Independence

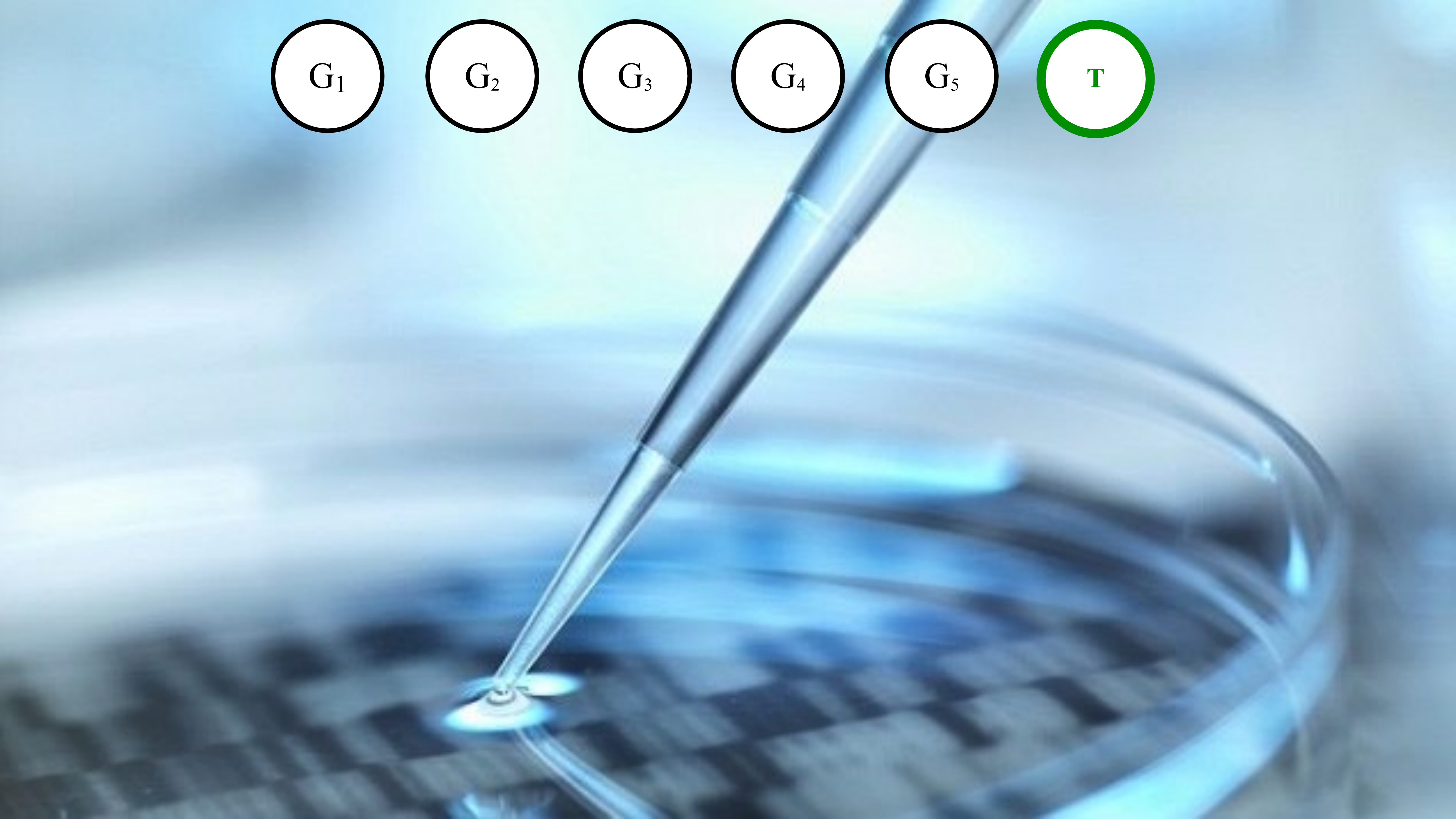
Chris Piech, CS109

Announcements

- **Sections** start today. Wahoo! Enjoy.
- **PSet #1** is due Friday 1p. Recall grace period.

Today, start with a cool program

G_1 G_2 G_3 G_4 G_5 **T**



G_1 G_2 G_3 G_4 G_5 T

dna.txt — dna

dna.txt

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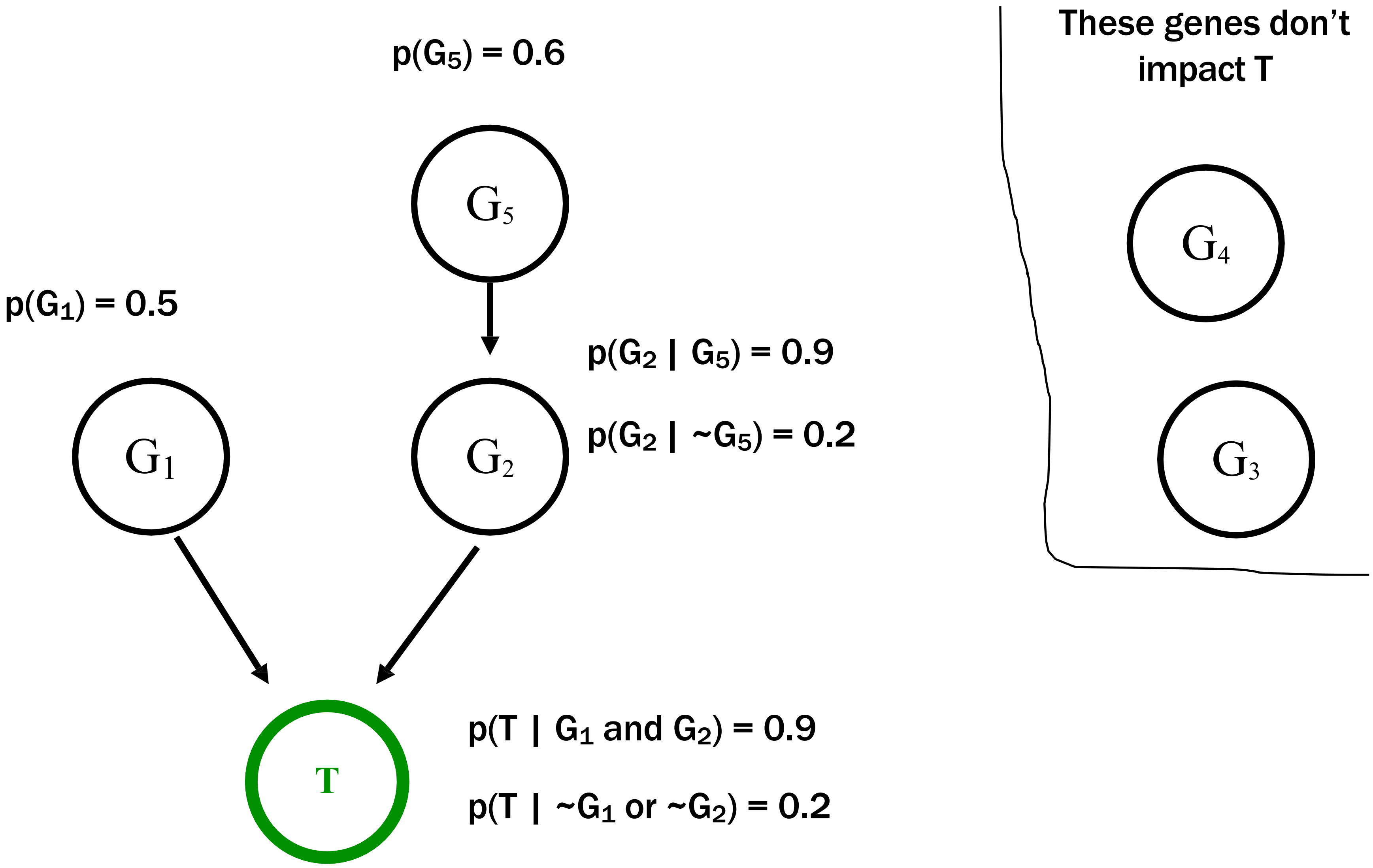
100,000 samples

6 observations per sample

Stanford University

Discovered Hypothesis



We’ve gotten ahead of ourselves

Source: The Hobbit

Start at the beginning

Source: The Hobbit

Review

Review: Conditional Probability

$$p(AB) \quad \text{vs} \quad P(A|B)$$

$$P(AB) = P(A|B)P(B)$$

Review: Chain Rule

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

$$P(EF) = P(E|F)P(F)$$

Relationship Between Probabilities

$P(E \text{ and } F)$

Chain rule (Product rule)

Definition of conditional probability

$P(E|F)$

Law of Total Probability

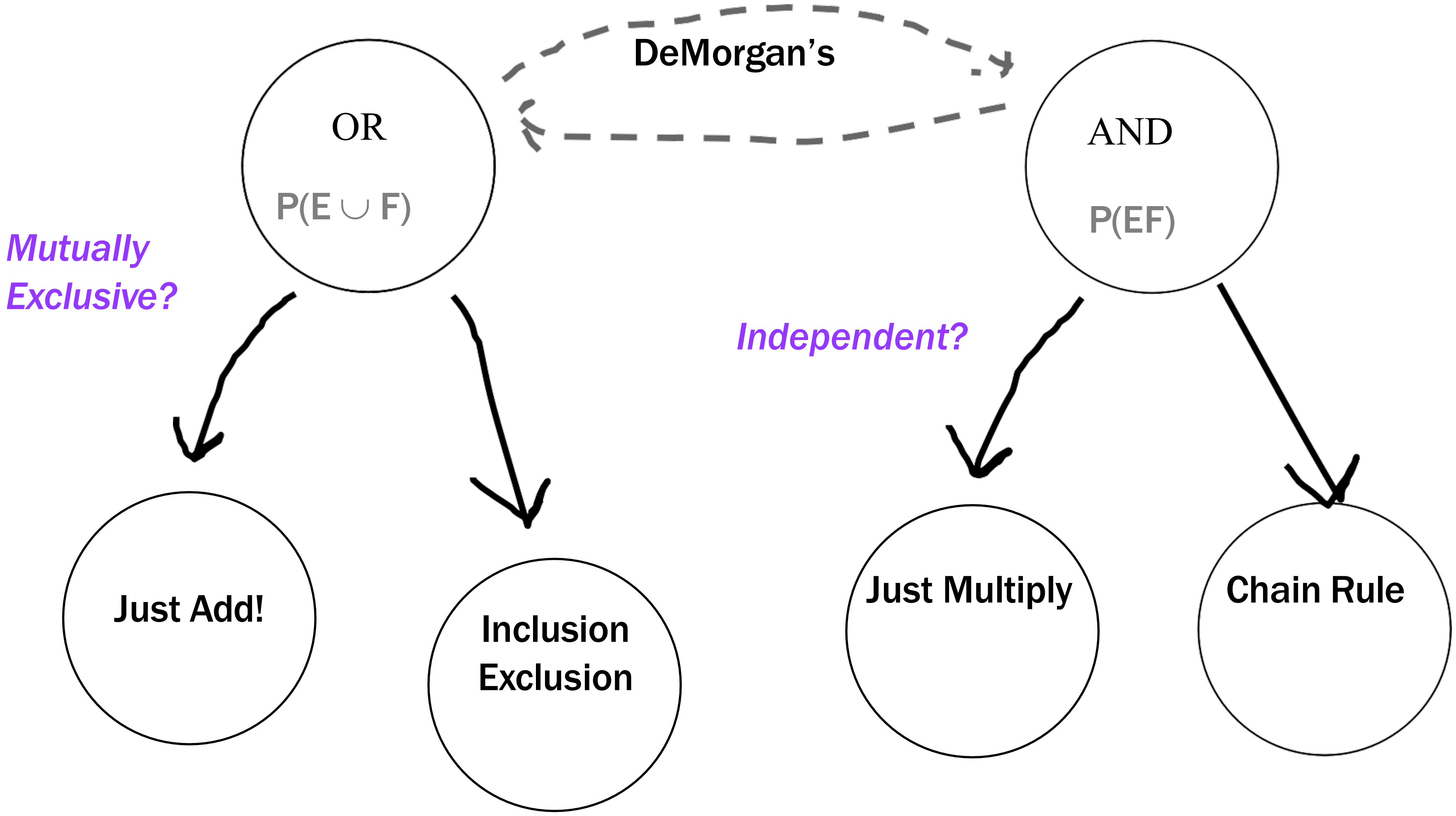
Bayes' Theorem

$P(E)$

$P(F|E)$

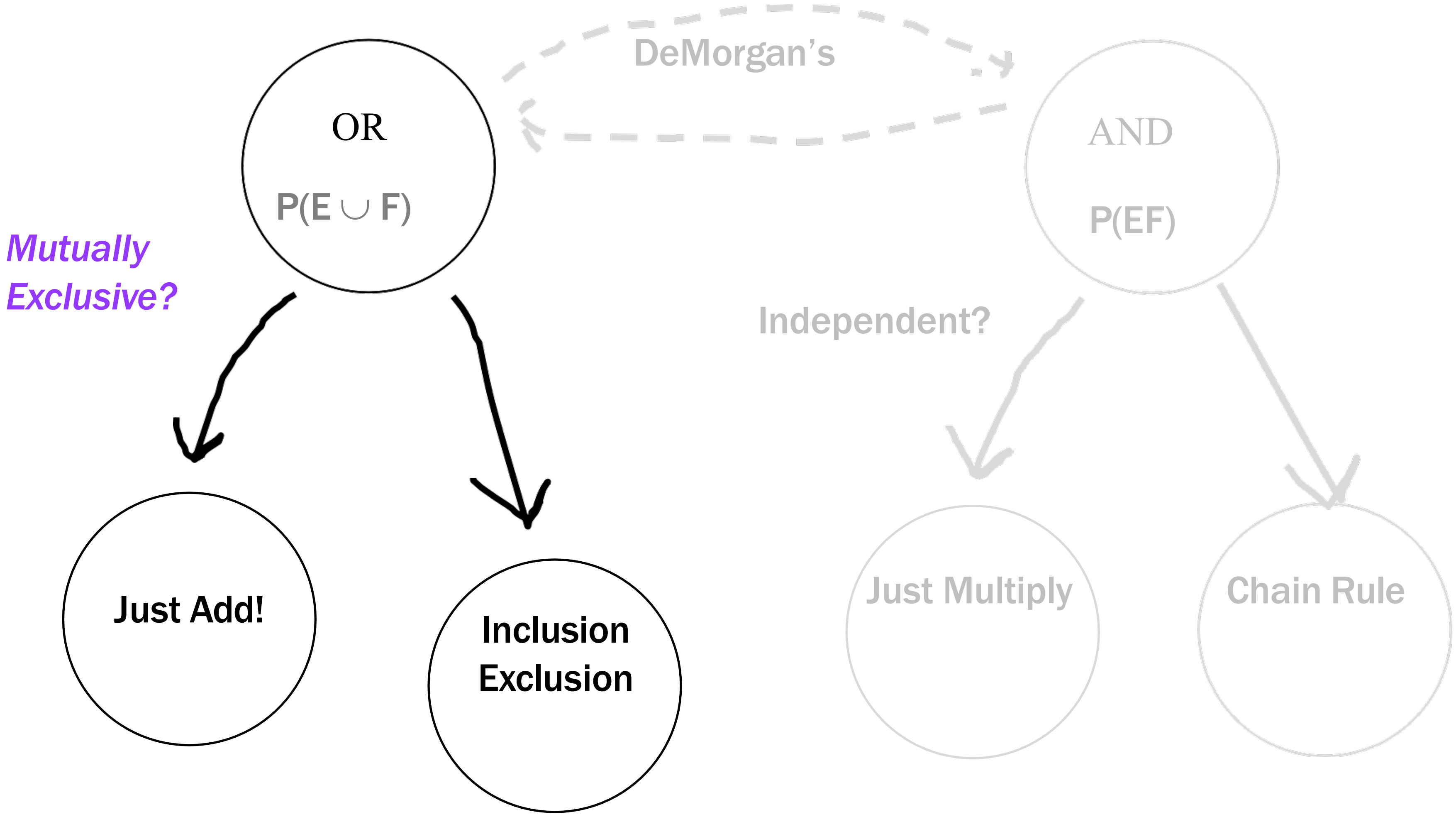
End Review

Today



DeMorgan's
OR
P(E ∪ F)
AND
P(EF)
Mutually Exclusive?
Independent?
Just Add!
Inclusion Exclusion
Just Multiply
Chain Rule

Today



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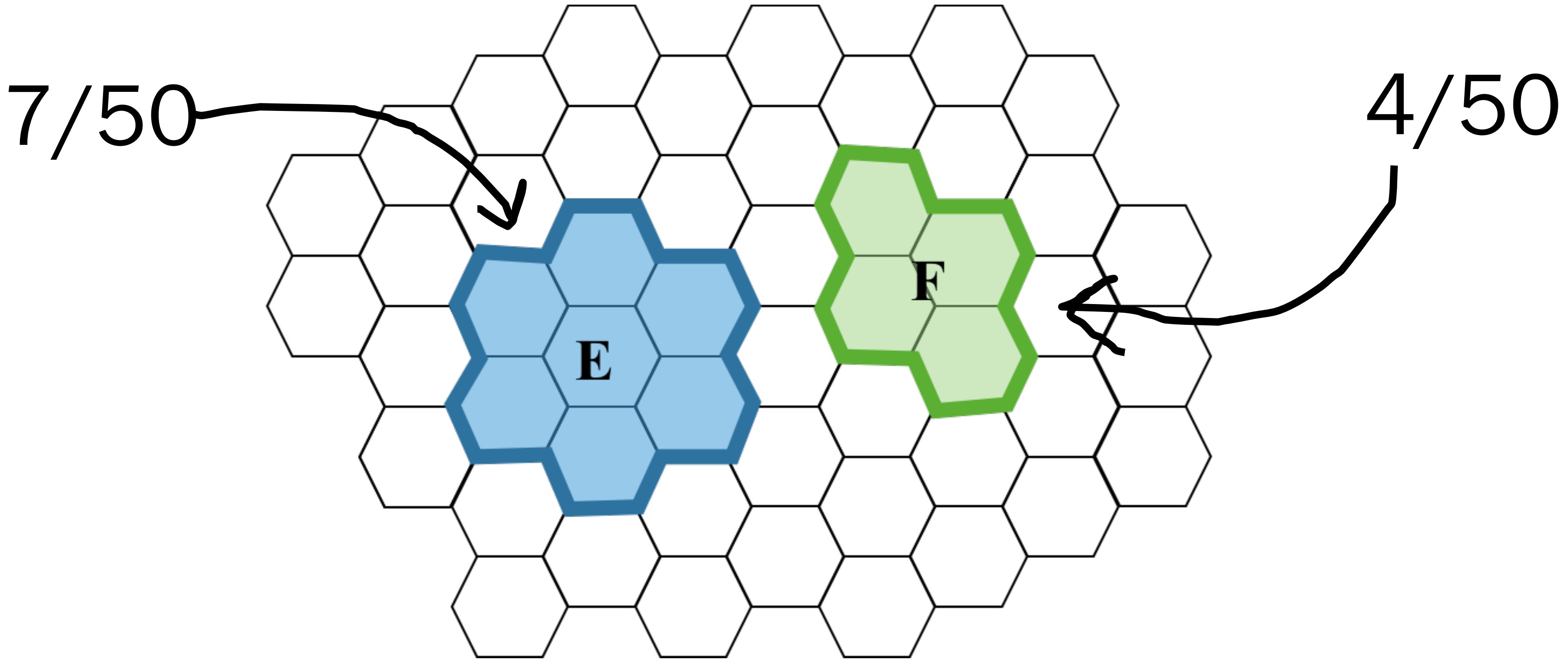
Probability of “OR”

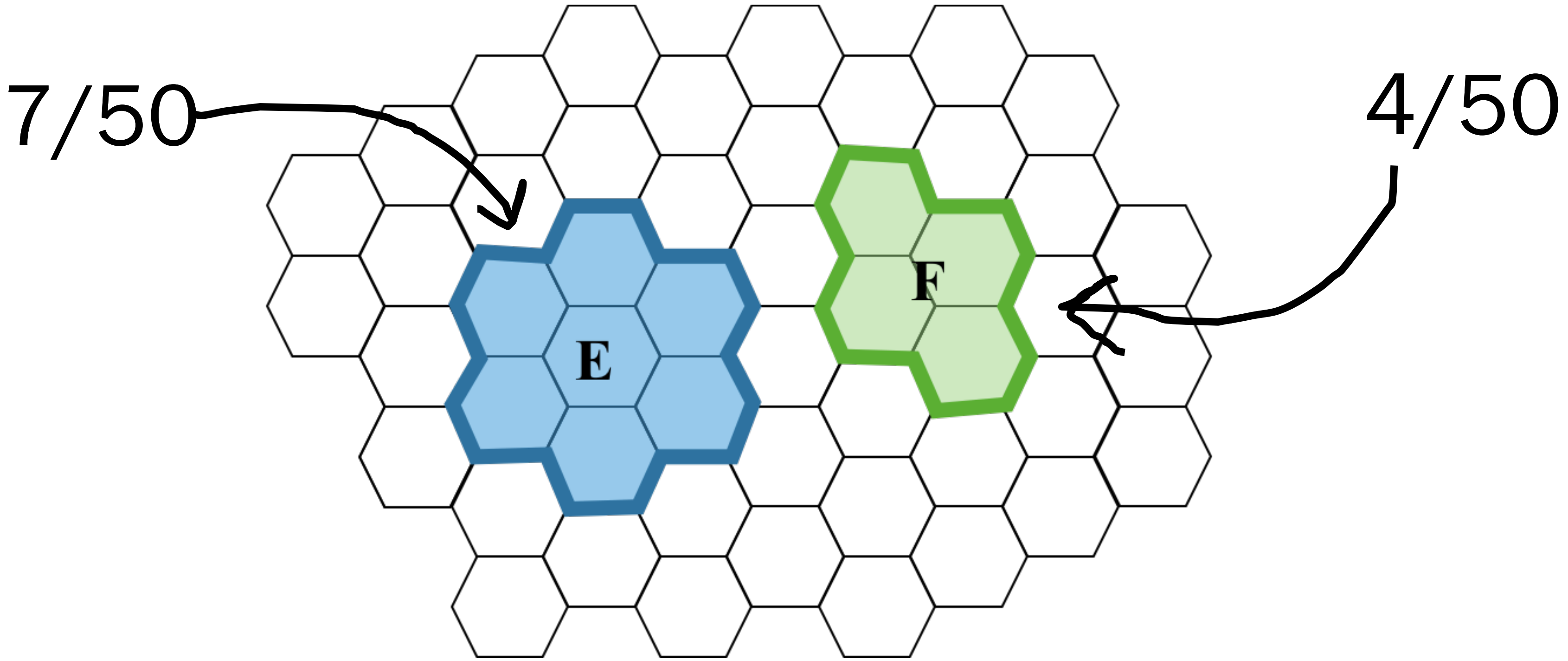
Review: OR with Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

Review: OR with Mutually Exclusive Events

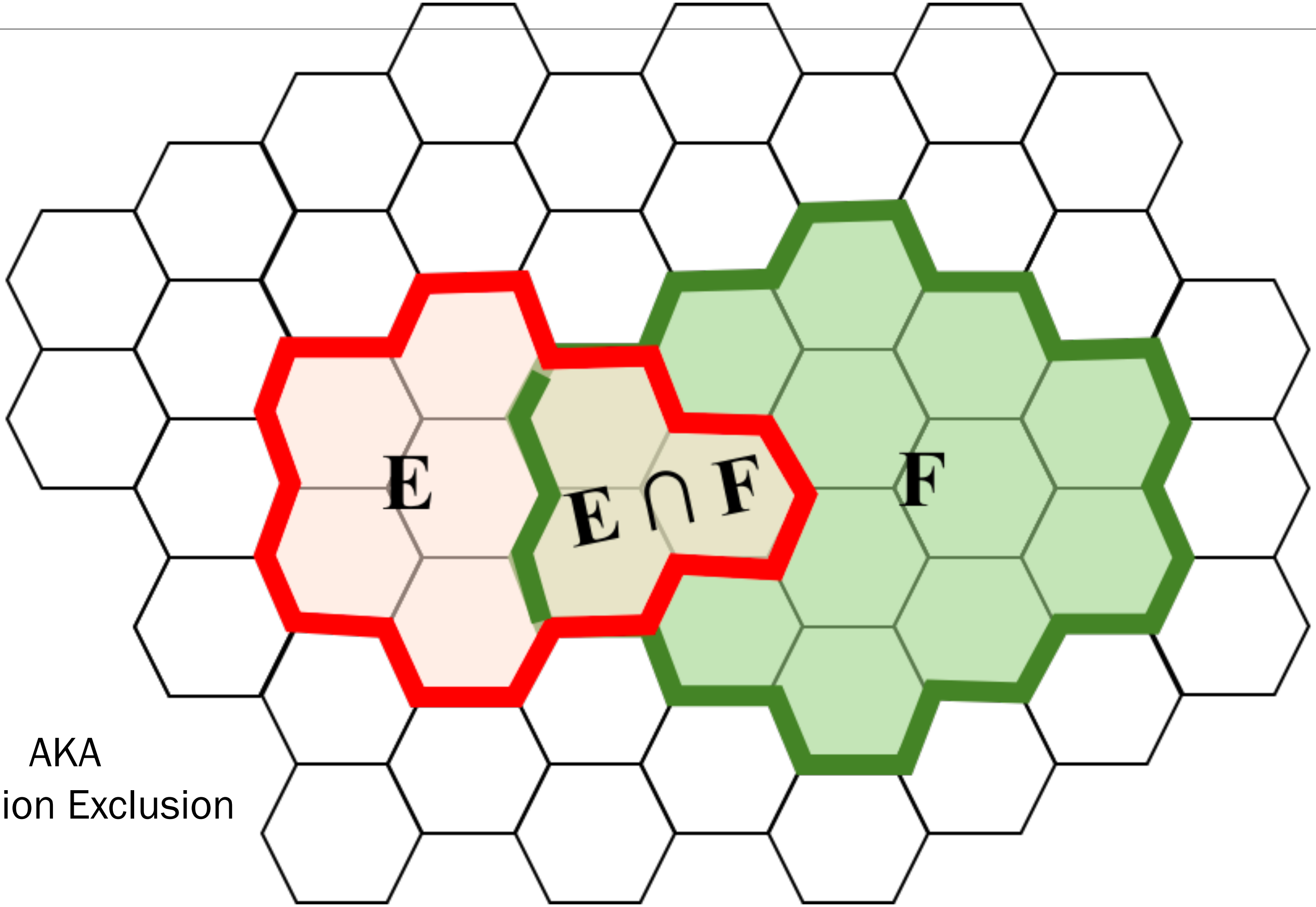


If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = \frac{7}{50} + \frac{4}{5} = \frac{11}{50}$$

What about when they are not *Mutually exclusive*?

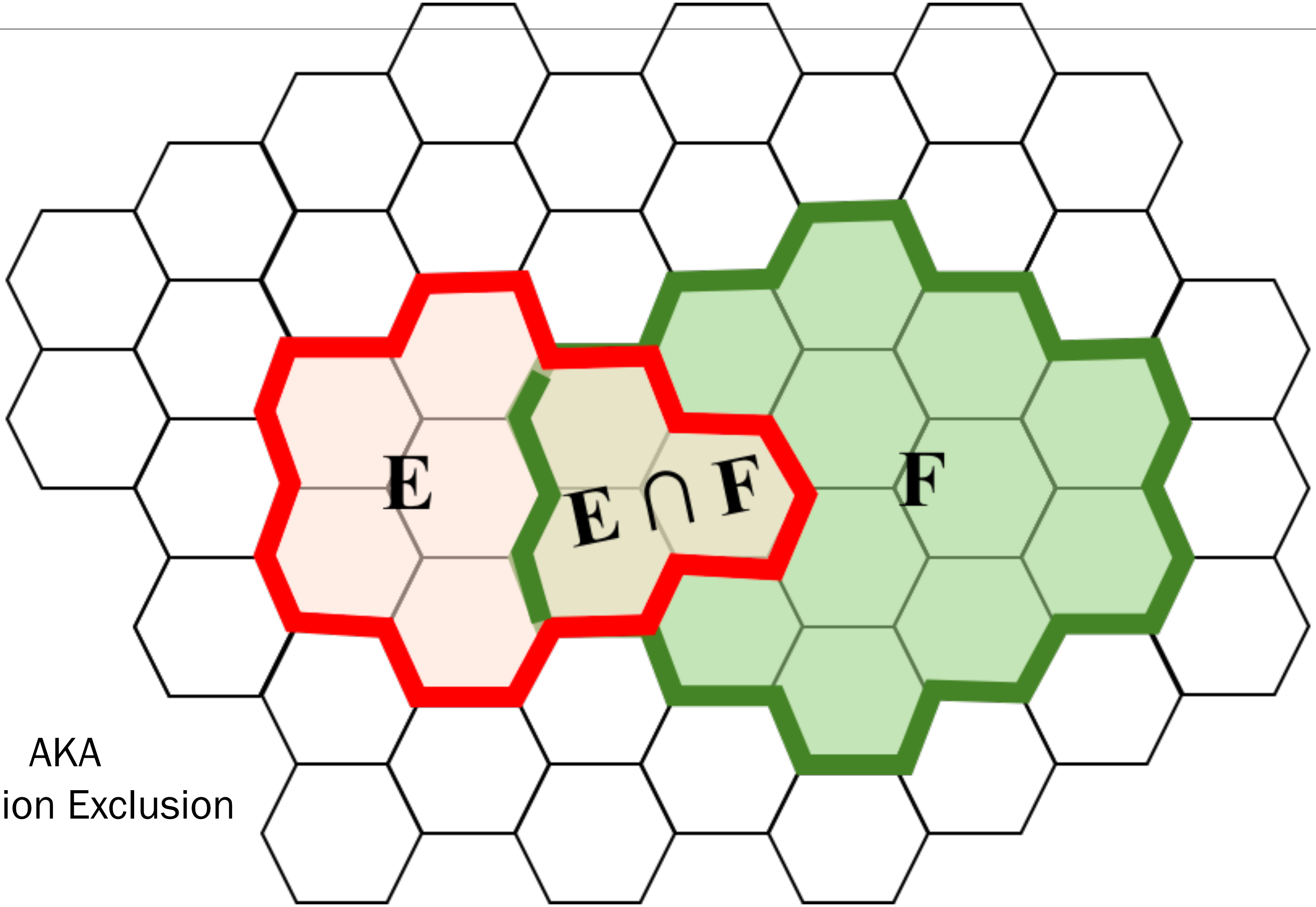
OR *without* Mutually Exclusive Events



AKA
Inclusion Exclusion

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

OR *without* Mutually Exclusive Events



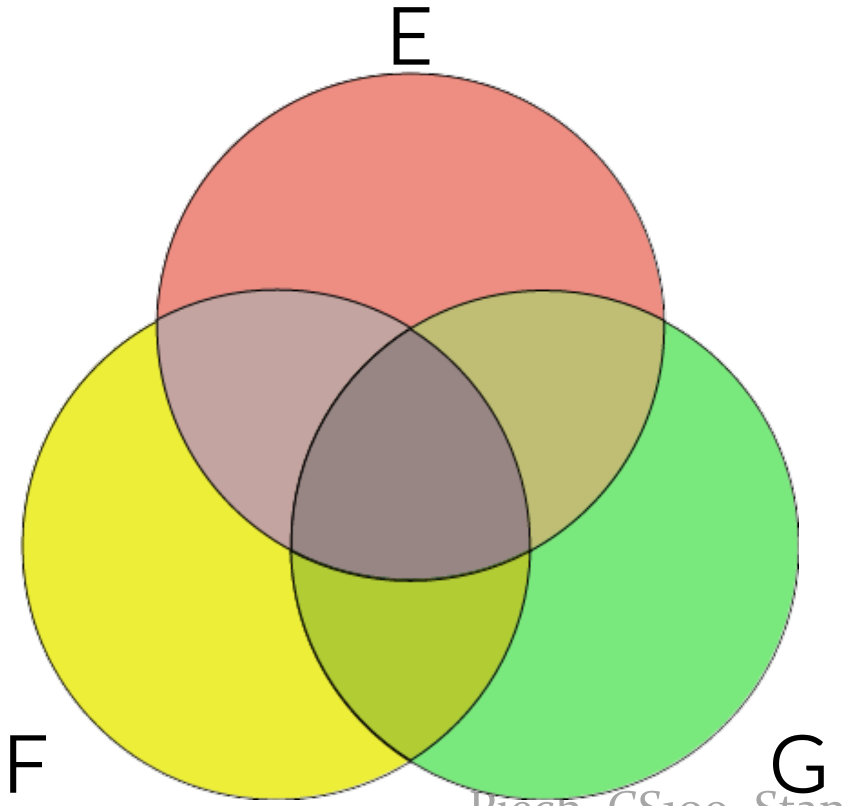
AKA
Inclusion Exclusion

$$P(E \cup F) = \frac{8}{50} + \frac{14}{50} - \frac{3}{50} = \frac{19}{50}$$

More than two sets?

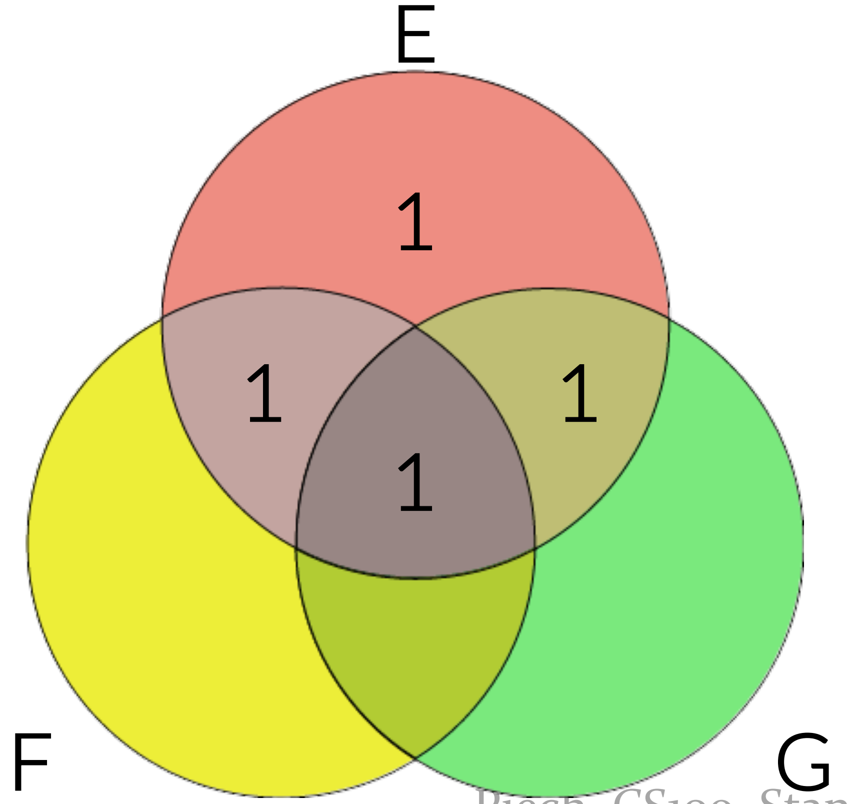
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) =$$



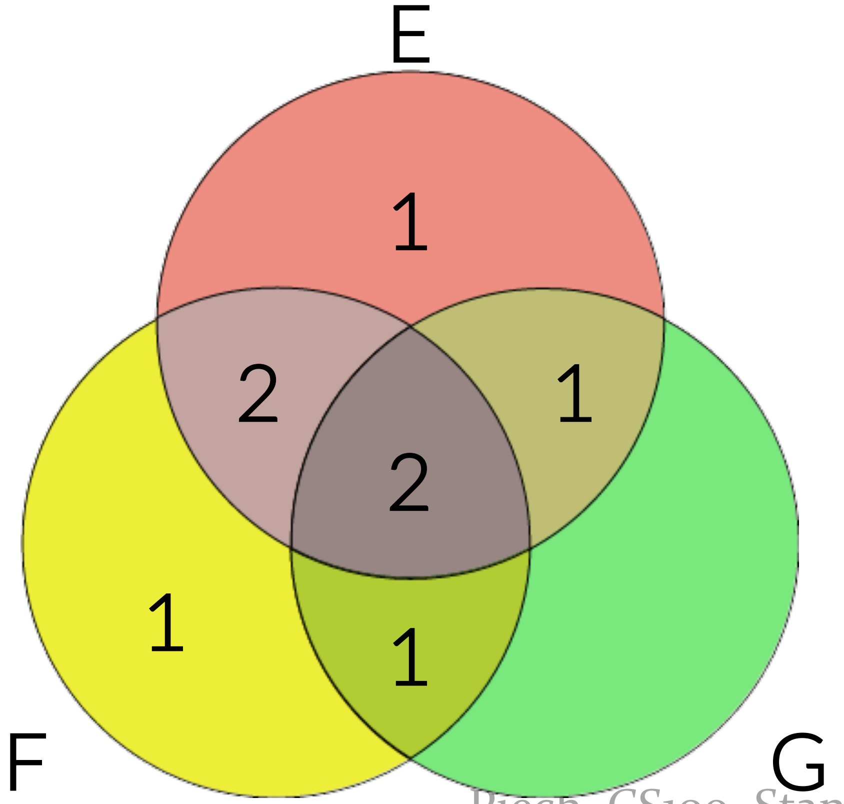
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E)$$



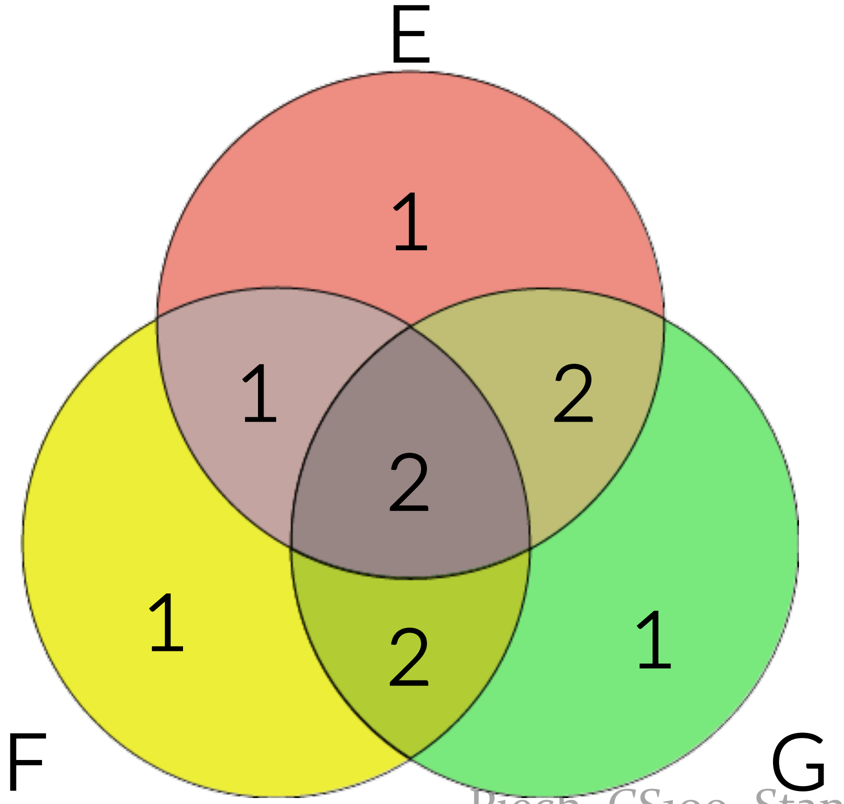
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F)$$



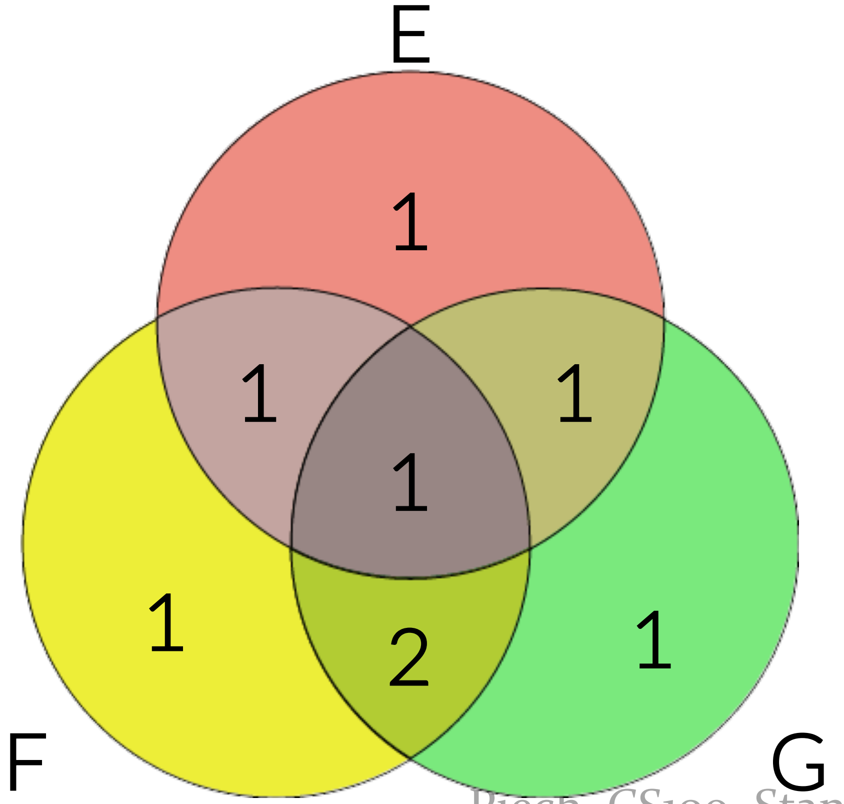
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF)$$



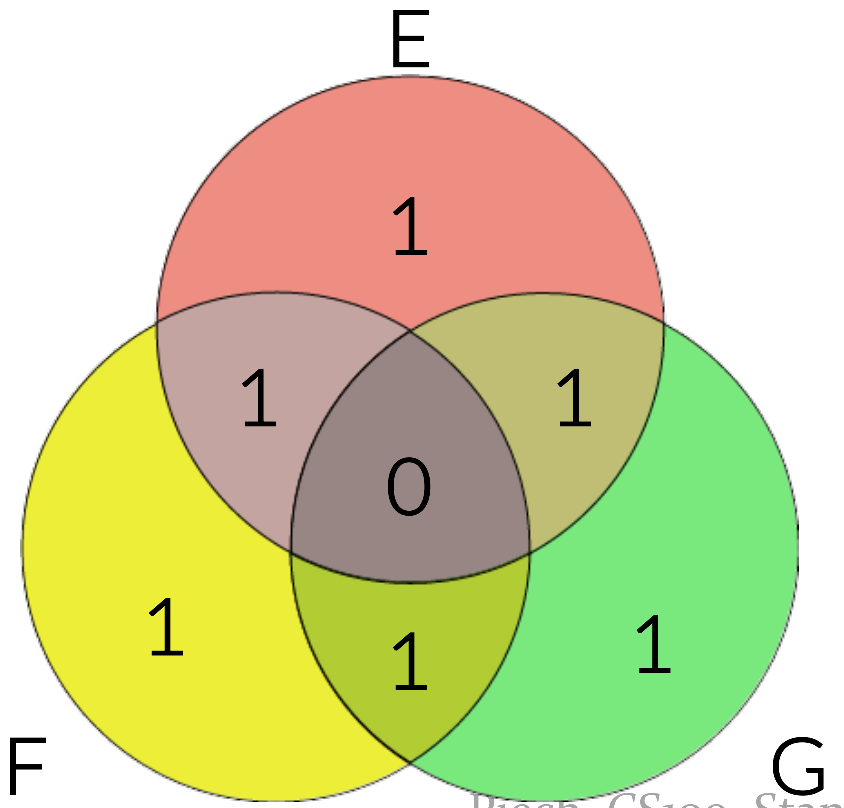
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG)$$



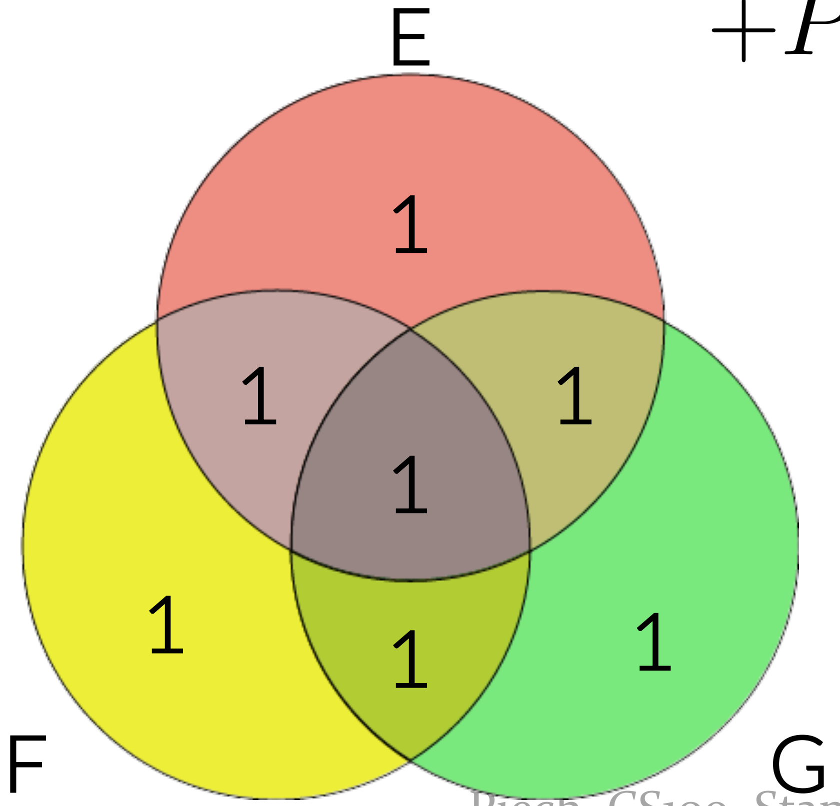
Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG) - P(FG)$$



Inclusion / Exclusion with Three Events

$$P(E \cup F \cup G) = P(E) + P(F) + P(G)$$
$$-P(EF) - P(EG) - P(FG)$$
$$+P(EFG)$$



General Inclusion / Exclusion

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} Y_r$$

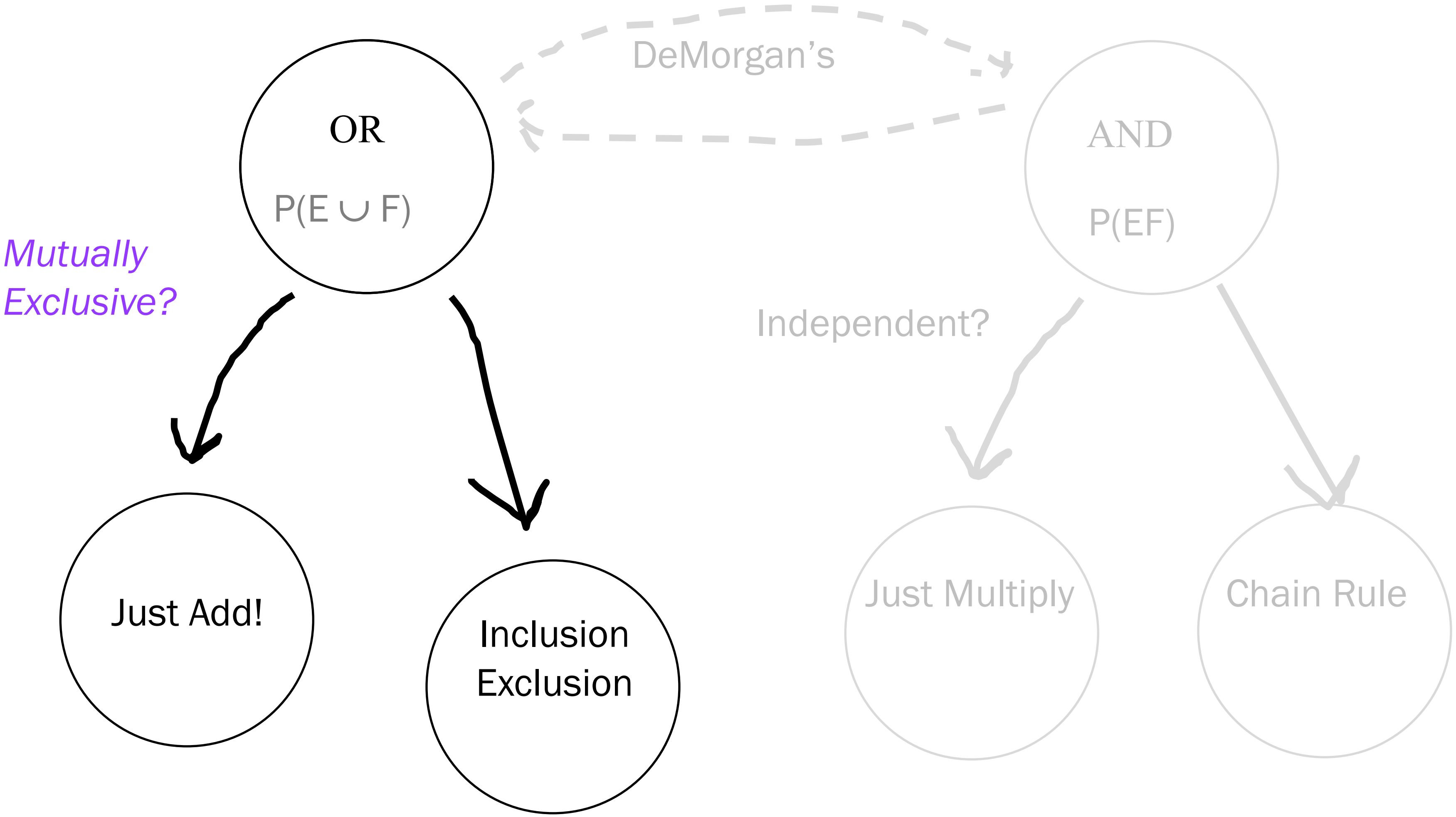
Y_1 = Sum of all events on their own $\quad \sum_i P(E_i)$

Y_2 = Sum of all pairs of events $\quad \sum_{i,j} P(E_i \cap E_j) \quad \text{s.t.} i \neq j$

Y_3 = Sum of all triples of events $\quad \sum_{i,j,k} P(E_i \cap E_j \cap E_k) \quad \text{s.t.} i \neq j, j \neq k, i \neq k$

* Where Y_r is the sum, for all combinations of *r* events, of the probability of the union those events.

Today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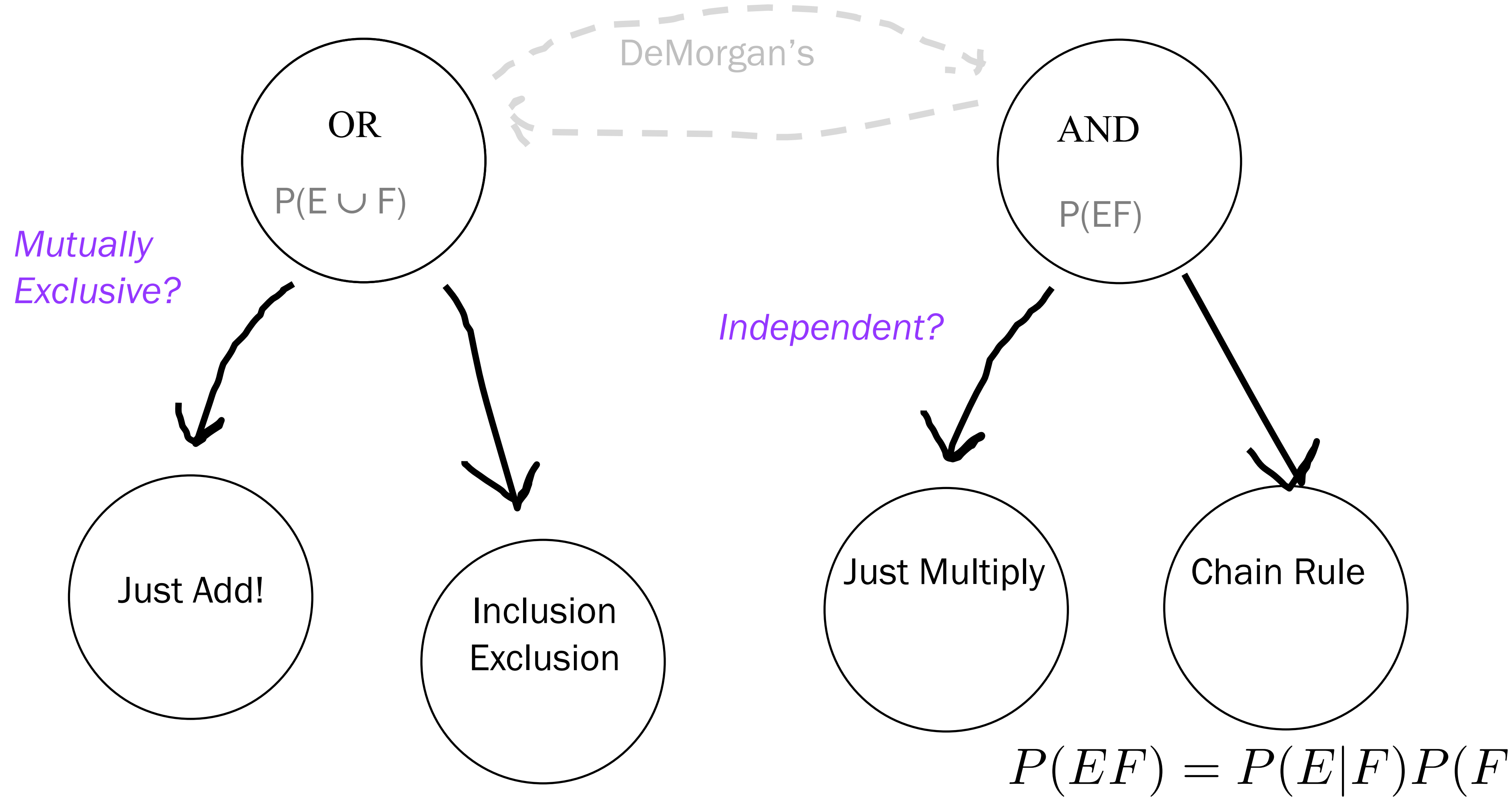


$$P(EF) = P(E|F)P(F)$$

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Probability of “AND”


We the People
insure domestic Tranquility, provide for the common
and our Posterity, do ordain and establish this
Article. I

Independence

Two events A and B are called **independent** if:

$$P(A) = P(A|B)$$

Knowing that event B happened, doesn't change our belief that A will happen.

Otherwise, they are called **dependent** events

Independence is reciprocal

If A is independent of B, then B is independent of A

$$P(A) = P(A|B)$$

$$P(B|A) = P(B)$$

Proof:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Bayes' Thm.

$$= \frac{P(A)P(B)}{P(A)}$$

Because A is independent of B

$$= P(B)$$

Alternative Definition of Independence

$$P(A,B) = P(A) \cdot P(B|A)$$ Probability of and

$$= P(A) \cdot P(B)$$ Since B is independent of A

If you show this is true, you have proved the two events are independent!

If events are *independent* probability of AND is easy!

*You will need to use this "trick" with high probability

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2

- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 1$

What is P(E), P(F), and P(EF)?

- $P(E) = 1/6$, $P(F) = 1/6$, $P(EF) = 1/36$
- $P(EF) = P(E)\,P(F)$ → E and F ***independent***

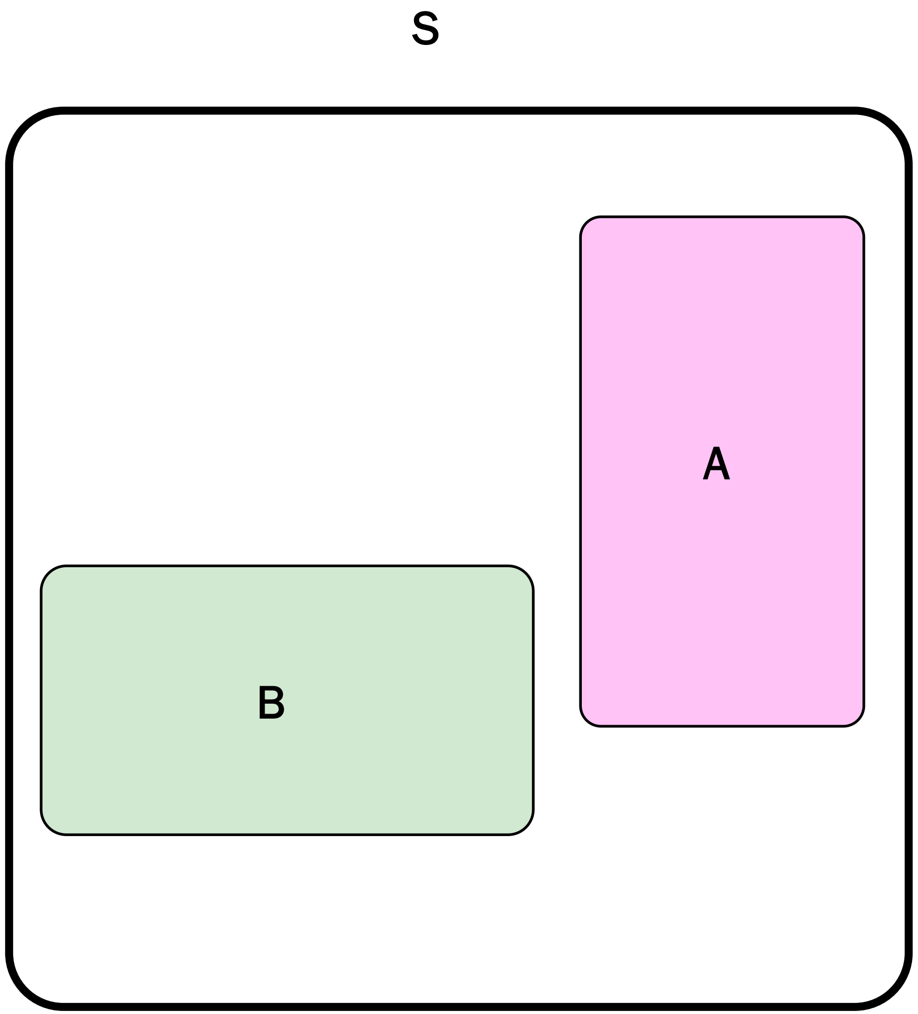
Let G be event: $D_1 + D_2 = 5$ {(1, 4), (2, 3), (3, 2), (4, 1)}

What is P(E), P(G), and P(EG)?

- $P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(EG) = 1/36$
- $P(EG) \neq P(E)\,P(G)$ → E and G ***dependent***

What does independence look like?

Independence



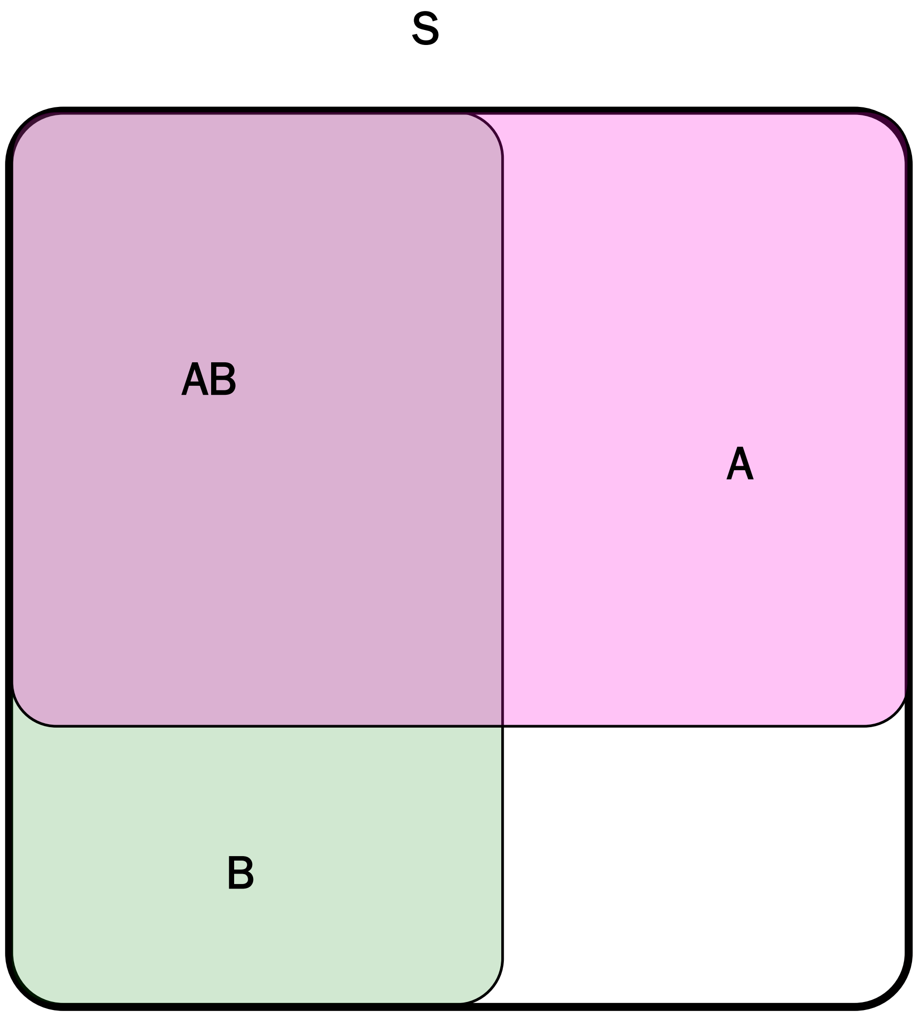
Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

0

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

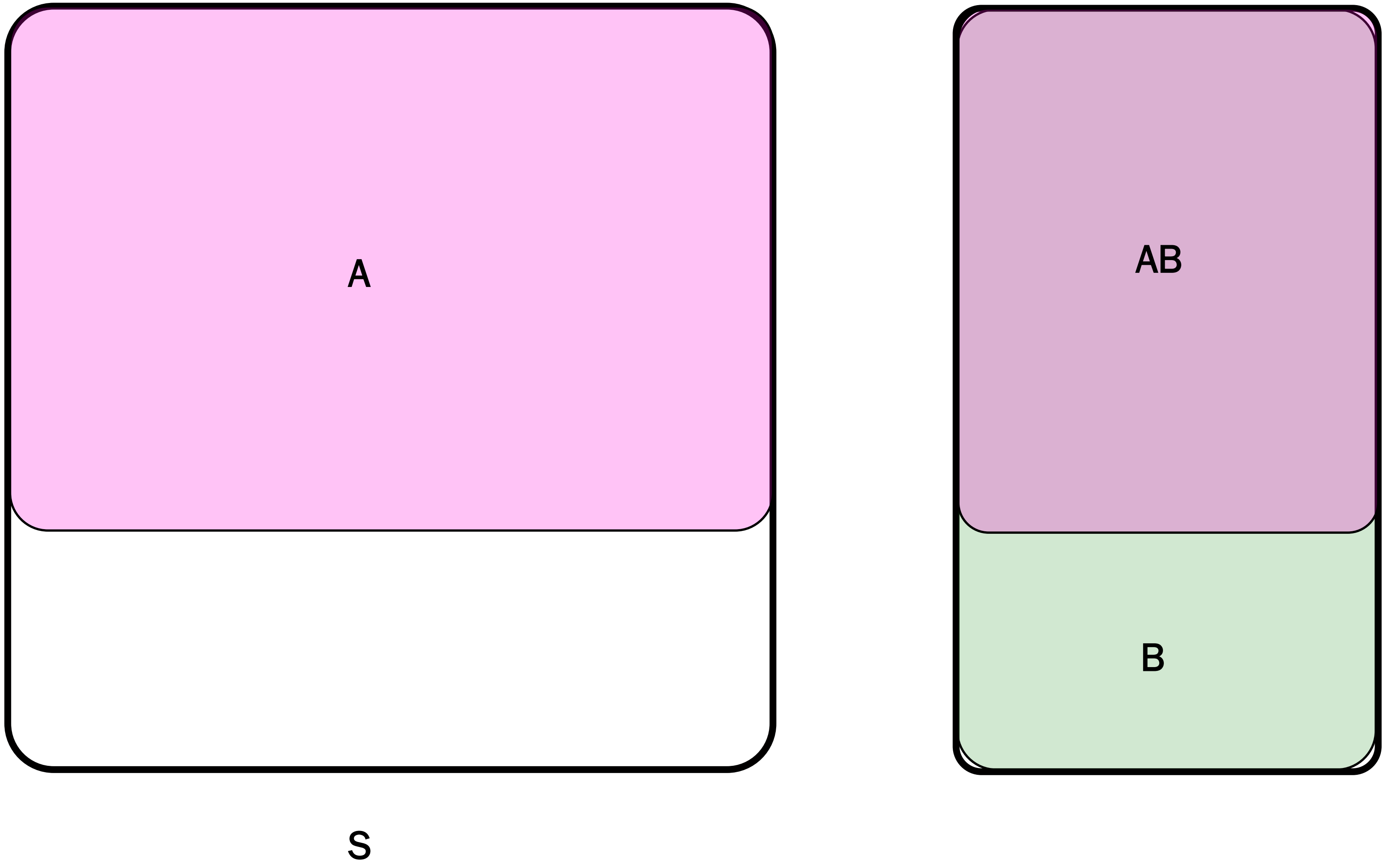
Independence Definition 2:

$$P(A|B) = P(A)$$

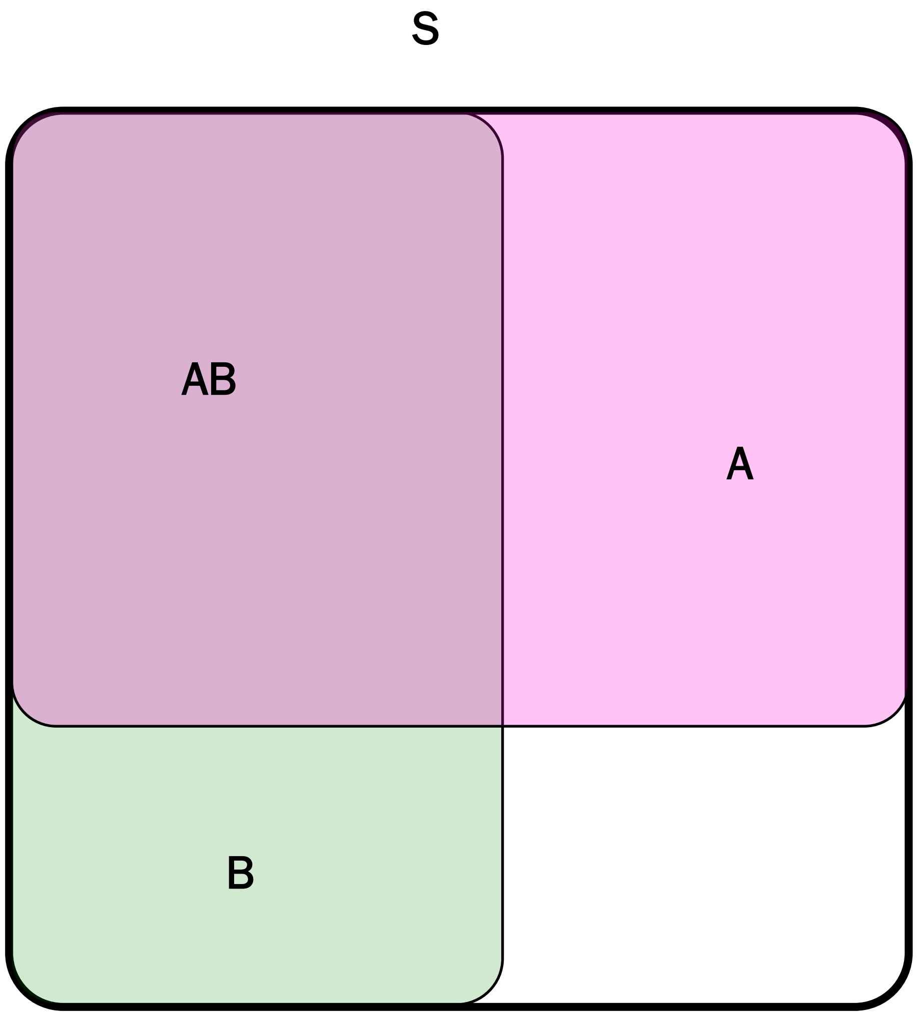
$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Independence

This ratio, P(A)... ... is the same as this one, P(A|B)



Independence



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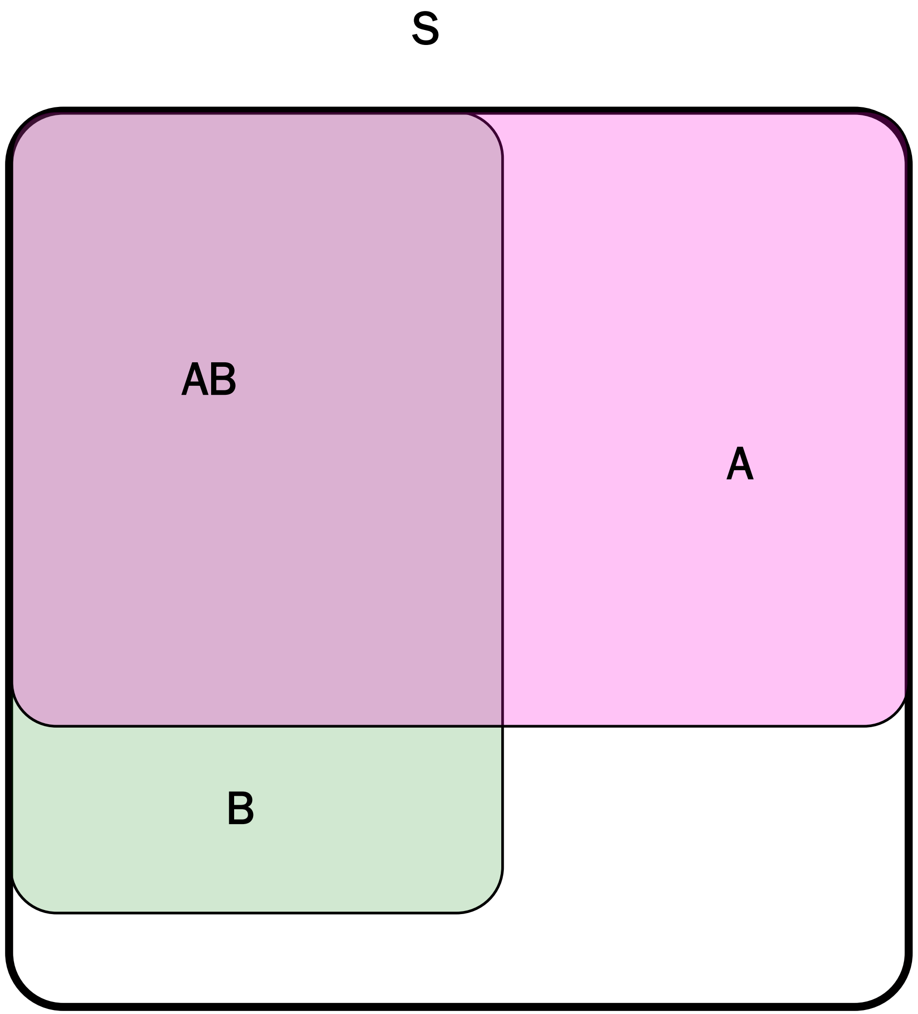
Dependence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

More Intuition through proofs:

Independence

Given independent events A and B, prove that A and B^C are independent

We want to show that $P(AB^C) = P(A)P(B^C)$

$$
\begin{aligned}
P(AB^C) &= P(A) - P(AB) && \text{By Total Law of Prob.} \\
&= P(A) - P(A)P(B) && \text{By independence} \\
&= P(A)[1 - P(B)] && \text{Factoring} \\
&= P(A)P(B^C) && \text{Since } P(B) + P(B^C) = 1
\end{aligned}
$$

So if A and B are independent A and B^C are also independent

Generalization

Generalized Independence

General definition of Independence:

Events E_1 , E_2 , ..., E_n are independent if **for every subset** with r elements (where $r \leq n$) it holds that:

$$P(E_{1'}E_{2'}E_{3'}\ldots E_{r'}) = P(E_{1'})P(E_{2'})P(E_{3'})\ldots P(E_{r'})$$

Example: outcomes of n separate flips of a coin are all independent of one another

- Each flip in this case is called a "trial" of the experiment

Math > Intuition

Two Dice

Roll two 6-sided dice, yielding values D_1 and D_2

- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 6$

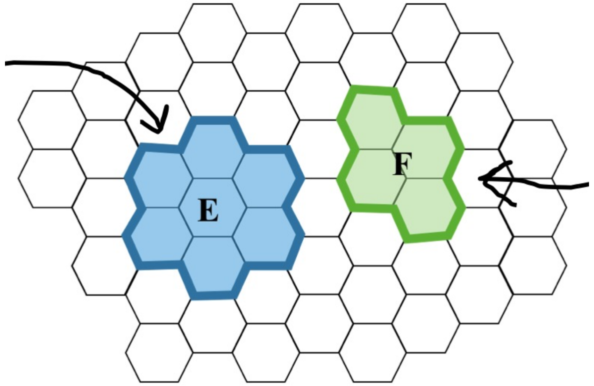
- Are E and F independent? Yes!

Let G be event: $D_1 + D_2 = 7$

- Are E and G independent? Yes!
- $P(E) = 1/6$, $P(G) = 1/6$, $P(E\,G) = 1/36$ [roll (1, 6)]
- Are F and G independent? Yes!
- $P(F) = 1/6$, $P(G) = 1/6$, $P(F\,G) = 1/36$ [roll (1, 6)]
- Are E, F and G independent? No!
- $P(EFG) = 1/36 \neq 1/216 = (1/6)(1/6)(1/6)$

New Ability

Properties of Pairs of Events



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

also:

$$P(A \text{ or } B) = P(A) + P(B)$$

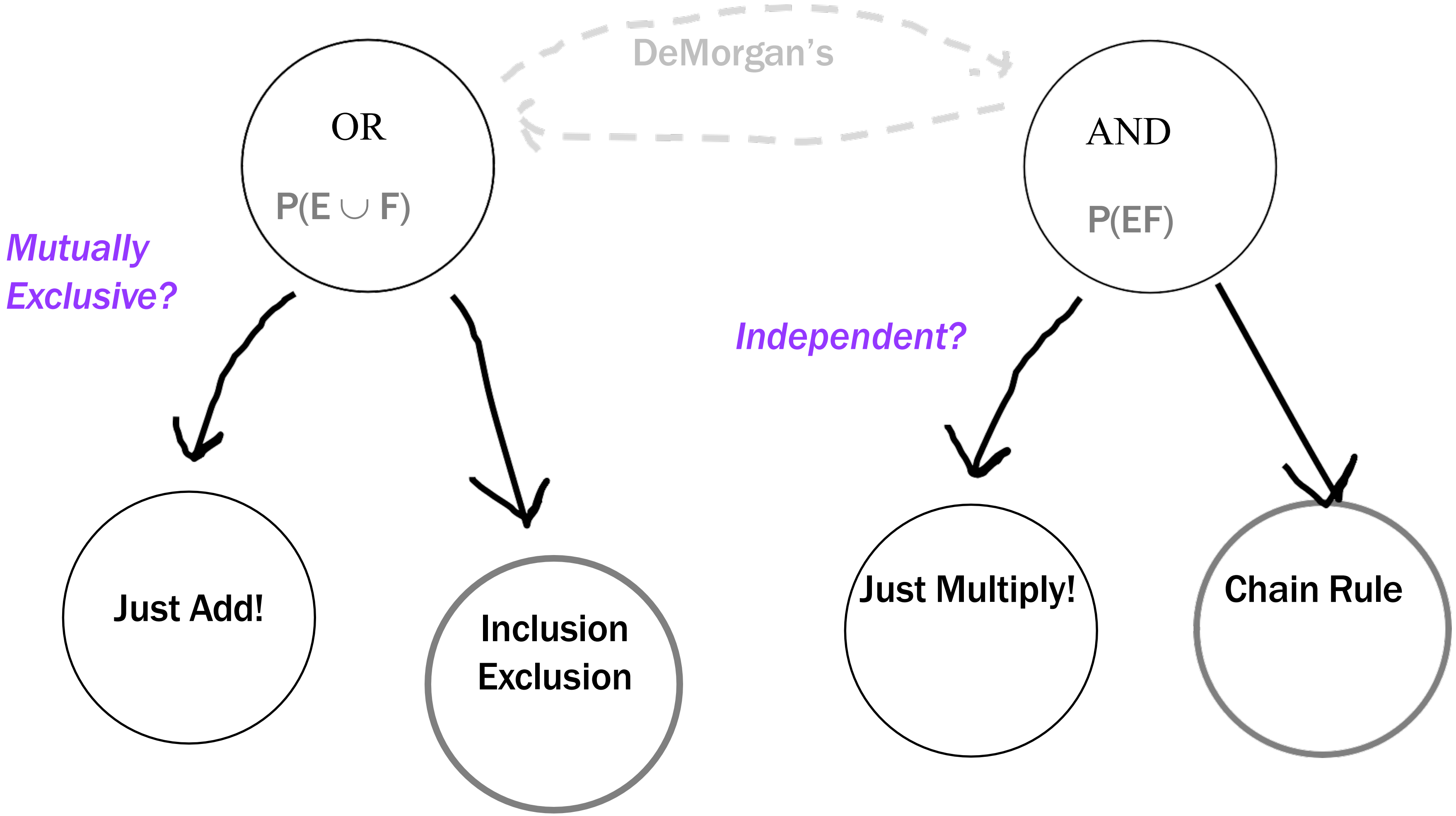
Independent

$$P(A) = P(A|B)$$

also:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Today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Chain Rule

Think of the children as independent trials

Two parents both have an (A, a) gene pair.

- Each parent will pass on one of their genes (each gene equally likely) to a child.
- The probability of **any single child** having curly hair (the recessive trait) is 0.25, independent of other siblings.
- There are three children.

What is the probability that all three children have curly hair?

Let E_1, E_2, E_3 be the events that child 1, 2, and 3 have curly hair, respectively.

$$P(E_1E_2E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)$$
$$= P(E_1)P(E_2)P(E_3)$$

Independence

Two events E and F are defined as independent if:

$$P(EF) = P(E)P(F)$$

For independent events E and F,

- $P(E|F) = P(E)$
- E and F^C are independent.

new

Independence of complements

Statement:

If E and F are independent, then E and F^C are independent.

Proof:

$P(EF^C) = P(E) - P(EF)$	Intersection
$= P(E) - P(E)P(F)$	Independence of E and F
$= P(E)[1 - P(F)]$	Factoring
$= P(E)P(F^C)$	Complement
E and F^C are independent	Definition of independence

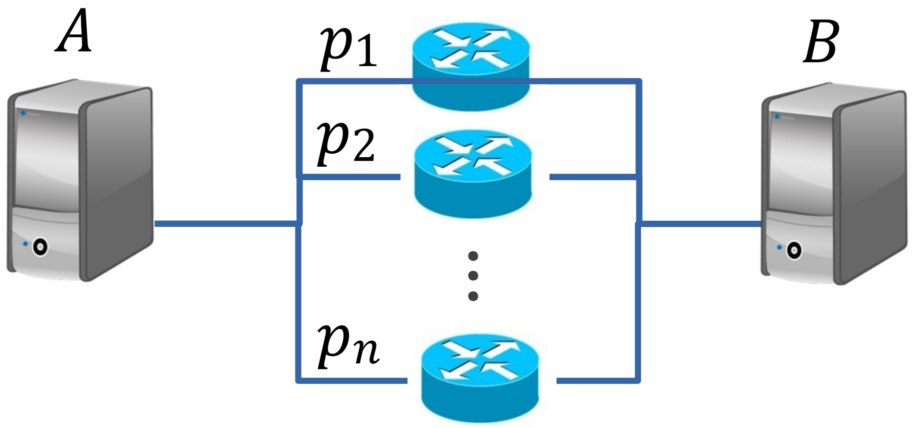
Knowing whether F does or doesn't happen doesn't change our belief about E happening.

Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.

What is $P(E)$?



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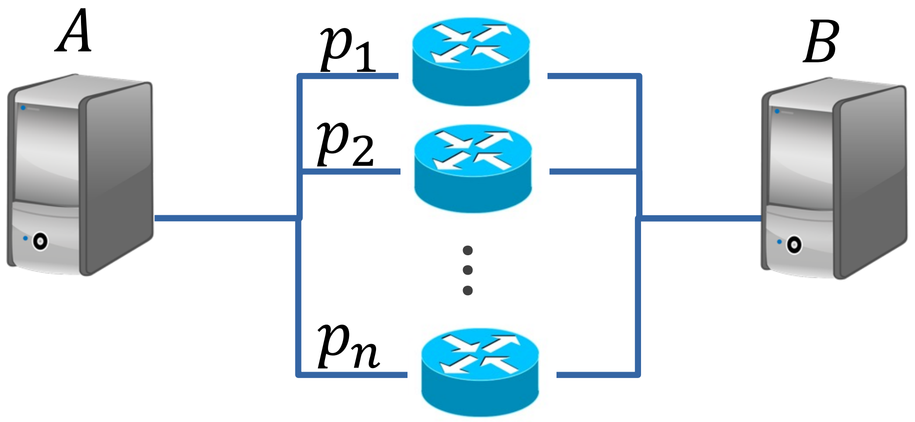


$$P(E) = P(\geq 1 \text{ one router works})$$
$$= 1 - P(\text{all routers fail})$$
$$= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n)$$
$$= 1 - \prod_{i=1}^{n} (1 - p_i)$$

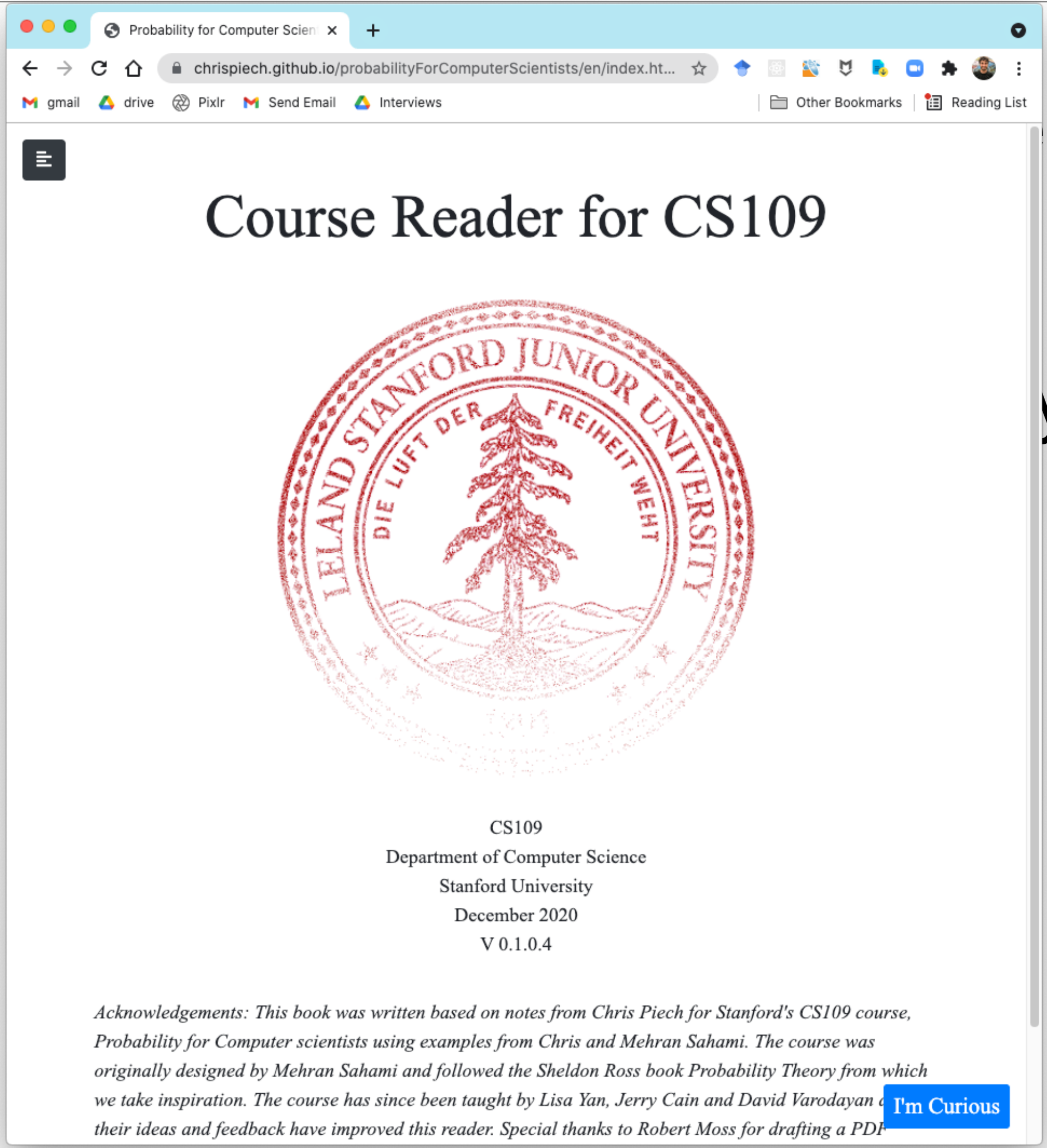
≥ 1 with independent trials: take complement

The Most Important Core Probability Question

Say a coin comes up heads with probability p

- Flip the coin n times
- Each coin flip is an **independent** trial
- What is the probability of exactly k heads?

The Most Important Core Probability Question



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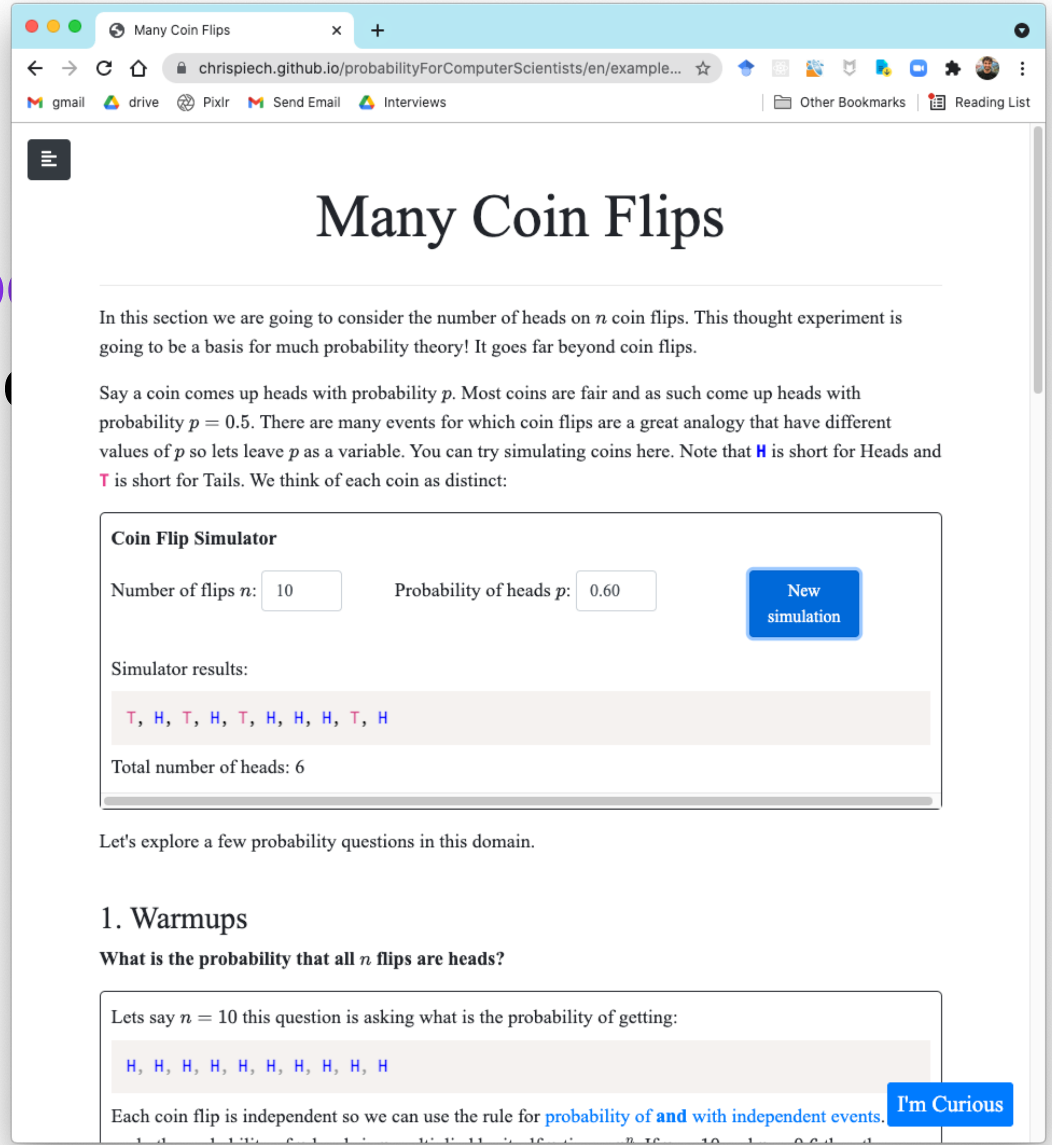
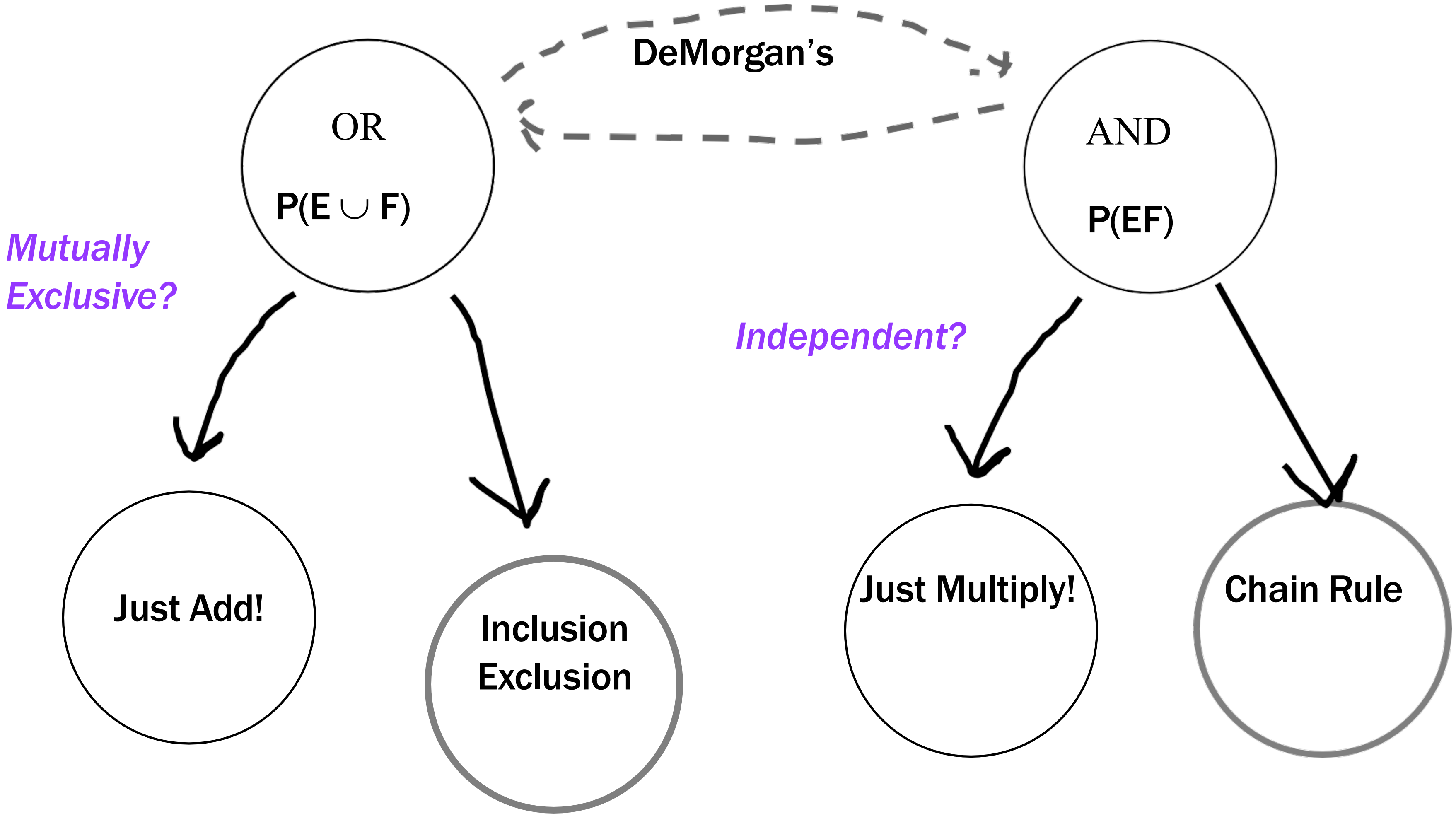


Pedagogical Pause



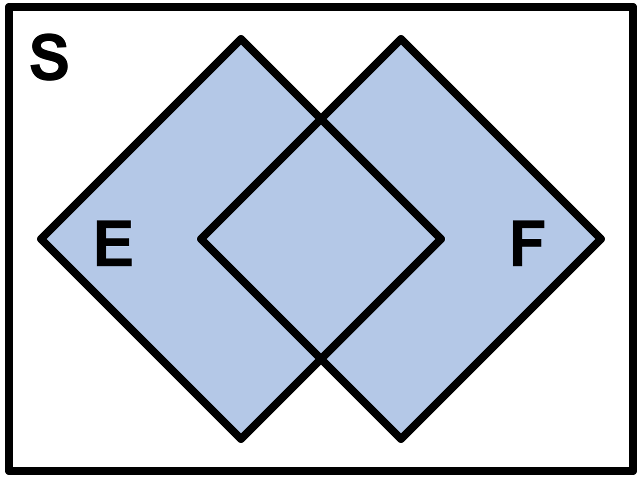
Sets Review

Sets Review

Say E and F are events in S

Event that is in E or F

E ∪ F



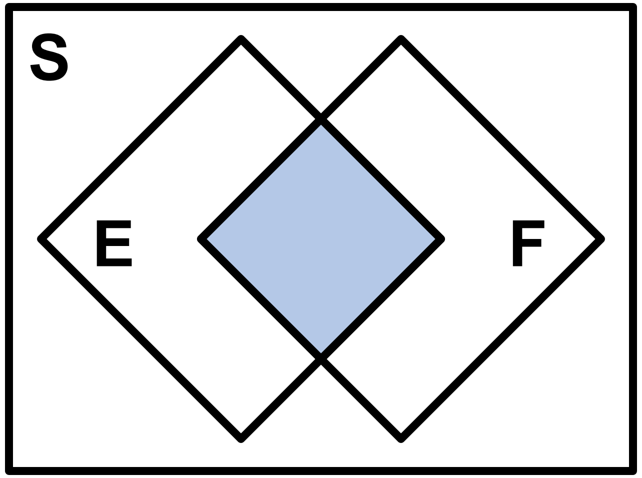
- S = {1, 2, 3, 4, 5, 6} die roll outcome
- E = {1, 2}   F = {2, 3}   E ∪ F = {1, 2, 3}

Sets Review

Say E and F are events in S

Event that is in E and F

$E \cap F$

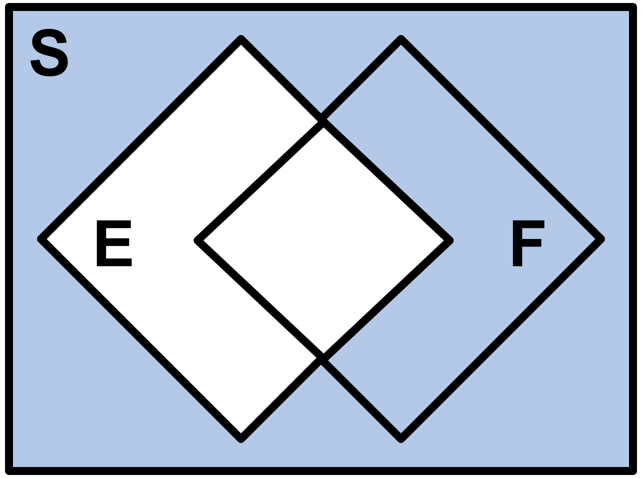


Sets Review

Say E and F are events in S

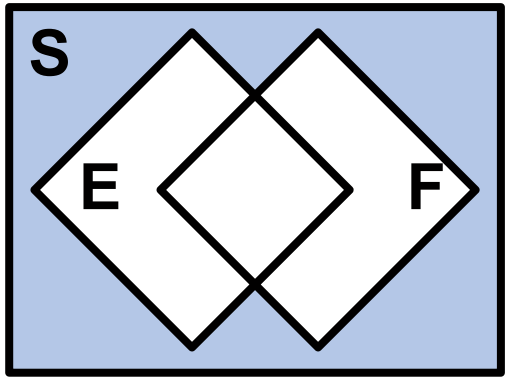
Event that is not in E (called complement of E)

E^c *or* **~E**



Sets Review

Say E and F are subsets of S



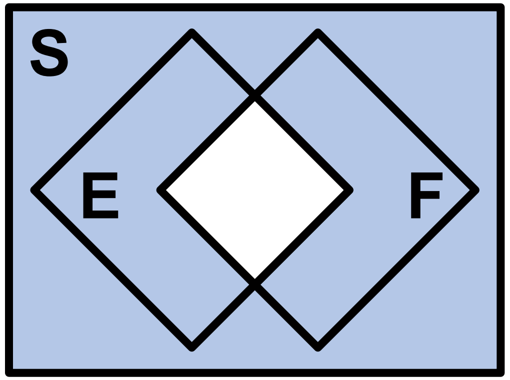
Which of these two is it?

a) $(E \text{ or } F)^C$

b) $(E^C \text{ and } F^C)$

Sets Review

Say E and F are subsets of S



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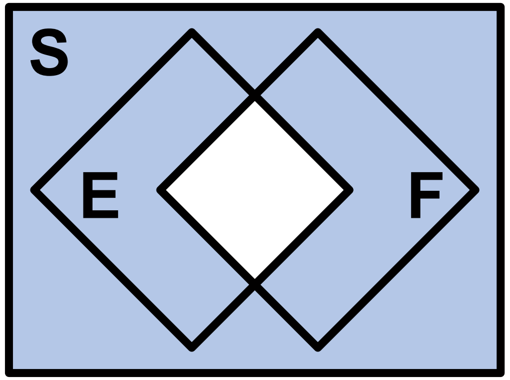
De Morgan's Laws

De Morgan's Law lets you alternate between AND and OR.

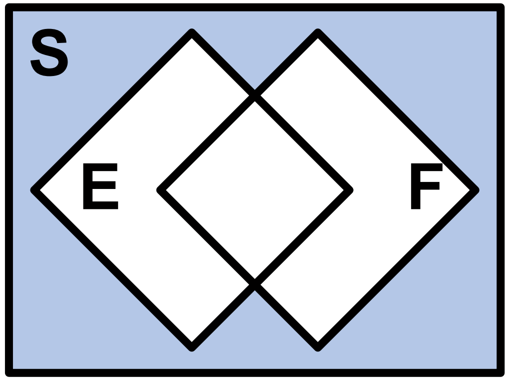


$(E \cap F)^C = E^C \cup F^C$

In probability:

$$P(E_1 E_2 \cdots E_n)$$
$$= 1 - P\left((E_1 E_2 \cdots E_n)^C\right)$$
$$= 1 - P(E_1^C \cup E_2^c \cup \cdots \cup E_n^c)$$

Great if E_i^C mutually exclusive!



$(E \cup F)^C = E^C \cap F^C$

In probability:

$$P(E_1 \cup E_2 \cup \cdots \cup E_n)$$
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$$= 1 - P(E_1^c E_2^c \cdots E_n^c)$$

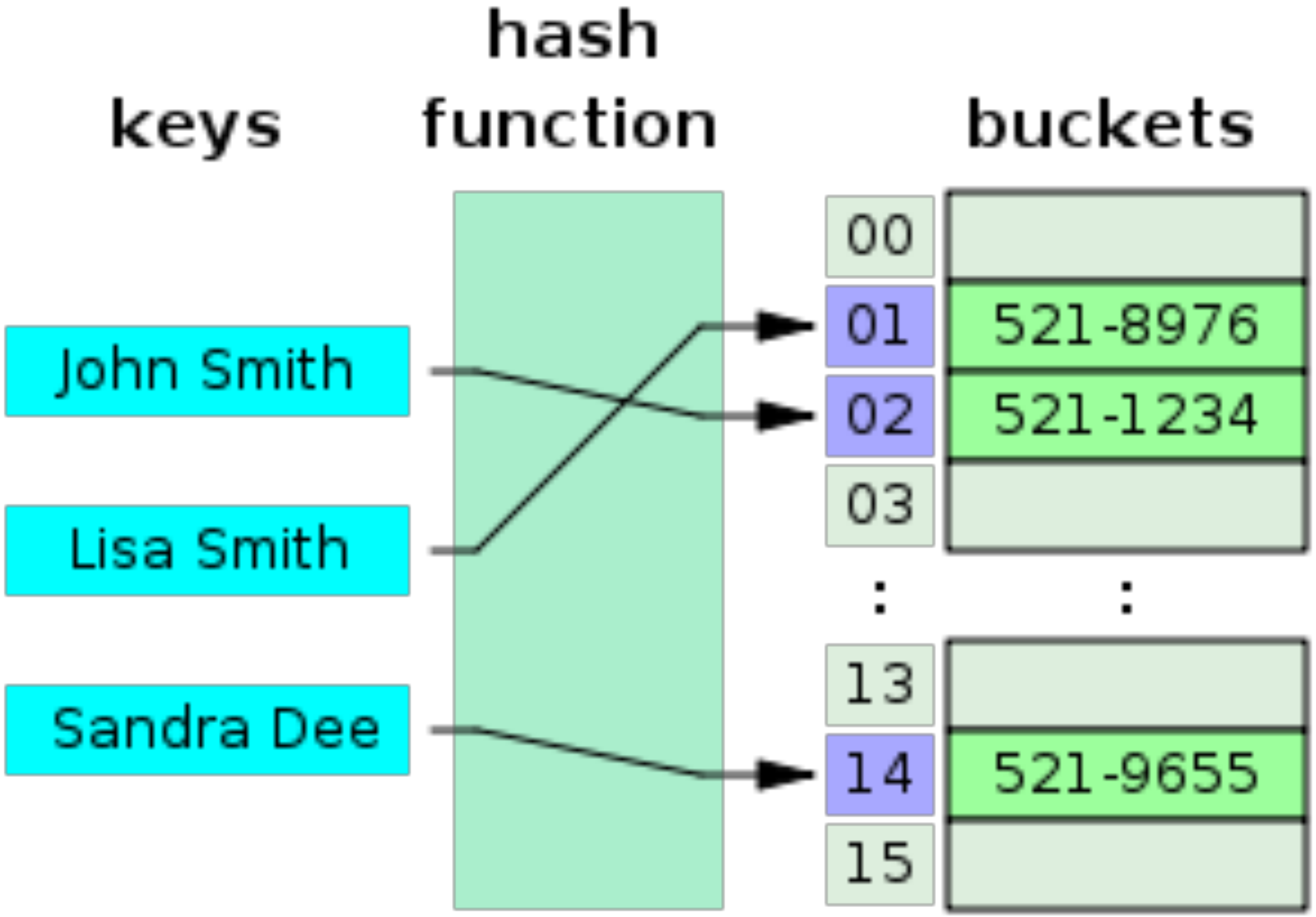
Great if E_i independent!

Augustin Demorgan

Jason Alexander

- British Mathematician who wrote the book “Formal Logic”in 1847
- Celebrity lookalike is Jason Alexander from Seinfeld.

Hash Tables. Hardest Core Probability Question



Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is **independent** with probability p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?

Define: S_i = **string** i **hashes to bucket** 1
S_i^C = **string** i **doesn't hash to bucket** 1

$$P(S_i) = p_1$$
$$P(S_i^C) = 1 - p_1$$

Hash table fun

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it?

Define: S_i = string i hashes to bucket 1
S_i^C = string i doesn't hash to bucket 1

WTF (not-real acronym for Want To Find):

$$P(E) = P(S_1 \cup S_2 \cup \cdots \cup S_m)$$

$$= 1 - P\left((S_1 \cup S_2 \cup \cdots \cup S_m)^C\right)$$ **Complement**

$$= 1 - P\left(S_1^C S_2^C \cdots S_m^C\right)$$ **De Morgan's Law**

$$= 1 - P(S_1^C)P(S_2^C) \cdots P(S_m^C) = 1 - \left(P(S_1^C)\right)^m$$ **S_i independent trials**

$$= 1 - (1 - p_1)^m$$

$$P(S_i) = p_1$$
$$P(S_i^C) = 1 - p_1$$

More hash table fun: Possible approach?

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$$P(E) \quad =$$

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Define F_i = bucket i has at least one string in it

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What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?

$$P(E) \quad = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$

Define F_i = bucket i has at least one string in it

⚠ F_i bucket events are *dependent*! So we cannot just add.

More hash table fun: Possible approach?

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Define F_i = bucket i has at least one string in it

$$P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$
$$= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right)$$
$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$
$$=$$

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$$= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right)$$
$$=$$

$= P($**buckets** 1 **to** k **all denied strings**$)$
$= \left(P(\text{each string hashes to } k+1 \text{ or higher})\right)$
$= (1 - p_1 - p_2 ... - p_k)^m$

⚠ F_i **bucket events are *dependent*! So we cannot just add.**

More hash table fun: Possible approach?

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Define F_i = bucket i has at least one string in it

$$
\begin{aligned}
P(E) &= P(F_1 \cup F_2 \cup \cdots \cup F_k) \\
&= 1 - P\left((F_1 \cup F_2 \cup \cdots \cup F_k)^C\right) \\
&= 1 - P\left(F_1^C F_2^C \cdots F_k^C\right) \\
&= 1 - (1 - p_1 - p_2 \ldots - p_k)^m
\end{aligned}
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\begin{aligned}
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\end{aligned}
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⚠ F_i **bucket events are *dependent*! So we cannot just add.**

The fun never stops with hash tables

- m strings are hashed (not uniformly) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it? ✅
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it? ✅

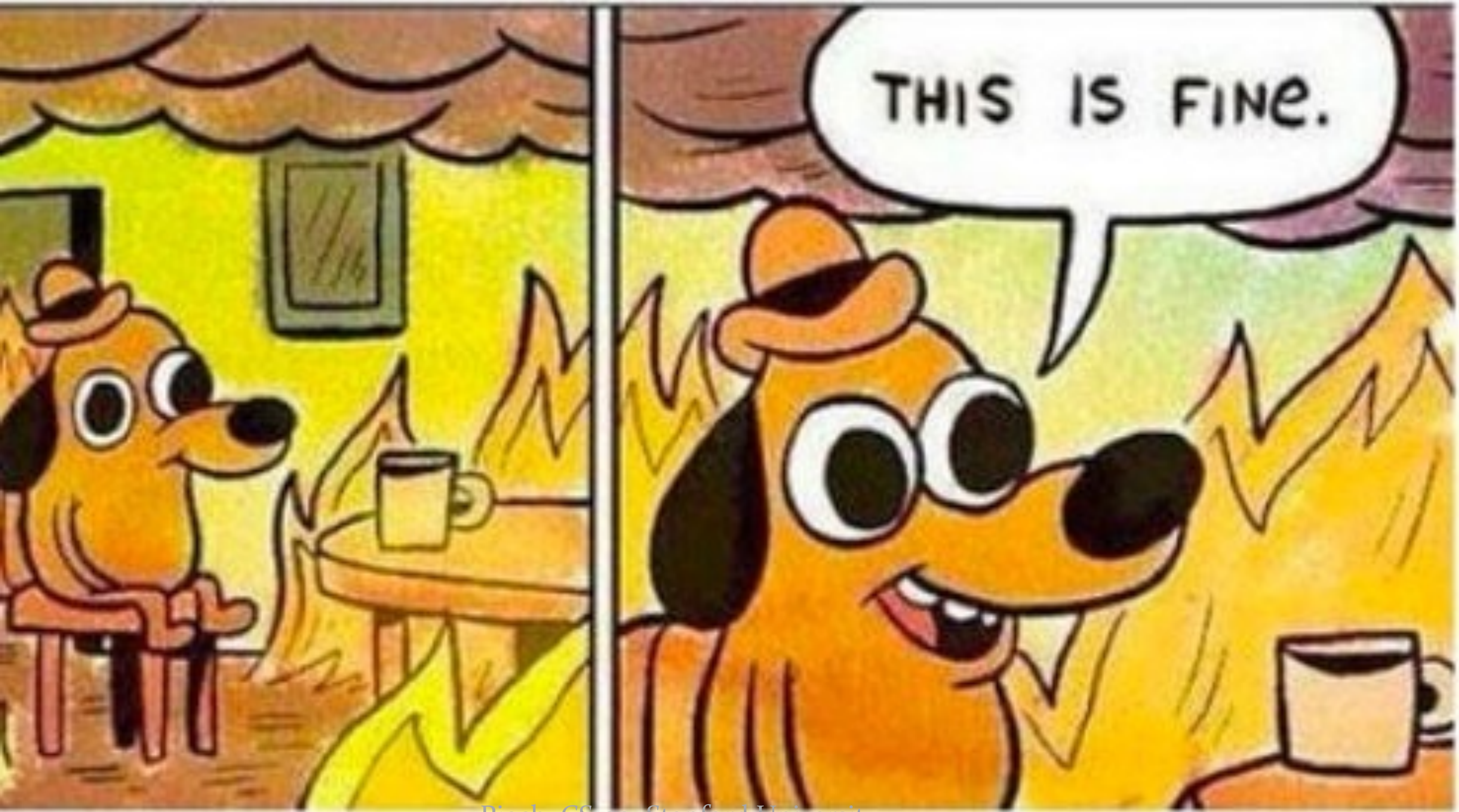
Looking for a challenge? ☺

The fun never stops with hash tables

- m strings are hashed (unequally) into a hash table with n buckets.
- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has $\geq$ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has $\geq$ 1 string hashed into it?
3. E = each of buckets 1 to k has $\geq$ 1 string hashed into it?


THIS IS FINE.

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- Each string hash is an **independent trial** w.p. p_i of getting hashed into bucket i.

What is $P(E)$ if

1. E = bucket 1 has ≥ 1 string hashed into it?
2. E = at least 1 of buckets 1 to k has ≥ 1 string hashed into it?
3. E = each of buckets 1 to k has ≥ 1 string hashed into it?

Hint: Use Part 2's event definition:

Define F_i **= bucket** i **has at least one string in it**

Hint: Try $k = 2$**, then** $k = 3$**, then generalize.**

The fun never stops with hash tables

Solution

- F_i = at least one string hashed into *i*-th bucket
- P(E) $= P(F_1 F_2 \ldots F_k) = 1 - P((F_1 F_2 \ldots F_k)^c)$

  $= 1 - P(F_1^c \cup F_2^c \cup \ldots \cup F_k^c)$ (DeMorgan's Law)

  $= 1 -$

where $$P\left(\bigcup_{i=1}^{k} F_i^c\right) = 1 - \sum_{r=1}^{k} (-1)^{(r+1)} \sum_{i_1 < \ldots < i_r} P(F_{i_1}{}^c F_{i_2}{}^c \ldots F_{i_r}{}^c)$$

$$P(F_{i_1}{}^c F_{i_2}{}^c \ldots F_{i_r}{}^c) = (1 - p_{i_1} - p_{i_2} - \ldots - p_{i_r})^m$$

Here we are

Source: The Hobbit


G_1
G_2
G_3
G_4
G_5
T

G_1 G_2 G_3 G_4 G_5 T

dna.txt — dna

```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
False,True,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,False,True,False,False,False
False,False,True,False,True,False
True,False,False,True,False,False
False,True,False,True,True,False
True,False,False,True,False,False
True,False,True,True,False,False
False,True,False,False,True,False
False,False,True,True,False,False
True,True,False,False,True,True
True,False,True,True,False,False
True,True,True,True,True,True
True,False,True,False,False,True
False,True,False,True,True,True
False,False,True,False,False,False
False,False,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,True,False,True,True,False
True,False,False,False,False,True
False,False,True,True,False,True
False,False,False,True,False,False
False,True,True,False,False,True
False,True,False,False,True,True
False,False,False,False,False,True
False,True,False,True,True,False
True,False,False,True,False,False
True,True,False,True,True,True
True,True,False,False,True,True
True,True,False,True,True,True
False,False,False,True,False,False
```

100,000 samples

6 observations per sample

Stanford University

Discovered Pattern

```
Piech-2:dna piech$ python findStructure.py
size data =  100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

...

```
p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
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Only Causal Structure That Fits