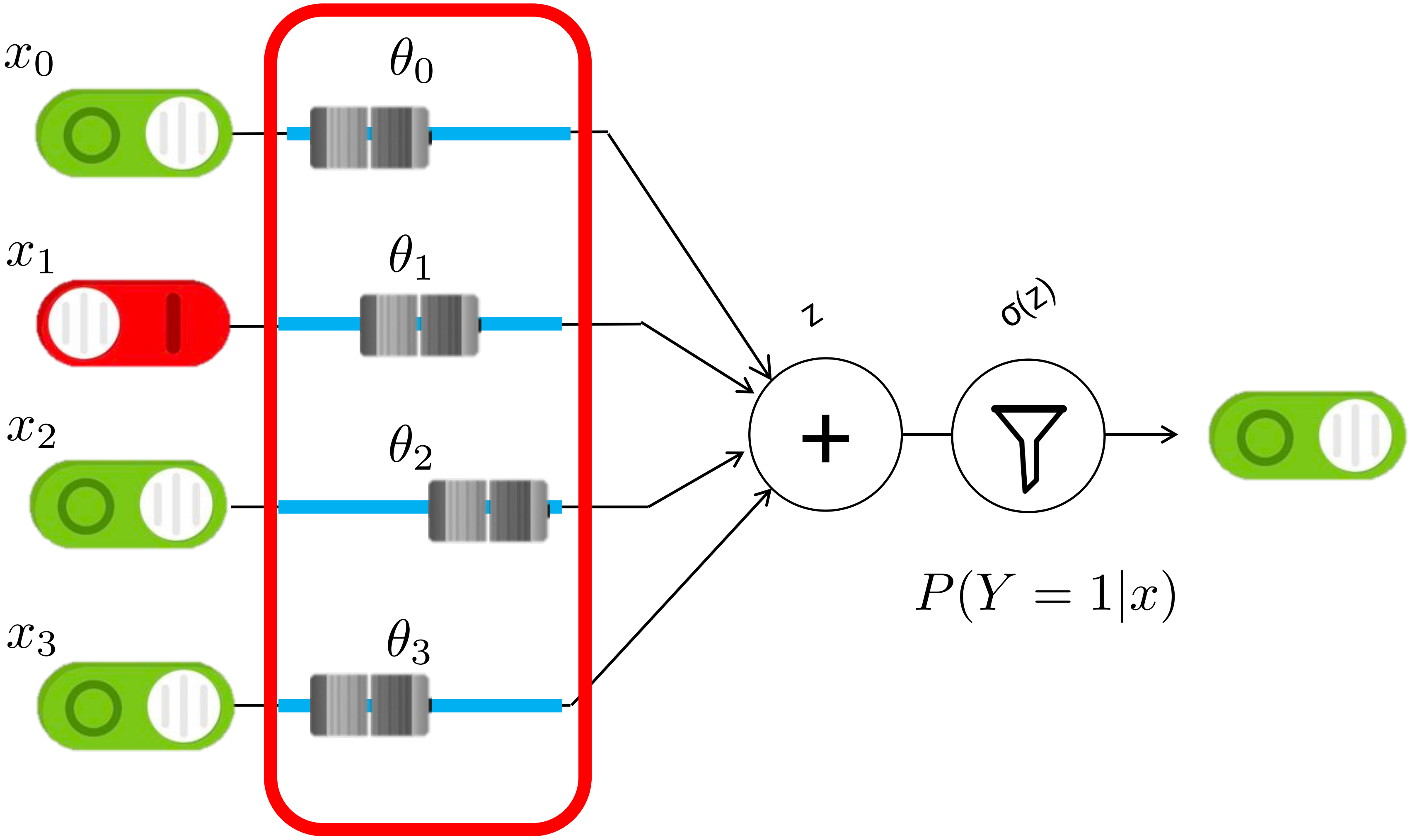
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

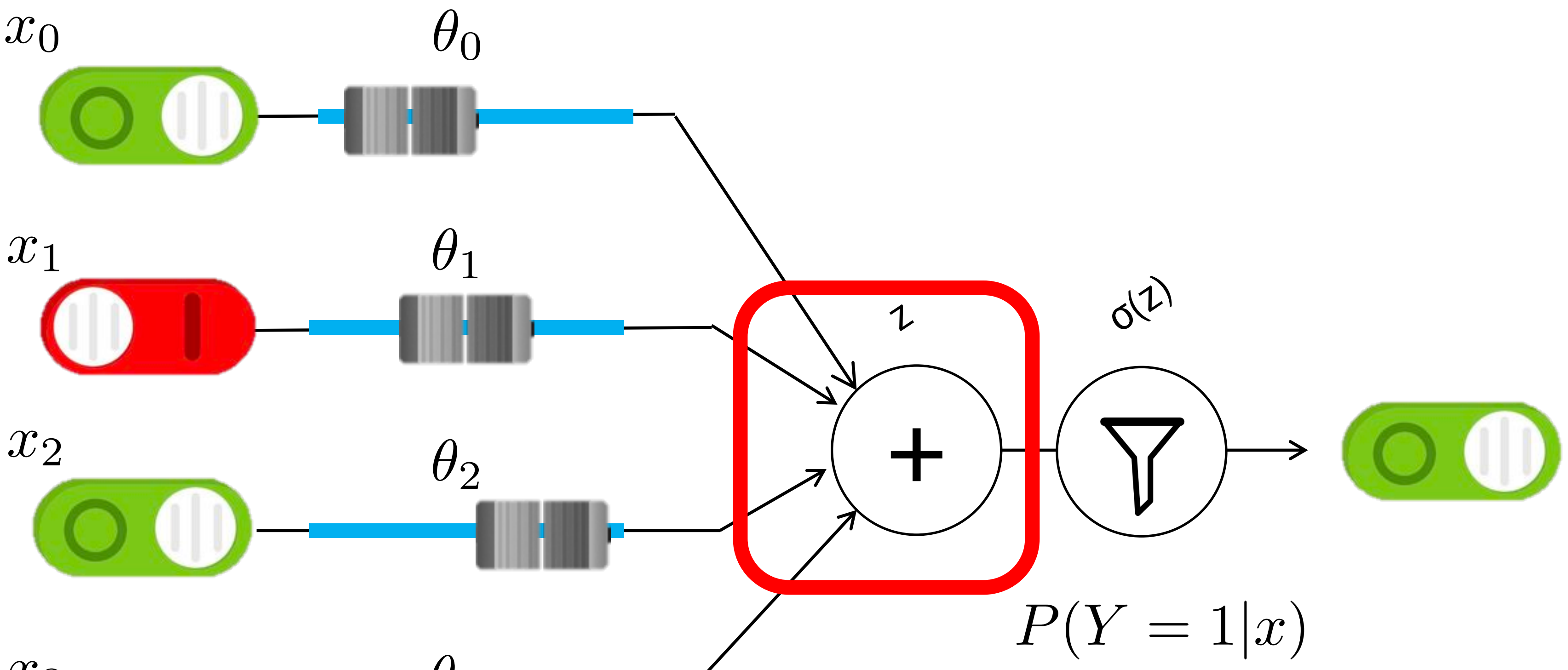
Weights



$P(Y = 1|x)$

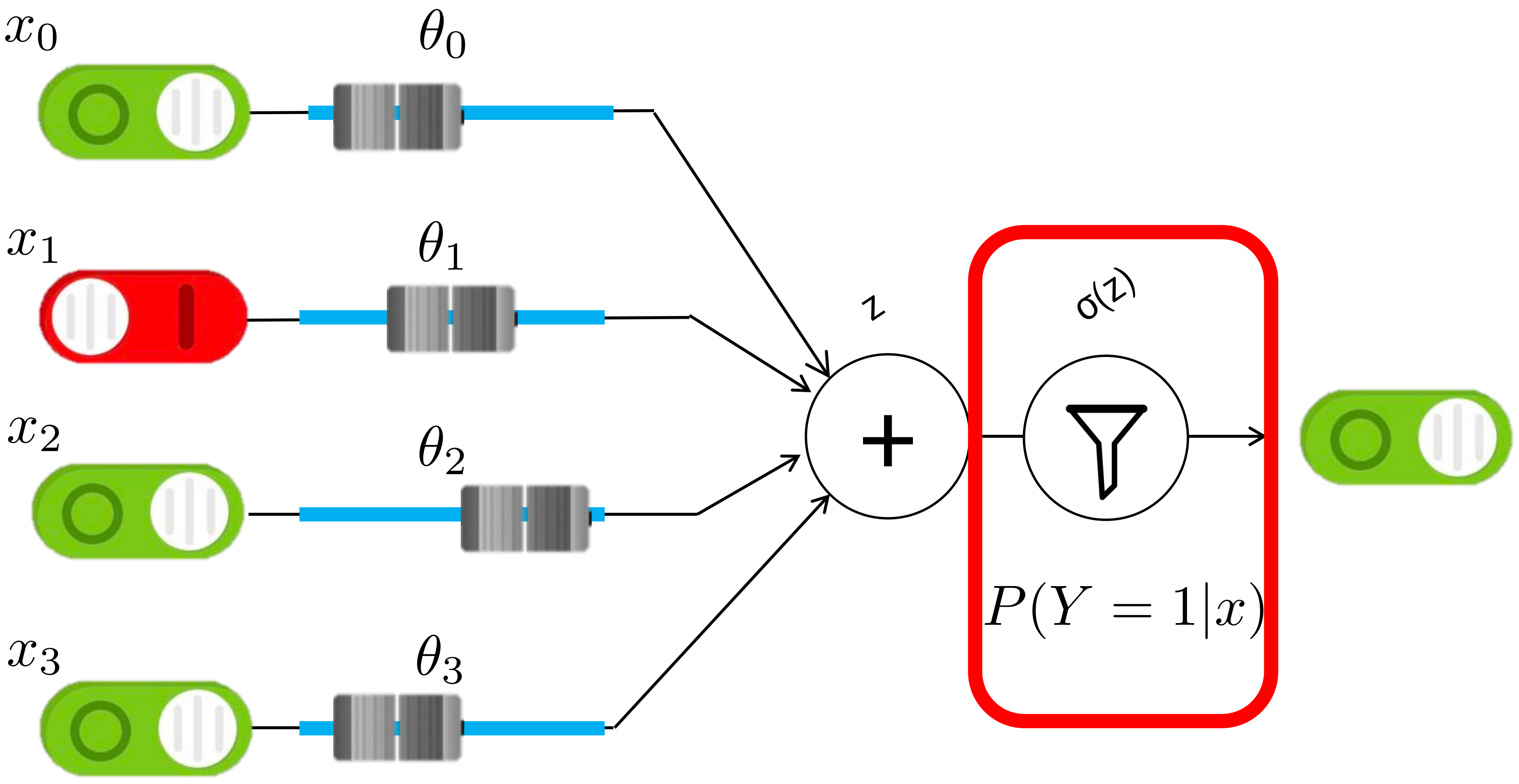
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Weighed Sum



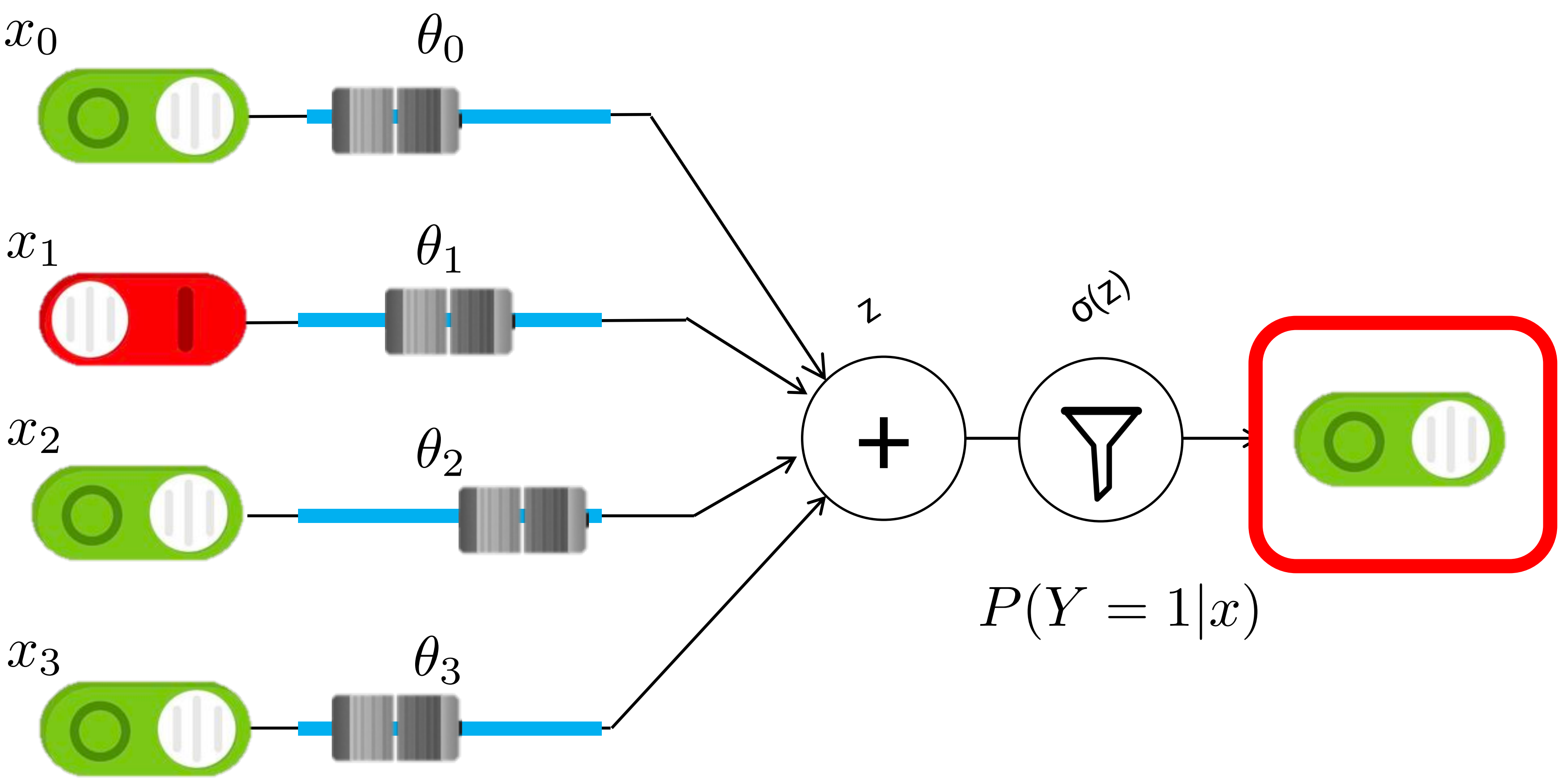
$$P(Y=1|\mathbf{X}=\mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Squashing Function



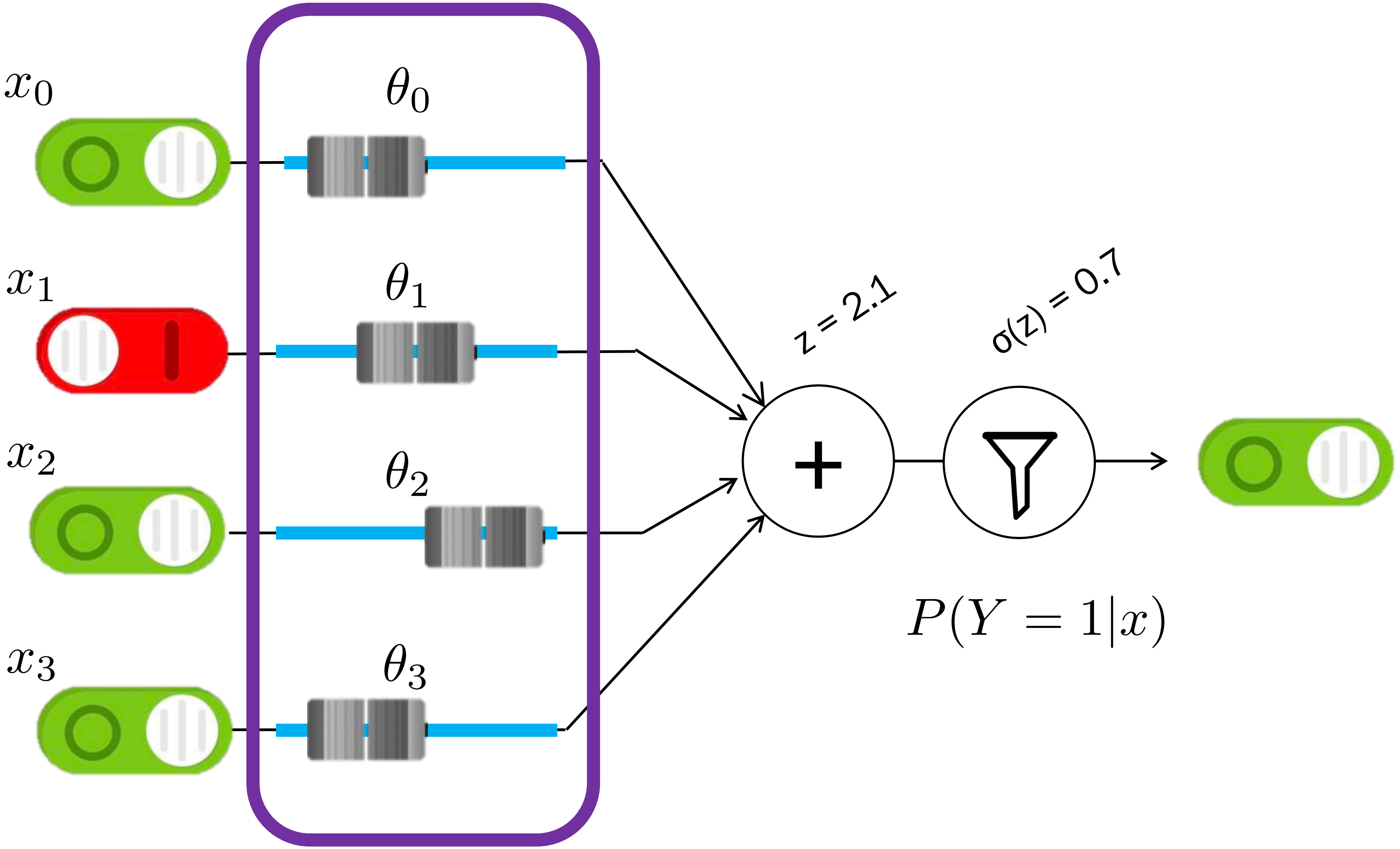
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Prediction



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Parameters Affect Prediction



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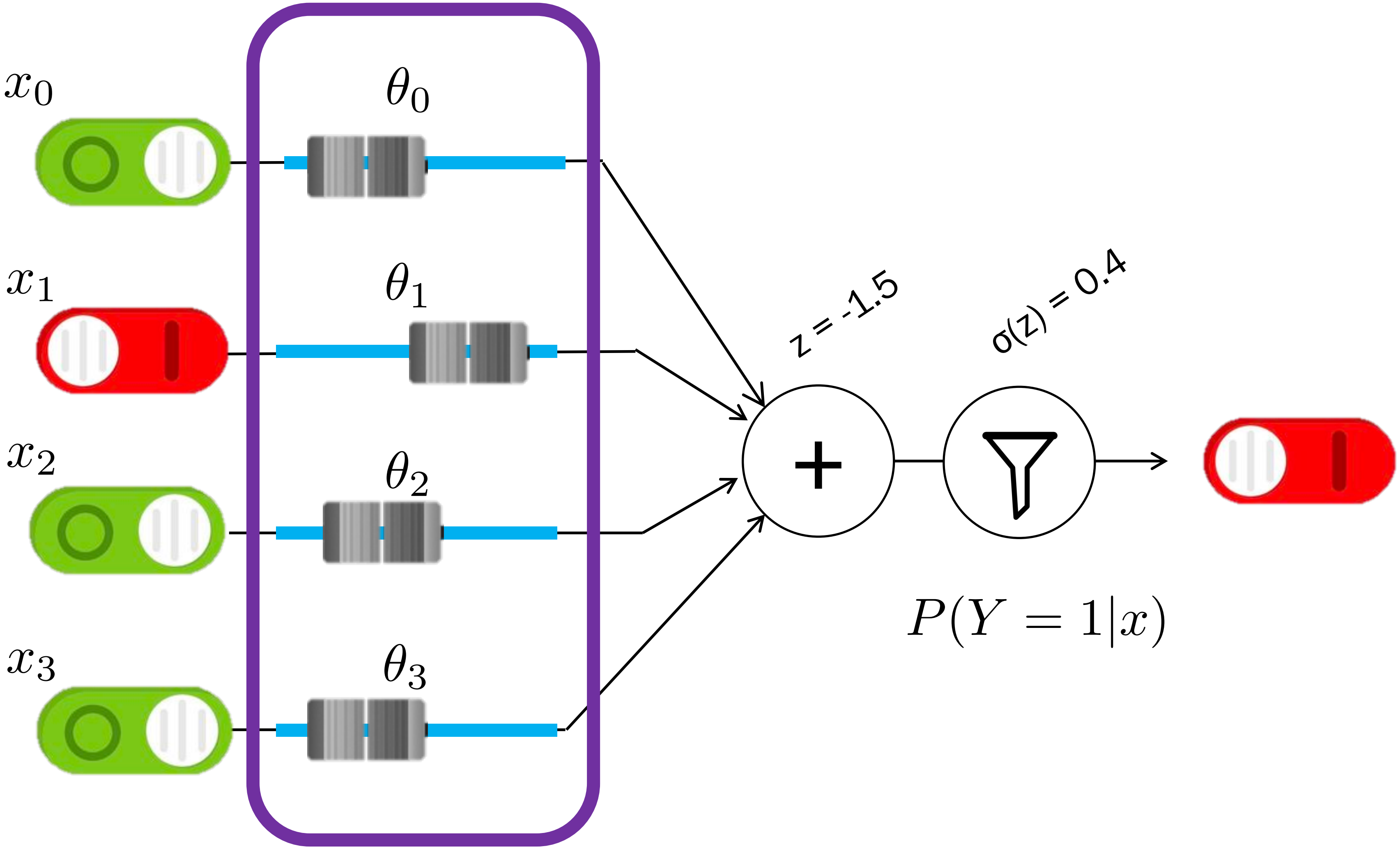
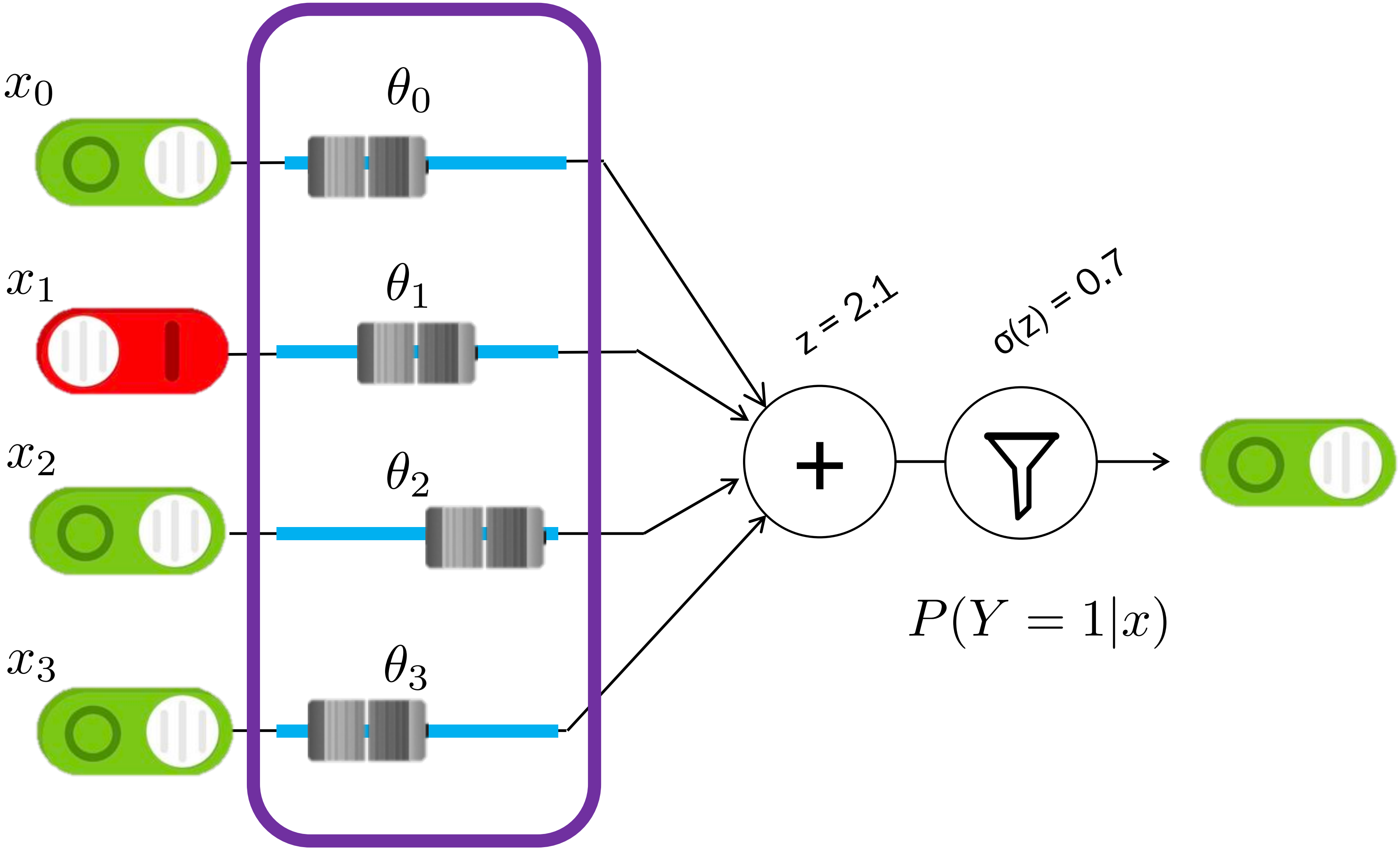
Parameters Affect Prediction



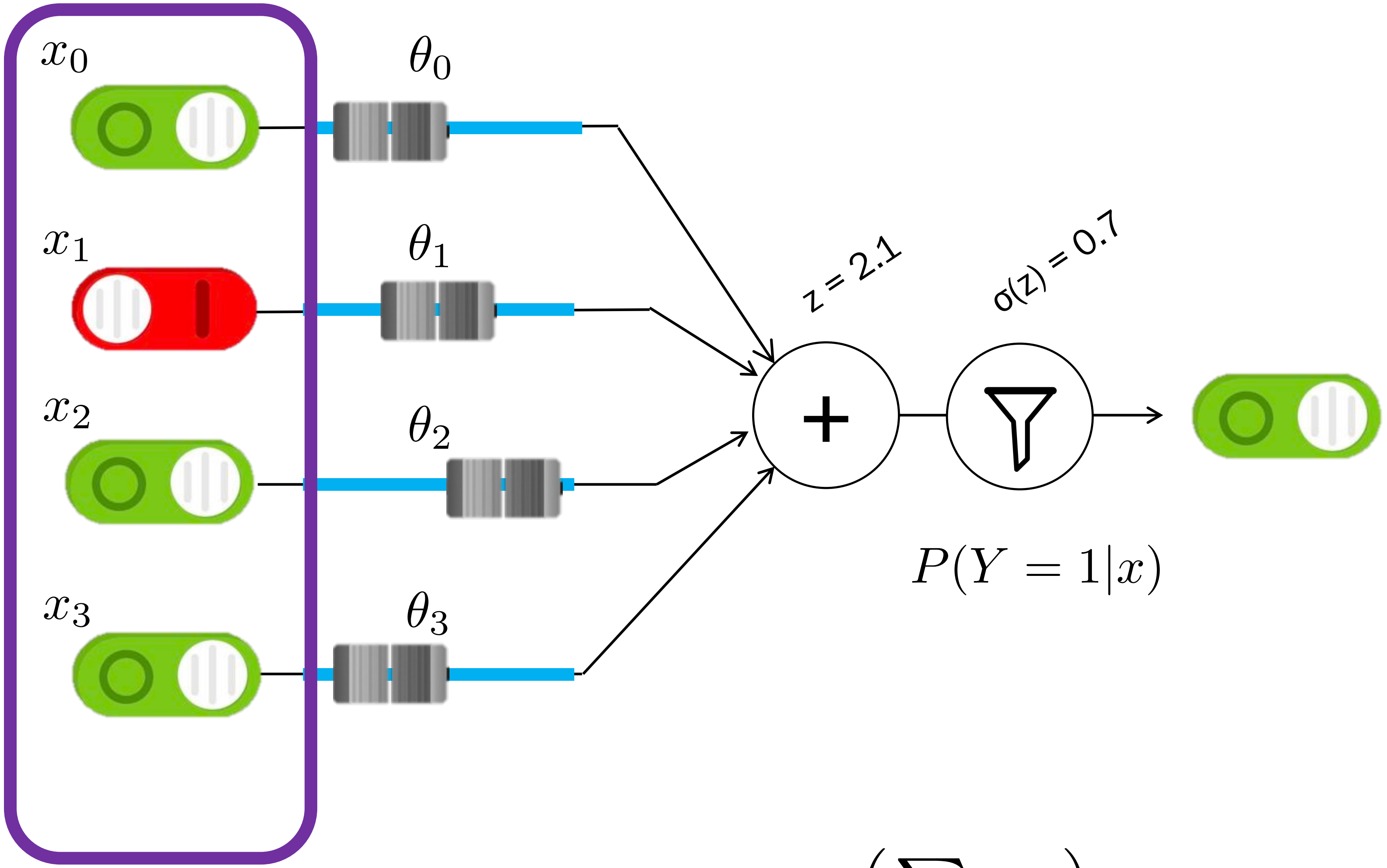
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Parameters Affect Prediction



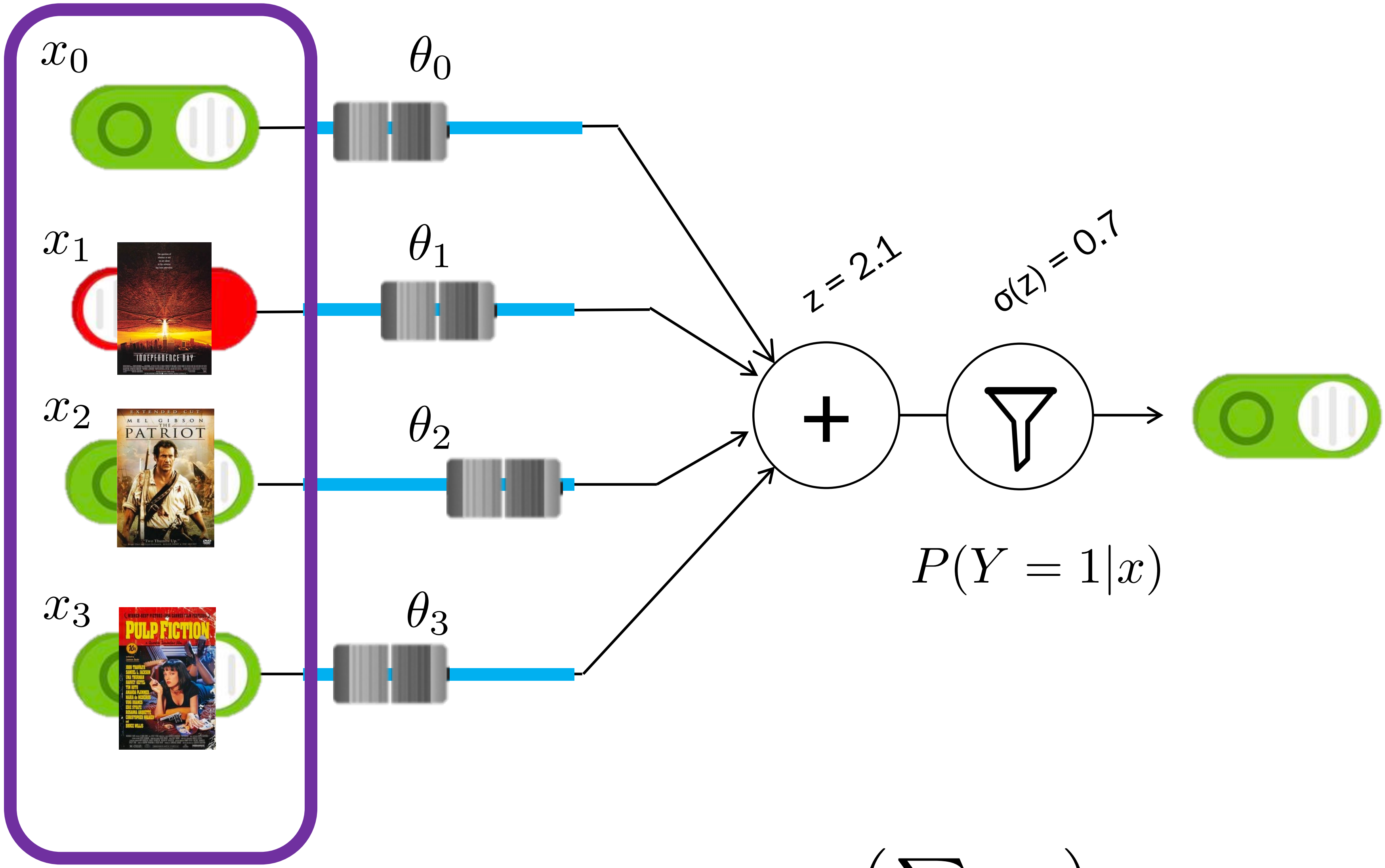
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Different Predictions for Different Inputs



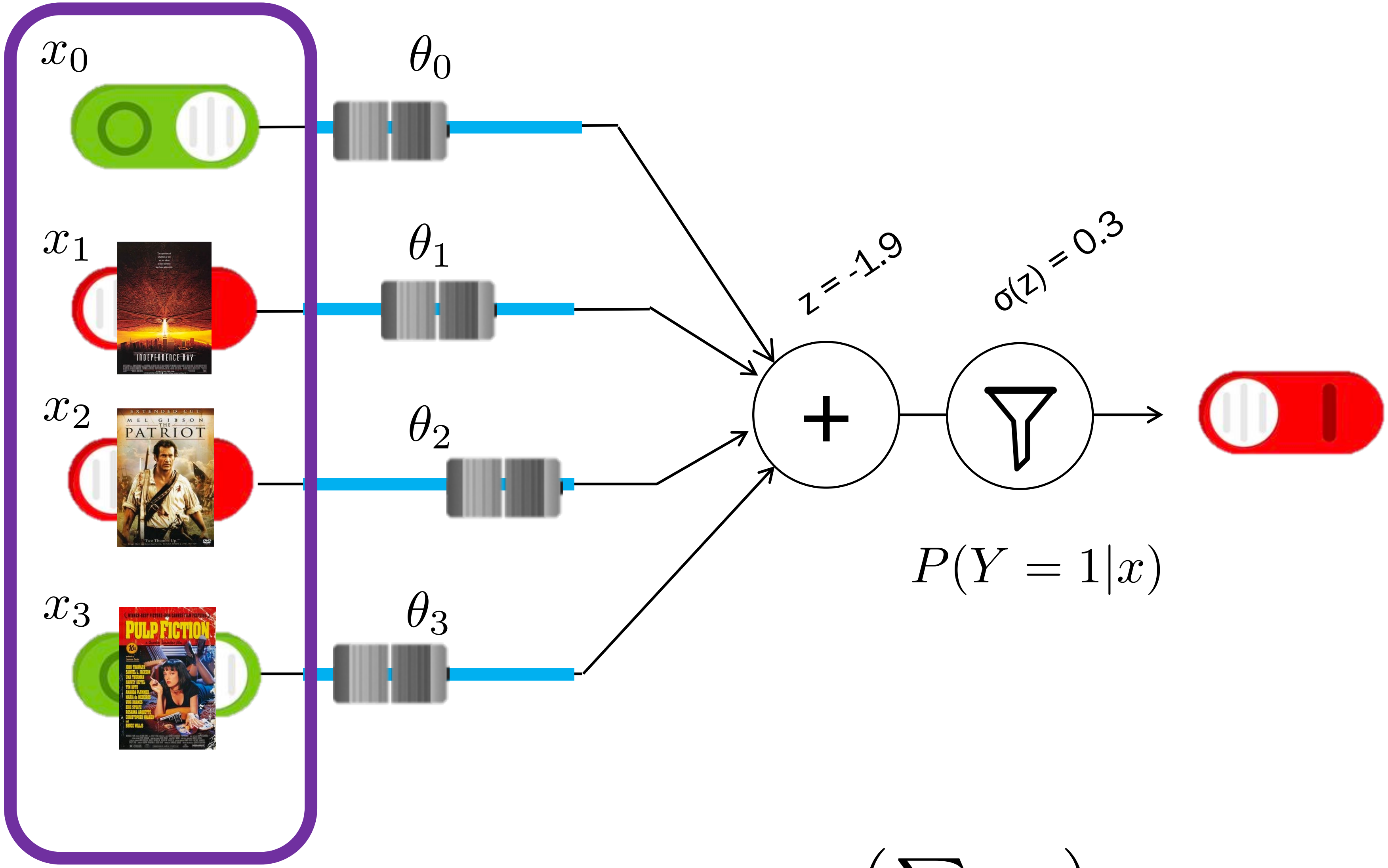
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Different Predictions for Different Inputs



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Different Predictions for Different Inputs



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression Assumption

Model *conditional* likelihood P(Y | **X**) directly

- Model this probability with *logistic* function:

$$P(Y=1|\mathbf{X}) = \sigma(z) \text{ where } z = \theta_0 + \sum_{i=1}^{m} \theta_i x_i$$

- For simplicity define $x_0 = 1$ so $z = \theta^T \mathbf{x}$

- Since $\mathrm{P}(Y = 0 \mid \mathbf{X}) + \mathrm{P}(Y = 1 \mid \mathbf{X}) = 1$:

$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

$$P(Y=0|X=\mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

Recall:
Sigmoid function

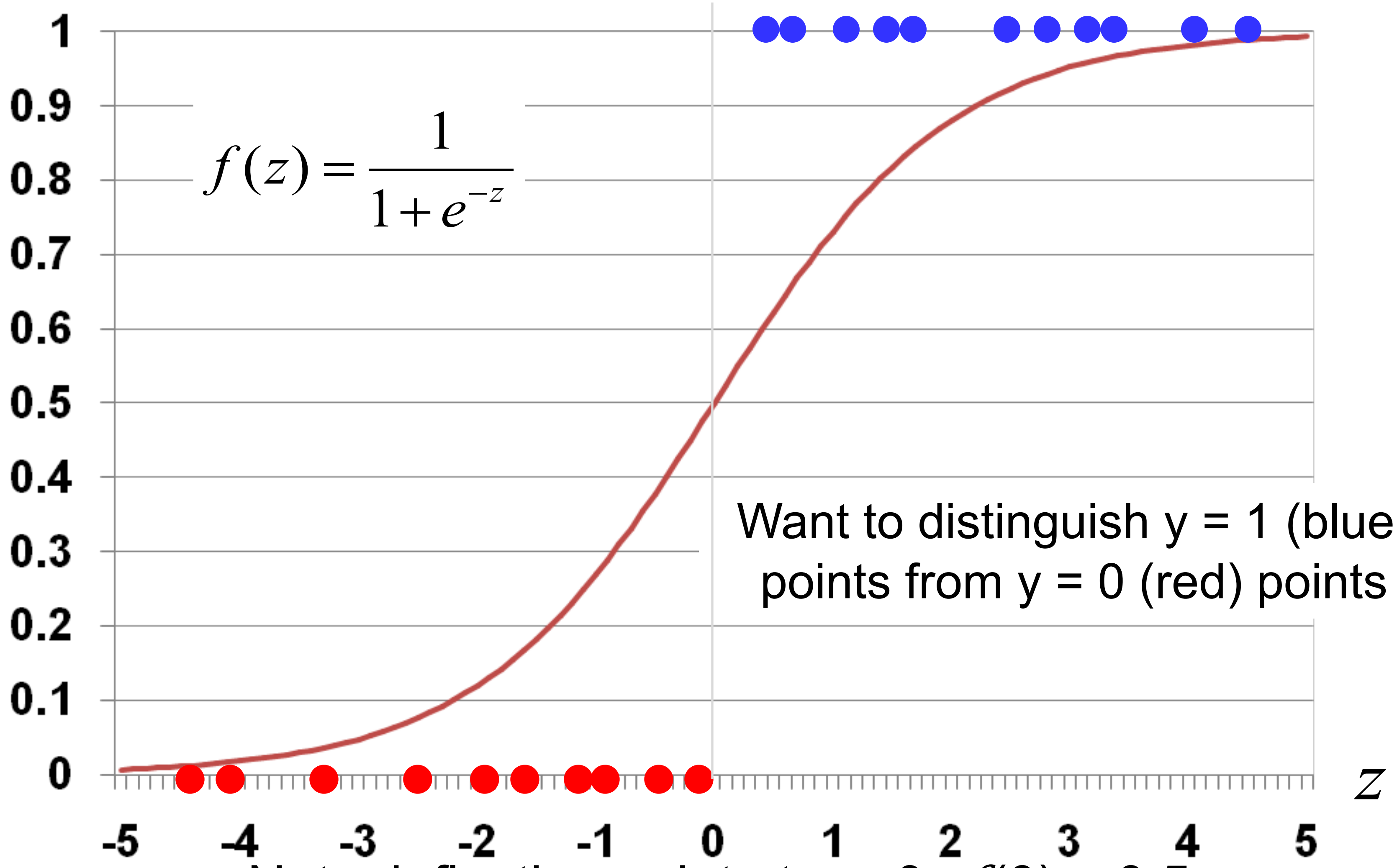
$$\sigma(z) = \frac{1}{1+e^{-z}}$$

Big Assumption

Logistic Regression Assumption:

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

The Sigmoid Function



$$f(z) = \frac{1}{1+e^{-z}}$$

Want to distinguish y = 1 (blue) points from y = 0 (red) points

Note: inflection point at z = 0. $f(0) = 0.5$

What is in a Name

Regression Algorithms

Linear Regression

Classification Algorithms

Naïve Bayes

Logistic Regression

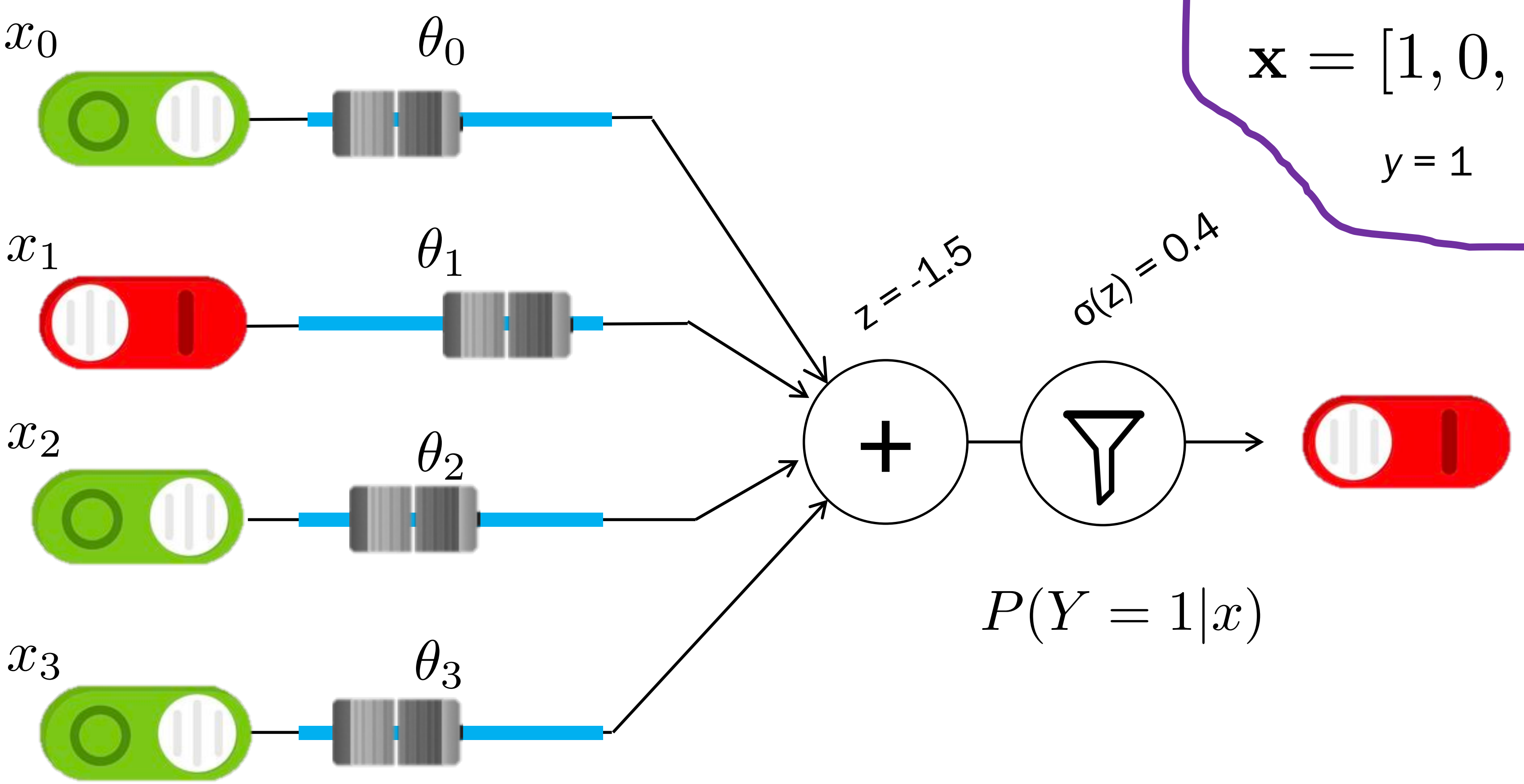
Awesome classifier,
terrible name

If Chris could rename it he would call it: Sigmoidal Classification

What makes for a “smart”
logistic regression algorithm?

Logistic regression gets its ***intelligence*** from its thetas (aka its parameters)

How Do We Learn Parameters?



Let's say that:

$$\mathbf{x} = [1, 0, 1, 1]$$

$y = 1$

$$P(Y=1|\mathbf{X}=\mathbf{x}) = \sigma\Big(\sum_i \theta_i x_i\Big) = 0.4$$

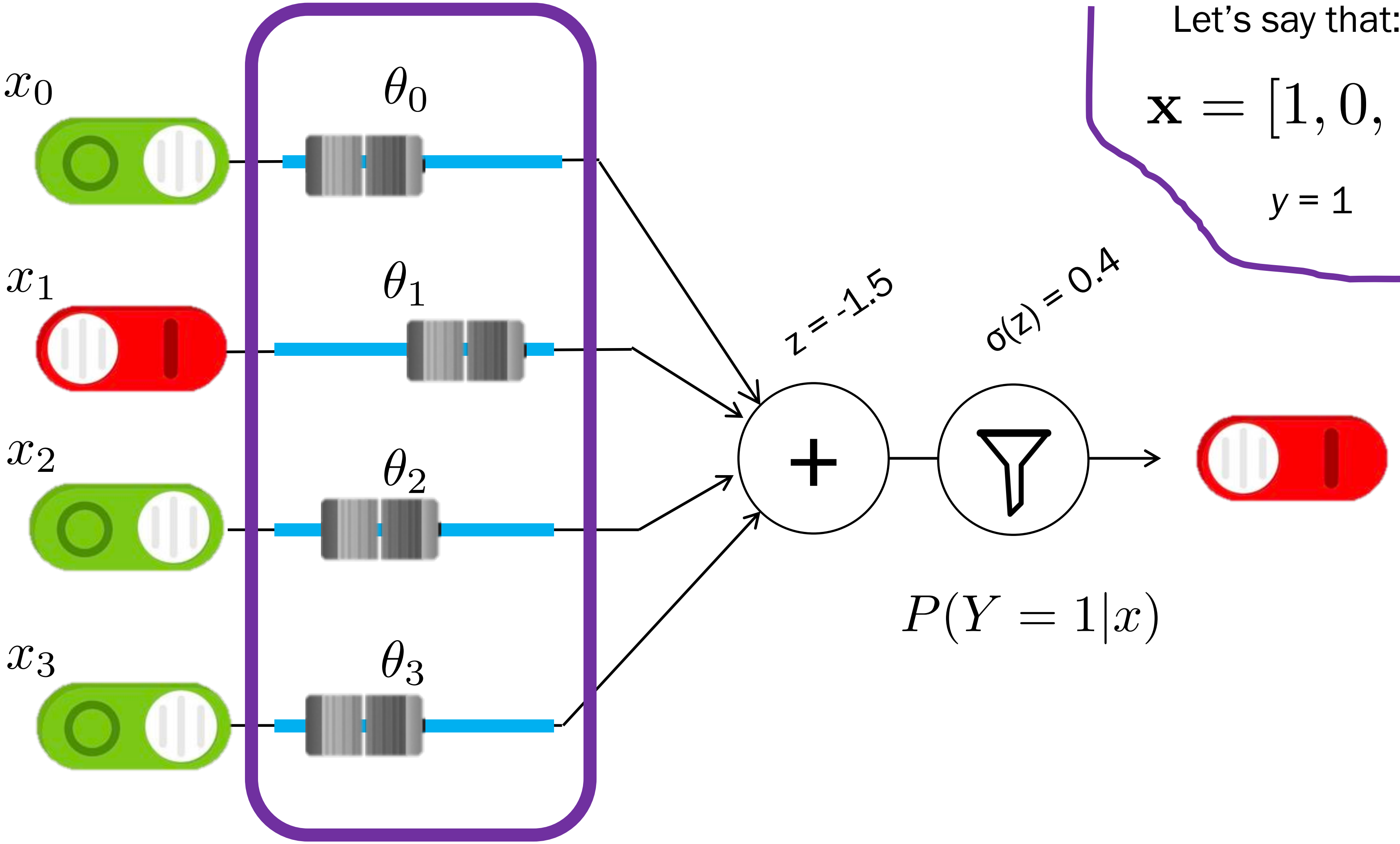
Data looks unlikely

How Do We Learn Parameters?

Let's say that:

$$\mathbf{x} = [1, 0, 1, 1]$$

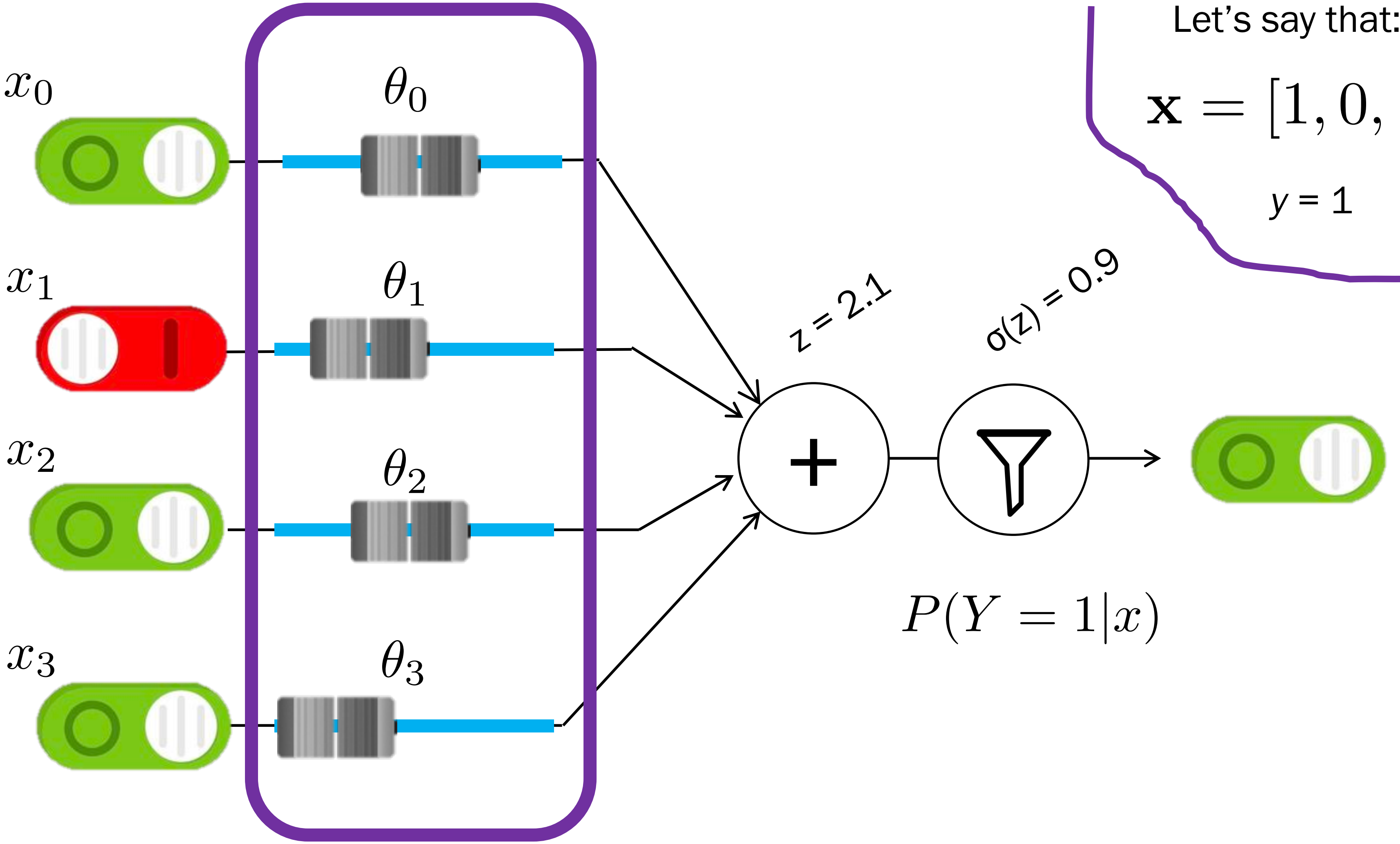
$y = 1$



Data looks unlikely

$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\Big(\sum_i \theta_i x_i\Big) = 0.4$$

How Do We Learn Parameters?



Let's say that:

$$\mathbf{x} = [1, 0, 1, 1]$$

$y = 1$

$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right) = 0.9$$

Data is much more likely!

Maximum Likelihood Estimation

Chose your parameter estimates

Parameter μ: 5 Parameter σ: 1.1

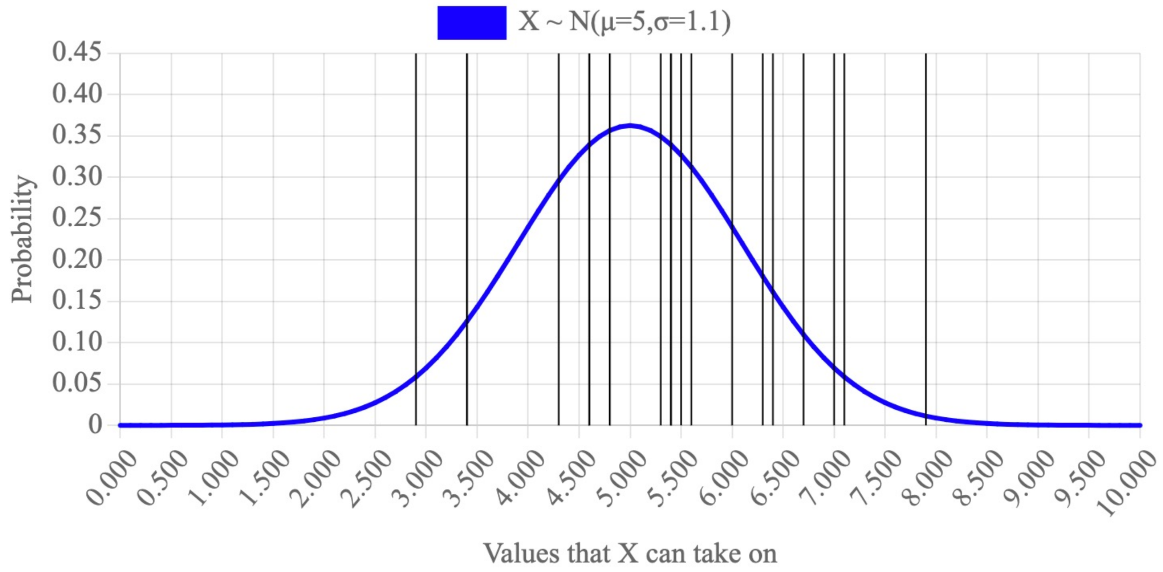
Likelihood of the data given your params

Likelihood: 5.204152095194613e-16

Log Likelihood: -314.1

Best Seen: -311.2

Your Gaussian



Math for Logistic Regression

1. Make logistic regression assumption

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

Often call this $\hat{y}$

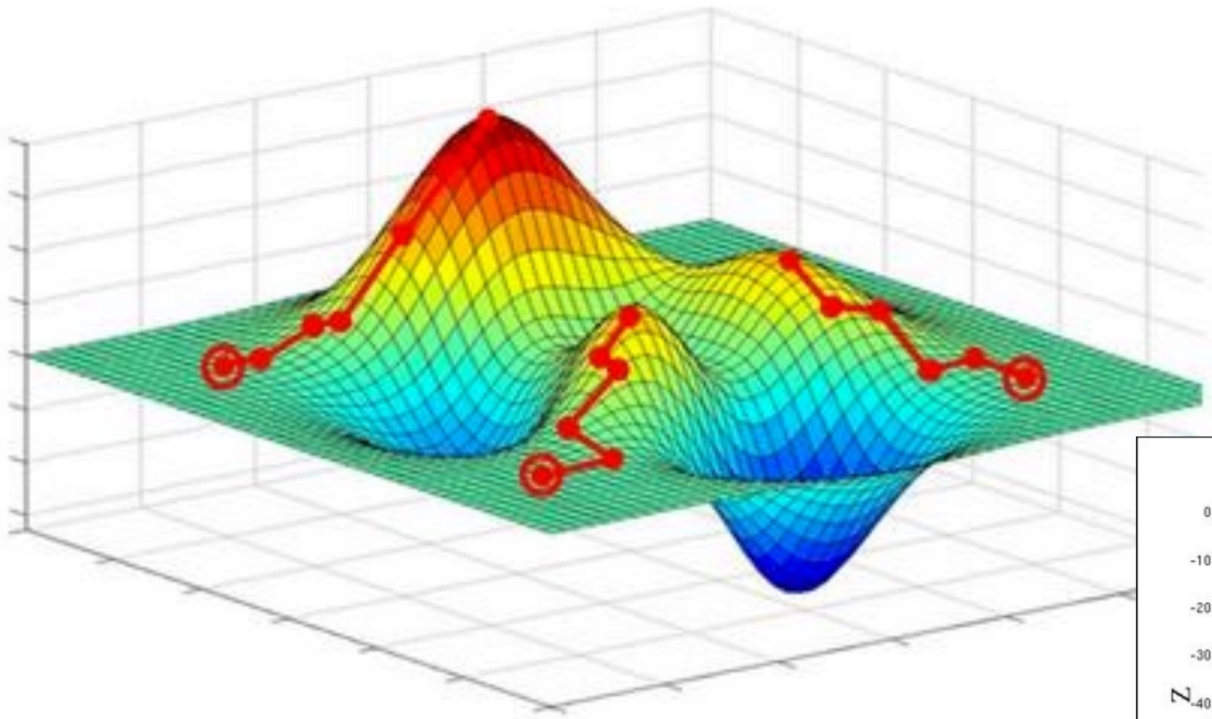
2. Calculate the log likelihood for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

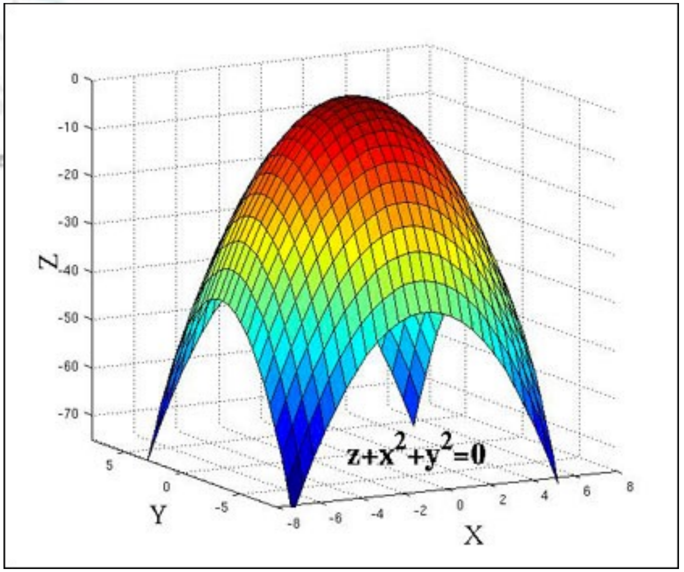
3. Get derivative of log likelihood with respect to thetas

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Gradient Ascent



Logistic regression LL function is convex



Walk uphill and you will find a local maxima (if your step size is small enough)

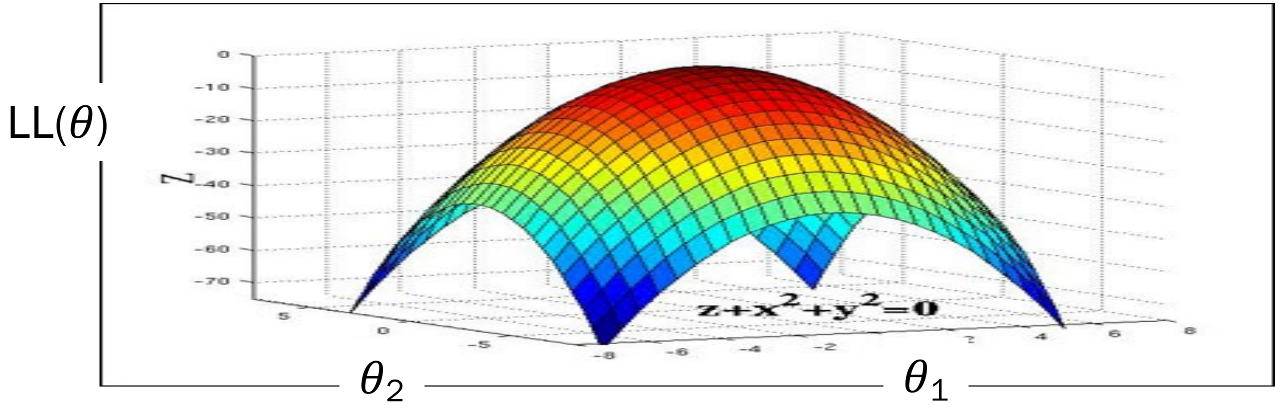
Gradient Ascent Step

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

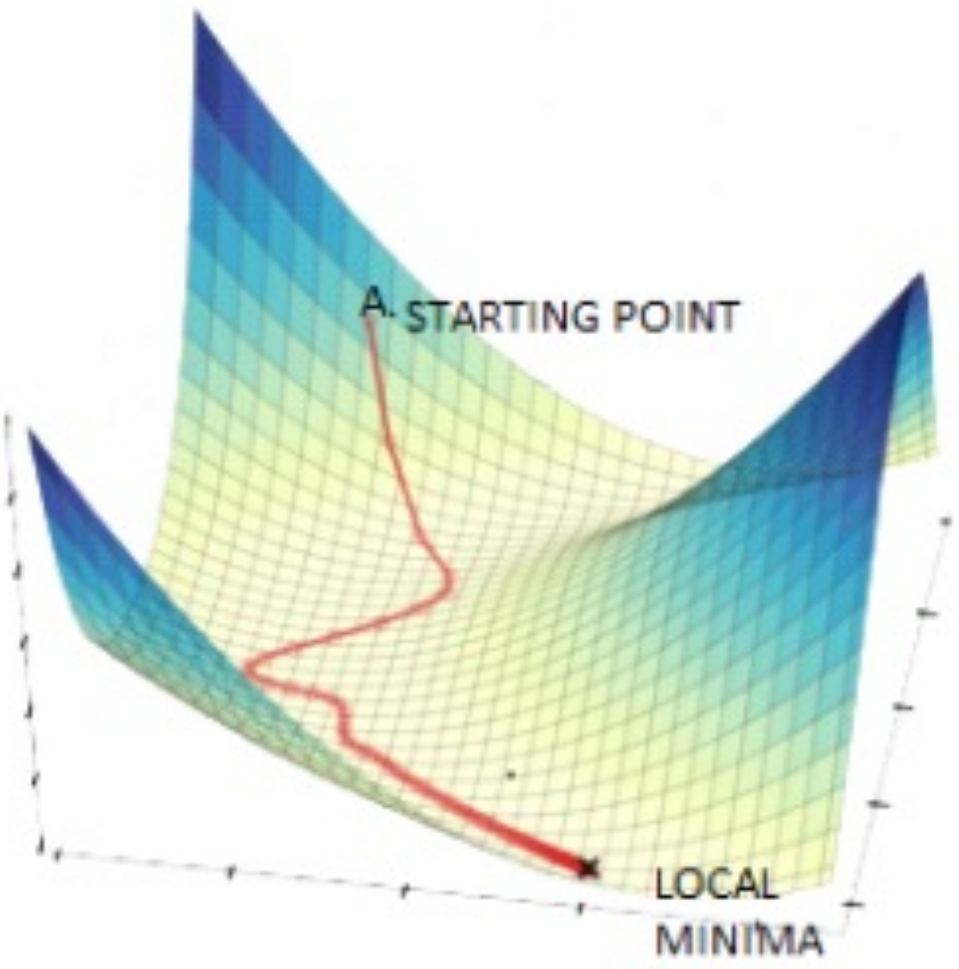
$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \frac{\partial LL(\theta^{\text{old}})}{\partial \theta_j^{\text{old}}}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Do this for all thetas!



Gradient Decent



Walk downhill and you will find a local maxima
(if your step size is small enough)

Gradient Descent with Negative LL

Assume some loss function with known derivative $\dfrac{\partial \text{ Loss}}{\partial \theta_j}$

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Loss}}{\partial \theta_j}$$

Gradient Descent with Negative LL

Assume some loss function with known derivative $\dfrac{\partial \text{ Loss}}{\partial \theta_j}$

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Loss}}{\partial \theta_j}$$

$$= \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Negative}LL}{\partial \theta_j}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Gradient Descent with Negative LL

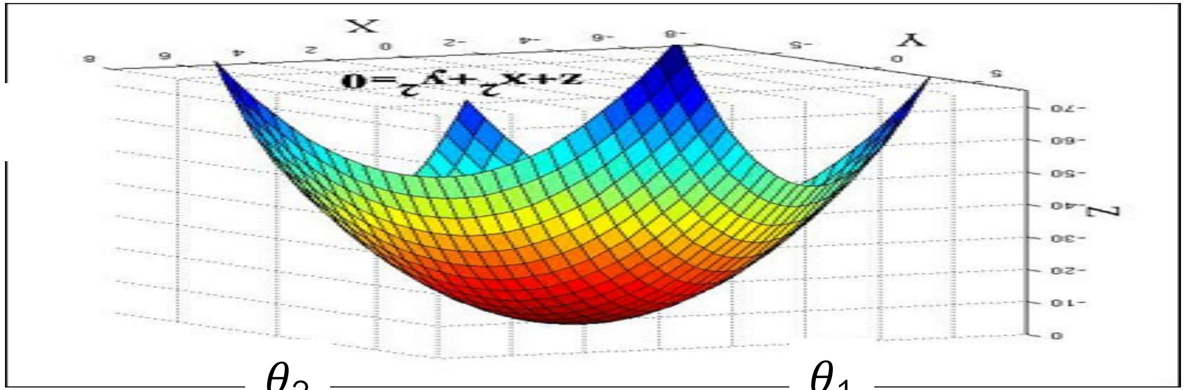
Assume some loss function with known derivative $\dfrac{\partial \text{ Loss}}{\partial \theta_j}$

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Loss}}{\partial \theta_j}$$

$$= \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Negative}LL}{\partial \theta_j}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

...exactly the same



What does this look like in code?

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \frac{\partial LL(\theta^{\text{ old}})}{\partial \theta_j^{\text{old}}}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Logistic Regression Training

Initialize: $\theta_j = 0$ for all $0 \leq j \leq m$

Calculate all θ_j

Logistic Regression Training

Initialize: θ_j **= 0 for all 0 ≤** j **≤** m

Repeat many times:

gradient[j] = 0 for all 0 ≤ j **≤** m

Calculate all **gradient[j]*****'s based on data***

θ_j **+= η * gradient[j] for all 0 ≤** j **≤** m

Logistic Regression Training

Initialize: $\theta_j = 0$ **for all** $0 \leq j \leq m$

Repeat many times:

`gradient[j] = 0` **for all** $0 \leq j \leq m$

For each training example (x, y):

For each parameter j:

Update `gradient[j]` *for current training example*

θ_j `+=` η `* gradient[j]` **for all** $0 \leq j \leq m$

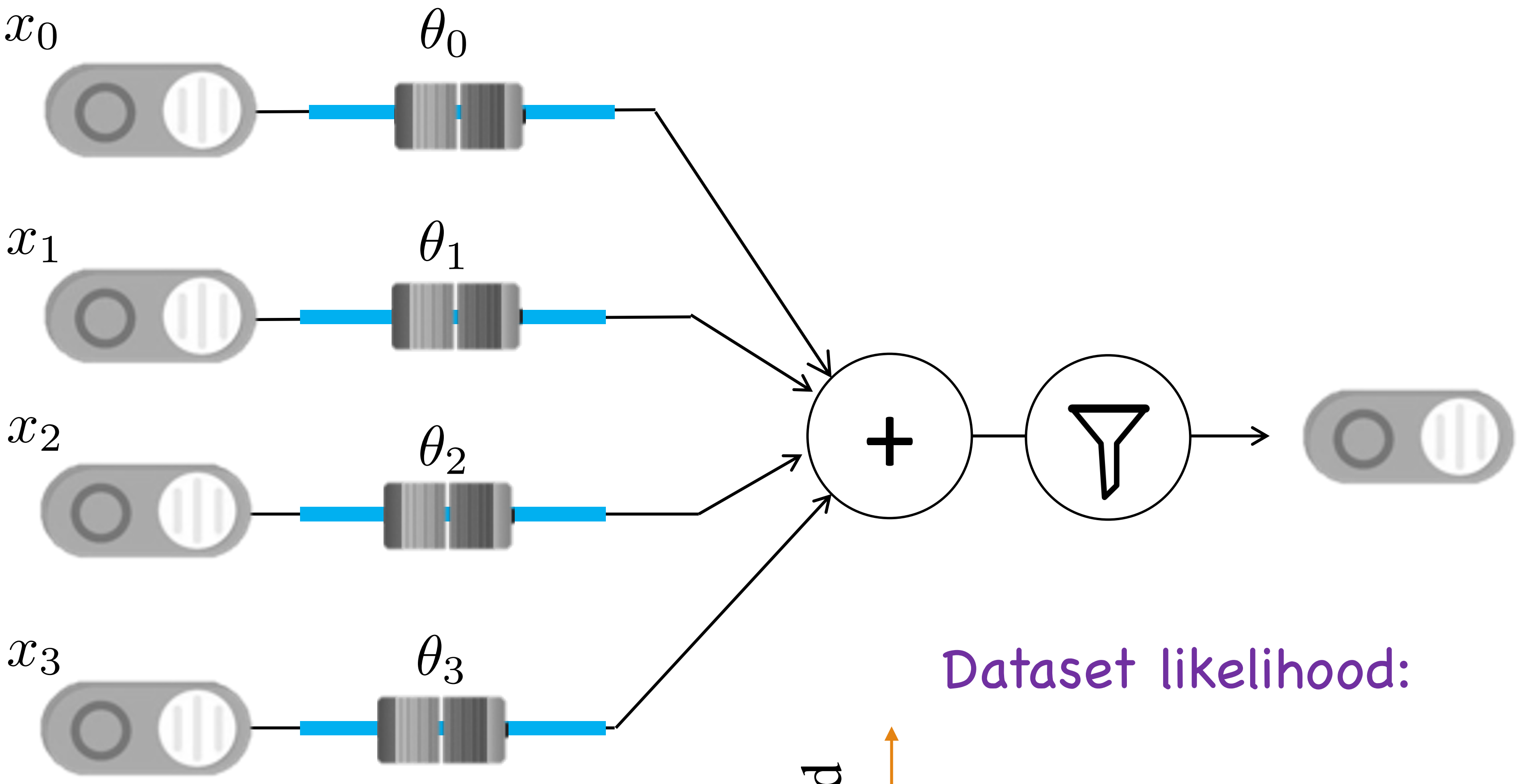
Logistic Regression Training

Initialize: $\theta_j = 0$ **for all** $0 \le j \le m$

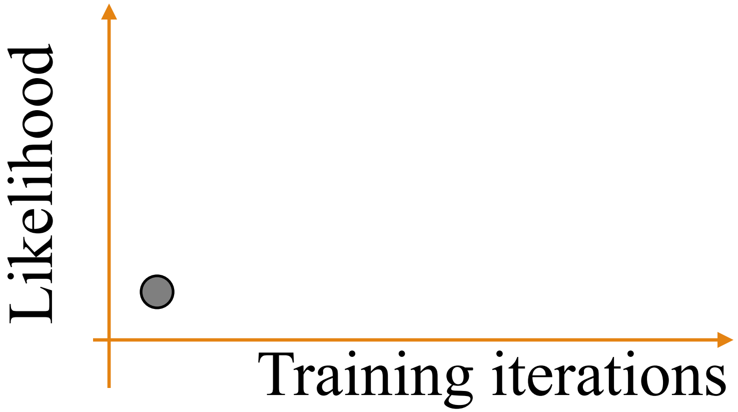
Repeat many times:

> **gradient[j] = 0 for all** $0 \le j \le m$
>
> **For each training example (x, y):**
>
> > **For each parameter j:**
> >
> > > **gradient[j]** $\mathrel{+}= x_j \left(y - \dfrac{1}{1 + e^{-\theta^{\mathrm{T}} \mathbf{x}}} \right)$
>
> θ_j **+=** η *** gradient[j] for all** $0 \le j \le m$

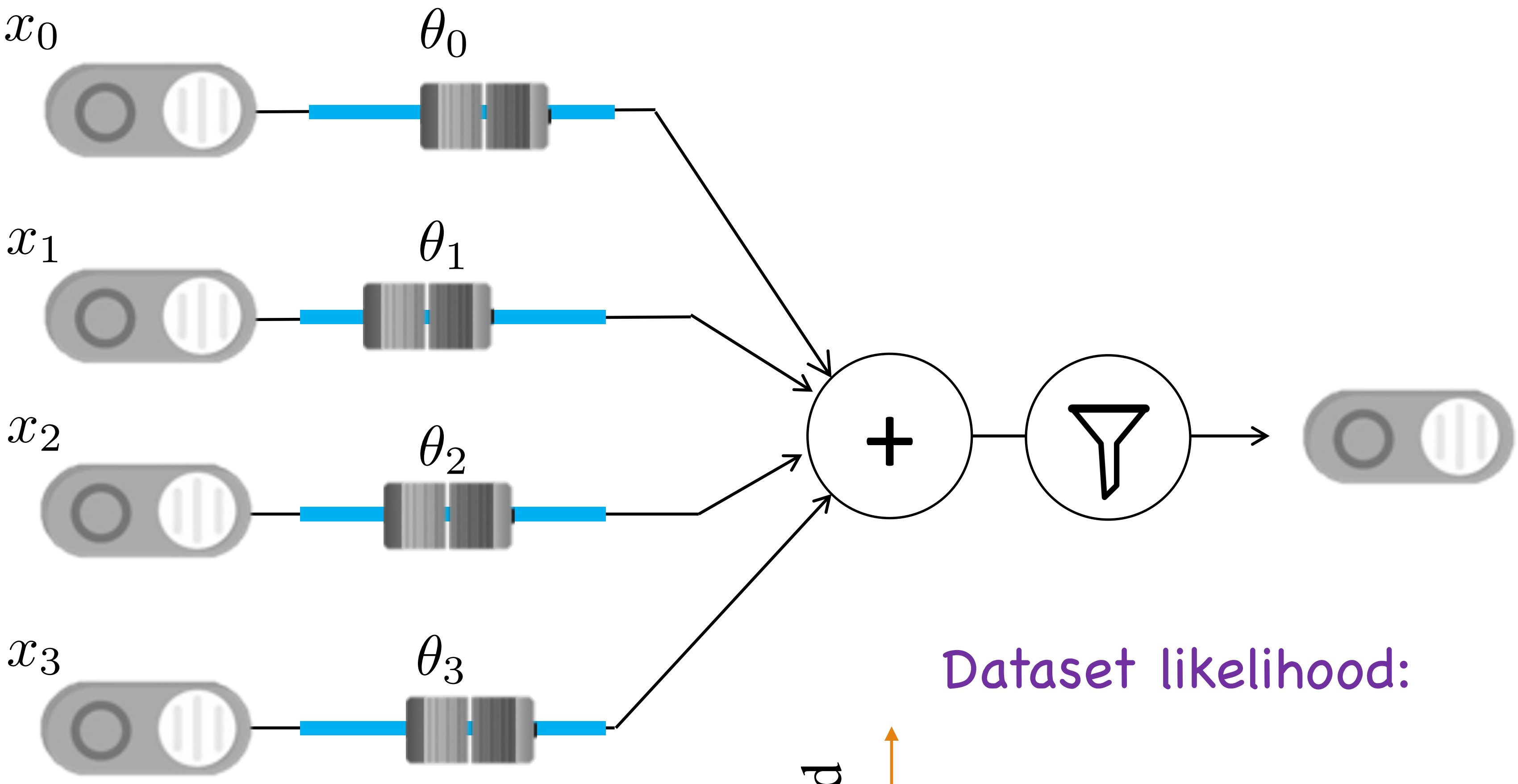
Training



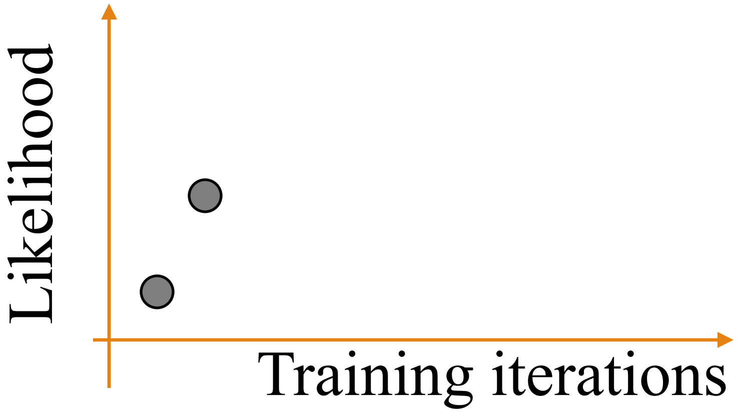
Dataset likelihood:



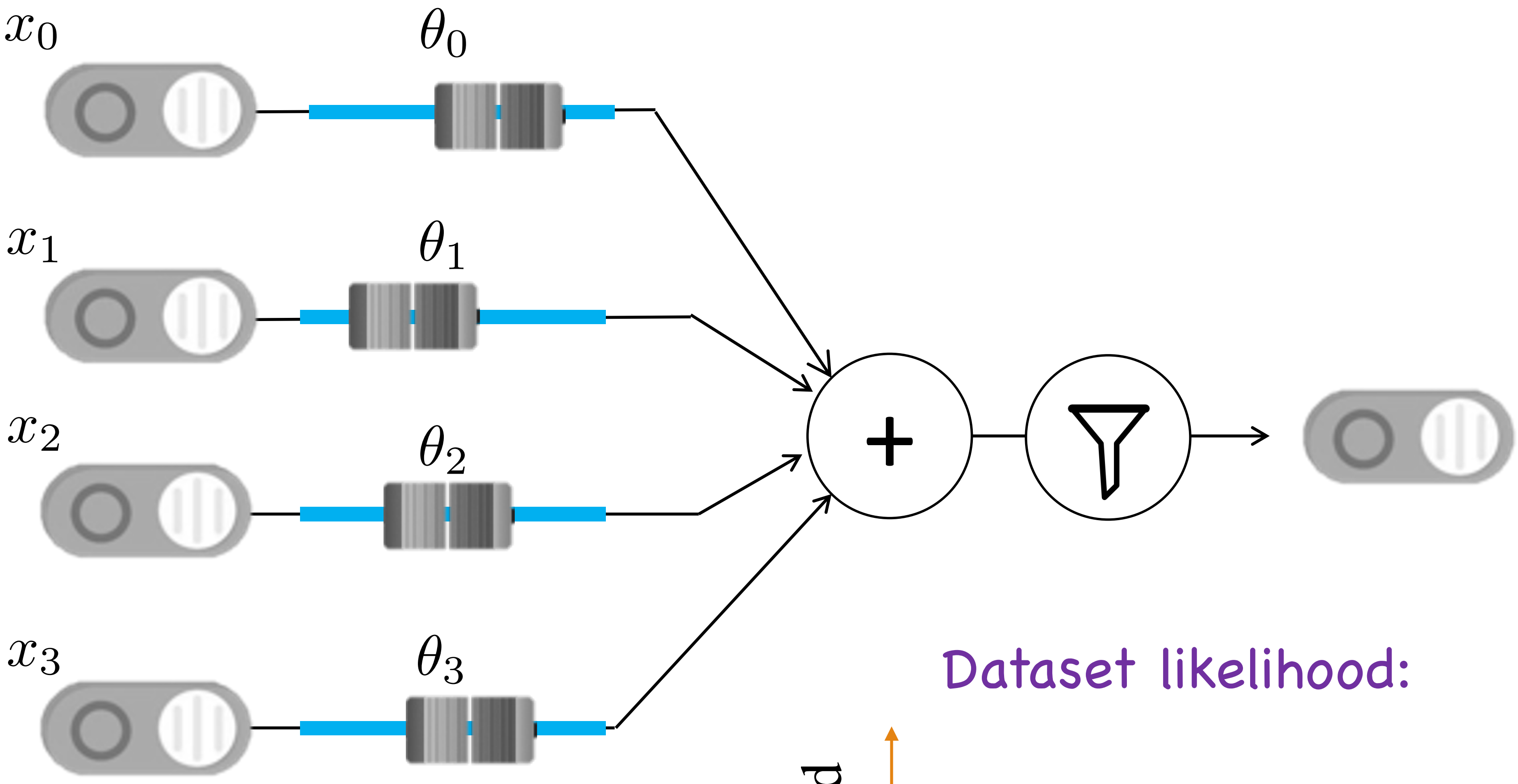
Training



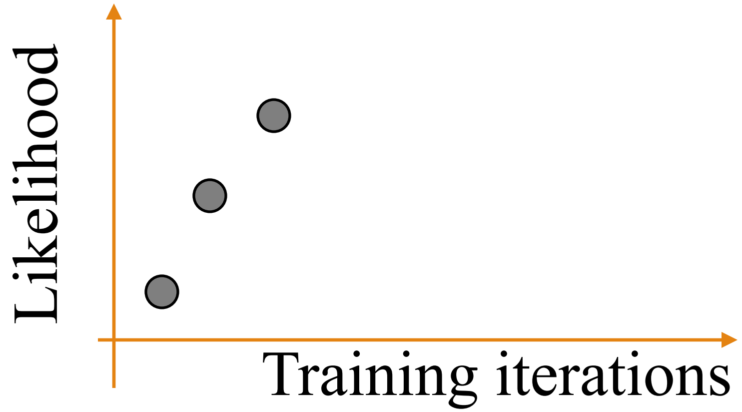
Dataset likelihood:



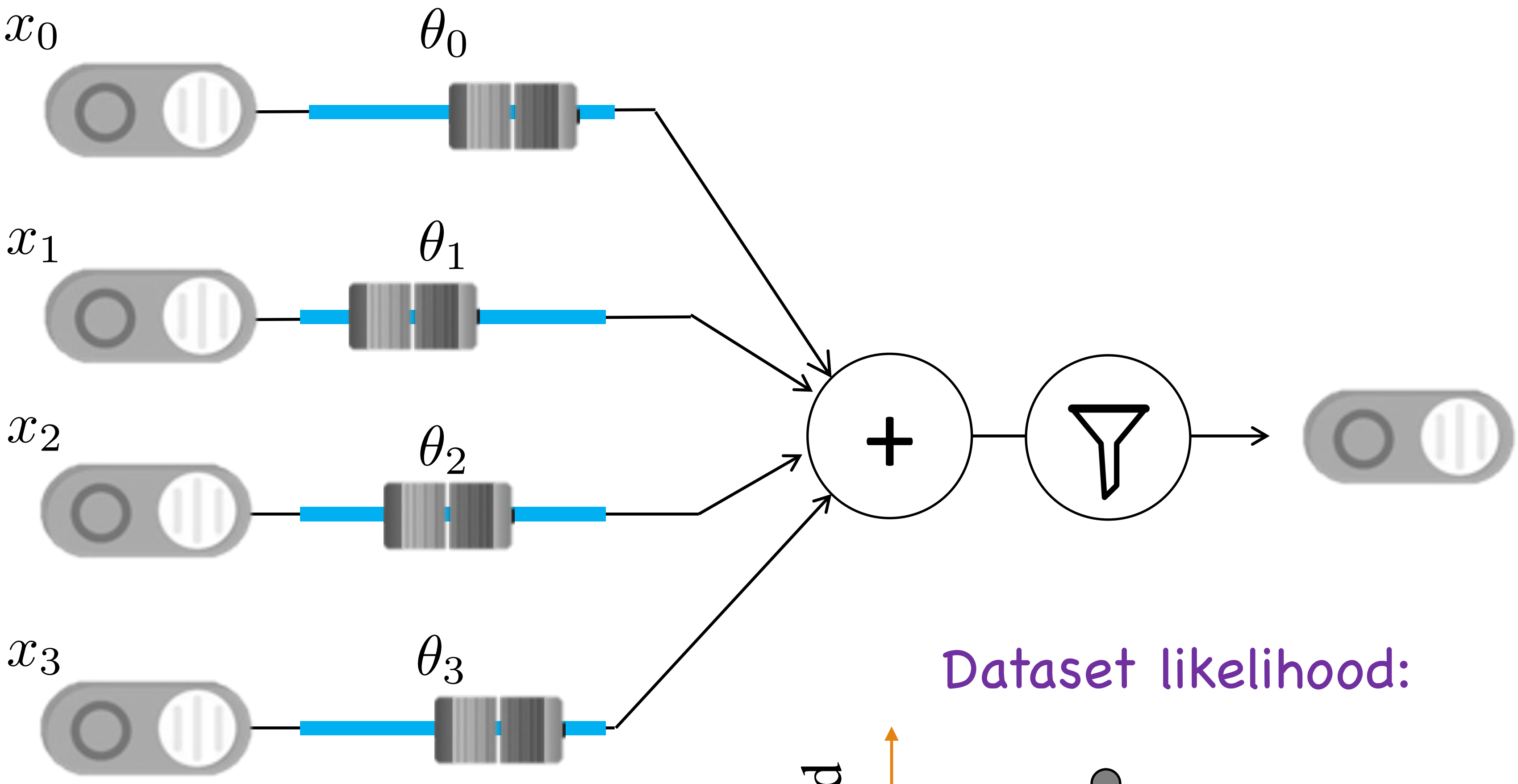
Training



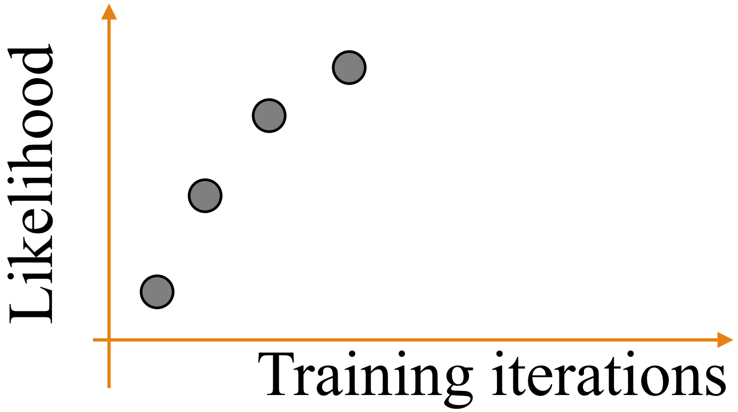
Dataset likelihood:



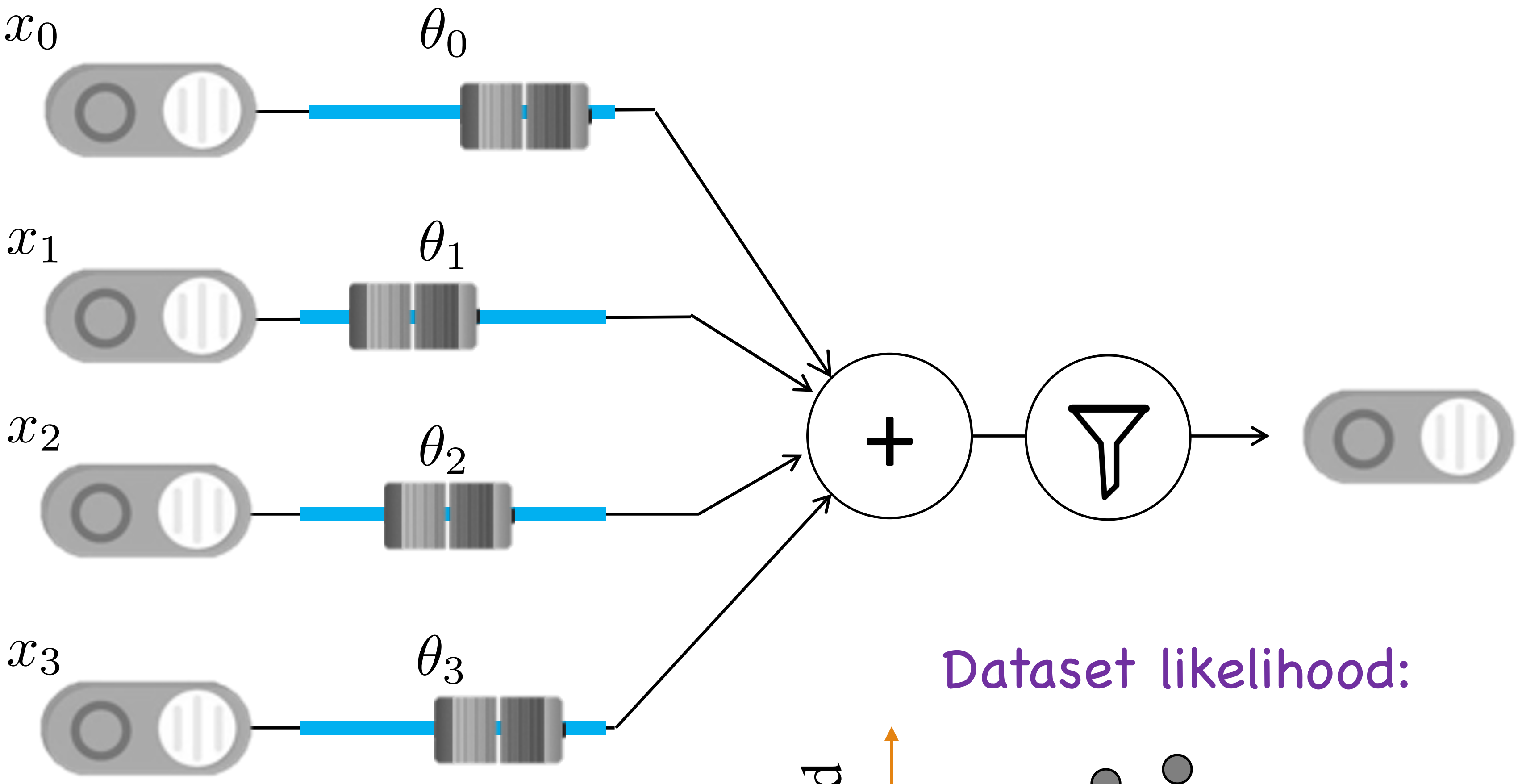
Training



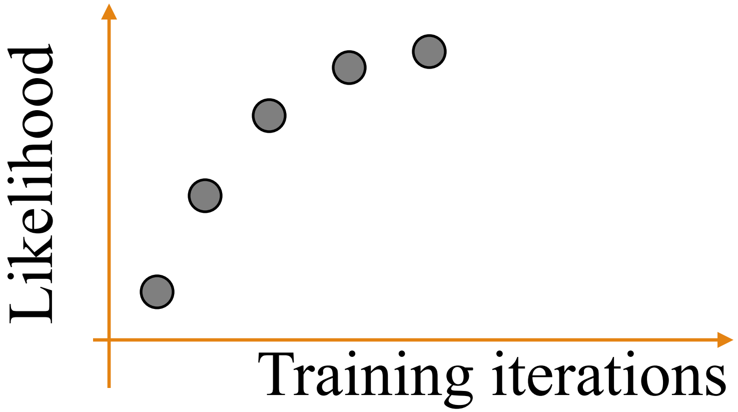
Dataset likelihood:



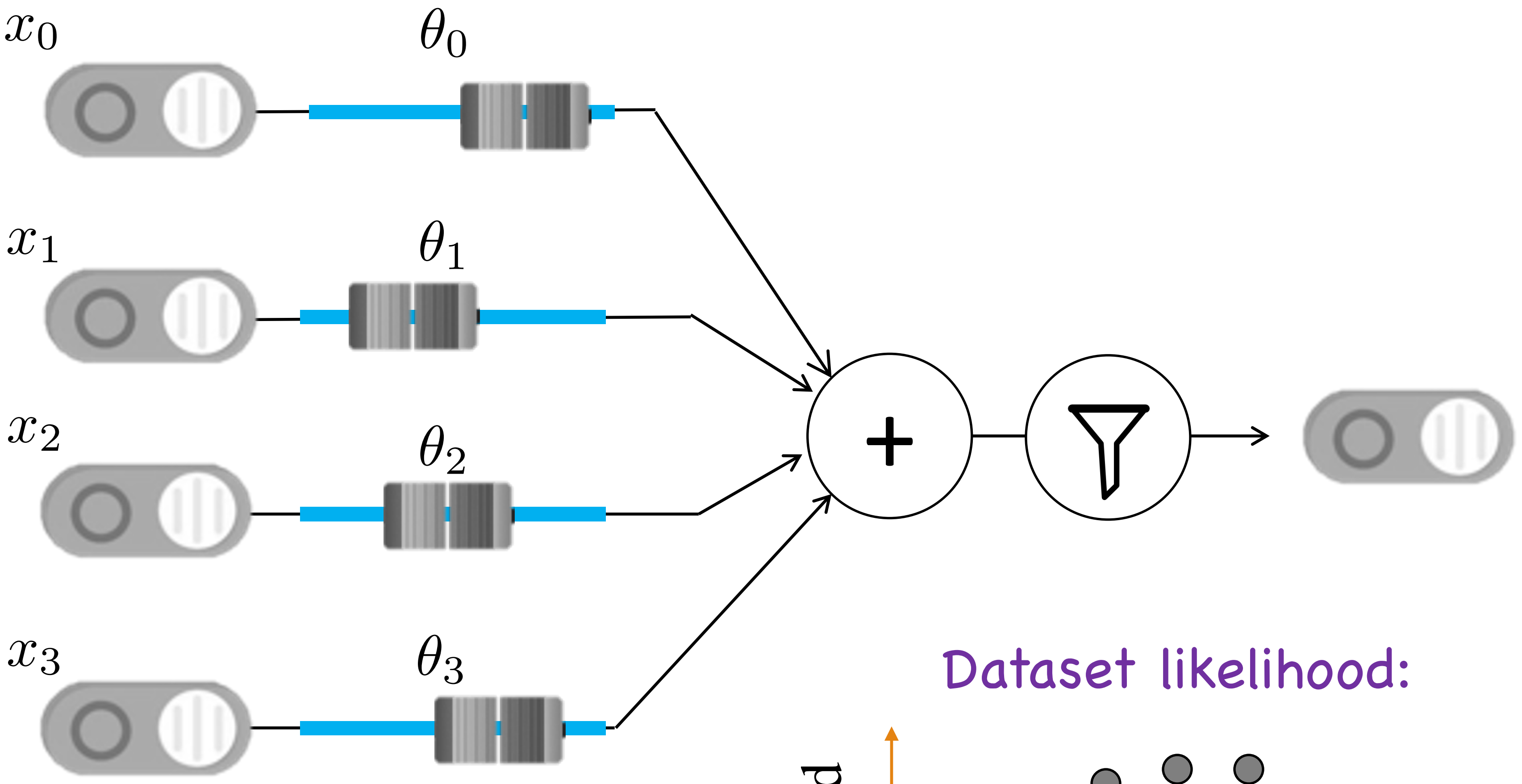
Training



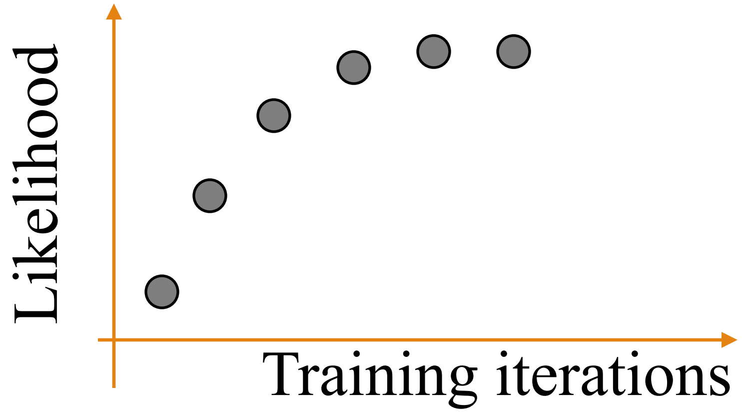
Dataset likelihood:



Training



Dataset likelihood:



Don't forget:

x_j is j-th input variable and $x_0 = 1$.

Allows for θ_0 to be an intercept.

Classification with Logistic Regression

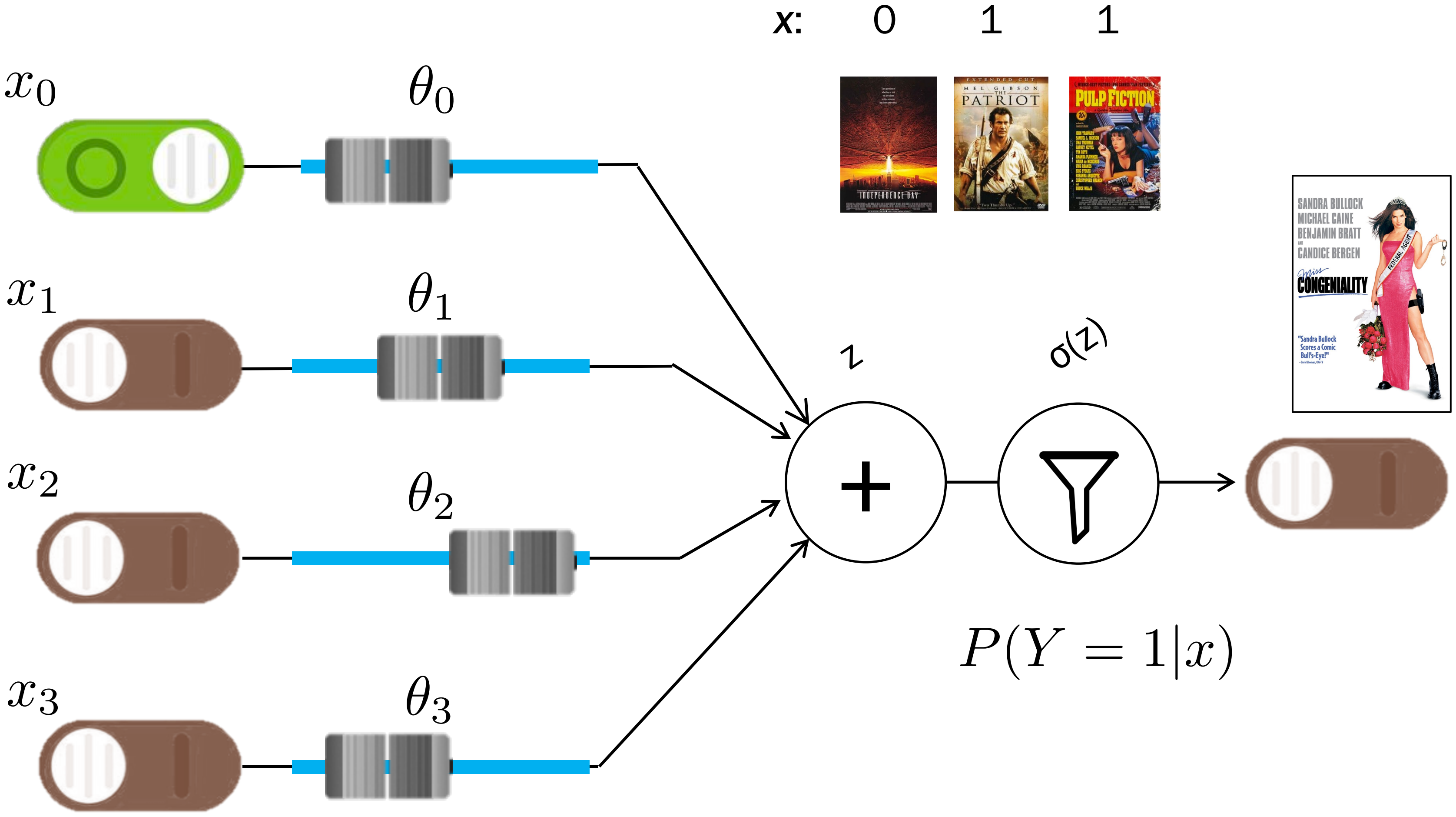
Training: determine parameters θ_j (for all $0 \le j \le m$)

- After parameters θ_j have been learned, test classifier

To test classifier, for each new (test) instance ***X***:

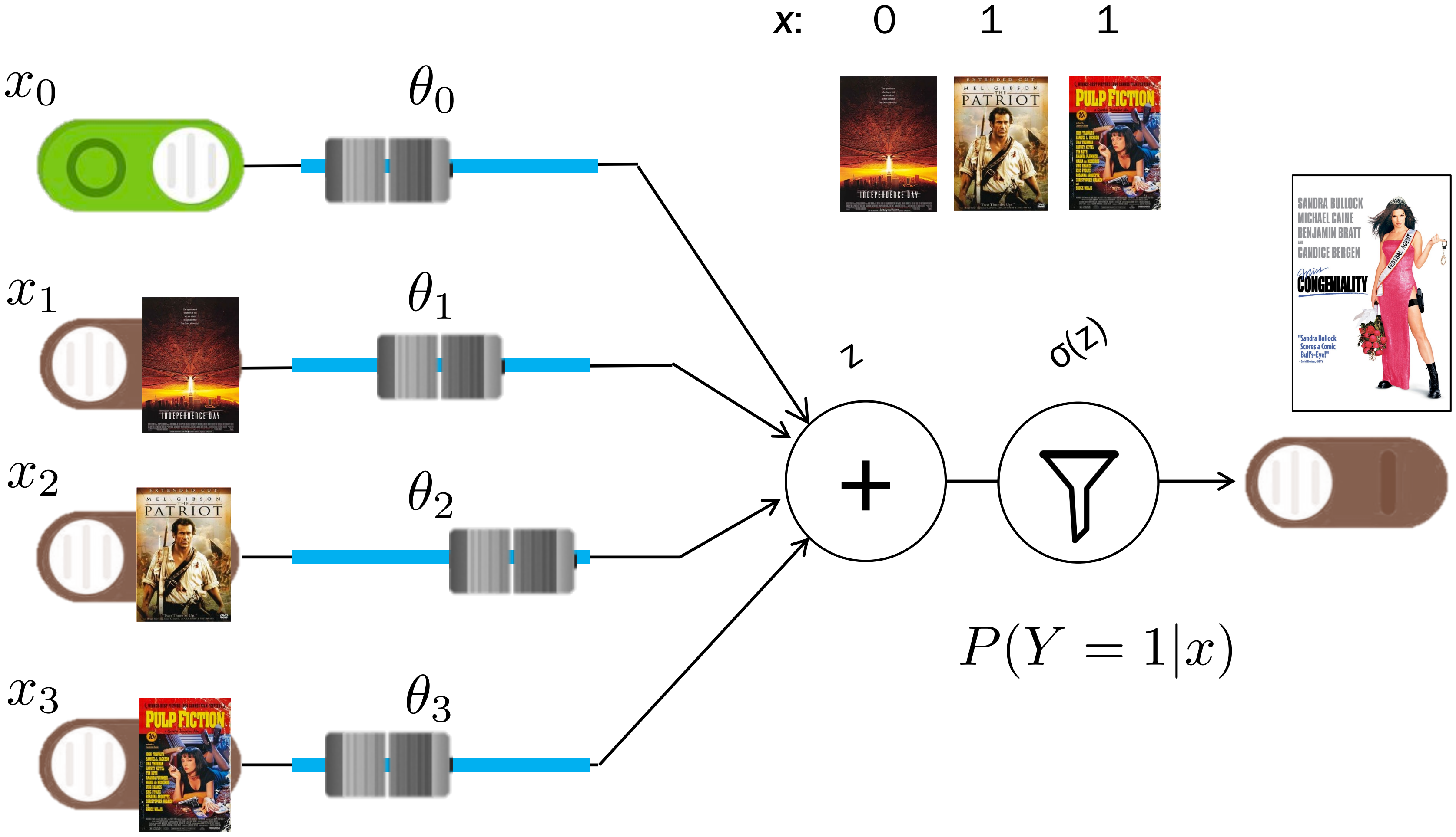
- Compute: $p = P(Y = 1 \mid \boldsymbol{X}) = \dfrac{1}{1+e^{-z}}$, where $z = \theta^T \mathbf{x}$
- Classify instance as: $\hat{y} = \begin{cases} 1 & p > 0.5 \\ 0 & \text{otherwise} \end{cases}$
- Note about evaluation set-up: parameters θ_j are **not** updated during "testing" phase

Prediction



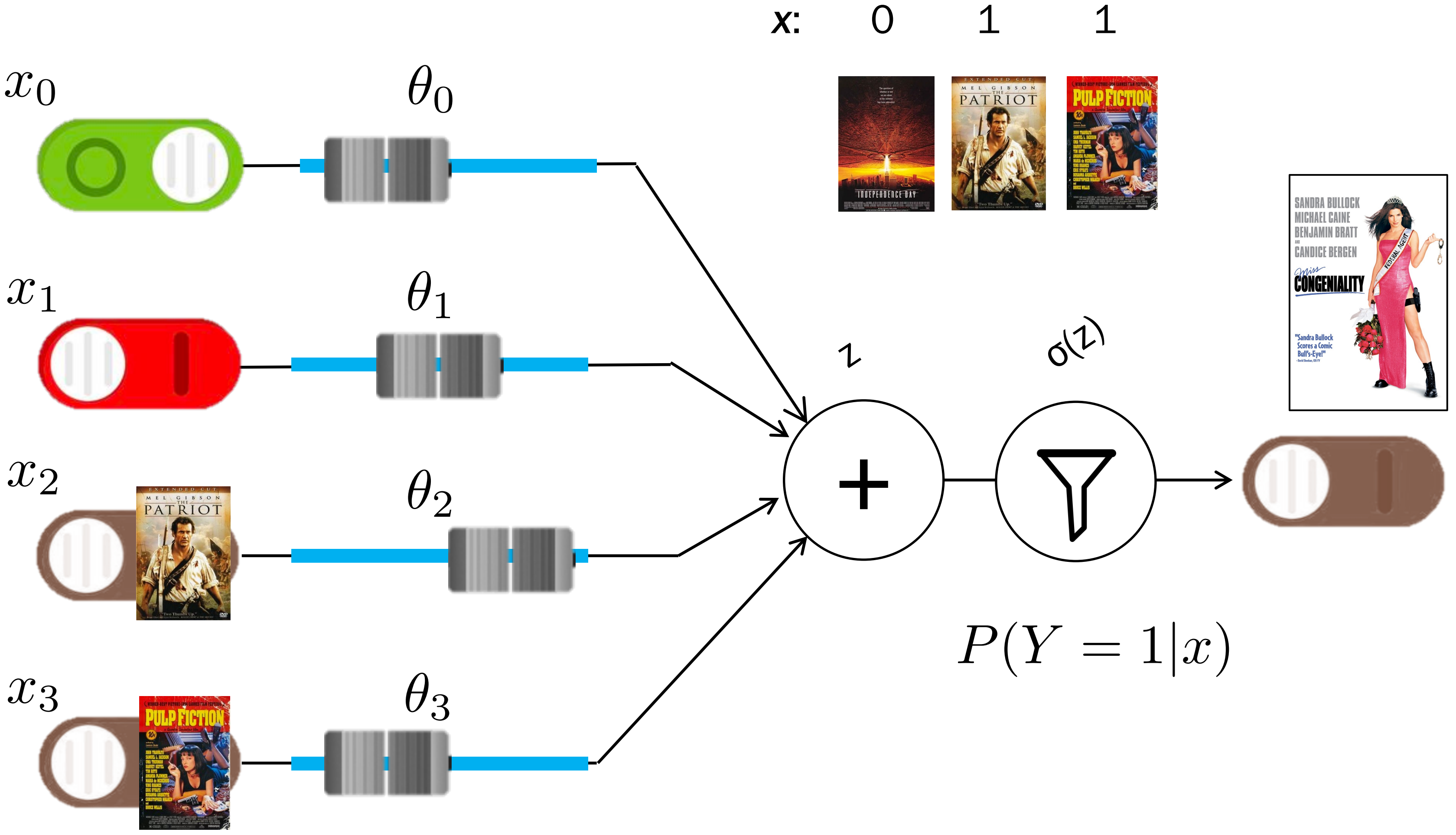
$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Prediction



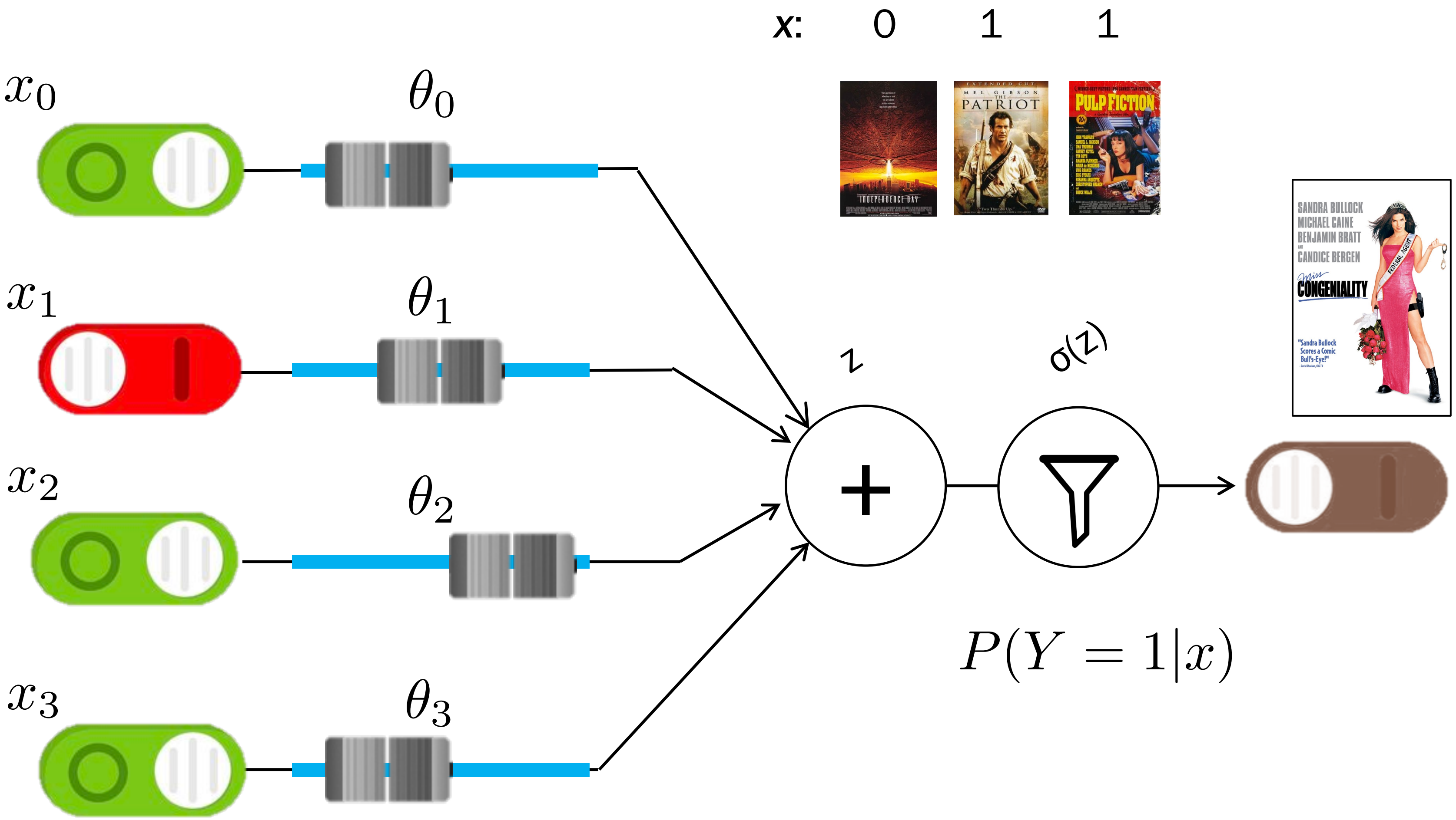
$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Prediction



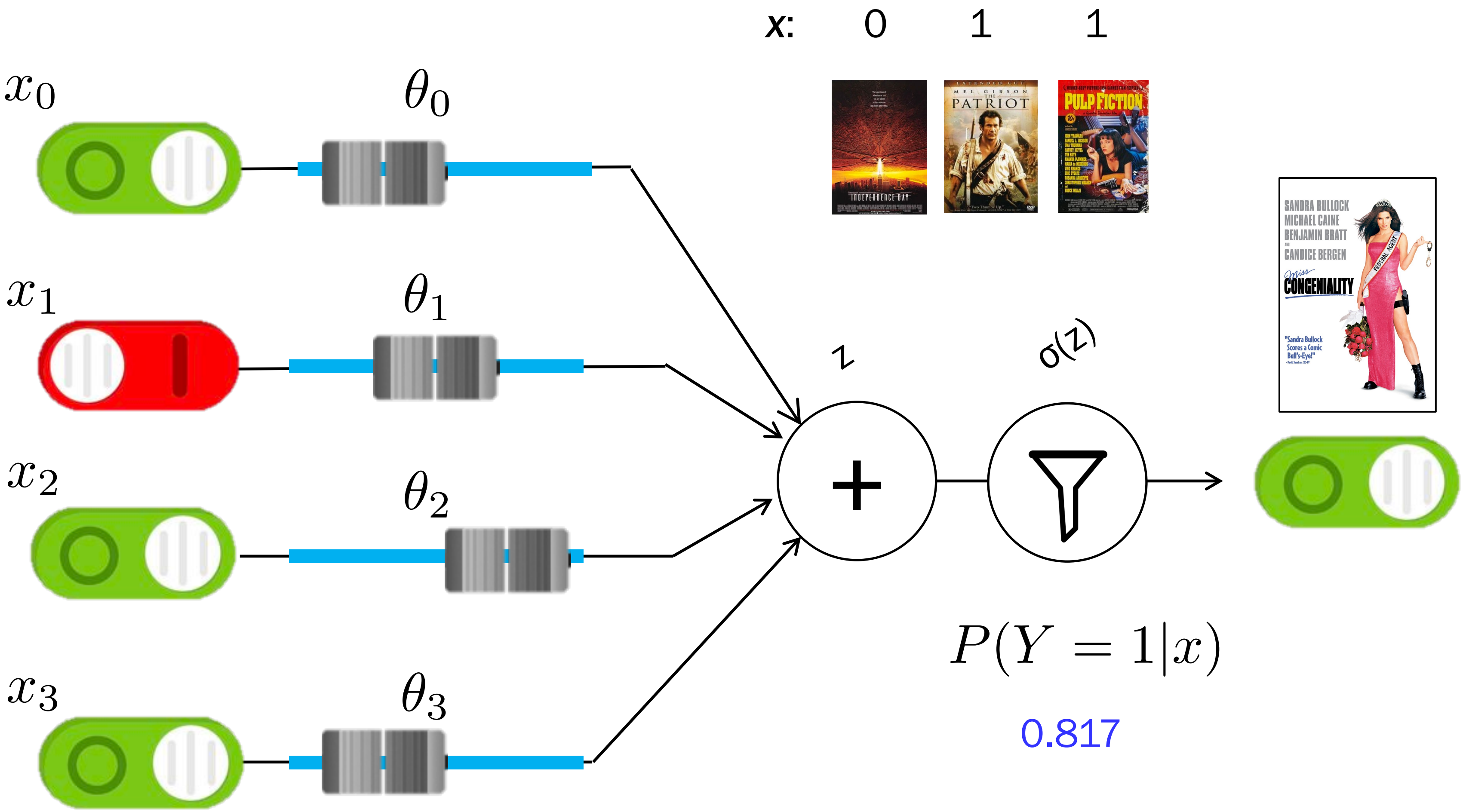
$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Prediction



$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Prediction



$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Chapter 2: How Come?

Logistic Regression

1 Make logistic regression assumption

$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T\mathbf{x})$$
$$P(Y=0|X=\mathbf{x}) = 1-\sigma(\theta^T\mathbf{x})$$

Often call this $\hat{y}$

2 Calculate the log probability for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log\sigma(\theta^T\mathbf{x}^{(i)}) + (1-y^{(i)})\log[1-\sigma(\theta^T\mathbf{x}^{(i)})]$$

3 Get derivative of log probability with respect to thetas

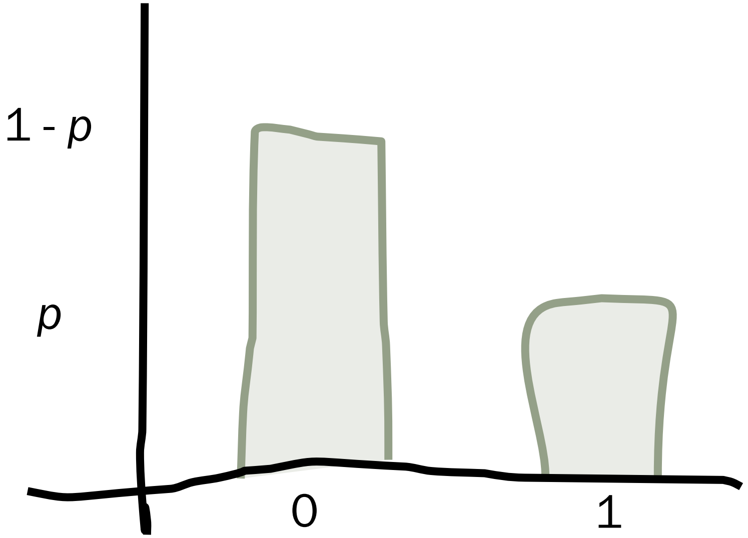
$$\frac{\partial LL(\theta)}{\partial\theta_j} = \sum_{i=1}^{n}\left[y^{(i)} - \sigma(\theta^T\mathbf{x}^{(i)})\right]x_j^{(i)}$$

How did we get that LL function?

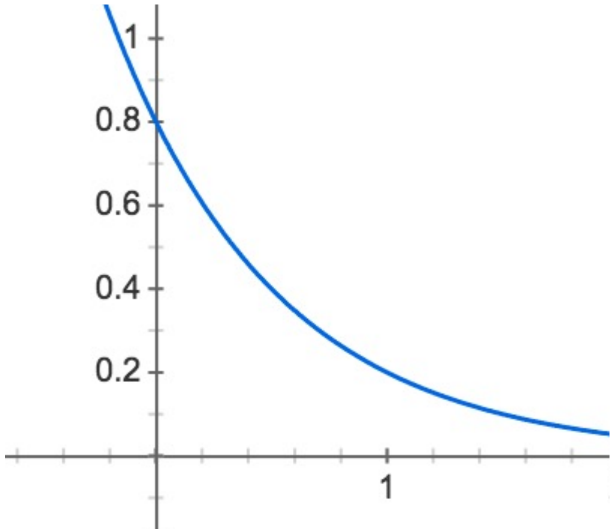
Recall: PMF of Bernoulli

- Y ~ Ber(*p*)
- Probability mass function: $P(Y = y)$

PMF of Bernoulli



PMF of Bernoulli (*p* = 0.2)



$$P(Y = y) = p^y(1-p)^{1-y}$$

$$P(Y = y) = 0.2^y(0.8)^{1-y}$$

Log Probability of Data

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$
$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

Implies

$$P(Y = y|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})^y \cdot \left[1 - \sigma(\theta^T \mathbf{x})\right]^{(1-y)}$$

For IID data

$$L(\theta) = \prod_{i=1}^{n} P(Y = y^{(i)}|X = \mathbf{x}^{(i)})$$
$$= \prod_{i=1}^{n} \sigma(\theta^T \mathbf{x}^{(i)})^{y^{(i)}} \cdot \left[1 - \sigma(\theta^T \mathbf{x}^{(i)})\right]^{(1-y^{(i)})}$$

Take the log

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

How did we get that gradient?

Sigmoid has a Beautiful Slope

True fact about
sigmoid functions

$$\frac{\partial}{\partial z}\sigma(z) = \sigma(z)[1 - \sigma(z)]$$

Sigmoid has a Beautiful Slope

$$\frac{\partial}{\partial \theta_j} \sigma(\theta^T x)? \qquad \frac{\partial}{\partial z} \sigma(z) = \sigma(z)[1 - \sigma(z)]$$

where $z = \theta^T x$

$$\frac{\partial}{\partial \theta_j} \sigma(\theta^T x) = \frac{\partial}{\partial z} \sigma(z) \cdot \frac{\partial z}{\partial \theta_j}$$

Chain rule!

$$\frac{\partial}{\partial \theta_j} \sigma(\theta^T x) = \sigma(\theta^T x)[1 - \sigma(\theta^T x)] x_j$$

Plug and chug

Sigmoid, you should be a ski hill

Sigmoid has a Beautiful Slope

$$\hat{y} = \sigma(\theta^T x)$$

$$\frac{\partial \hat{y}}{\partial \theta_j} = \sigma(\theta^T x)[1 - \sigma(\theta^T x)]x_j$$

$$= \hat{y}(1 - \hat{y})x_j$$

ARE YOU READY???

I think I'm Ready…

$$\frac{\partial LL(\theta)}{\partial \theta_j}$$

Where

$$LL(\theta) \;= \sum_{i=1}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

This is Sparta!!!!!

This is ~~Sparta~~!!!!!

Stanford

Think About Only One Training Instance

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}]$$

We only need to calculate the gradient for one training example!

$$\frac{\partial}{\partial x} \sum_i f(x, i) = \sum_i \frac{\partial}{\partial x} f(x, i)$$

We will pretend we only have one example

$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

We can sum up the gradients of each example to get the correct answer

First, imagine only one example

$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

Where $\hat{y} = \sigma(\theta^T \mathbf{x})$

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \frac{\partial LL(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial \theta_j}$$

CHAIN RULZ!

First, imagine only one example

$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

Where $$\hat{y} = \sigma(\theta^T \mathbf{x})$$

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \frac{\partial LL(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial \theta_j}$$

CHAIN RULZ!

$$= \frac{\partial LL(\theta)}{\partial \hat{y}} \hat{y}(1 - \hat{y}) x_j$$

Already did that one

$$= \left[\frac{y}{\hat{y}} - \frac{1 - y}{1 - \hat{y}}\right] \hat{y}(1 - \hat{y}) x_j$$

Derive this one

$$= (y - \hat{y}) x_j$$

Simplify

Make it Simple

$$LL(\theta) = y \log \hat{y} + (1-y)\log[1-\hat{y}]$$

Where $$\hat{y} = \sigma(\theta^T \mathbf{x})$$

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \frac{\partial LL(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial \theta_j}$$

CHAIN RULZ!

$$= \frac{\partial LL(\theta)}{\partial \hat{y}} \hat{y}(1-\hat{y})x_j$$

Already did that one

$$= \left[\frac{y}{\hat{y}} - \frac{1-y}{1-\hat{y}}\right]\hat{y}(1-\hat{y})x_j$$

Derive this one

$$= (y-\hat{y})x_j$$

Simplify

Now, all the data

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}]$$

$$\hat{y}^{(i)} = \sigma(\theta^T \mathbf{x}^{(i)})$$

Derivative of sum…

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_j} \left[y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}] \right]$$

$$= \sum_{i=1}^{n} [y^{(i)} - \hat{y}^{(i)}] x_j^{(i)}$$

See last slide

$$= \sum_{i=1}^{n} [y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)})] x_j^{(i)}$$

Some people don't like hats…

Now, all the data

$$\frac{\partial LL(\theta)}{\partial \theta_j}$$

$$= \sum_{i=1}^{n} [y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)})] x_j^{(i)}$$

Logistic Regression

1. Make logistic regression assumption

$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T\mathbf{x})$$
$$P(Y=0|X=\mathbf{x}) = 1 - \sigma(\theta^T\mathbf{x})$$

2. Calculate the log probability for all data

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \sigma(\theta^T\mathbf{x}^{(i)}) + (1-y^{(i)}) \log[1-\sigma(\theta^T\mathbf{x}^{(i)})]$$

3. Get derivative of log probability with respect to thetas

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma(\theta^T\mathbf{x}^{(i)})\right] x_j^{(i)}$$

The Hard Way

$$LL(\theta) = y \log \sigma(\theta^T \mathbf{x}) + (1-y) \log[1 - \sigma(\theta^T \mathbf{x})]$$

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \frac{\partial}{\partial \theta_j} y \log \sigma(\theta^T \mathbf{x}) + \frac{\partial}{\partial \theta_j}(1-y) \log[1 - \sigma(\theta^T \mathbf{x}]$$

$$= \left[\frac{y}{\sigma(\theta^T x)} - \frac{1-y}{1-\sigma(\theta^T x)}\right] \frac{\partial}{\partial \theta_j} \sigma(\theta^T x)$$

$$= \left[\frac{y}{\sigma(\theta^T x)} - \frac{1-y}{1-\sigma(\theta^T x)}\right] \frac{\partial}{\partial \theta_j} \sigma(\theta^T x)$$

$$= \left[\frac{y - \sigma(\theta^T x)}{\sigma(\theta^T x)[1-\sigma(\theta^T x)]}\right] \sigma(\theta^T x)[1 - \sigma(\theta^T x)] x_j$$

$$= \left[y - \sigma(\theta^T x)\right] x_j$$

Phew!

Chapter 3: Philosophy
(if time)

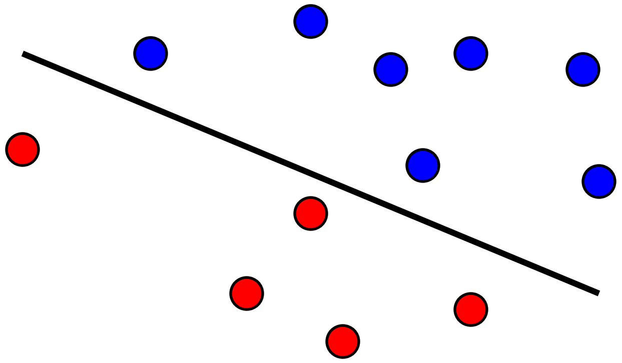
Choosing an Algorithm?

Many trade-offs in choosing learning algorithm

- Continuous input variables
  - Logistic Regression easily deals with continuous inputs
  - Naive Bayes needs to use some parametric form for continuous inputs (e.g., Gaussian) or "discretize" continuous values into ranges (e.g., temperature in range: <50, 50-60, 60-70, >70)
- Discrete input variables
  - Naive Bayes naturally handles multi-valued discrete features by using multinomial distribution for $P(X_i \mid Y)$
  - Logistic Regression requires some sort of representation of multi-valued discrete data (e.g., one hot vector)
  - Say $X_i \in \{A, B, C\}$. Not necessarily a good idea to encode X_i as taking on input values 1, 2, or 3 corresponding to A, B, or C.

Discrimination Intuition

- Logistic regression is trying to fit a **line** that separates data instances where *y* = 1 from those where *y* = 0



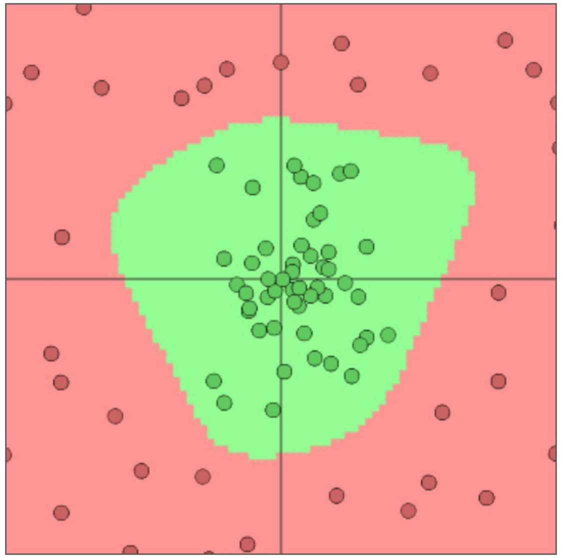
$$\theta^T \mathbf{x} = 0$$

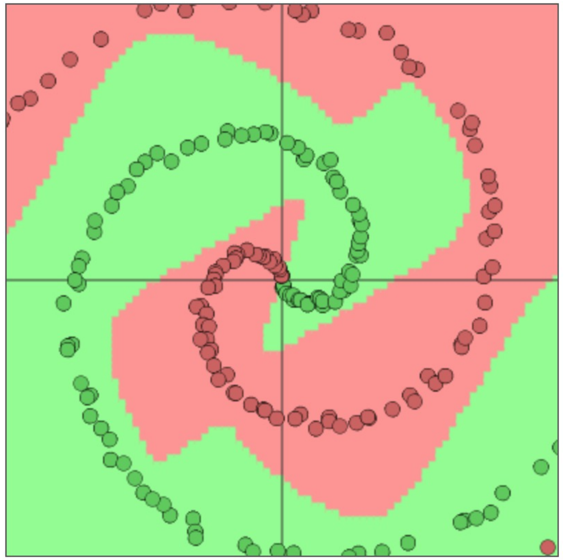
$$\theta_0 x_0 + \theta_1 x_1 + \cdots + \theta_m x_m = 0$$

- We call such data (or the functions generating the data) "**linearly separable**"
- **Naïve bayes is linear too** as there is no interaction between different features.

Some Data Not Linearly Seperable

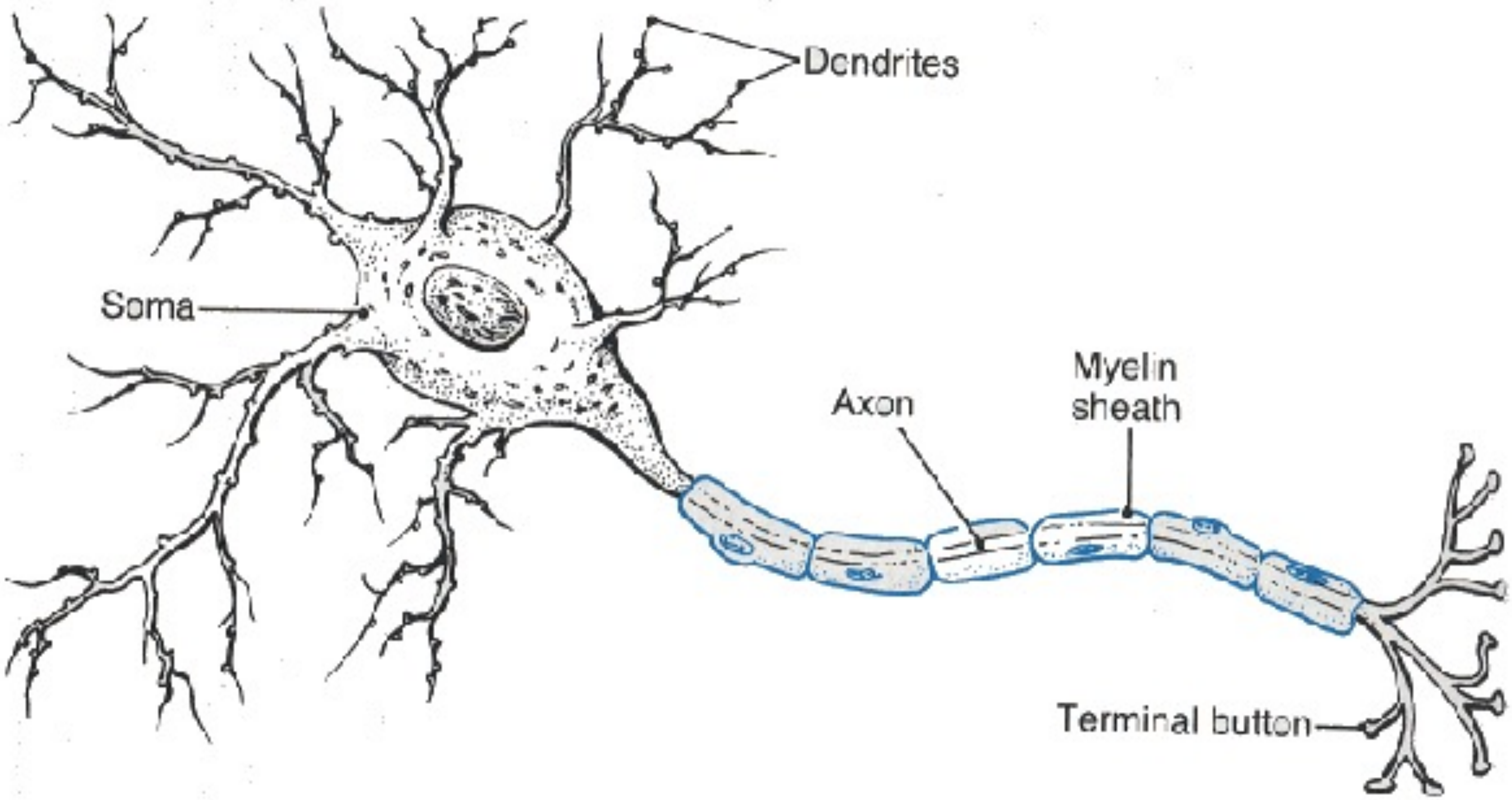
Some data sets/functions are not separable



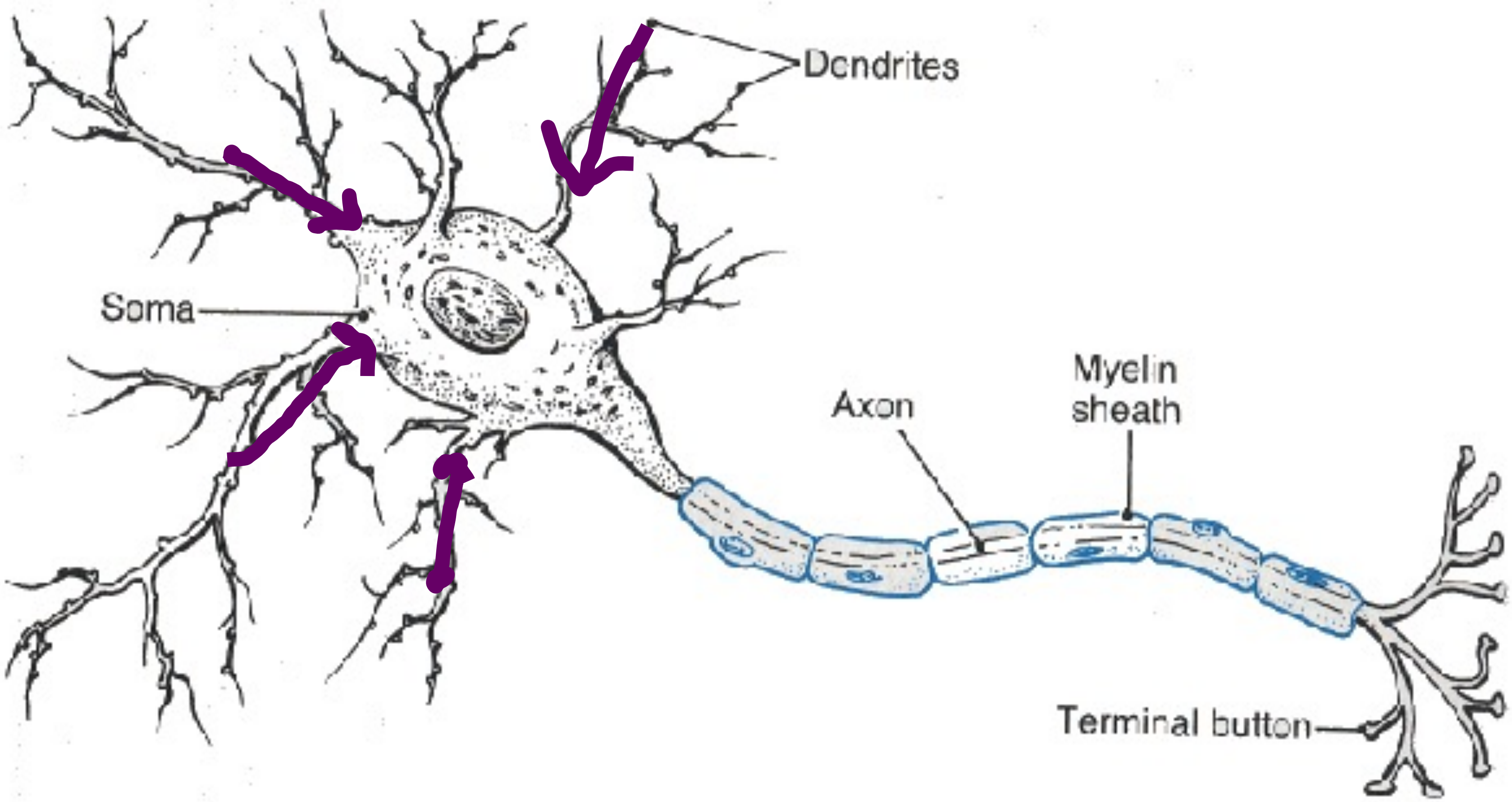


- Not possible to draw a line that successfully separates all the $y = 1$ points (green) from the $y = 0$ points (red)
- Despite this fact, logistic regression and Naive Bayes still often work well in practice

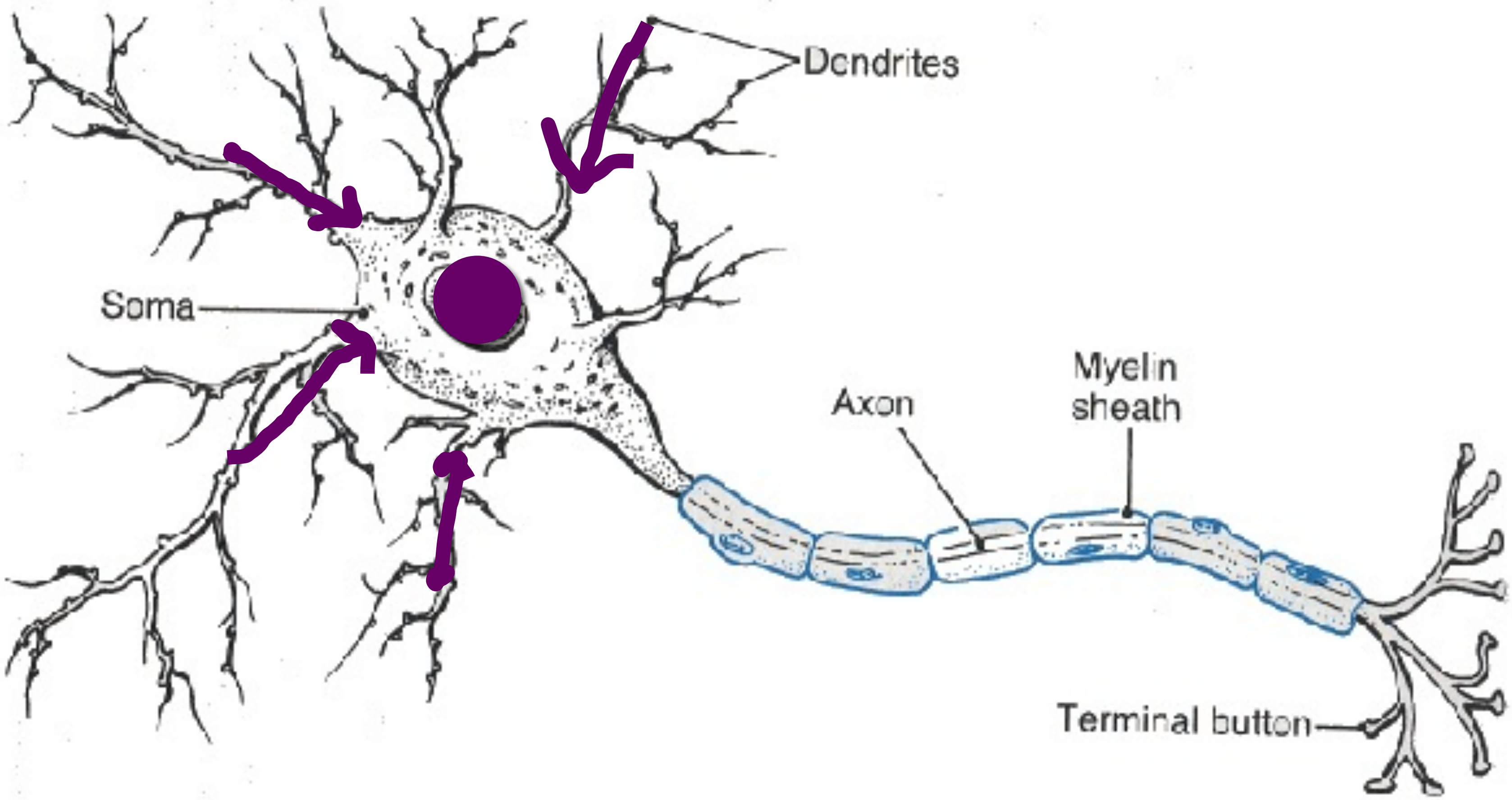
Neuron



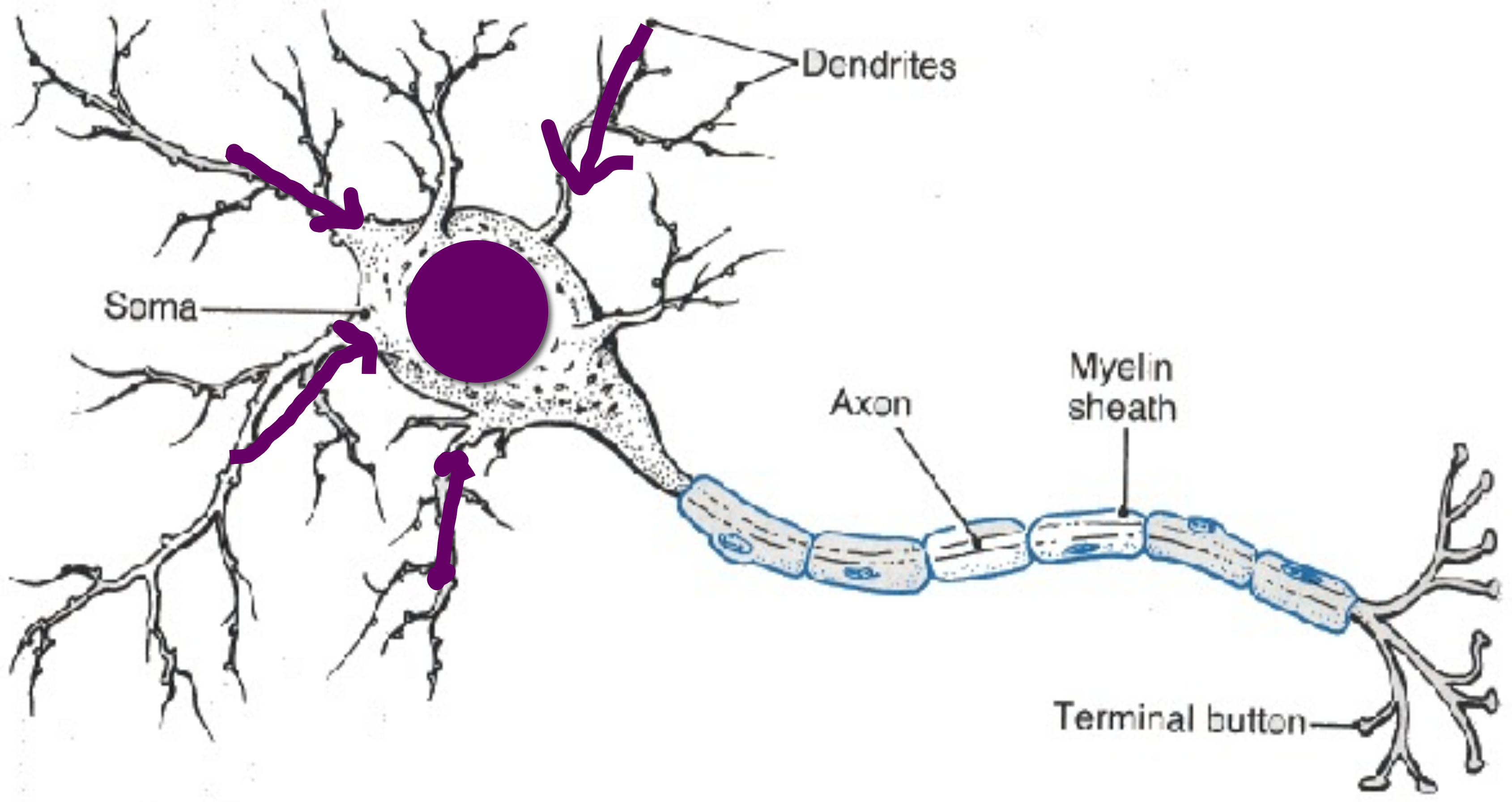
Neuron



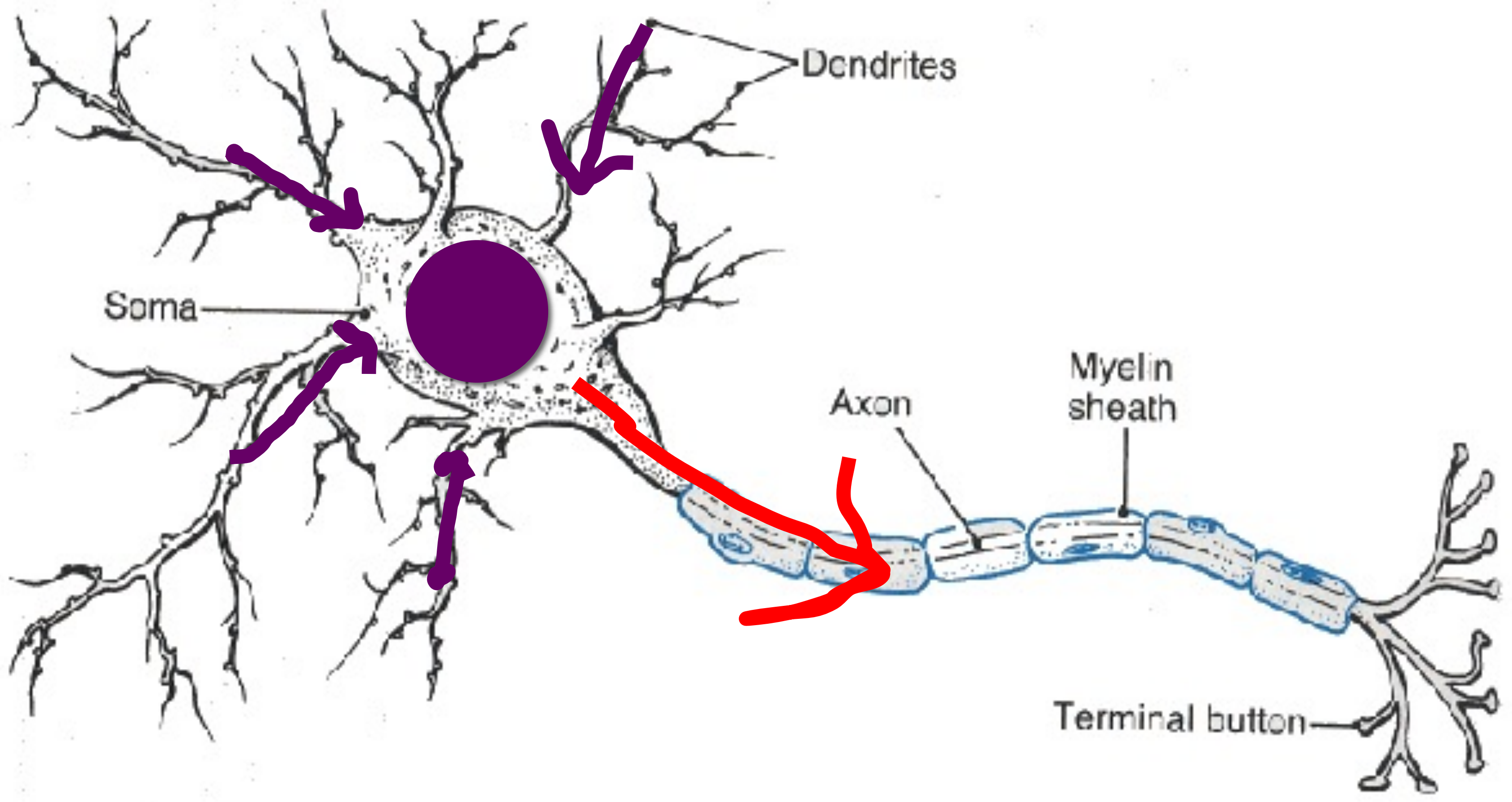
Neuron



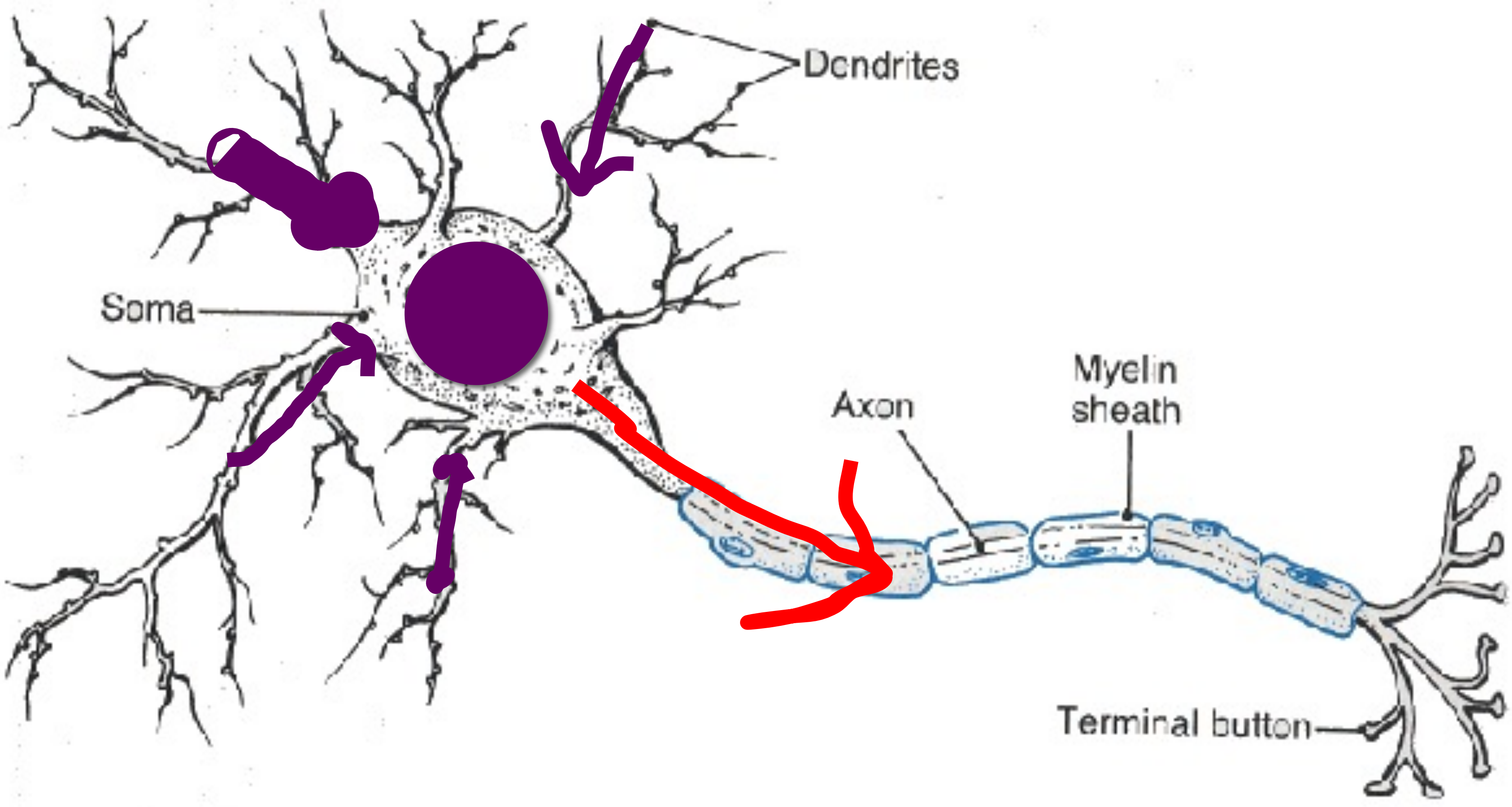
Neuron



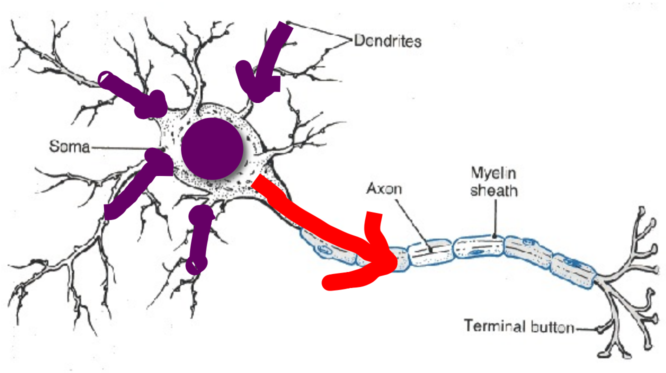
Neuron

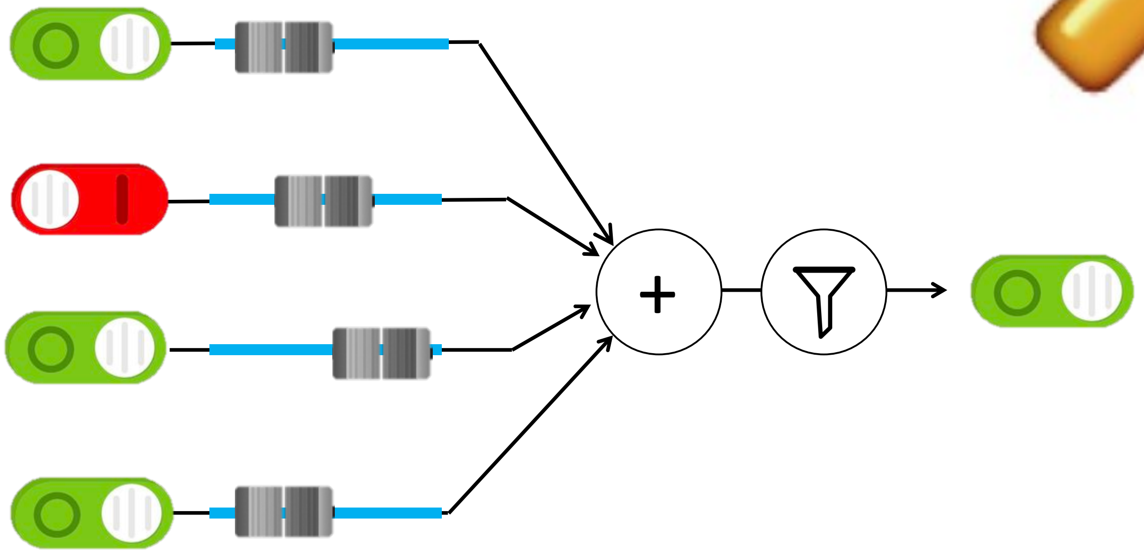


Some inputs are more important



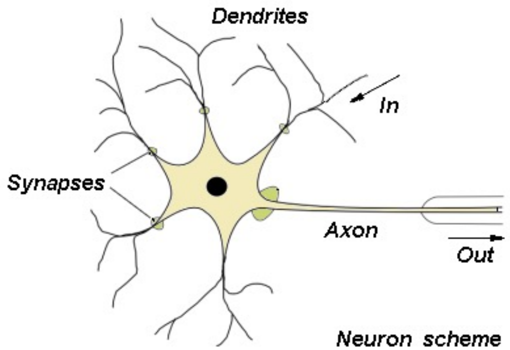
Artificial Neurons

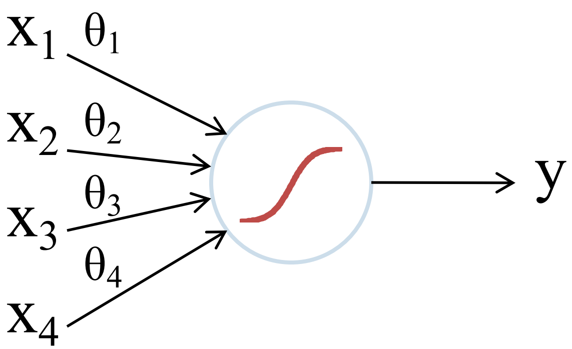




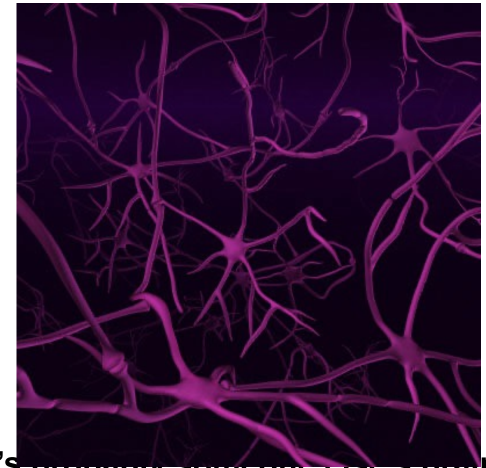
Biological Basis for Neural Networks

A neuron

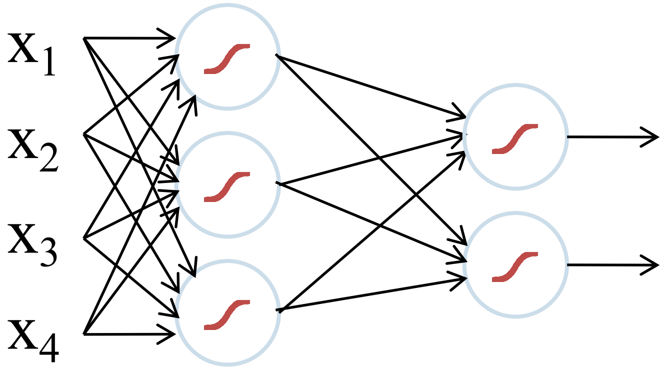




Your brain



Actually, it's probably someone else's brain



(aka Neural Networks)

Deep learning is (at its core) many logistic regression pieces stacked on top of each other.

Computer Vision

Alpha GO

Revolution in AI

Computers Making Art

Basically just many logistic regression cells
And lots of chain rule…

Next up: Ethics