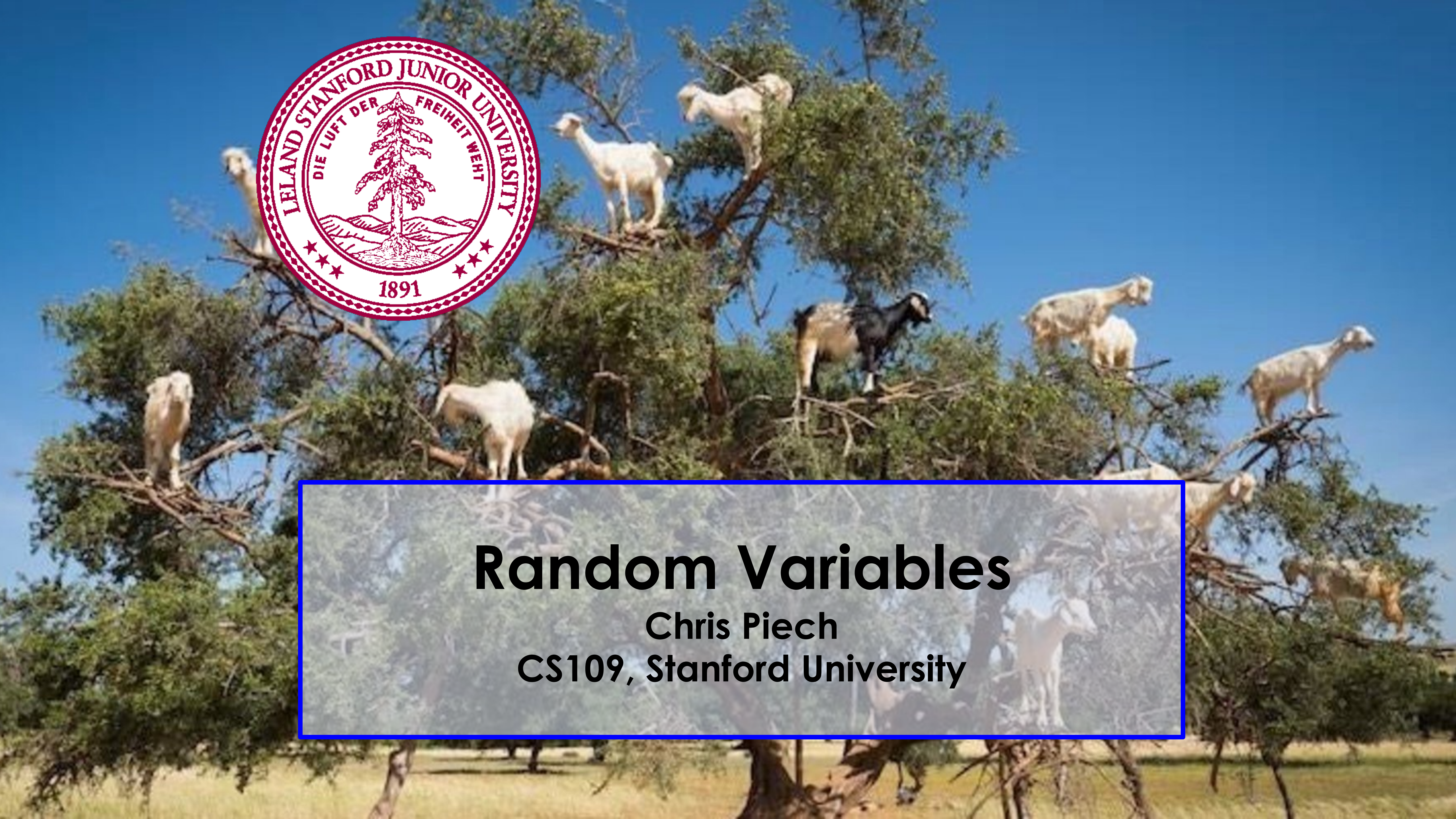


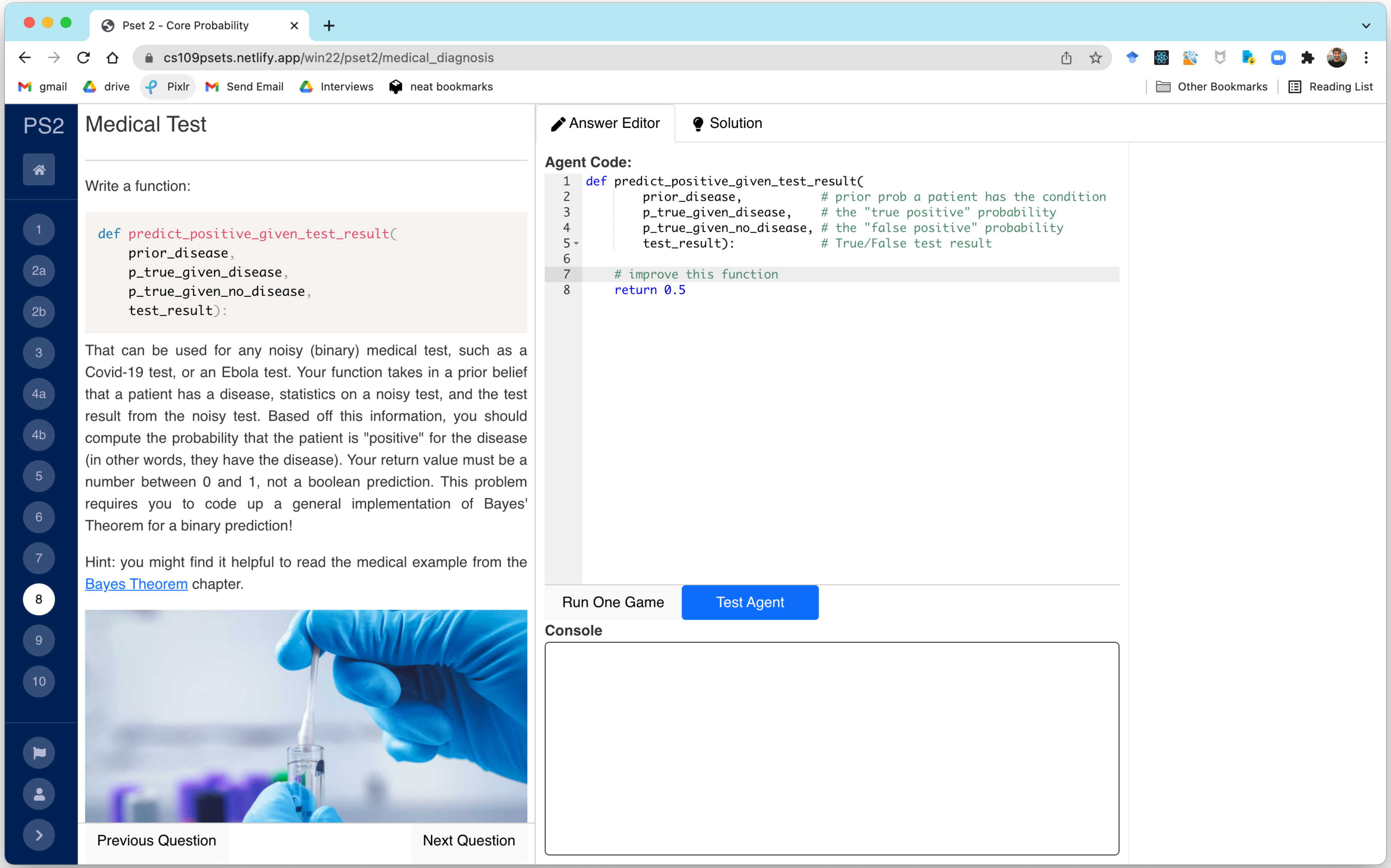
Random Variables

Chris Piech
CS109, Stanford University

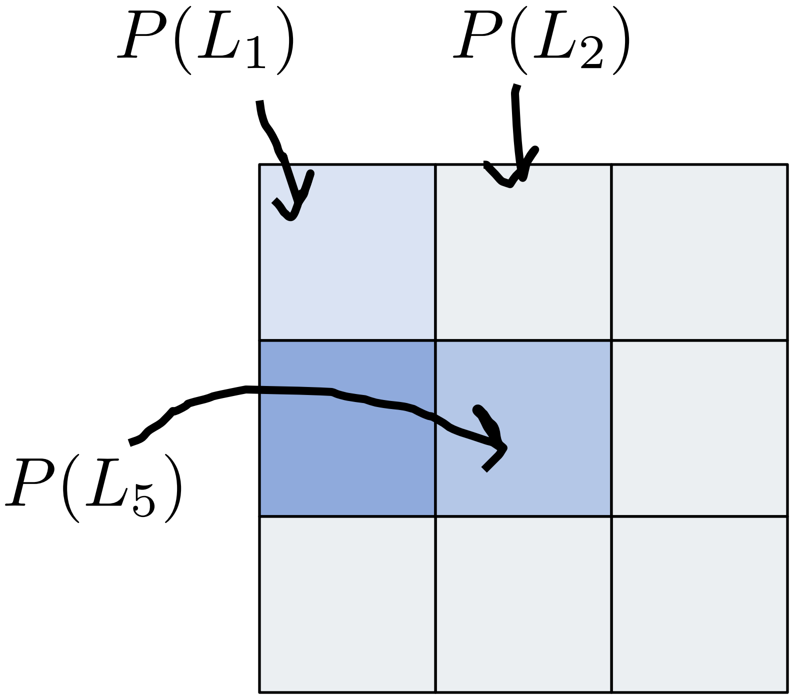
Announcements

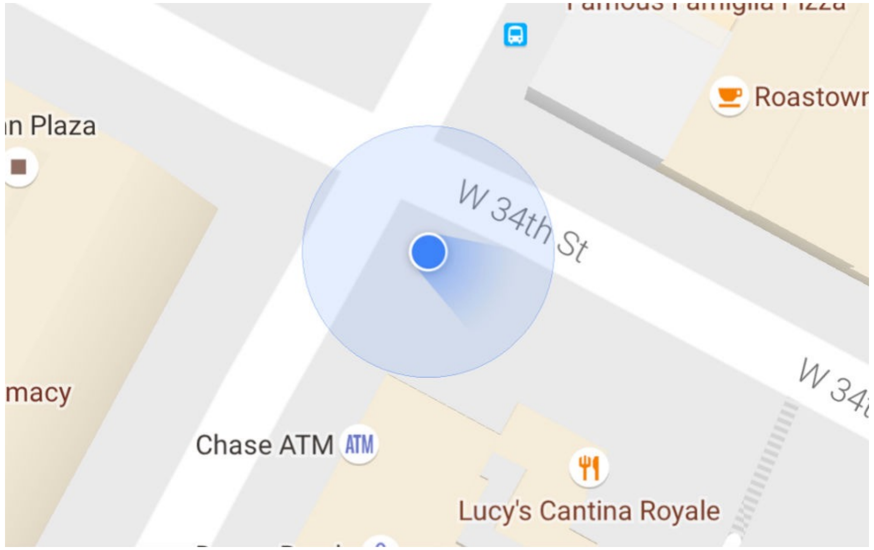
- Pset #1 in today
- Pset #2 out today
- MLK Day
- Happy Friday!

Problem Set #2



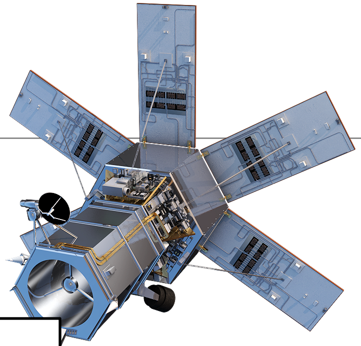
Bayes' Theorem and Location



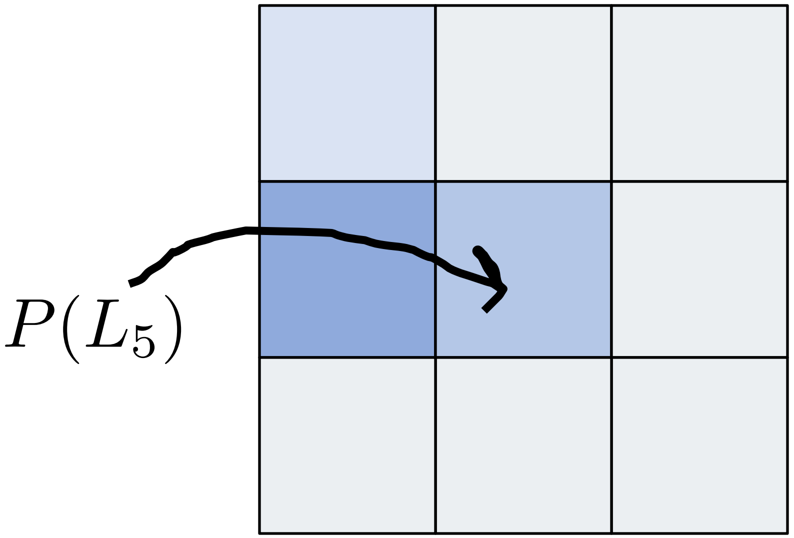


Before Observation

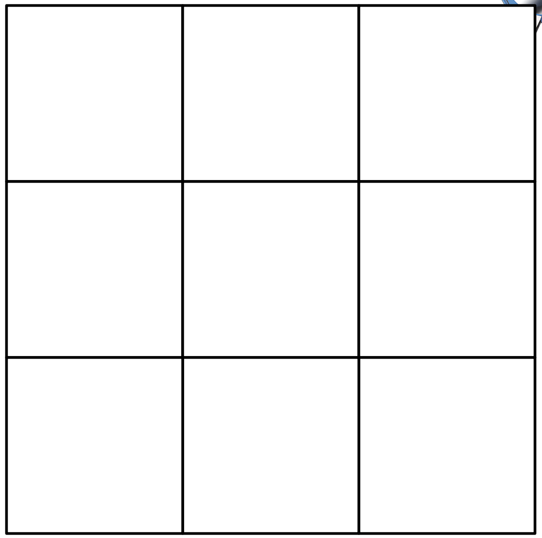
Bayes' Theorem and Location



Know: $P(O|L_i)$

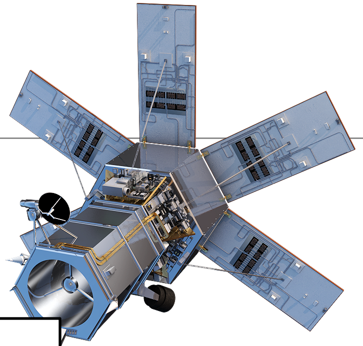


Before Observation

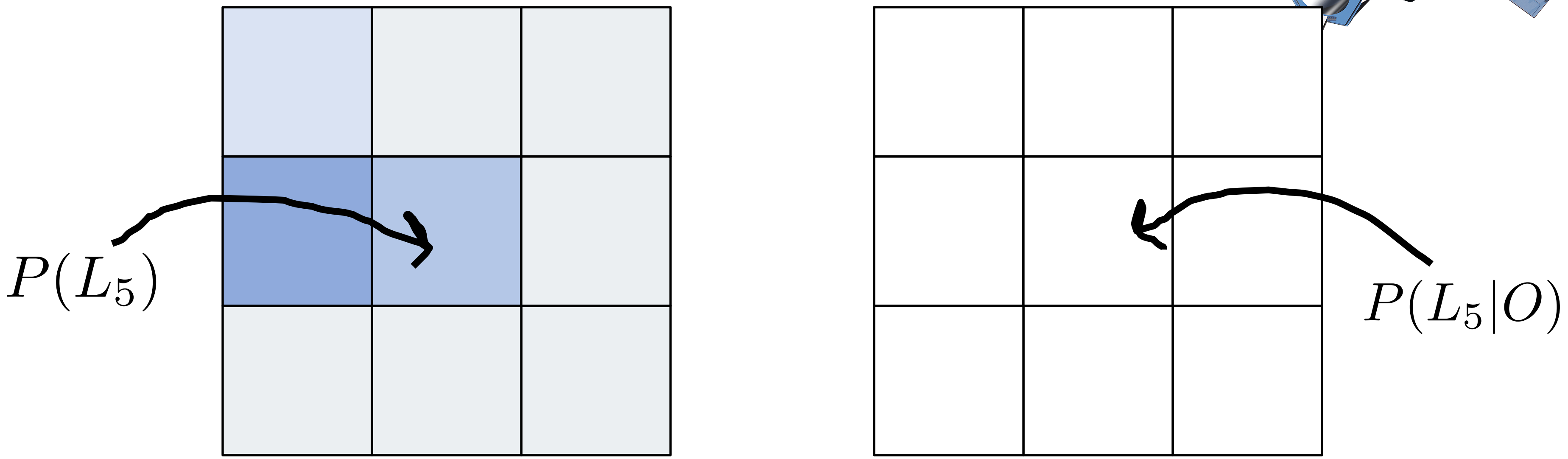


After Observation

Bayes' Theorem and Location



Know: $P(O|L_i)$

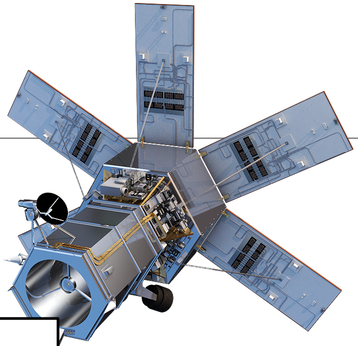


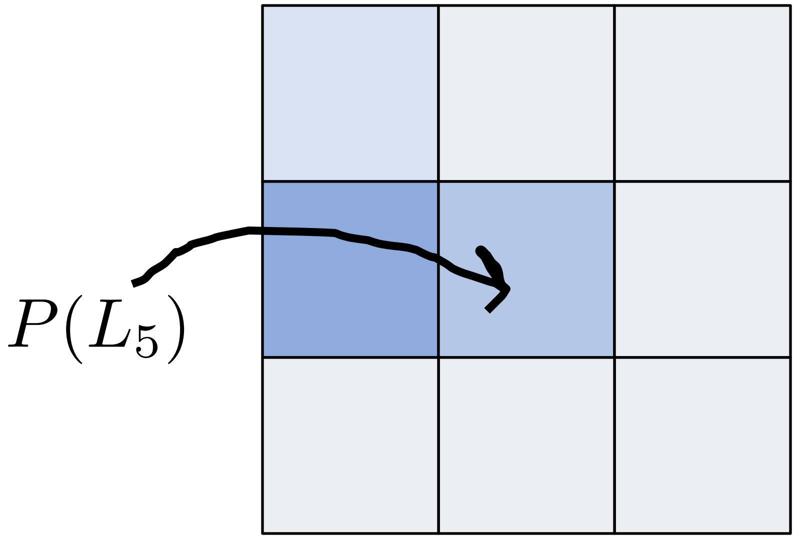
Before Observation

After Observation

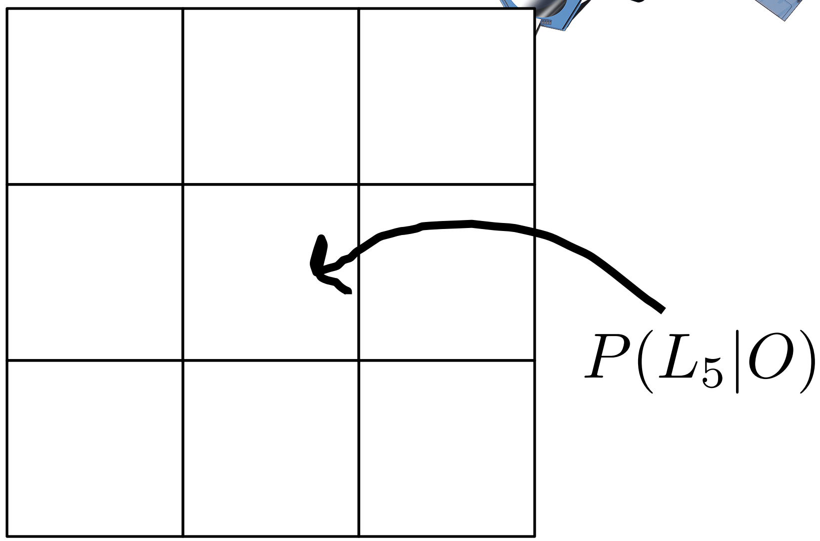
$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{P(O)}$$

Bayes' Theorem and Location





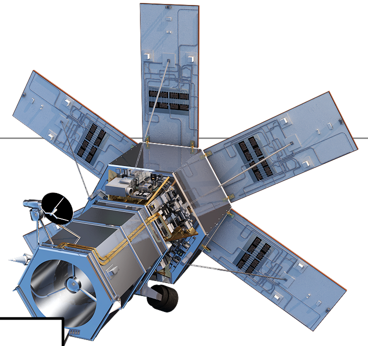
Before Observation

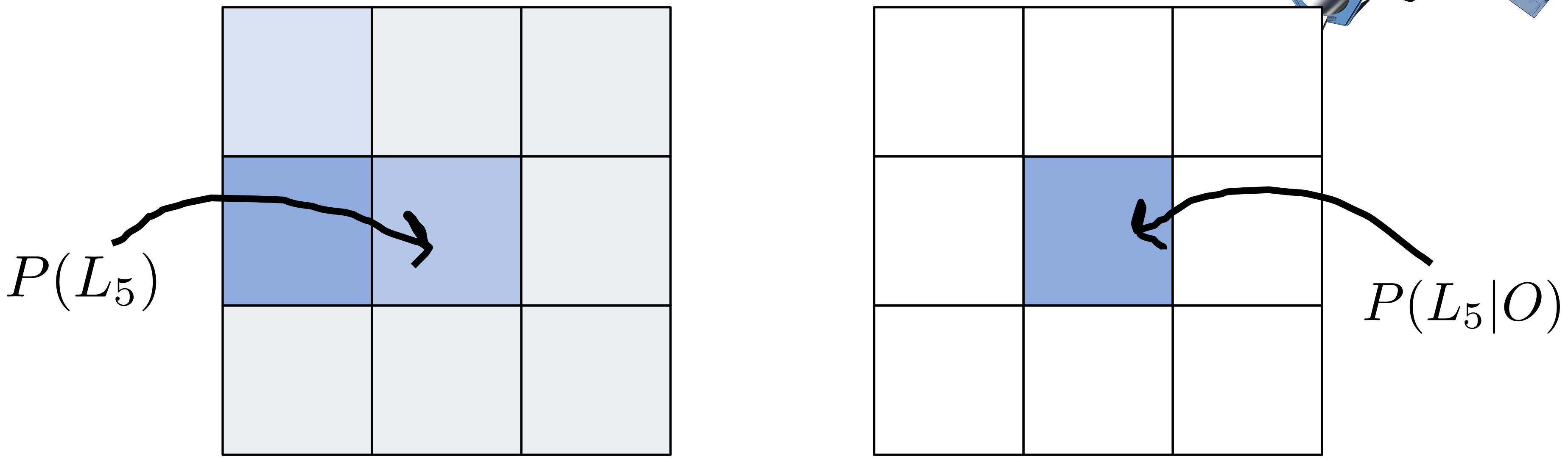


After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Bayes' Theorem and Location





Before Observation

After Observation

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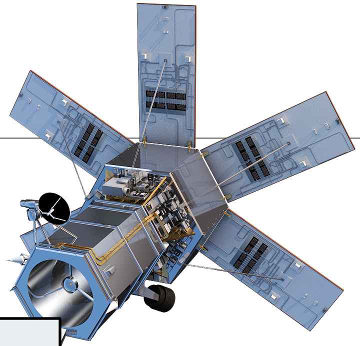
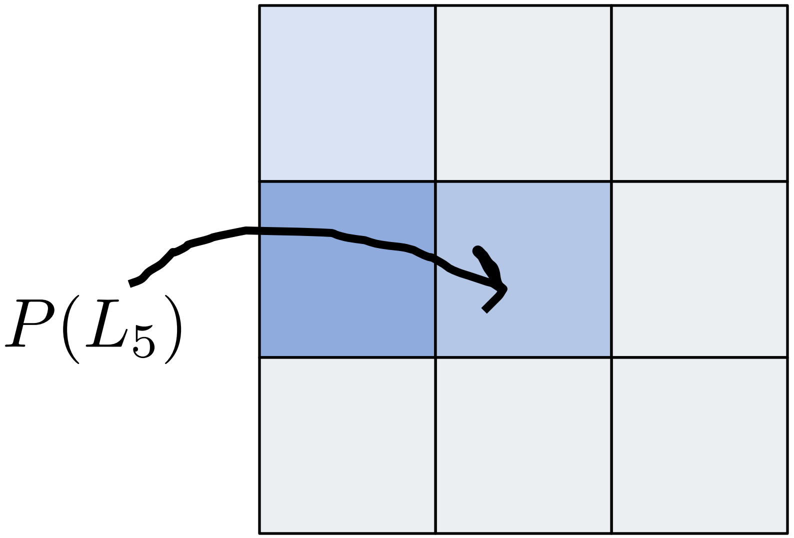
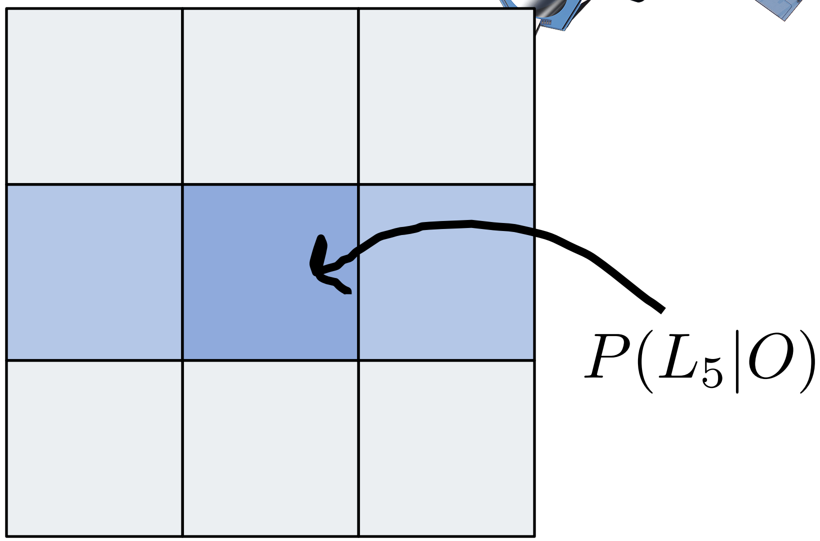
Bayes' Theorem and Location



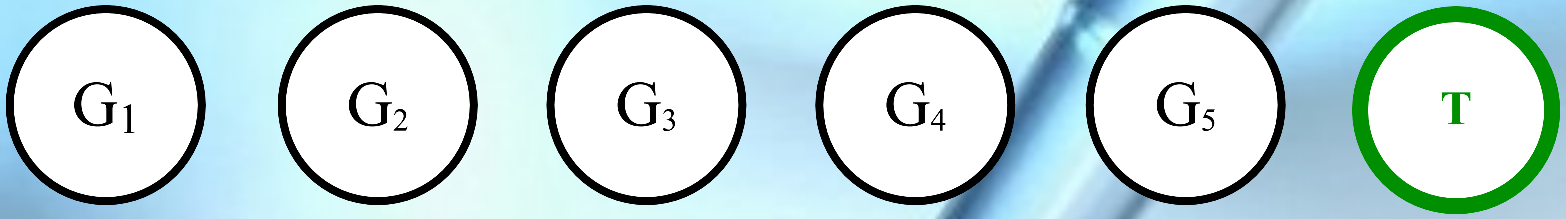


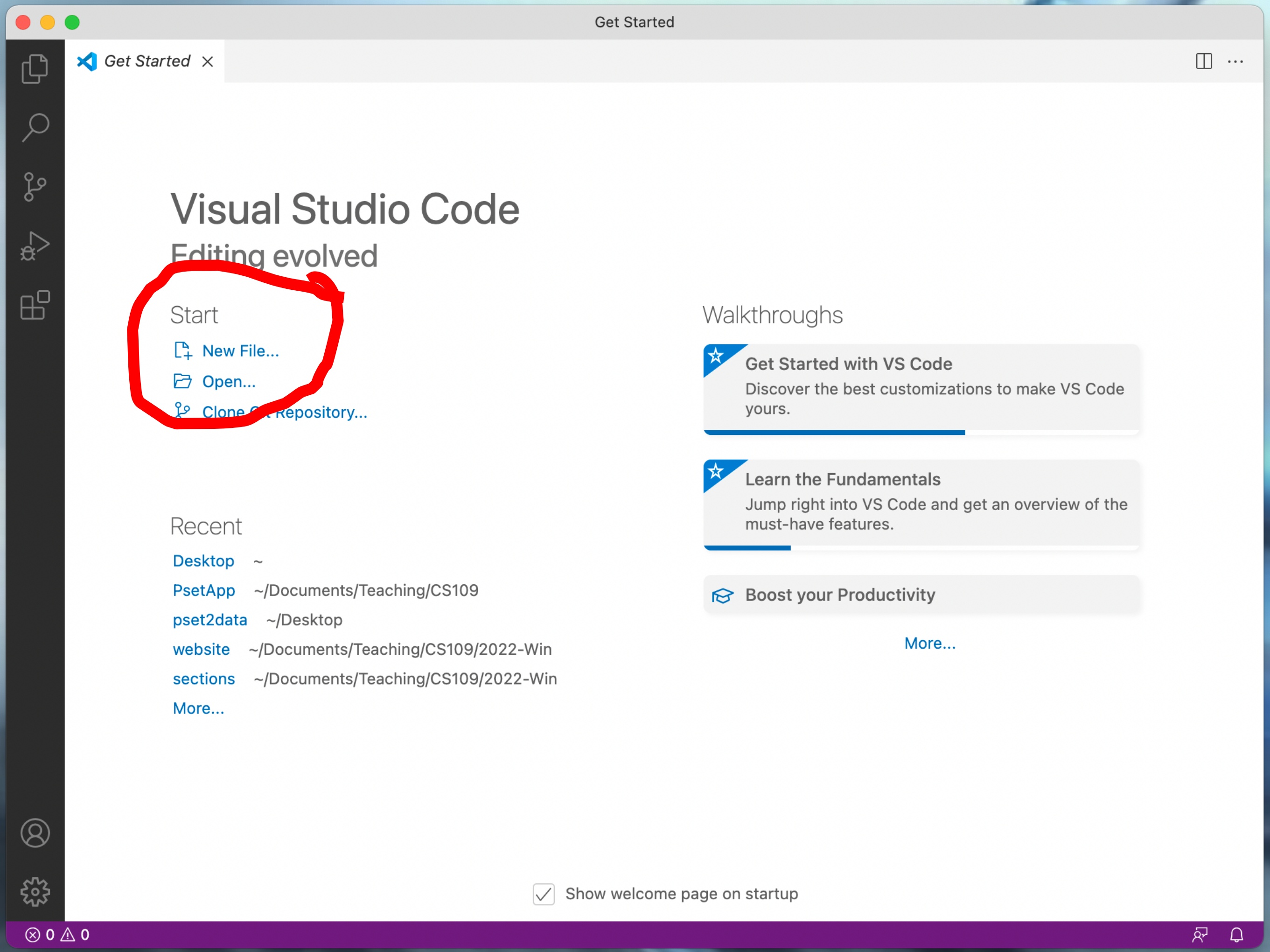
Before Observation



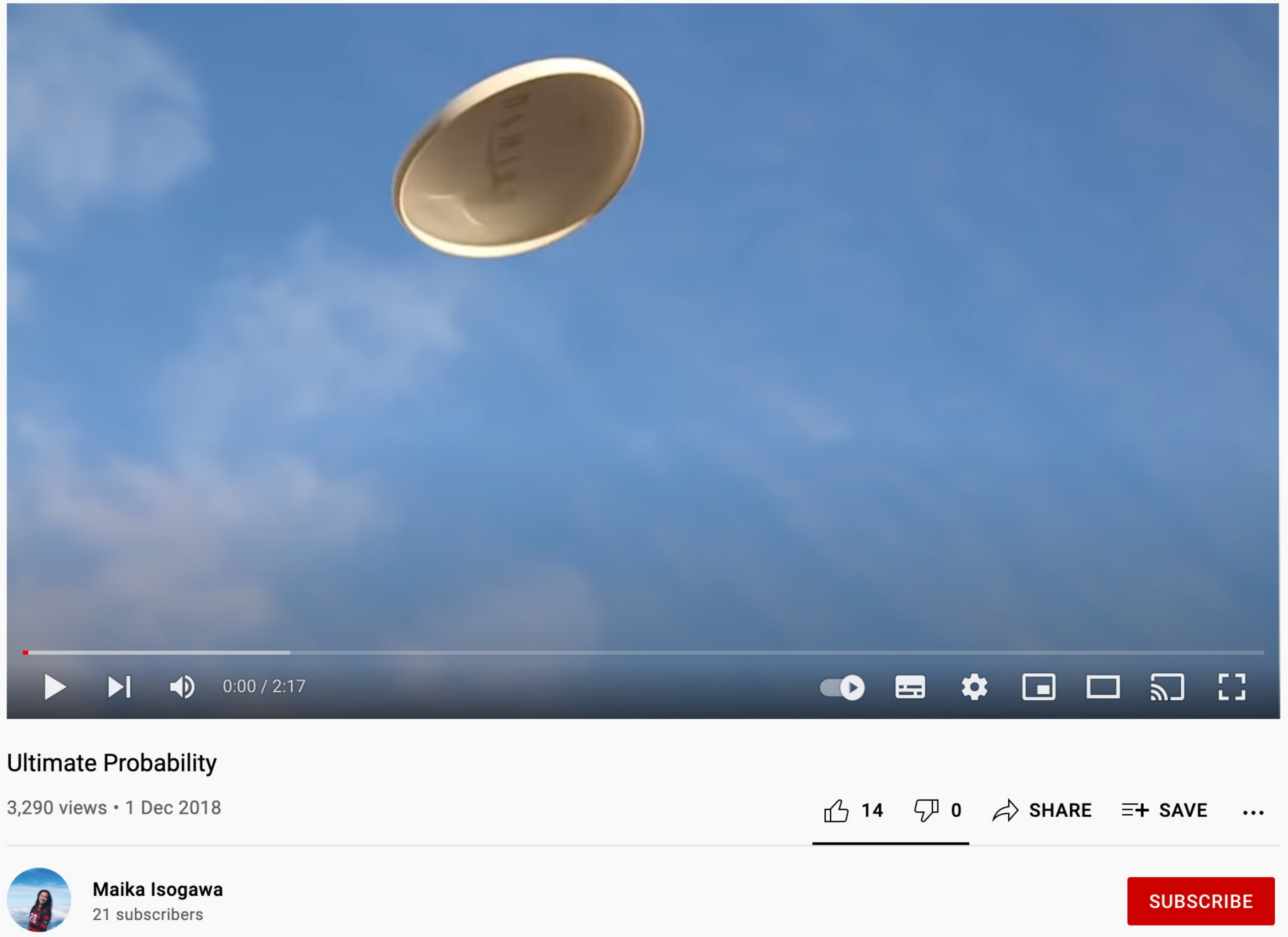
After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$


G1
G2
G3
G4
G5
T


Get Started
Get Started
Visual Studio Code
Editing evolved
Start
New File...
Open...
Repository...
Recent
Desktop ~
PsetApp ~/Documents/Teaching/CS109
pset2data ~/Desktop
website ~/Documents/Teaching/CS109/2022-Win
sections ~/Documents/Teaching/CS109/2022-Win
More...
Walkthroughs
Get Started with VS Code
Discover the best customizations to make VS Code yours.
Learn the Fundamentals
Jump right into VS Code and get an overview of the must-have features.
Boost your Productivity
More...
Show welcome page on startup
0
0

Story: Ultimate Probability



https://www.maikaisogawa.com/ultimate-frisbee-probability/

https://www.youtube.com/watch?v=H2IfTwGisOg

Let's Play a Game

Game set-up

- We have a fair coin (come up "heads" with p = 0.5)
- Let n = number of coin flips ("heads") before first "tails"
- You win $\$2^n$

How much would you pay to play?

Review!

1. Review: Axiom Probability Tools

- Axiom 1: $0 \leq P(E) \leq 1$
- Axiom 2: $P(S) = 1$
- Identity 3: $P(E^c) = 1 - P(E)$

2. Review: Conditional Probability Tools

$P(E \text{ and } F)$

Chain rule (Product rule)

Definition of conditional probability

$P(E|F)$

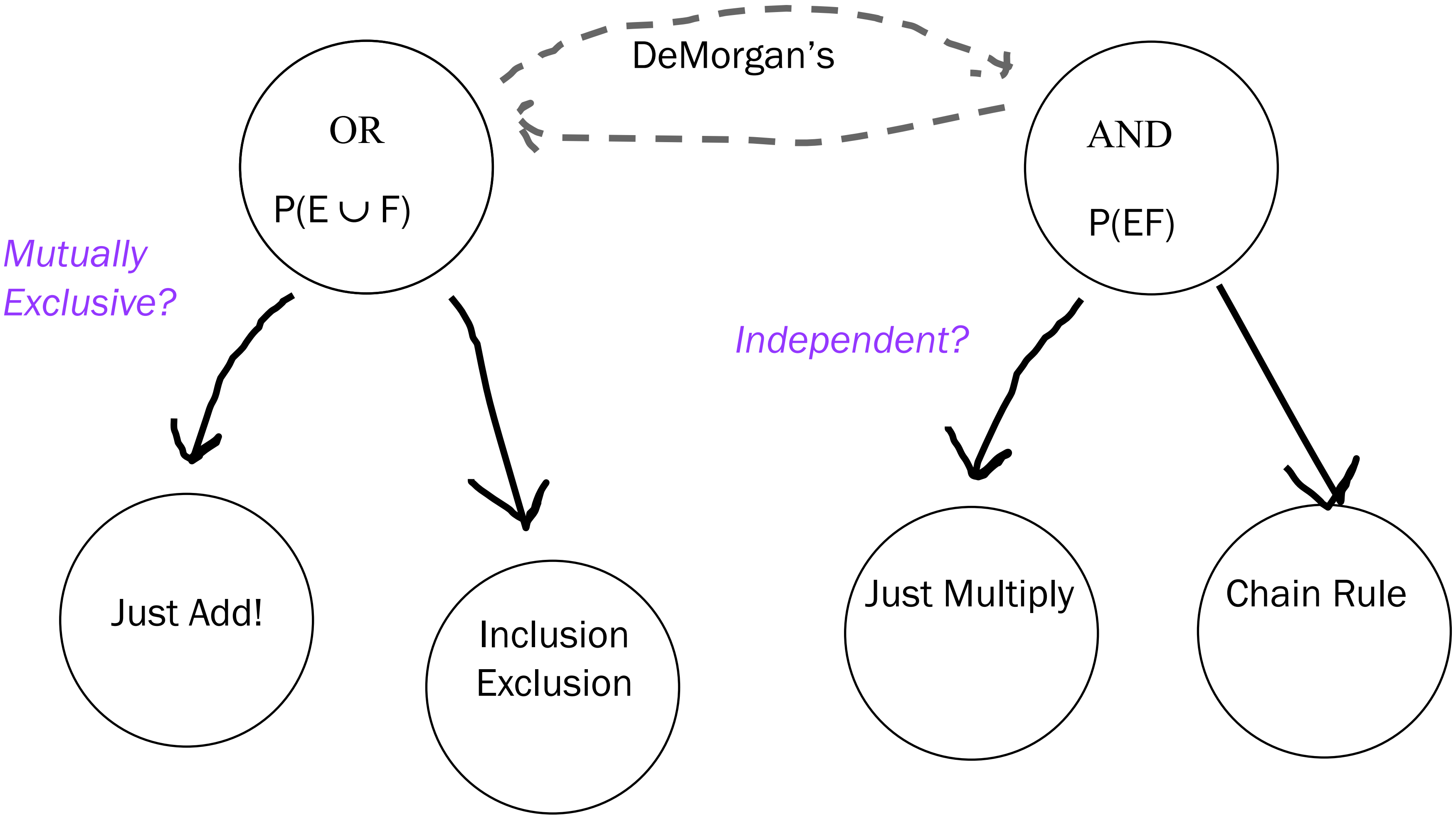
Law of Total Probability

Bayes' Theorem

$P(E)$

$P(F|E)$

3. Review: Probability of Or and And



Skill: Art form of Defining Events

m strings are hashed (not uniformly) into a hash table with n buckets. Each string hash is an **independent trial** with probability p_i of getting hashed into bucket i

What is the probability that **at least 1** of buckets 1 to k has ≥ 1 string hashed into it?

Define: F_i = bucket i has at least one string in it

$$P(E) = P(F_1 \cup F_2 \cup \cdots \cup F_k)$$

Practice: Lets do the Frisbee Problem.

You flip **two frisbees**. For each frisbee, the probability that it lands "**heads**" is **0.6**. The two frisbees are considered "**even**" if both frisbees are heads **or** both frisbees are tails.

What is the probability that the frisbees are even?

Lets go solve it!

Let p be the probability of a 1 from unknown_random.
What is the probability of a 1 from fair_random if p =.45?

```
def fair_random():
    """
    There are four outcomes for assignments to r1 and r2:
      [0, 0], [0, 1], [1, 0], [1, 1]. Return 1 if the
      outcomes are [0, 0] or [1, 1]
    """
    r1 = unknown_random()
    r2 = unknown_random()
    return r1 == r2
```

https://cs109psets.netlify.app/win22/pset2/fairRandom__a

Early Pedagogical Pause!

Advanced Concept:
Conditional Independence

Conditional Paradigm: In the conditional paradigm, the ***formulas*** of probability are preserved.

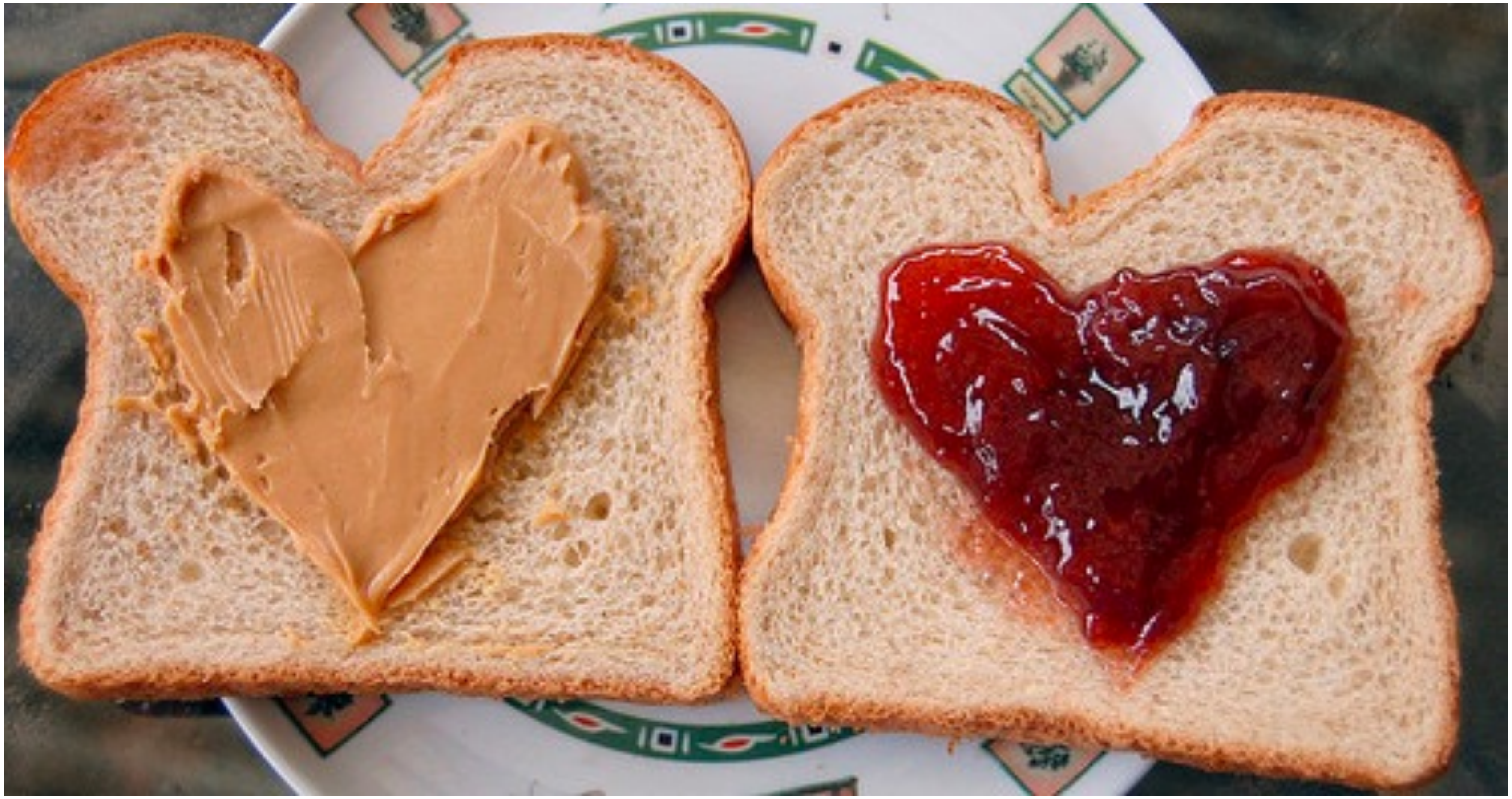
BAE's Theorem

$$P(A \mid B\ E) = \frac{P(B \mid A\ E)\ P(A \mid E)}{P(B \mid E)}$$

Two Great Tastes

Conditional Probability

Independence



Conditional Paradigm: In the conditional paradigm, the ***formulas*** of probability are preserved.

Independence relationships can change with conditioning.

If E and F are independent, that does not mean they will still be independent given another event G.

G_1 G_2 G_3 G_4 G_5 T

G_1 G_2 G_3 G_4 G_5 T

dna.txt — dna

dna.txt

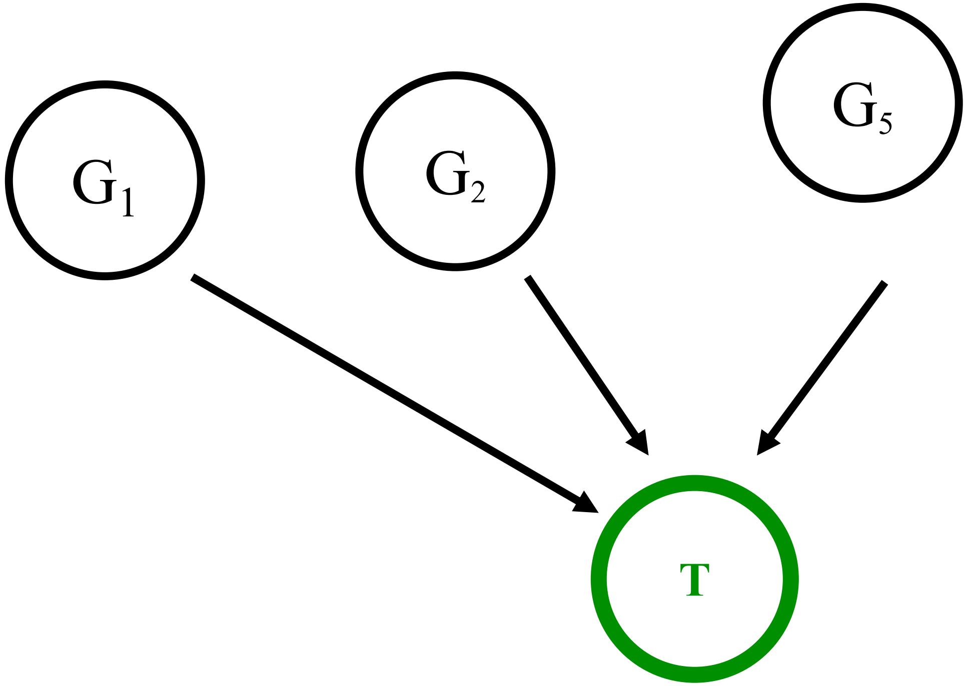
```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
False,True,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,False,True,False,False,False
False,False,True,False,True,False
True,False,False,True,False,False
False,True,False,True,True,False
True,False,False,True,False,False
True,False,True,True,False,False
False,True,False,False,True,False
False,False,True,True,False,False
True,True,False,False,True,True
True,False,True,True,False,False
True,True,True,True,True,True
True,False,True,False,False,True
False,True,False,True,True,True
False,False,True,False,False,False
False,False,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,True,False,True,True,False
True,False,False,False,False,True
False,False,True,True,False,True
False,False,False,True,False,False
False,True,True,False,False,True
False,True,False,False,True,True
False,False,False,False,False,True
False,True,False,True,True,False
True,False,False,True,False,False
True,True,False,True,True,True
True,True,False,False,True,True
True,True,False,True,True,True
False,False,False,True,False,False
```

100,000 samples

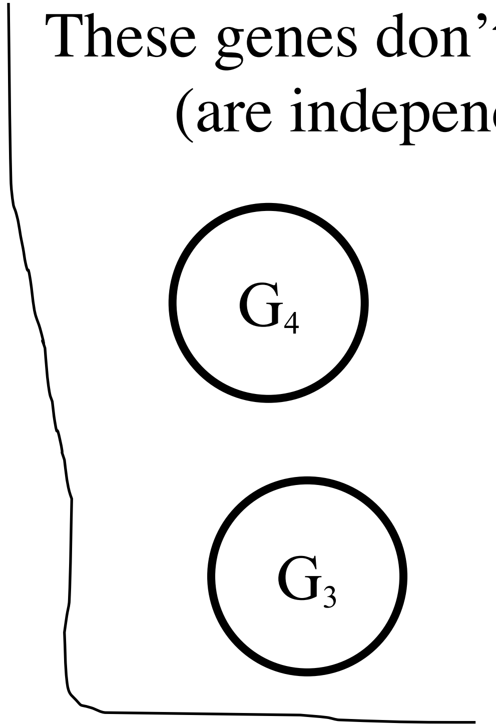
6 observations per sample

Independence Rules out Causality Structures

These genes do impact T (are not independent)



These genes don't impact T (are independent)



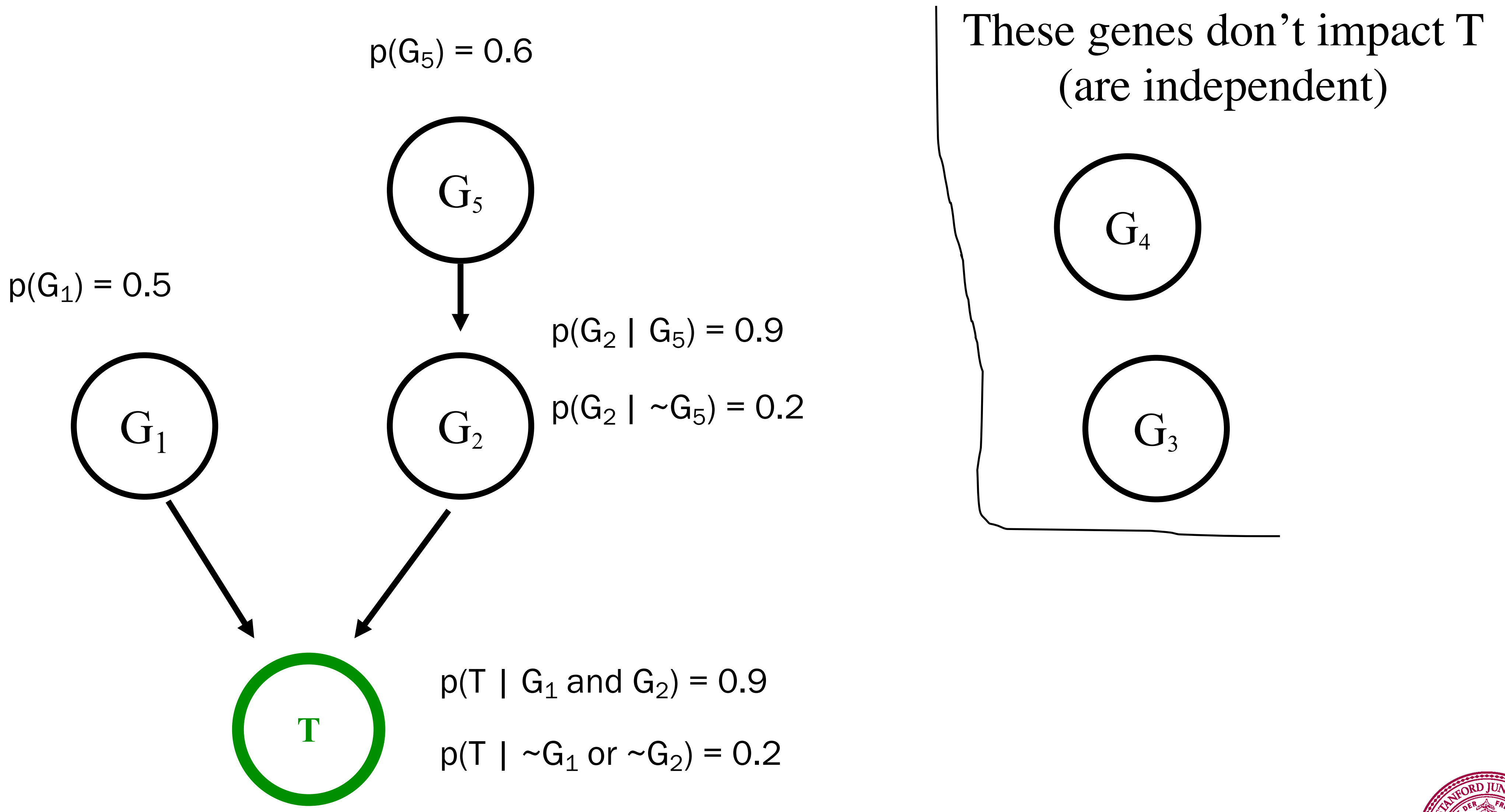
Teaser for Future Classes: Conditional Independence

```
Piech-2:dna piech$ python findStructure.py
size data =  100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

...

```
p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
```

Only Causal Structure That Fits



Advanced: Conditional Independence

Two events E and F are called **conditionally independent given G**, if

$$P(EF|G) = P(E|G)P(F|G)$$

Or, equivalently if:

$$P(E|FG) = P(E|G)$$

NETFLIX

And Learn

Netflix Learning

What is the probability that a user will watch Life is Beautiful?

$P(E)$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\#\text{people who watched movie}}{\#\text{people on Netflix}}$$

$P(E) = 10{,}234{,}231 \;/\; 50{,}923{,}123 = 0.20$

Netflix Learning

What is the probability that a user will watch Life is Beautiful, given they watched Coda?

$P(E|F)$

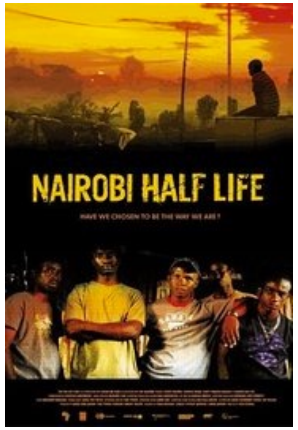
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\#\text{people who watched both}}{\#\text{people who watched } F}$$

$P(E|F) = 0.42$

Conditioned on liking a set of movies?

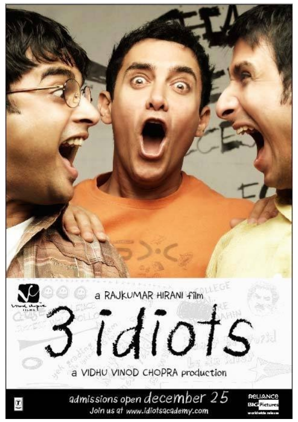
Netflix Learning

Each event corresponds to liking a particular movie



E_1

E_2



E_3

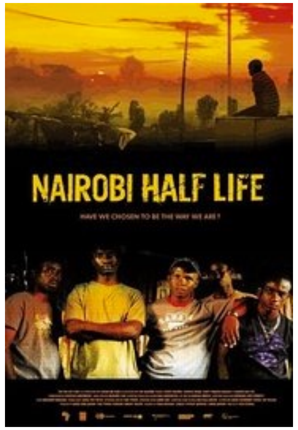
E_4

$$P(E_4 | E_1, E_2, E_3)?$$

Is E_4 independent of E_1,E_2,E_3?

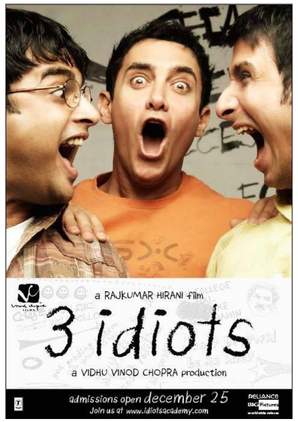
Netflix Learning

Is E_4 independent of E_1,E_2,E_3?



E_1

E_2



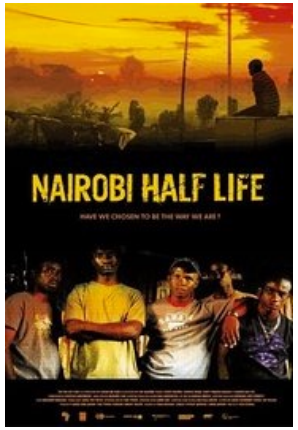
E_3

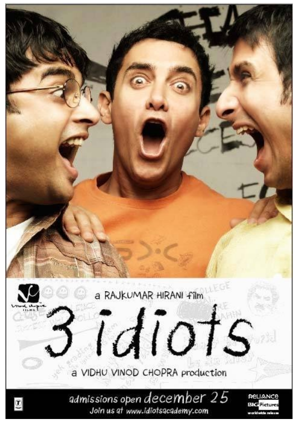
E_4

$$P(E_4|E_1, E_2, E_3) \stackrel{?}{=} P(E_4)$$

Netflix Learning

Is E_4 independent of E_1,E_2,E_3?





E_1 E_2 E_3 E_4

$$P(E_4|E_1, E_2, E_3) = \frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)}$$

Netflix Learning

What is the probability that a user watched four particular movies?

- There are 13,000 titles on Netflix
- The user watches 30 random titles.
- E = movies watched include the given four.

Solution:

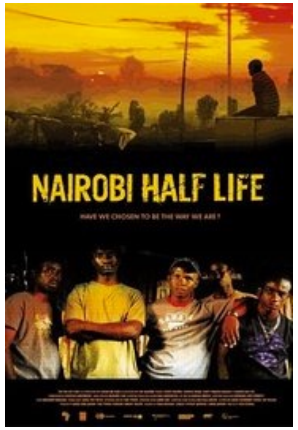
Watch those four

Choose 24 movies not in the set

$$P(E) = \frac{\binom{4}{4}\binom{12996}{24}}{\binom{13000}{30}} = 10^{-11}$$

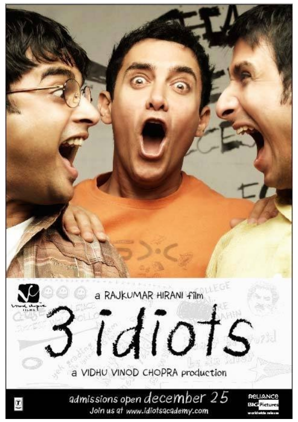
Choose 30 movies from netflix

Netflix Learning



E_1

E_2



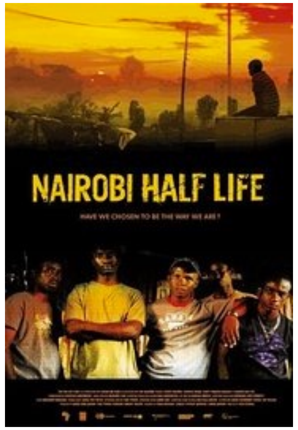
E_3

E_4

Netflix Learning: Advanced, Conditional Independence

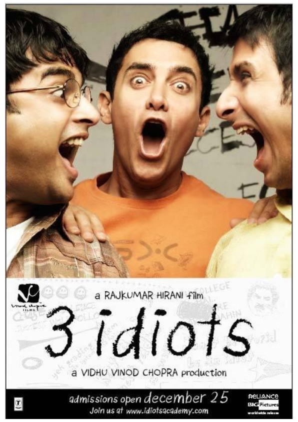
K_1

Like foreign emotional comedies



E_1

E_2

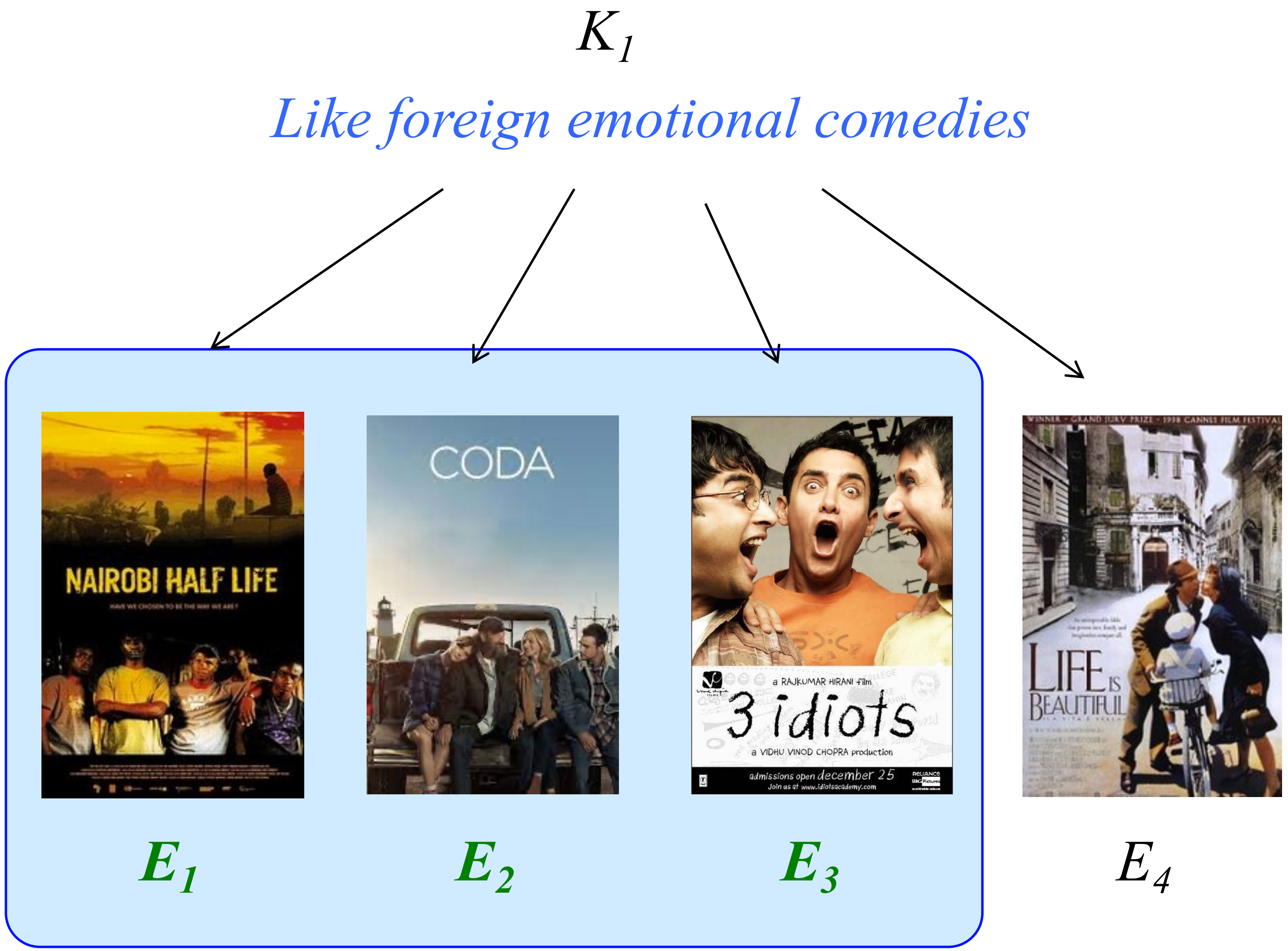


E_3

E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix Learning: Advanced, Conditional Independence

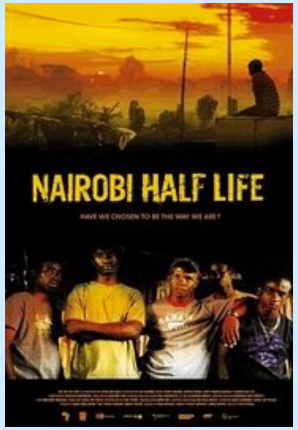


Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Netflix Learning: Advanced, Conditional Independence

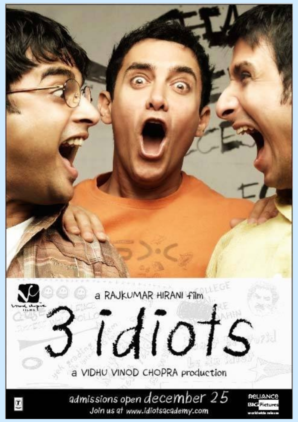
K_1

Like foreign emotional comedies



E_1

E_2



E_3

E_4

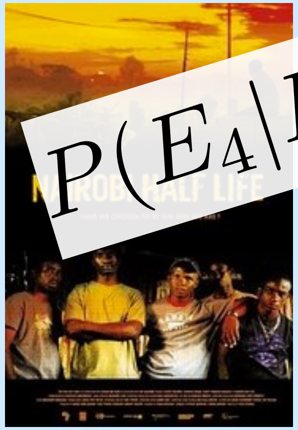
Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

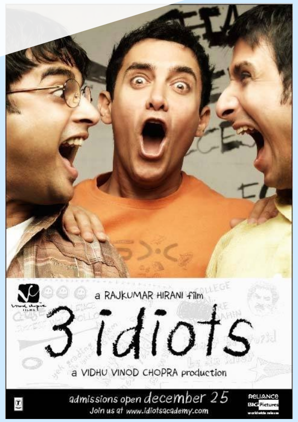
Netflix Learning: Advanced, Conditional Independence

K_1

Like foreign emotional comedies

$$P(E_4|E_1, E_2, E_3, K_1) = P(E_4|K_1)$$





E_1 ***E_2*** ***E_3*** *E_4*

Assume E_1, E_2, E_3 and E_4 are conditionally independent given K_1

Conditional independence is a practical, real world way of decomposing hard probability questions.

Conditional Independence

If E and F are dependent,

that does not mean E and F will be dependent when another event is observed.

Conditional Dependence

If E and F are independent,

that does not mean E and F will be independent when another event is observed.

Big Deal!

"Exploiting ***conditional independence*** to generate fast probabilistic computations is one of the main contributions CS has made to probability theory"

-Judea Pearl wins 2011 Turing Award, *"For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning"*

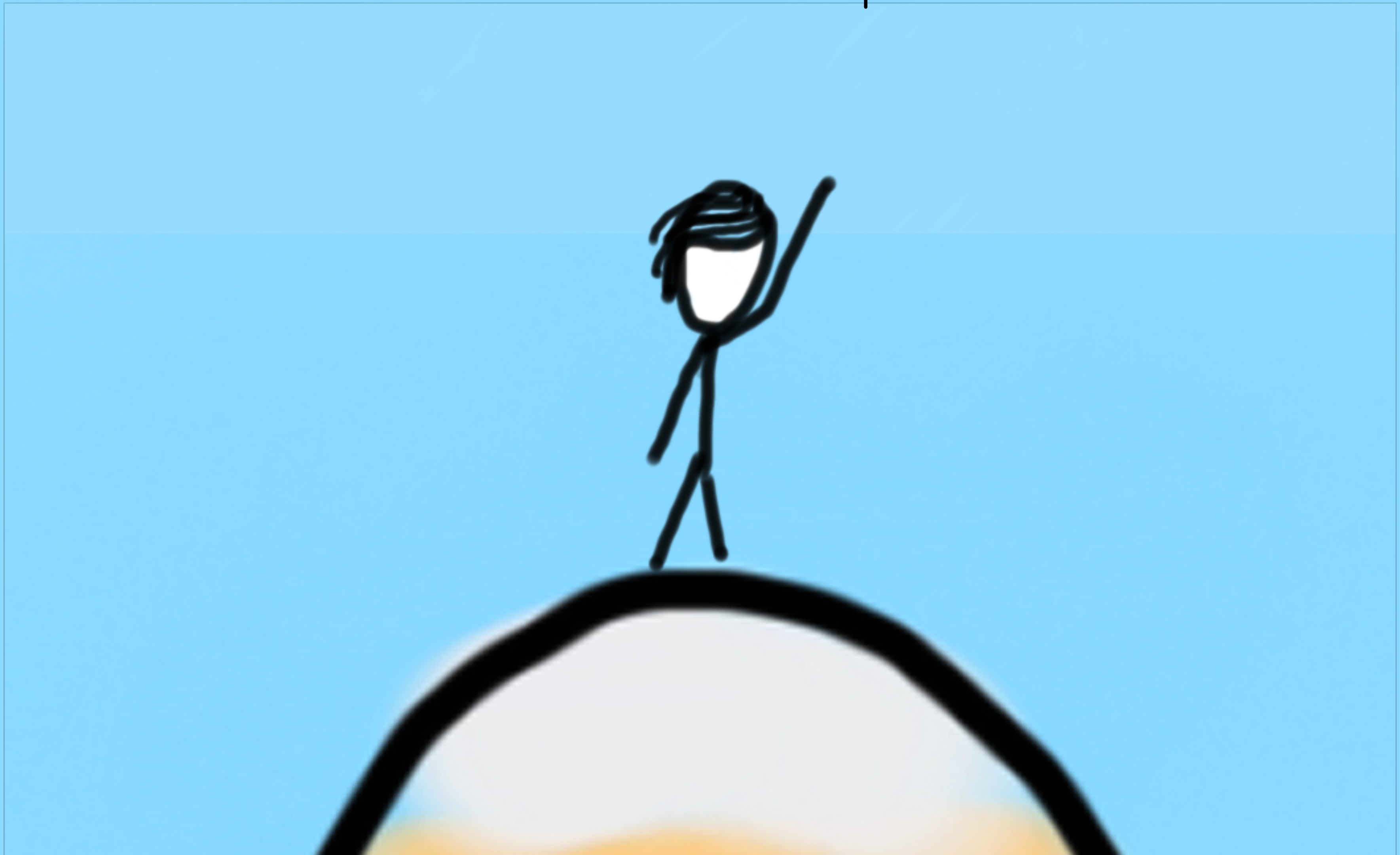
Ready for the next (cs109) episode

Another Pause!

Random Variables

Learning Goals

1. Be able to define a random variable (R.V.)
2. Be able to use and produce a PMF of a R.V.
3. Be able to calculate the expectation of the R.V.



Remember Learning to Code

name

value

type

```
int a = 5;
double b = 4.2;
bit c = 1;
choice d = medium;
```

$$z \in \{\text{high}, \text{medium}, \text{low}\}$$

Random variables are like programming variables, with uncertainty

Pirates of the Variables

```
int a = 5;
```

A is the number of pirate ships in our *future* armada.

$$A \in \{1, 2, \ldots, 10\}$$

```
double b = 4.2;
```

B is the amount of money we get *after* we defeat Blackbeard.

$$B \in \mathbb{R}^+$$

```
bit c = 1;
```

C is 1 *if* we successfully raid Isla de Muerta. 0 otherwise.

$$C \in \{0, 1\}$$

Random Variable

A **Random Variable** is a variable will have a value.
But there is uncertainty as to what value.

Example:

- 3 fair coins are flipped.
- Y = number of “heads” on 3 coins
- **Y is a random variable**
- $P(Y = 0) = 1/8$ (T, T, T)
- $P(Y = 1) = 3/8$ (H, T, T), (T, H, T), (T, T, H)
- $P(Y = 2) = 3/8$ (H, H, T), (H, T, H), (T, H, H)
- $P(Y = 3) = 1/8$ (H, H, H)
- $P(Y \geq 4) = 0$

It is confusing that both random variables and events use the same notation

Random variables and events are two *different* things

We can define an event to be a particular assignment to a random variables

Example of Random Variables

Consider 5 coin flips, each which independently come up heads with probability *p*

- Recall:

$$P(2 \text{ heads}) = \binom{5}{2} p^2 (1-p)^3$$

$$P(3 \text{ heads}) = \binom{5}{3} p^3 (1-p)^2$$

- Y = number of "heads" on 5 flips

$$Y \in \{1, 2, \ldots, 5\}$$

$$P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$$

* Pro tip: Not really about coins. Many real world binary events work like this.

Properties of Random Variables

Probability Mass Function:

$$P(X = a)$$

Expectation:

$$E[X]$$

Variance:

$$\text{Var}(X)$$

Learning goals for today

1. Probability Mass Function

The relationship between values a random variable can take on, and the corresponding probability, is a ***function***!

Let Y be a random variable

$$Y$$

For example Y is the number of heads in 5 coin flips

$$Y = 2$$

It is an event when
Y takes on a value

For example Y is the number of heads in 5 coin flips

If this is a number

$$P(Y = 2)$$

Then this is a number (between 0 and 1)

For example Y is the number of heads in 5 coin flips

If this is a variable

$$P(Y = k)$$

Then this is a function

For example Y is the number of heads in 5 coin flips

Random Variables -> Functions

$$P(Y = k)$$

$$k = 5$$

$$0.03125$$

For example Y is the number of heads in 5 coin flips

Random Variables -> Functions

$$P(Y = k)$$

```
def event_probability(k):
    # probability mass function of Y in python
    N = 5     # number of coin flips
    P = 0.5  # probability of heads

    ways = math.comb(N, k);
    prob_heads = math.pow(P, k)
    prob_tails = math.pow(P, N-k)
    return ways * a * b
```

For example Y is the number of heads in 5 coin flips

If a random variable is discrete we call this function the **Probability Mass** ***Function***

Probability Mass Function (PMF)

Let X be a random variable that represents the result of a **single dice roll**. X can take on the values $\{1, 2, 3, 4, 5, 6\}$

$$P(X = x)$$

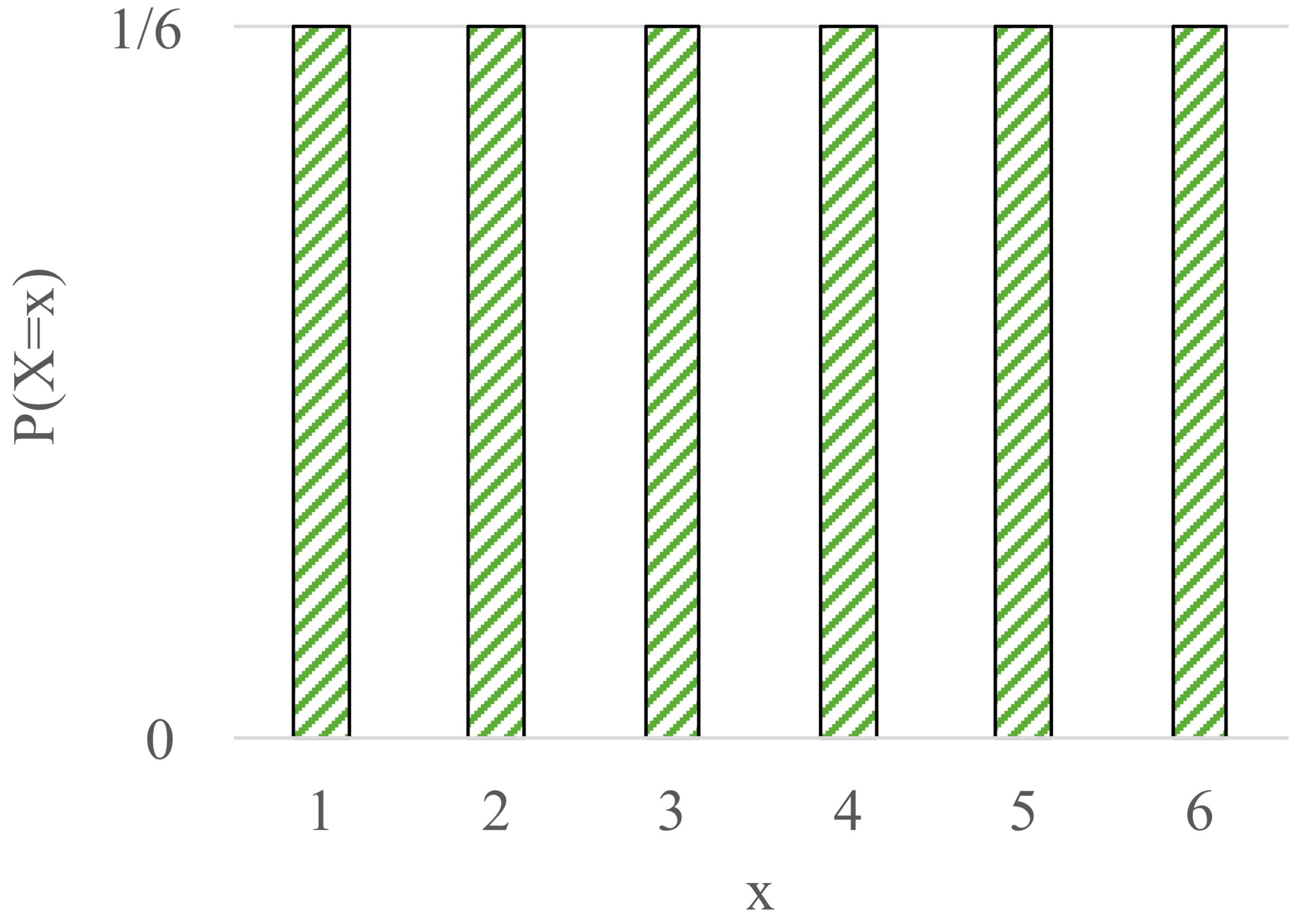
$$p(x)$$

This is shorthand notation for the PMF

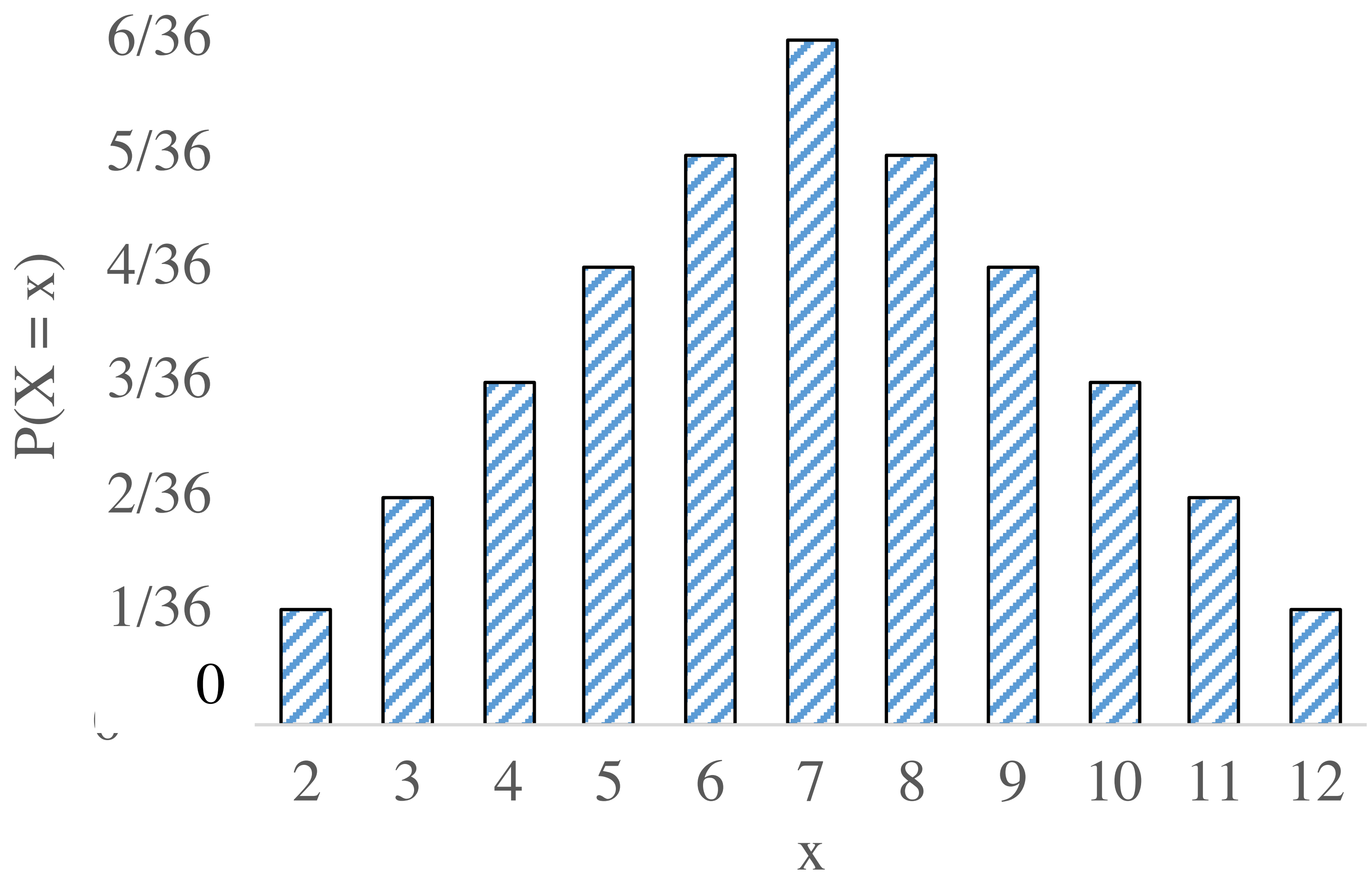
$$p_X(x)$$

This is also shorthand notation for the PMF

PMF for *X the outcome of a die roll*



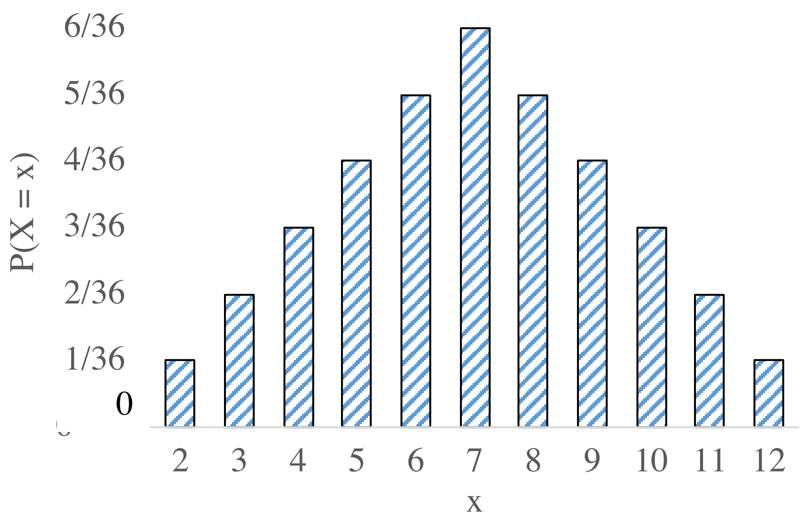
PMF for *X the sum of two dice rolls*



PMF as an equation

$$p(X = x) = \begin{cases} \frac{x-1}{36} & \text{if } x \in \mathbb{Z}\ ,\ 1 \leq x \leq 6 \\ \frac{13-x}{36} & \text{if } x \in \mathbb{Z}\ ,\ 7 \leq x \leq 12 \\ 0 & \text{else} \end{cases}$$

Again, this is the probability for the sum of two dice



This is Fine Check

$$\sum_{\text{all } k} P(Y = k) \stackrel{?}{=}$$

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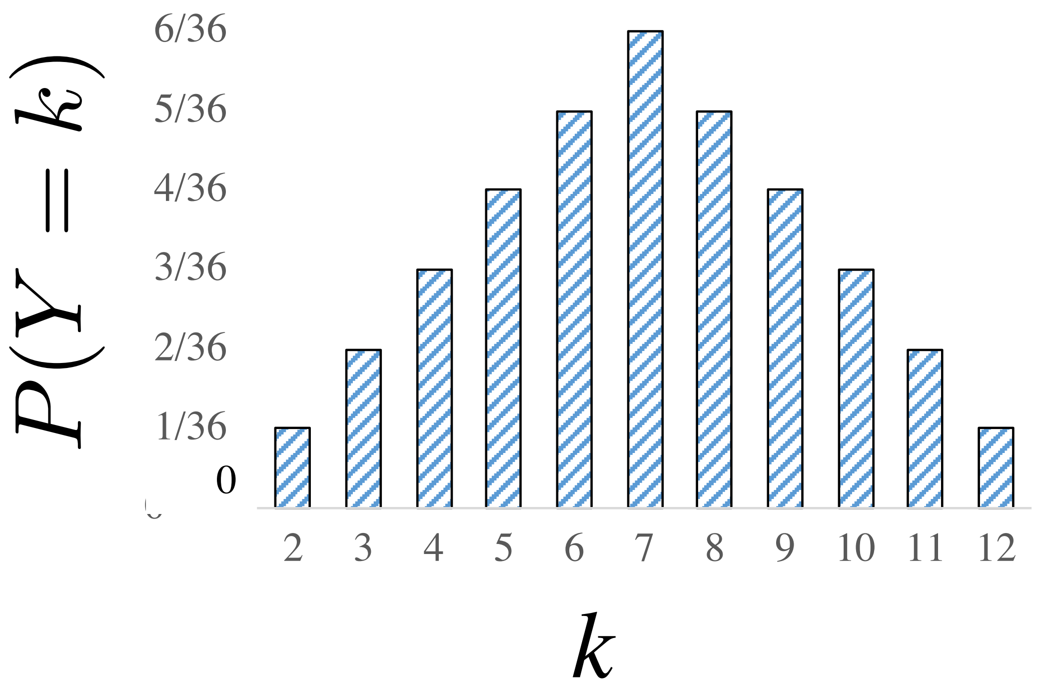
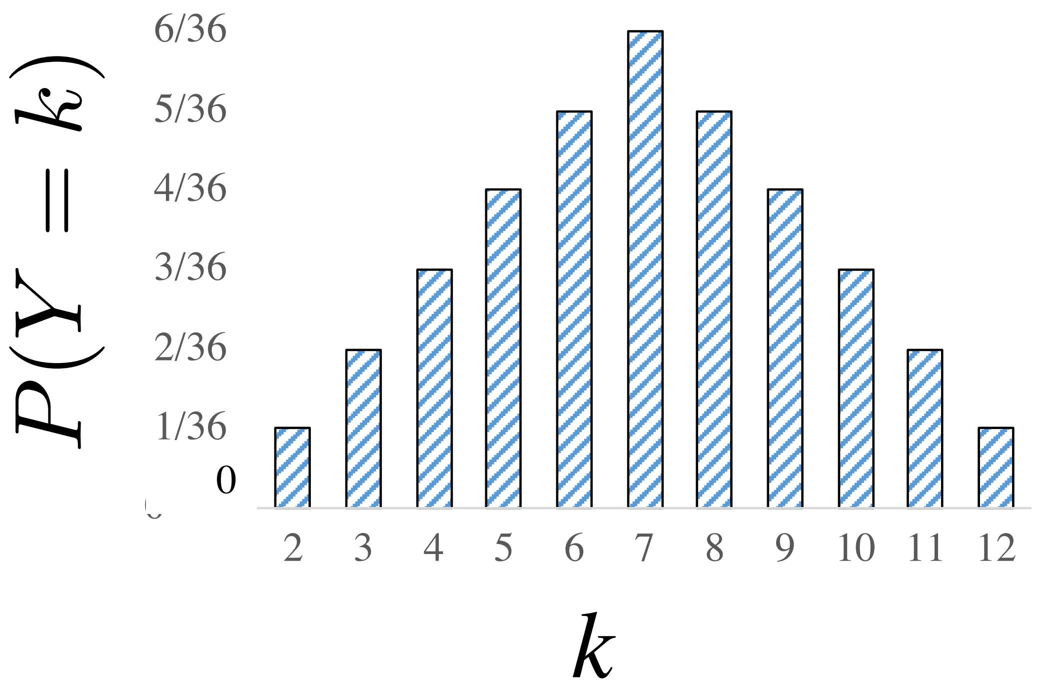


This is Fine Check

$$\sum_k P(Y = k) = 1$$



2. Expectation

Properties of Random Variables

Probability Mass Function:

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Learning goals for today

Expected Value

The Expected Values for a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} x \cdot p(x)$$

Note: sum over all values of x that have $p(x) > 0$.

Expected value also called: ***Mean***, *Expectation*, ***Weighted Average***, ***Center of Mass***, *1st Moment*

Expected Value

Roll a 6-Sided Die. X is outcome of roll

- p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \frac{7}{2}$$

Y is random variable

- P(Y = 1) = 1/3,   P(Y = 2) = 1/6,   P(Y = 3) = 1/2

E[Y] = 1 (1/3) + 2 (1/6) + 3 (1/2) = 13/6

Lying With Statistics

"There are three kinds of lies:
lies, damned lies, and statistics"

– *Mark Twain*

School has 3 classes with 5, 10 and 150 students

Randomly choose a class with equal probability

X = size of chosen class

What is E[X]?

- E[X] = 5 (1/3) + 10 (1/3) + 150 (1/3)
  = 165/3 = 55

Lying With Statistics Part 2

"There are three kinds of lies:
lies, damned lies, and statistics"

– *Mark Twain*

School has 3 classes with 5, 10 and 150 students

Randomly choose a <u>student</u> with equal probability

Y = size of class that student is in

What is E[Y]?

- $E[Y] = 5\,(5/165) + 10\,(10/165) + 150\,(150/165)$
  $= 22635/165 \approx 137$

Note: E[Y] is students' perception of class size

- But E[X] is what is usually reported by schools!

Properties of Expectation (more on this later)

Linearity:

$$E[aX + b] = aE[X] + b$$

- Consider X = 6-sided die roll, Y = 2X – 1.
- E[X] = 3.5 E[Y] = 6

Expectation of a sum is the sum of expectations

$$E[X + Y] = E[X] + E[Y]$$

Unconscious statistician:

$$E[g(x)] = \sum_x g(x)p(x)$$

Properties of Random Variables

Probability Mass Function:

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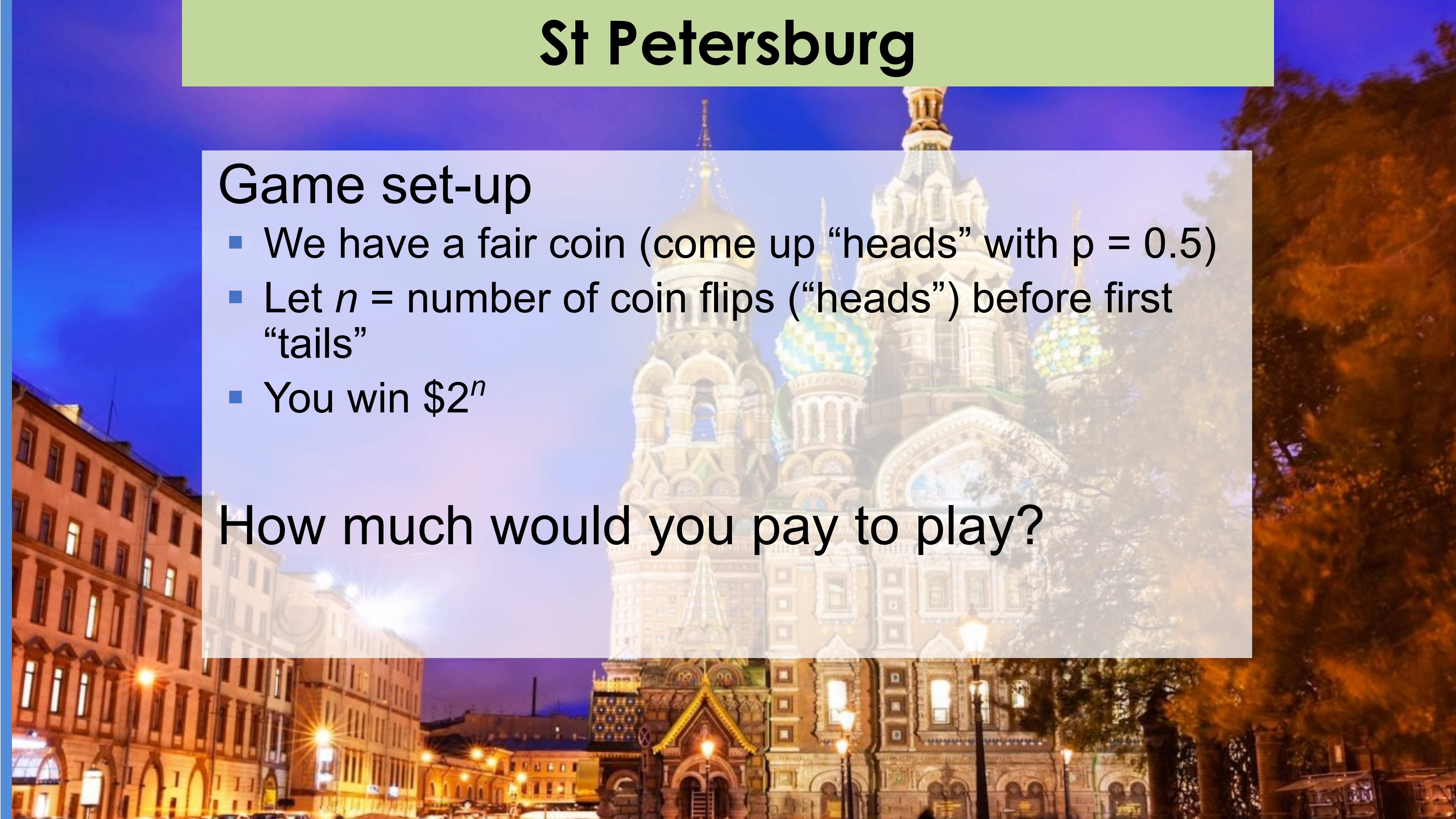
Learning goals for today

Wonderful



St Petersburg

Game set-up

- We have a fair coin (come up “heads” with p = 0.5)
- Let n = number of coin flips (“heads”) before first “tails”
- You win $\$2^n$

How much would you pay to play?

St Petersburg

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- We have a fair coin (come up "heads" with p = 0.5)
- Let *n* = number of coin flips ("heads") before first "tails"
- You win $\$2^n$

How much would you pay to play?

Solution

- Let X = your winnings
- E[X] = $\left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \ldots = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} 2^i$

$$= \sum_{i=0}^{\infty} \frac{1}{2} = \infty$$

- I'll let you play for $1 thousand... but just once! Takers?

St Petersburg + Reality

What if Chris has only $65,536?

- Same game
- If you win over $65,536 I leave the country.

Solution

- Let X = your winnings
- E[X]

$$= \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \dots$$

$$= \sum_{i=0}^{k} \left(\frac{1}{2}\right)^{i+1} 2^i \text{ s.t. } k = \log_2(65,536)$$

$$= \sum_{i=0}^{16} \frac{1}{2} = 8.5$$

Learning Goals

1. Know what is meant by Conditional Independence
2. Be able to define a random variable (R.V.)
3. Be able to use + produce a PMF of a R.V.
4. Be able to calculate the expectation of the R.V.