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CS109

Section 1: Analytic Probability

Problems by three through six by Chris Piech. Rest by past CS109 teaching teams

Overview of Section Materials

The warmup questions provided will help students practice concepts introduced in lectures. The section problems are meant to apply these concepts in more complex scenarios similar to what you will see in problem sets and exams.

Warmups

1. Fish Pond

Suppose there are 7 blue fish, 4 red fish, and 8 green fish in a large fishing tank. You drop a net into it and end up with 2 fish. What is the probability you get 2 blue fish?

For the full sample space, we consider the number of ways we can select 2 fish from all 19 fish without regard for blue, red, or green. We treat all of the fish as distinct in order to make sure that each event is equally likely. We don't consider order of fish to matter. The size of our sample space is thus $\binom{19}{2}$. For our event space, we consider the number of ways we can select 2 blue fish. There are 7 distinct blue fish from which we can chose two. Recall that the event space must be consistent with the sample space:

$$p = \frac{\binom{7}{2}}{\binom{19}{2}} = \frac{21}{171} \approx 0.123$$

The same answer could be arrived at by using the chain rule. The probability that the first fish is blue is 7/19. The probability that the second fish is blue, given that the first fish was blue is 6/18. The probability that both fish are blue is $7/19 \cdot 6/18 \approx 0.123$

2. Axioms of Probability

Decide whether each of the three statements below is true or false:

a. $P(A) + P(A^C) = 1$. Recall that A^C means A "complement" or "not" A

b. $P(A \cap B) + P(A \cap B^C) = 1$. Recall that $\cap$ means "and"

c. If $P(A) = 0.4$ and $P(B) = 0.6$ then it must be the case that $A = B^C$

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The first one is simply saying that an event either falls inside of an event space A or outside of it. The second is false, as the left hand side, in general, is $P(A)$, which isn't guaranteed to be 1. And the fact that $P(A)$ and $P(B^C)$ are each 0.4 doesn't require the event spaces to be the same.

3. Conditional Probability

What is the difference between these two terms $P(B|A)$ and $P(A \cap B)$? Imagine that B is the event that a student "correctly answer a multiple choice question" and A is the event that the same student "guesses randomly". Provide an explanation as well as a mathematical relationship between the two.

They are very different concepts! Students often confuse the two. $P(B|A)$ is the chance that the student gets the problem correct **given** that we **already know** that they are guessing. $P(A \cap B)$ is very different. means that we are curious if both events will occur and that we don't know if the student has guessed. The relationship between the two is given by the chain rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

The key piece of information which would allow you to go from one value to the other is $P(A)$: the probability that a student guesses randomly.

Problems

4. The Birthday Problem

When solving a counting problem, it can often be useful to come up with a generative process, a series of steps that "generates" examples. A correct generative process to count the elements of set A will (1) *generate every element of A* and (2) *not generate any element of A more than once*. If our process has the added property that (3) *any given step always has the same number of possible outcomes*, then we can use the product rule of counting.

Example: Say we want to count the number of ways to roll two (distinct) dice where one die is even and one die is odd. Our process could be: (1) roll the first die and note if the value is even or odd, then (2) count the number of ways the second die can be rolled for a value of the opposite parity. Since the first step has 6 options and the second step has 3 options regardless of the outcome of the first step, the number of possibilities is $6 * 3 = 18$.

Problem: Assume that birthdays happen on any of the 365 days of the year with equal likelihood (we'll ignore leap years).

a. What is the probability that of the n people in class, at least two people share the same birthday?

Computing $P(\text{at least 2 people share a birthday})$ is difficult. We realize that this can be thought of as

$$P(\text{exactly 2 people share birthday} \cup \text{exactly 3} \cup \text{exactly 4} \cup \ldots \cup \text{exactly } n \text{ people share birthday})$$

Using the additivity axiom of probability, we realize that this can be split up because the events are mutually exclusive.

$$P(\text{exactly 2 people share birthday}) + P(\text{exactly 3}) + P(\text{exactly 4}) + \ldots + P(\text{exactly } n \text{ people share birthday})$$

However, this is very tedious!

It is much easier to calculate $1 - P(\text{no one shares a birthday})$. Let our sample space, S be the set of all possible assignments of birthdays to the students in section. By the assumptions of this problem, each of those assignments is equally likely, so this is a good choice of sample space. We can use the product rule of counting to calculate $|S|$:

$$|S| = (365)^n$$

Our event E will be the set of assignments in which there are no matches (i.e. everyone has a different birthday). We can think of this as a generative process where there are 365 choices of birthdays for the first student, 364 for the second (since it can't be the same birthday as the first student), and so on. Verify for yourself that this process satisfies the three conditions listed above. We can then use the product rule of counting:

$$|E| = (365) \cdot (364) \cdot \dots \cdot (365 - n + 1)$$

$$\begin{aligned} P(\text{birthday match}) &= 1 - P(\text{no matches}) \\ &= 1 - \frac{|E|}{|S|} \\ &= 1 - \frac{(365) \cdot (364) \dots (365 - n + 1)}{(365)^n} \end{aligned}$$

A common misconception is that the size of the event E can be computed as $|E| = \binom{365}{n}$ by choosing n distinct birthdays from 365 options. However, outcomes in this event (n unordered distinct dates) cannot recreate any outcomes in the sample space $|S| = 365^n$ (n distinct dates, one for each distinct person). However if we compute the size of event E as $|E| = \binom{365}{n} n!$ (equivalent to the number above), then we can assign the n birthdays to each person in a way consistent with the sample space. The expression $\binom{365}{n} n!$ is equivalent to $\frac{365!}{(365-n)!}$ which is known as a "falling factorial" and also as "365 permute n" outside of this class.

Interesting values: $(n = 13 : p \approx 0.19)$, $(n = 23 : p \approx 0.5)$, $(n = 70 : p \geq 0.99)$.

b. What is the probability that this class contains exactly one pair of people who share a birthday?

We can use the same sample space, but our event is a little bit trickier. Now E is the set of birthday assignments in which exactly two students share a birthday and the rest have different birthdays. One generative process that works for this is (1) choose the two students who share a birthday, (2) choose $n - 1$ birthdays in the same manner as in part a (i.e. one for the pair of students and one for each of the remaining students). We then have:

$$P(\text{exactly one match}) = \frac{|E|}{|S|} = \frac{\binom{n}{2}(365) \cdot (364) \cdot \ldots \cdot (365 - n + 2)}{(365)^n}$$

Many other generative processes work for this problem. Try to think of some other ones and make sure you get the same answer!

5. Self Driving Car

A self driving car has a 60% belief that there is a motorcycle to its left based on all the information it has received up until this point in time. Then, it receives a new, independent report from its left camera. The camera reports that there is **no** motorcycle. What is the updated belief that there is a motorcycle to the left of the car?

The camera is an imperfect instrument. When there is truly no motorcycle, the camera will report “no motorcycle" 90% of the time. When there actually is a motorcycle, the camera will report “no motorcycle" 5% of the time.

We need Bayes’ rule to solve this problem. Let’s define M to be the event that there is a motorcyle to our left. Let N be the event that the camera reports no motorcycle.

$$\begin{aligned} P(M|N) &= \frac{P(N|M)\cdot P(M)}{P(N)} && \text{Bayes' Theorem} \\ &= \frac{P(N|M)\cdot P(M)}{P(N|M)\cdot P(M) + P(N|M^C)\cdot P(M^C)} && \text{Law of Total Probability} \\ &= \frac{0.05\cdot 0.6}{0.05\cdot 0.6 + 0.9\cdot 0.4} \approx 0.077 && \text{Law of Plugging in} \end{aligned}$$

Extra Practice

6. Flipping Coins

One thing that students often find tricky when learning combinatorics is how to figure out when a problem involves permutations and when it involves combinations. Naturally, we will look at a problem that can be solved with both approaches. Pay attention to what parts of your solution represent distinct objects and what parts represent indistinct objects.

Problem: We flip a fair coin n times, hoping (for some reason) to get k heads.

a. How many ways are there to get exactly k heads? Characterize your answer as a *permutation* of H’s and T’s.

We want to know the number of sequences of n H’s and T’s such that there are k H’s and $n-k$ T’s. This is the same as permuting n objects of which one set of k is indistinguishable and one set of $n-k$ is indistinguishable. Using our formula for the permutation of indistinguishable objects, we get $\frac{n!}{k!(n-k)!}$

b. For what x and y is your answer to part (a) equal to $\binom{x}{y}$? Why does this *combination* make sense as an answer?

Our answer to part a is equal to $\binom{n}{k}$. This makes sense because we can come up with a valid sequence by *choosing* k flips to come out to heads (and implicitly define the other $n-k$ to be tails). The answer is also equivalent to $\binom{n}{n-k}$ for which the same logic applies except with choosing flips to be tails.

c. What is the probability that we get exactly k heads?

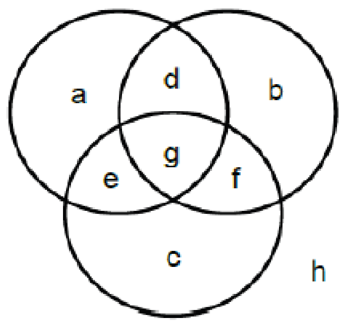
If we define our sample space to be all possible sequences of flips, then our event is the number of sequences where we get exactly k heads, meaning that $|E|$ is (conveniently) the answer to the previous two parts. Our probability is then $\frac{|E|}{|S|} = \frac{\binom{n}{k}}{2^n}$.

7. Counting

The Inclusion Exclusion Principle for three sets is:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Explain why in terms of a venn-diagram.



We want a, b, c, d, e, f, & g to each be accounted for exactly once. If we add A, B, and C together, we properly count a, b, & c exactly once. However we will double count d, e, & f, and triple count g. Removing $A \cap B$, $A \cap C$, & $B \cap C$ will reduce the counts for d, e, & f down to one. However, g went from being counted 3 times too many to being counted zero times. The expression corresponding to g is $A \cap B \cap C$, so we add it in to allow g to have a count of exactly one.

8. Combinatorial Proofs

Prove why $\binom{n}{k} = \binom{n}{n-k}$.

A fully algebraic proof is possible by simply showing the left hand side is equivalent to the right, as with:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!((n-n)+k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

An equally compelling proof is a combinatorial one, which relies on our ability to describe the counting problem in two, equivalent ways. In this case, we can simply say the above is true because choosing k items from a set of n items is equivalent to choosing the $n-k$ items to be excluded. These types of proofs are called **combinatorial proofs**, or **story proofs**.