Normal Distribution

Will Song
CS109, Stanford University

PS3

1
2
3
4
5
6
7
8
9a
9b
10
11
12
13

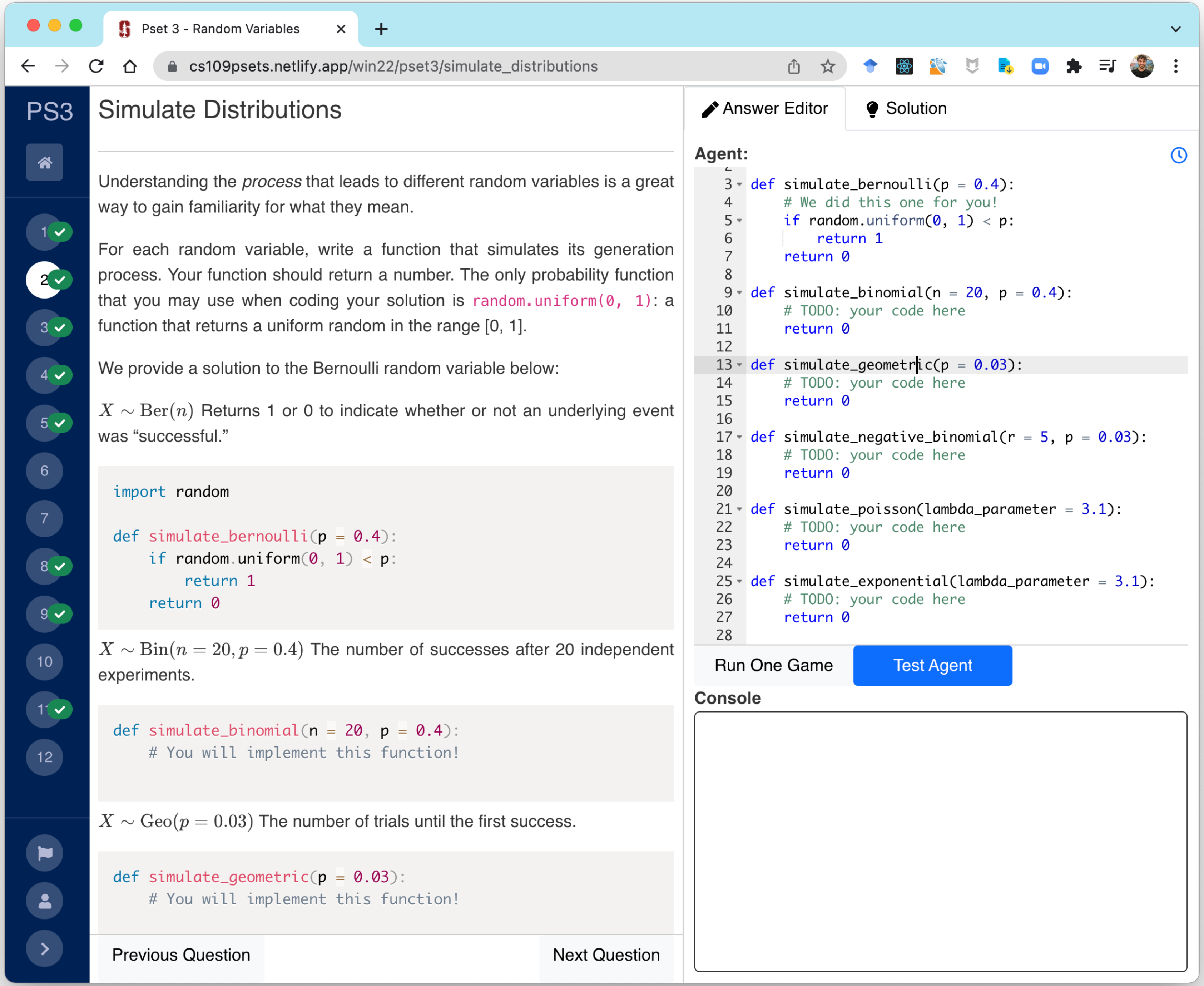
Uses discrete random variables

Uses continuous random variables

PSet 3 is out!

We put a lot of love into it,
and hope you enjoy

PSet 2 is in!
(Tonight)



Simulate!

Great way to make sure you really understand the random variables

Don't use numpy or scipy!

Think about the proof for poisson and exponential

Pset 3 - Random Variables

cs109psets.netlify.app/win22/pset3/measles_test

PS3

Measles Test

To determine whether they have measles, 60 people have their blood tested. However, rather than testing each individual separately, it is decided to first place the people into groups of 6. The blood samples of the 6 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 6 people, whereas if the test is positive, each of the 6 people will also be individually tested and, in all, 7 tests will be made on this group. Note that we assume that the pooled test will be positive if at least one person in the pool has measles. Assume that the probability that a person has measles is 5% for all people, independently of each other, and compute the expected number of tests necessary for each group of 6 people.

Previous Question | Next Question

Answer Editor | Solution

Numeric Answer: 2.589 ✓ Check Answer

Explanation:

Block LaTeX | Image | B | I | U

1. Define a random variable

2. Declare its distribution

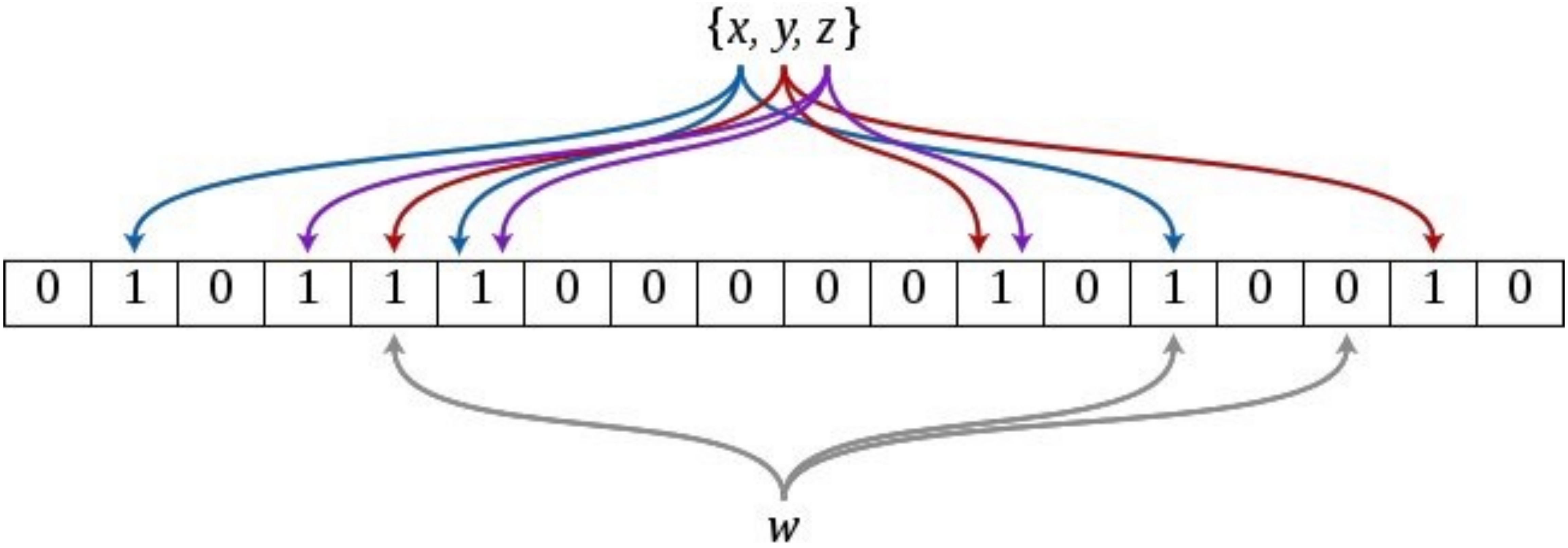
3. Write the question as a probability statement using the random variable

4. Solve

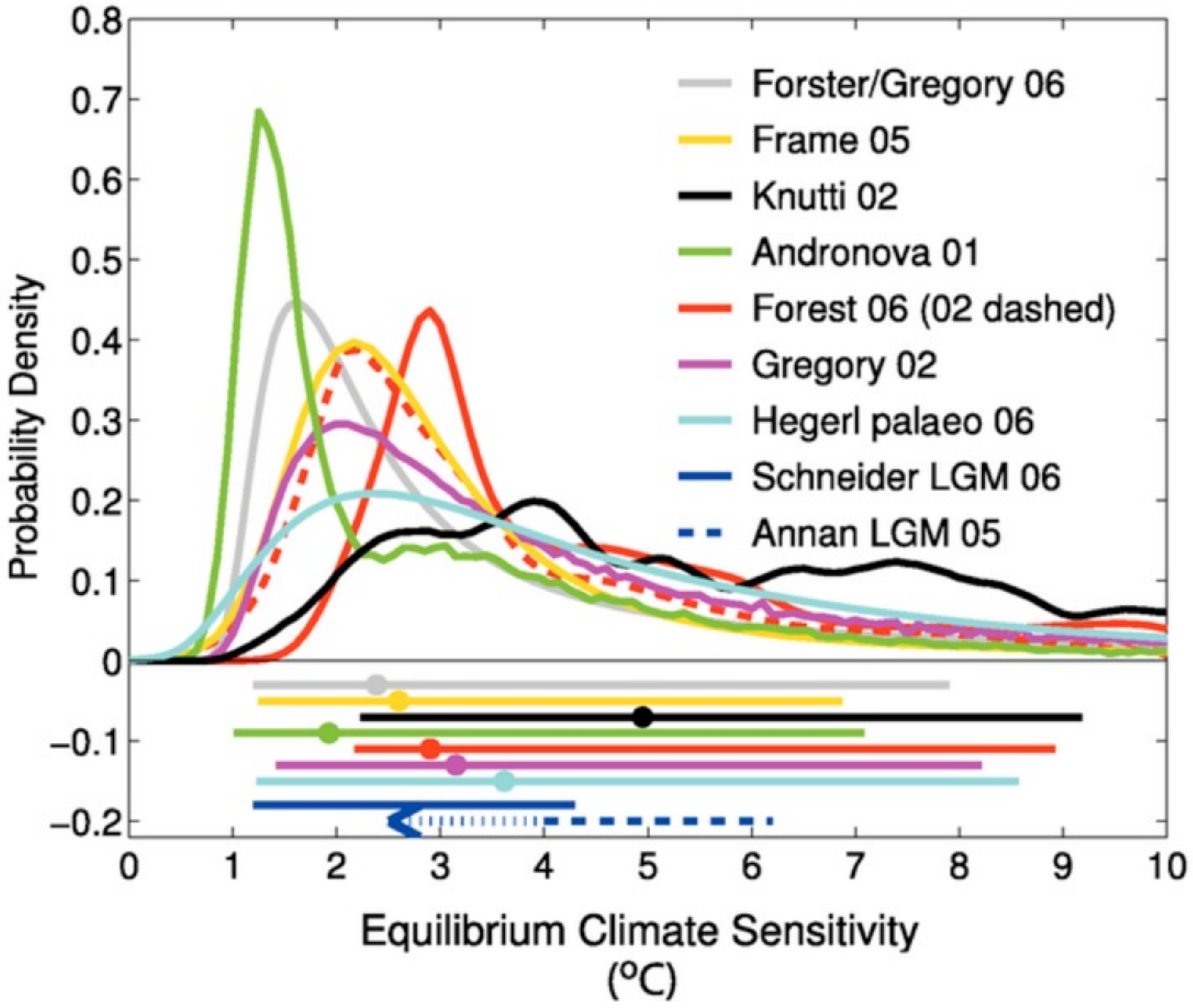
Many prototypical random variable questions.

Batch testing is a thing!

Bloom Filter



Climate Sensitivity



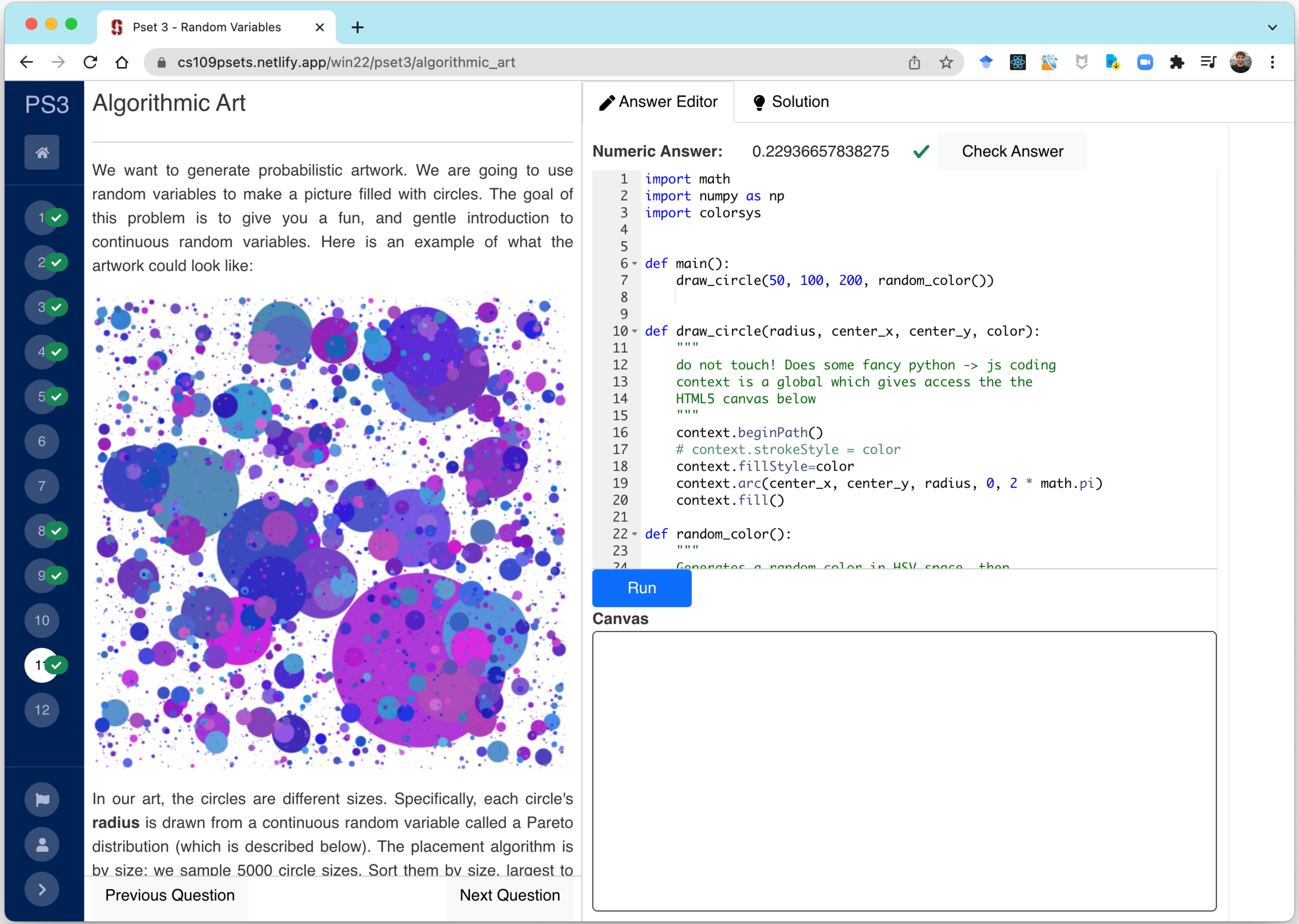
Which is Random?

Sequence 1:

TTHHTHTTHTTTHTTTHTTTHTTHTHHTHHTHTHHTTTHHTHTHTTHTHHTTHTHHTHTTTH
HTTHHTTHHHTHHTHTTHTHTTHHTHHHTTHTHTTTHHTTHTHTHTHTHTTHTHTHHHTTHT
HTHHTHHHTHTHTTHTTHHTHTHTHTTHHTTHTHTTHHHTHTHTHTTHTTHHTTHTHHTHHH
TTHHTHTTHTHTHTHTHTHTHHHTHTHTHTTHTHHTHTHTTHTTHHTHTTTHTHHTHHHHT
TTHHTHTHTHTHHHTTHHTHTTTHTHHTHTHTHHTHTTHTTHTHHTHTHTTT

Sequence 2:

HTHHHTHTTHHTTTTTTTTHHHTTTHHTTTTHHTTHHHTTHTHTTTTTTHTHTTTTHHHHTH
THTTHTTTHTTHTTTTHTHHTHHHHTTTTTHHHHTHHHTTTTHTHTTHHHHTHHHHHHHHHTT
HHTHHTHHHHHHHTTHTHTTTHHTTTTHTHHTTHTTHTHTHTTHHHHHTTHTTTHTHTHHTT
TTHTTTTTHHTHTHHHHTTTTHTHHHHHHTHTHTHTHHHTHTTHHHTHHHHHHTHHHTHTTT
HHHTTTHHTHTTHHTHHHTHTTHTTHTTTHHTHTHTTTTHTHTHTTHTHTHT



Final exam question from last year

Meant to be creative!

Regenerate

In our art, the circles are different sizes. Specifically, each circle's **radius** is drawn from a Pareto distribution (which is described below). The placement algorithm is greedy: we sample 1000 circle sizes. Sort them by size, largest to smallest. Loop over the circle sizes and place circles one by one.

To place a circle on the canvas, we sample the location of the center of the circle. Both the x and y coordinates are uniformly distributed over the dimensions of the canvas. Once we have selected a prospective location we then check if there would be a collision with a circle that has already been placed. If there is a collision we keep trying new locations until you find one that has no collisions.

The Pareto Distribution

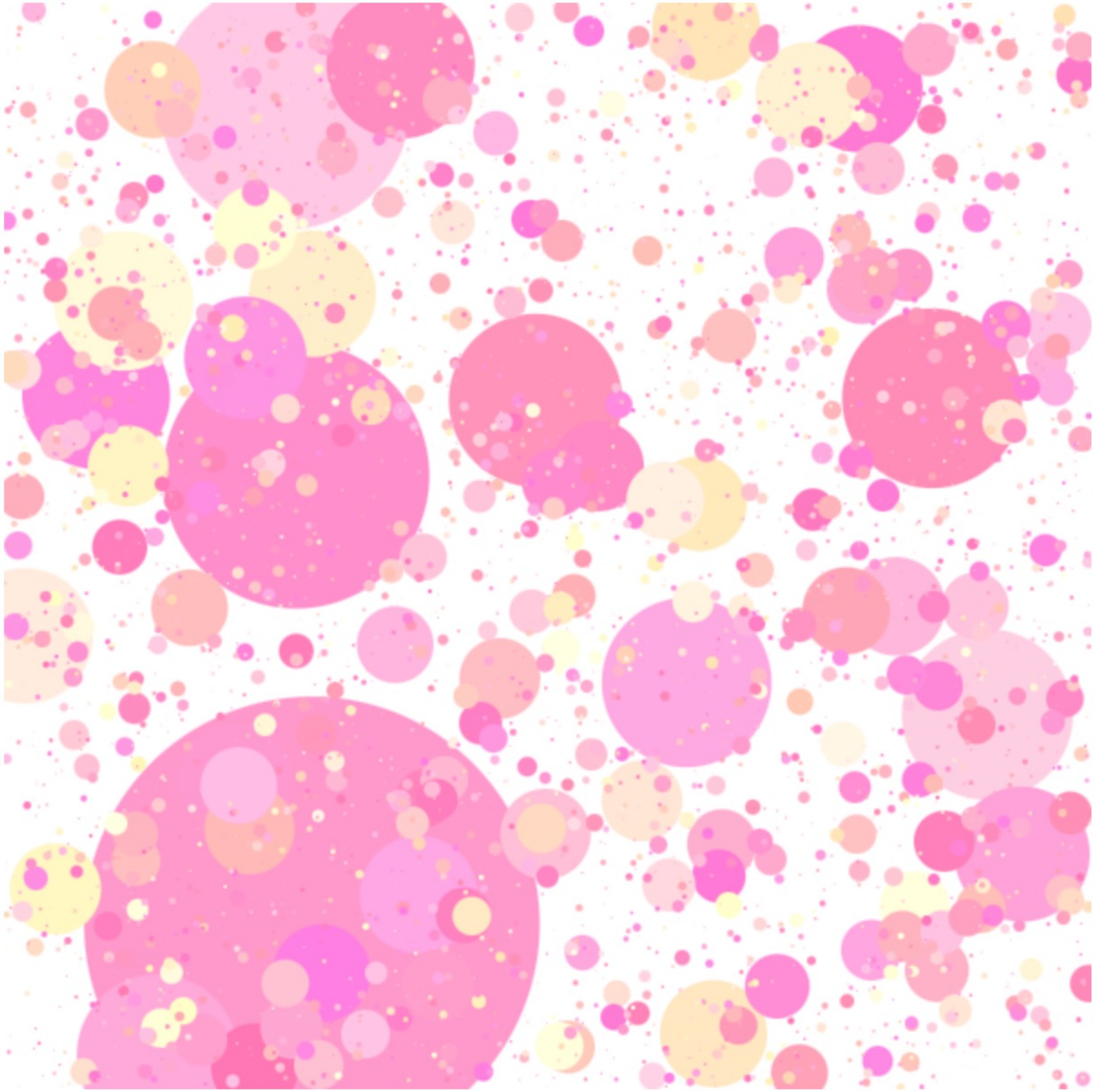
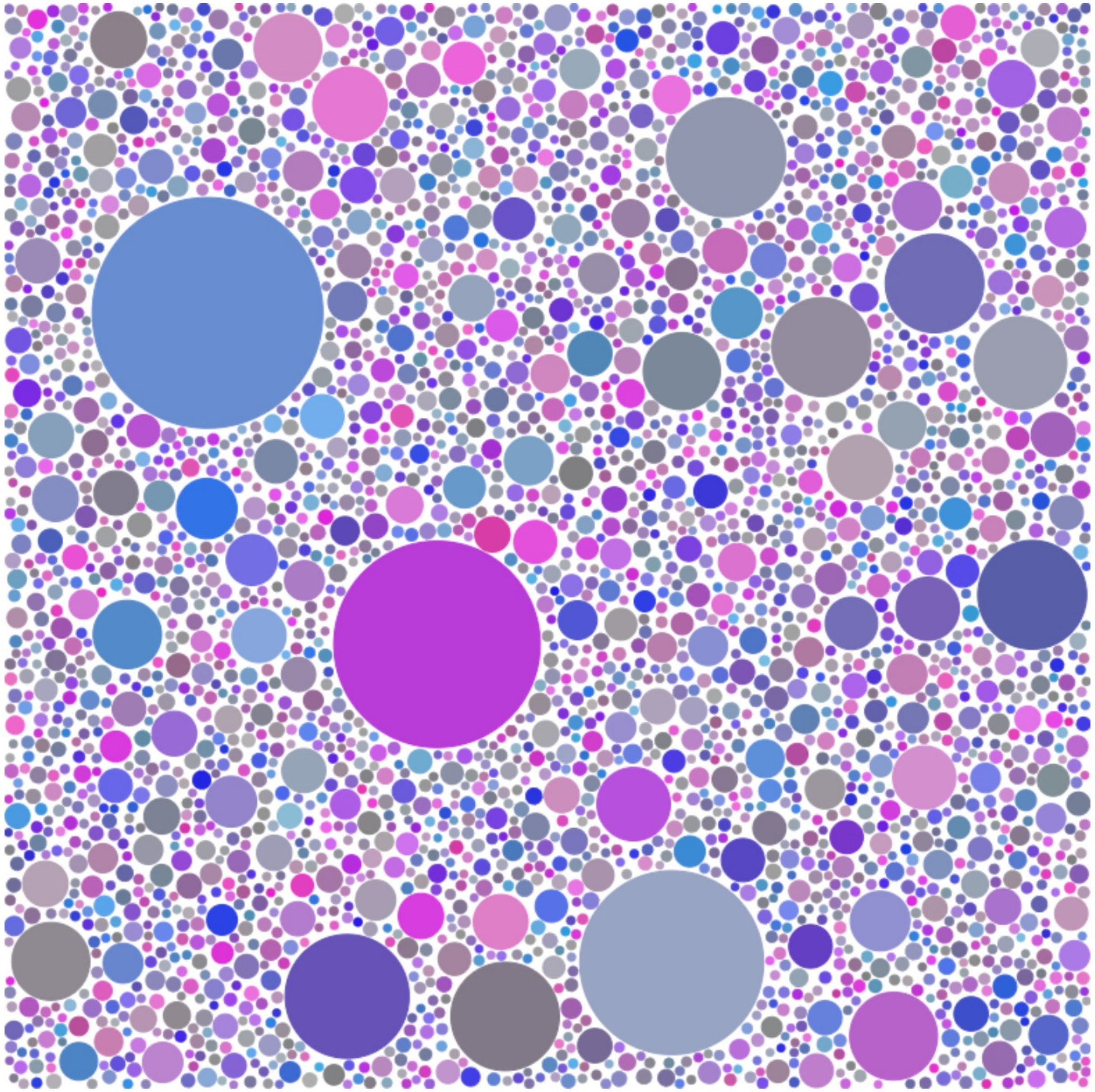
Pareto Random Variable

Notation:	$X \sim \text{Pareto}(a)$
Description:	A long tail distribution. Large values are rare and small values are common.
Parameters:	$a \geq 1$, the shape parameter Note there are other optional params. See wikipedia
Support:	$x \in [0, 1]$
PDF equation:	$f(x) = \frac{1}{x^{a+1}}$
CDF equation:	$F(x) = 1 - \frac{1}{x^a}$

Sampling from a Pareto Distribution

How can we draw samples from a pareto? In python its simple: `stats.pareto.rvs(a)` however in JavaScript, or other languages, it might not be made transparent

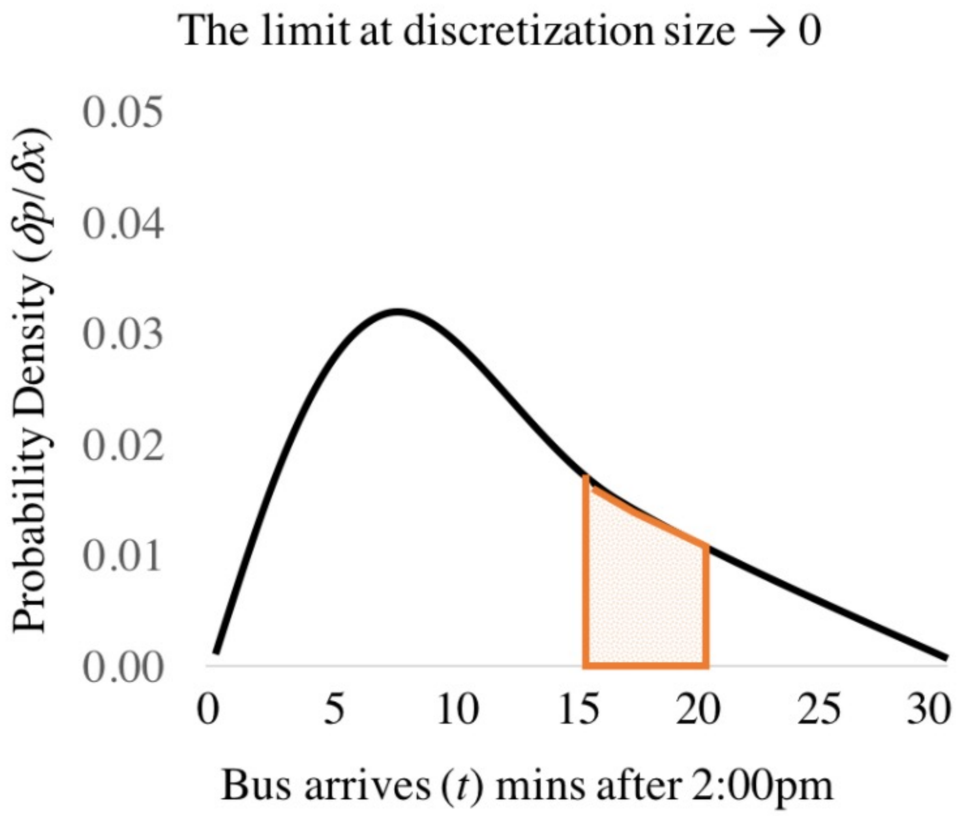
$$y = 1 - (\frac{\beta}{x})^\alpha$$
$$(\frac{\beta}{x})^\alpha = 1 - y$$

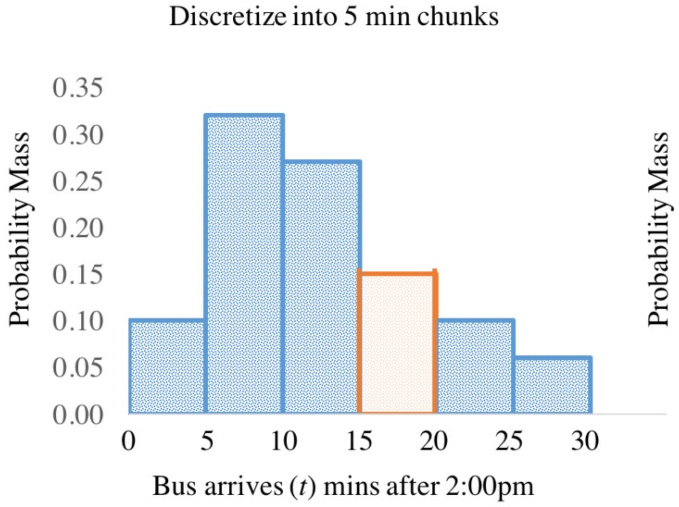
 

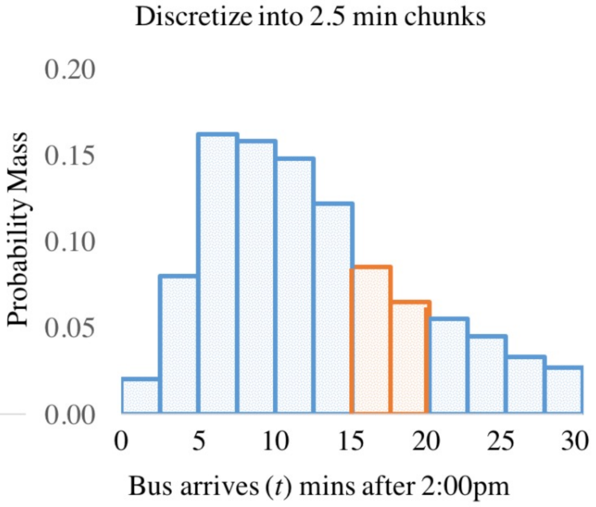
Please share your art on **Ed**

Review

Review: Probability Density Function







Continuous distributions!

Uniform Random Variable

Notation: $X \sim \text{Uni}(\alpha, \beta)$

Description: A continuous random variable that takes on values, with equal likelihood, between α and β

Parameters: $\alpha \in \mathbb{R}$, the minimum value of the variable.
$\beta \in \mathbb{R}$, $\beta > \alpha$, the maximum value of the variable.

Support: $x \in [\alpha, \beta]$

PDF equation: $f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{else} \end{cases}$

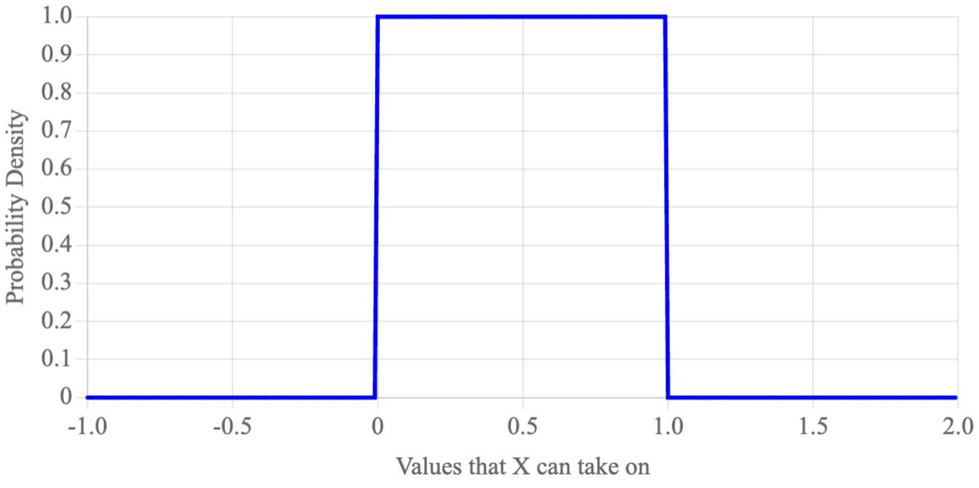
CDF equation: $F(x) = \begin{cases} \frac{x-\alpha}{\beta - \alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{for } x < \alpha \\ 1 & \text{for } x > \beta \end{cases}$

Expectation: $\text{E}[X] = \frac{1}{2}(\alpha + \beta)$

Variance: $\text{Var}(X) = \frac{1}{12}(\beta - \alpha)^2$

PDF graph:

Parameter α: 0 Parameter β: 1



Exponential Random Variable

Notation: $X \sim \text{Exp}(\lambda)$

Description: Time until next events if (a) the events occur with a constant mean rate and (b) they occur independently of time since last event.

Parameters: $\lambda \in \{0, 1, \ldots\}$, the constant average rate.

Support: $x \in \mathbb{R}^+$

PDF equation: $f(x) = \lambda e^{-\lambda x}$

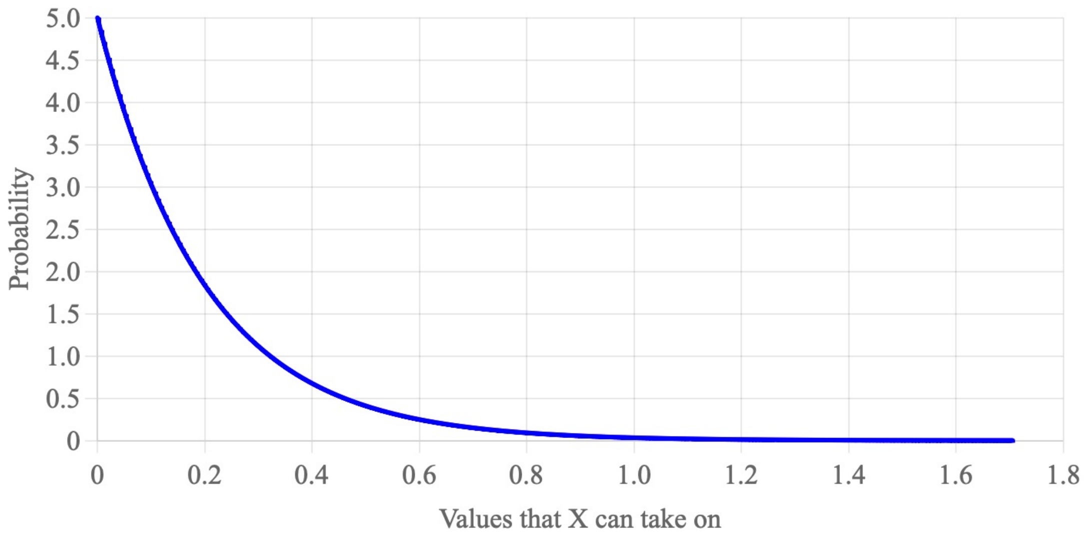
CDF equation: $F(x) = 1 - e^{-\lambda x}$

Expectation: $\text{E}[X] = 1/\lambda$

Variance: $\text{Var}(X) = 1/\lambda^2$

PDF graph:

Parameter λ: 5



Same as Poisson!!

What do you get if you integrate over a probability *density* function?

A probability!

Review: Probability Density Function

The **probability density function** (PDF) of a continuous random variable represents the **derivative** of probability at a given point.

Units of probability *divided by units of X*.
Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^{b} \boxed{f(X = x)}\, dx$$

Is there a way to avoid integrals?

Cumulative Density Function

A cumulative density function (CDF) is a “closed form” equation for the probability that a random variable is less than a given value

$$F(x) = P(X < x)$$

If you learn how to use a cumulative density function, you can avoid integrals!

$$F_X(x)$$

This is also shorthand notation for the PMF

Cumulative Density Function

$$F(x) = P(X < x)$$

$$x = 2$$

0.03125

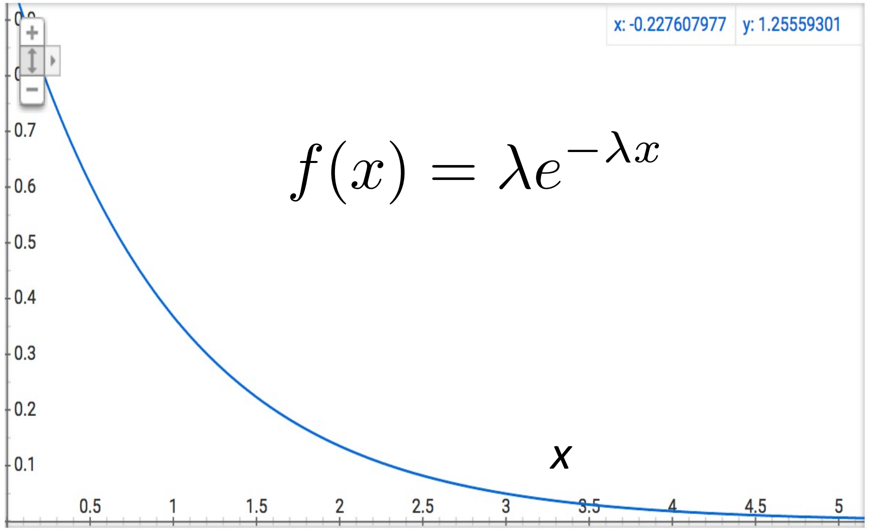
CDF of an Exponential

$$F_X(x) = 1 - e^{-\lambda x}$$

$$\begin{aligned}
P(X < x) &= \int_{y=-\infty}^{x} f(y)\, dy \\
&= \int_{y=0}^{x} \lambda e^{-\lambda y}\, dy \\
&= \frac{\lambda}{\lambda}\Big[-e^{-\lambda y}\Big]_0^x \\
&= [-e^{-\lambda x}] - [-e^{\lambda 0}] \\
&= 1 - e^{-\lambda x}
\end{aligned}$$

Using CDF Example. X is Exp(λ = 1)

Probability density function



$$f(x) = \lambda e^{-\lambda x}$$

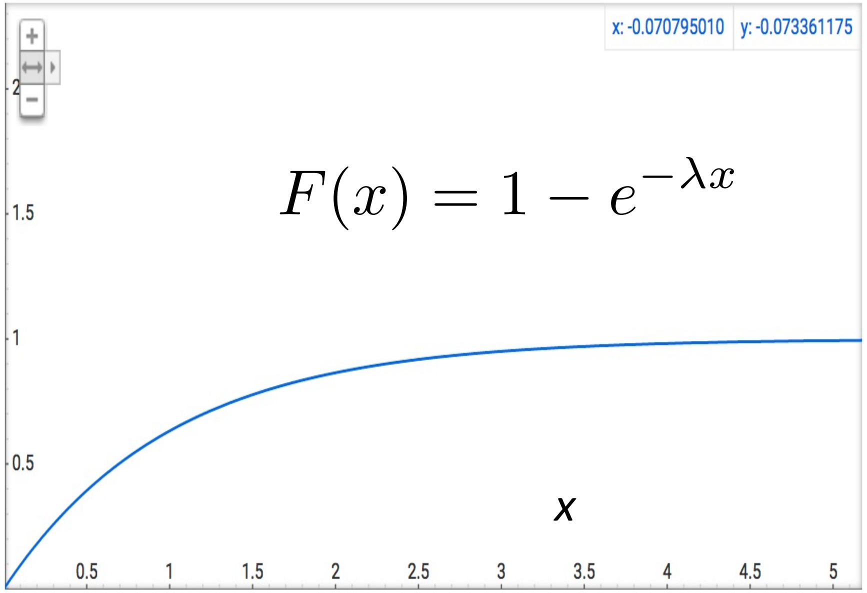
P(X < 2)

____

Cumulative density function

$$F_X(x) = P(X < x)$$

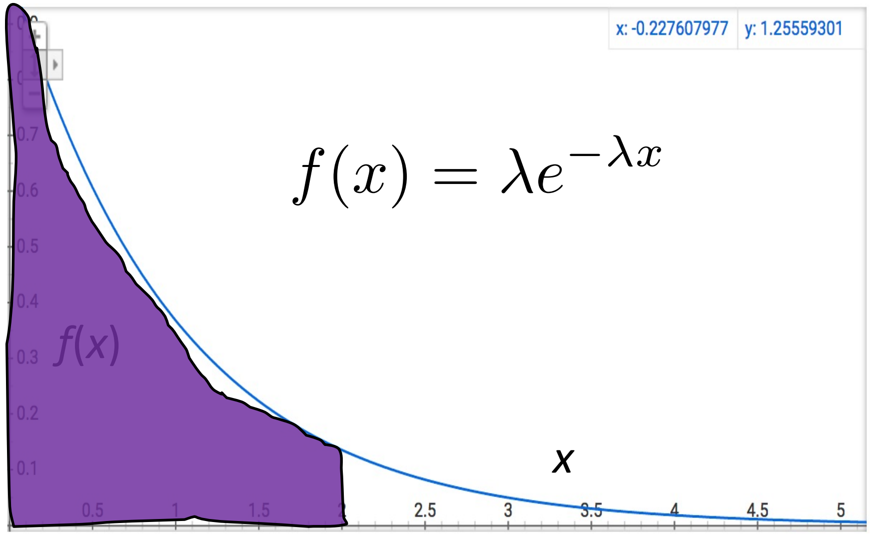
$$= \int_{y=-\infty}^{x} f(y)\ dy$$



$$F(x) = 1 - e^{-\lambda x}$$

Using CDF Example. X is Exp($\lambda = 1$)

Probability
density
function



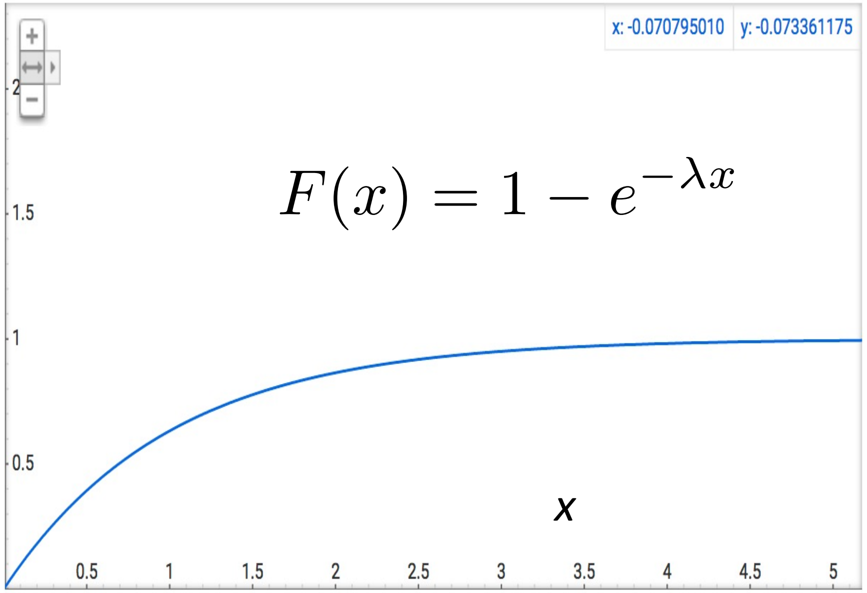
P(X < 2)

$$= \int_{x=-\infty}^{2} f(x)\, dx$$

Cumulative
density function

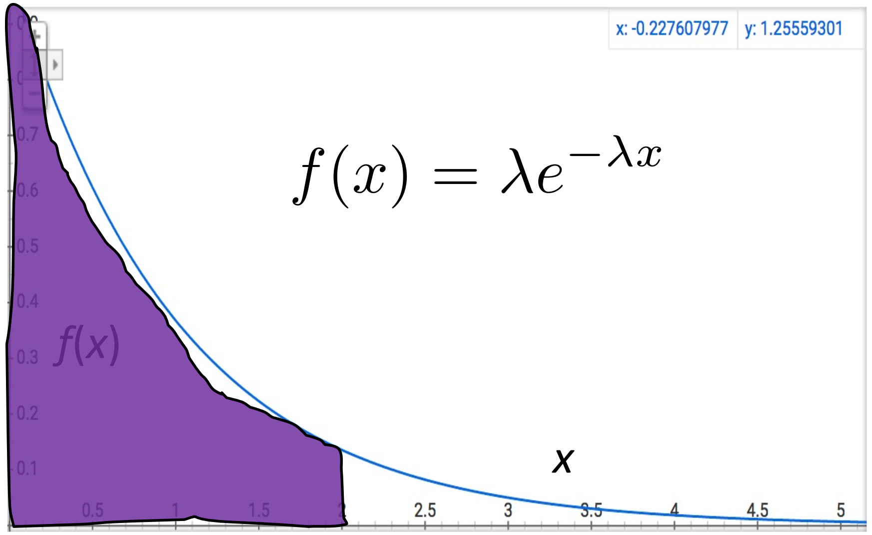
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability density function



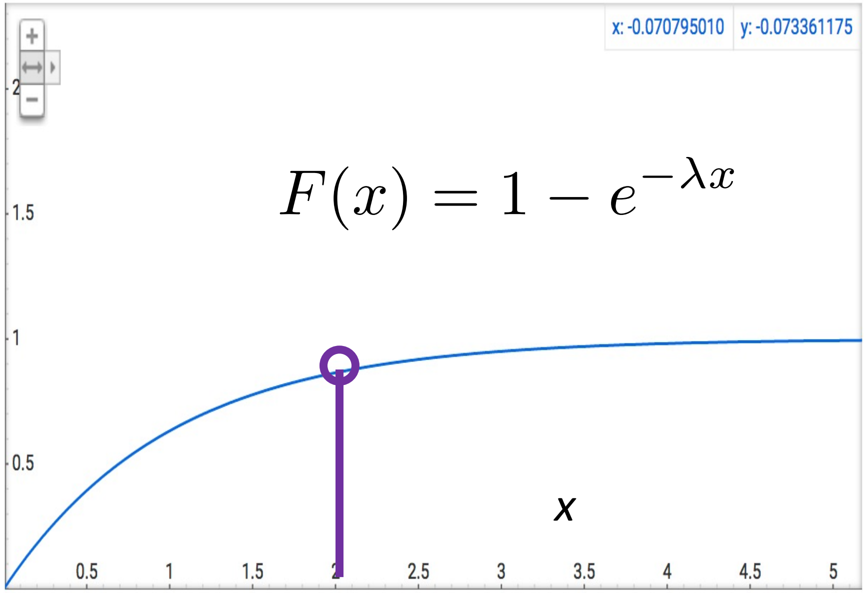
P(X < 2)

$$= \int_{x=-\infty}^{2} f(x)\, dx$$

Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



or

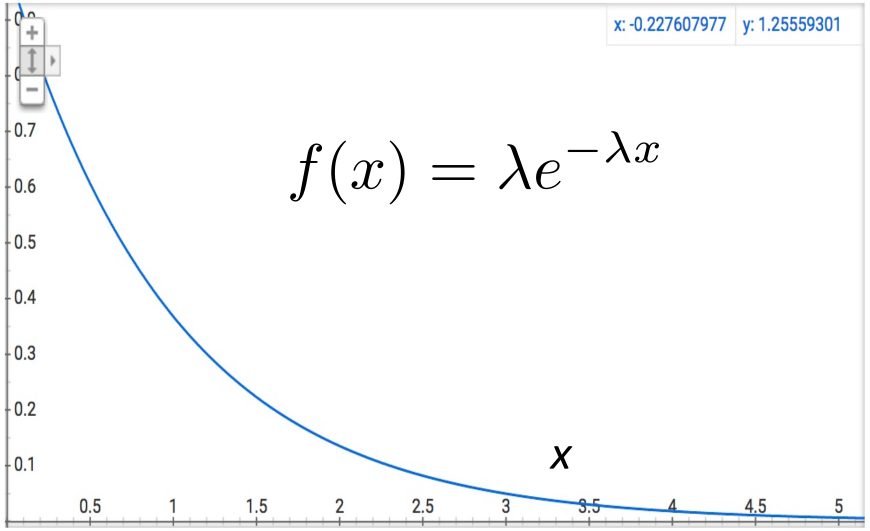
$$= F(2)$$

$$= 1 - e^{-2}$$

$$\approx 0.84$$

Using CDF Example. X is Exp($\lambda = 1$)

Probability
density
function

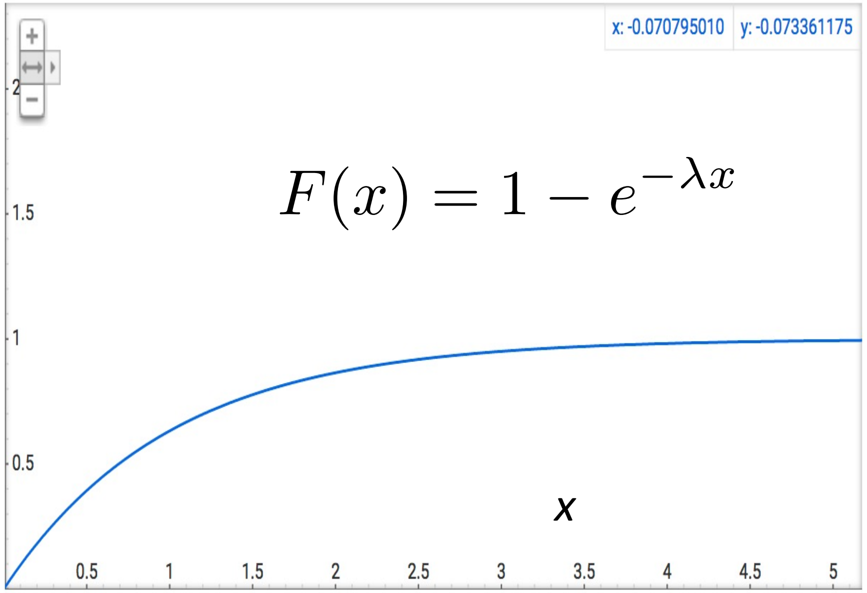


P(X > 1)

Cumulative
density function

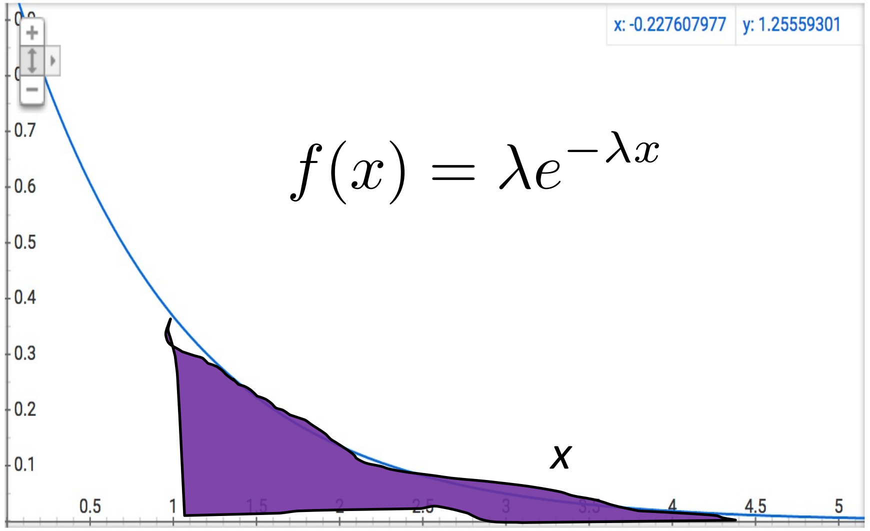
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp(λ = 1)

Probability density function



$$f(x) = \lambda e^{-\lambda x}$$

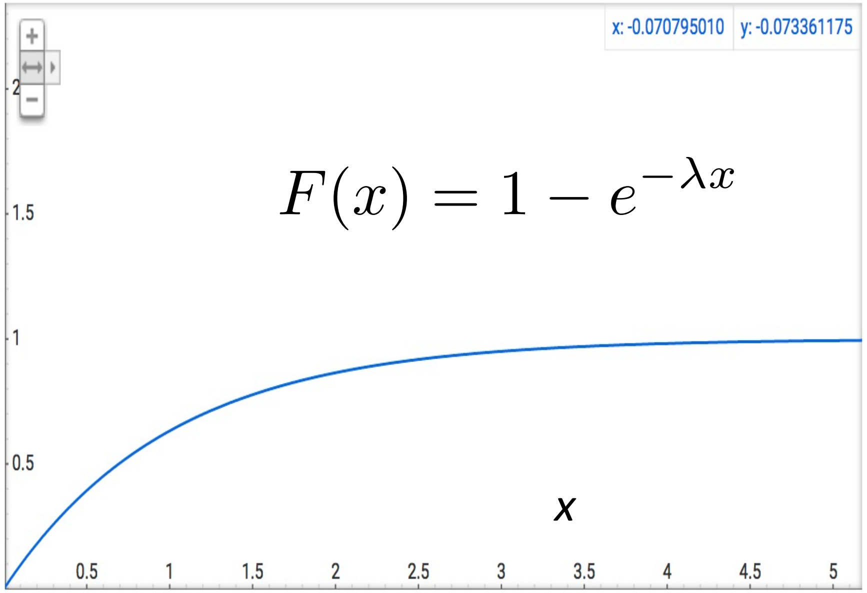
P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\, dx$$

Cumulative density function

$$F_X(x) = P(X < x)$$

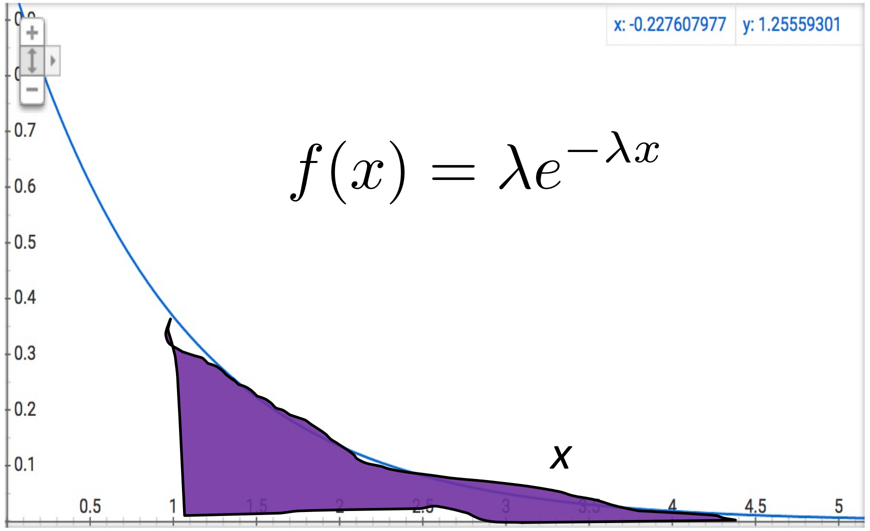
$$= \int_{y=-\infty}^{x} f(y)\, dy$$



$$F(x) = 1 - e^{-\lambda x}$$

Using CDF Example. X is Exp($\lambda = 1$)

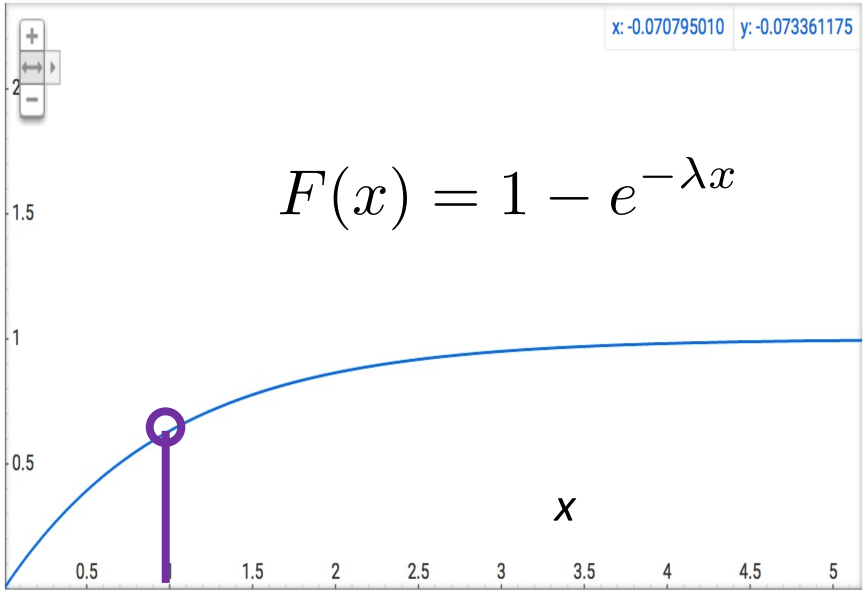
Probability density function



Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\ dy$$



P(X > 1)

$$= \int_{x=1}^{\infty} f(x)\ dx$$

or

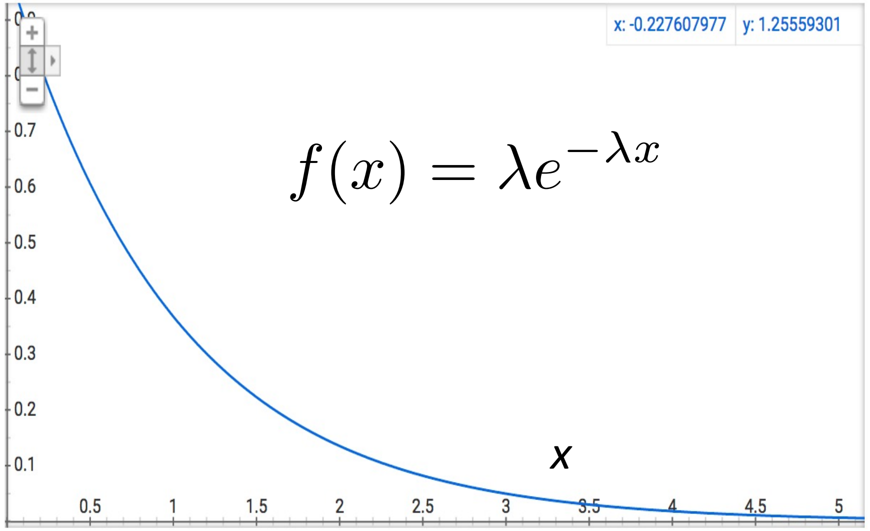
$$= 1 - F(1)$$

$$= e^{-1}$$

$$\approx 0.37$$

Using CDF Example. X is Exp($\lambda = 1$)

Probability density function

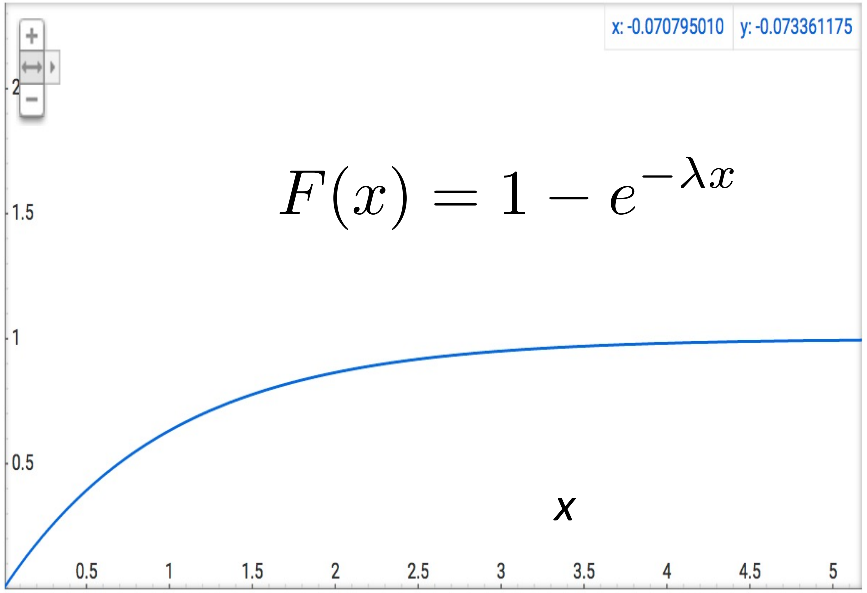


P(1 < X < 2)

Cumulative density function

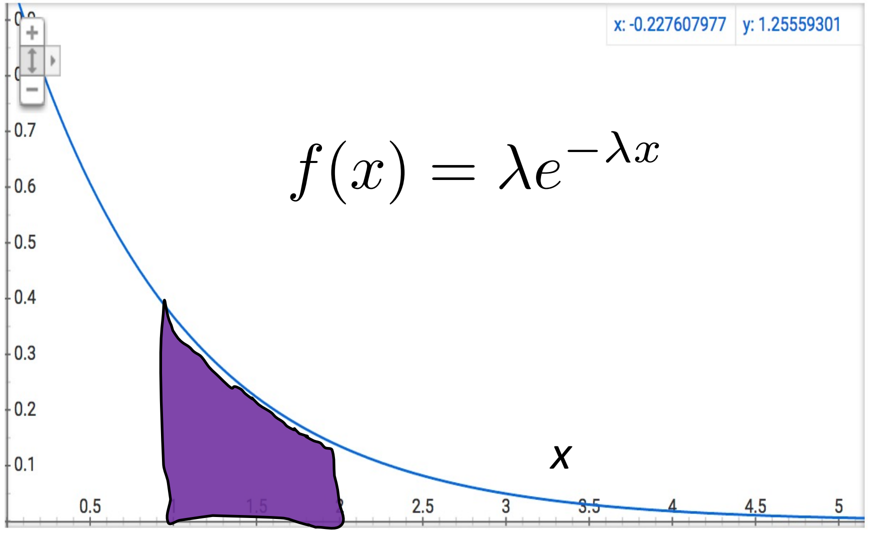
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp($\lambda = 1$)

Probability
density
function



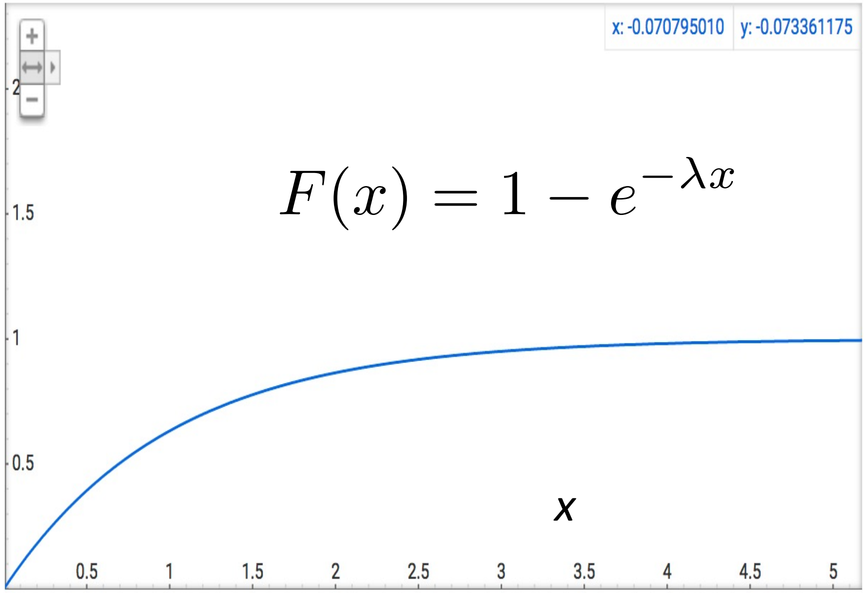
P(1 < X < 2)

$$= \int_{x=1}^{2} f(x)\, dx$$

Cumulative
density function

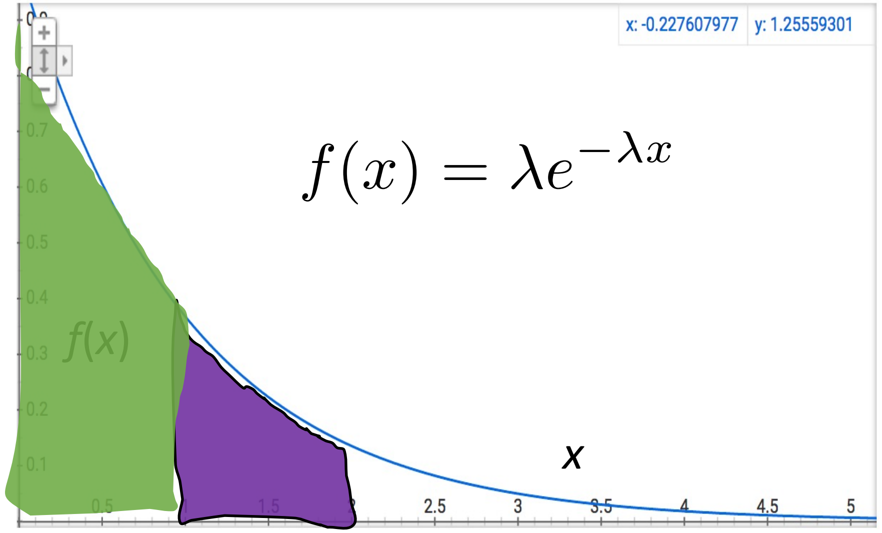
$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Using CDF Example. X is Exp($\lambda = 1$)

Probability density function



P(1 < X < 2)

$$= \int_{x=1}^{2} f(x)\, dx$$

or

$$= F(2) - F(1)$$

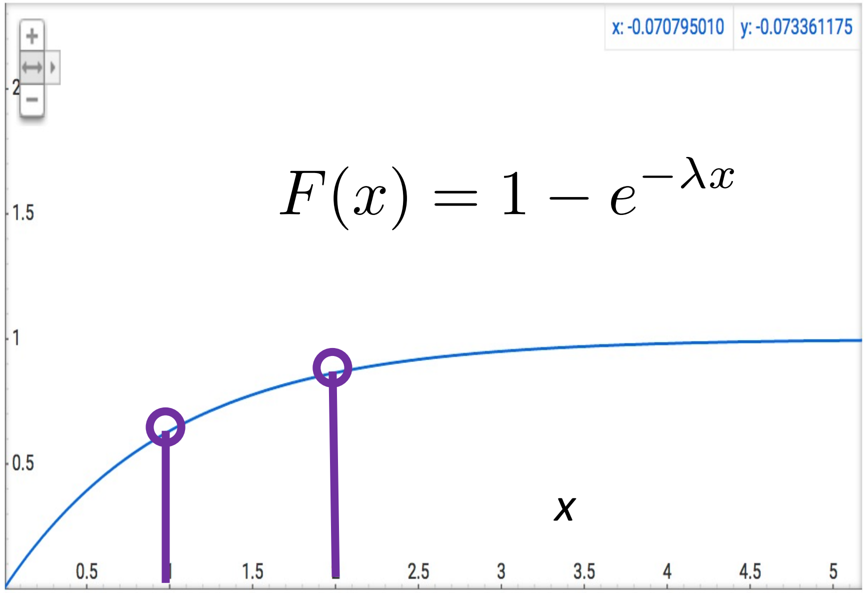
$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$\approx 0.23$$

Cumulative density function

$$F_X(x) = P(X < x)$$

$$= \int_{y=-\infty}^{x} f(y)\, dy$$



Probability of Earthquake in Next 4 Years?

Based on historical data, earthquakes of magnitude 8.0+ happen at a **rate of 0.002** per year*. What is the probability of **an major earthquake in the next 4 years?**

Y = Years until the next earthquake of magnitude 8.0+

$$Y \sim \text{Exp}(\lambda = 0.002) \qquad F(y) = 1 - e^{-0.002y}$$

$$\begin{aligned} P(Y < 4) &= F(4) \\ &= 1 - e^{-0.002 \cdot 4} \\ &\approx 0.008 \end{aligned}$$

Feeling lucky?

*According to USGS, 2015

CDF is available on Reader!

Uniform Random Variable

Notation: $X \sim \text{Uni}(\alpha, \beta)$

Description: A continuous random variable that takes on values, with equal likelihood, between α and β

Parameters: $\alpha \in \mathbb{R}$, the minimum value of the variable.
$\beta \in \mathbb{R}$, $\beta > \alpha$, the maximum value of the variable.

Support: $x \in [\alpha, \beta]$

PDF equation: $f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{else} \end{cases}$

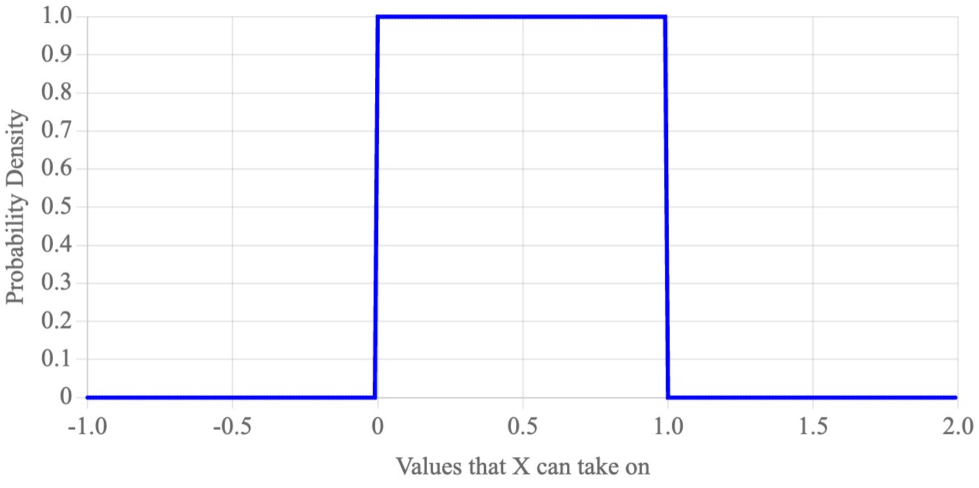
CDF equation: $F(x) = \begin{cases} \frac{x - \alpha}{\beta - \alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{for } x < \alpha \\ 1 & \text{for } x > \beta \end{cases}$

Expectation: $\text{E}[X] = \frac{1}{2}(\alpha + \beta)$

Variance: $\text{Var}(X) = \frac{1}{12}(\beta - \alpha)^2$

PDF graph:

Parameter α: 0 Parameter β: 1



Exponential Random Variable

Notation: $X \sim \text{Exp}(\lambda)$

Description: Time until next events if (a) the events occur with a constant mean rate and (b) they occur independently of time since last event.

Parameters: $\lambda \in \{0, 1, \ldots\}$, the constant average rate.

Support: $x \in \mathbb{R}^+$

PDF equation: $f(x) = \lambda e^{-\lambda x}$

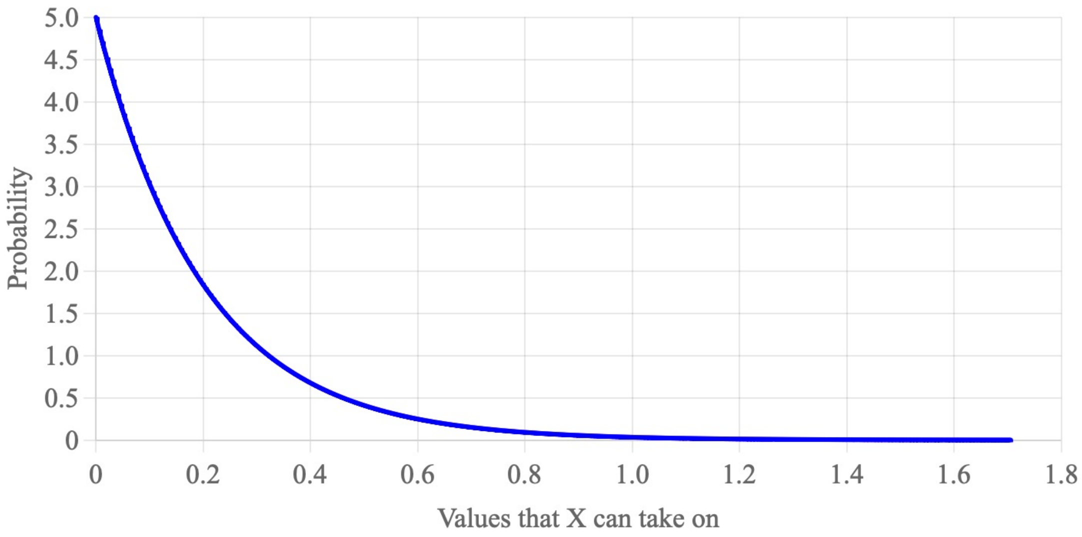
CDF equation: $F(x) = 1 - e^{-\lambda x}$

Expectation: $\text{E}[X] = 1/\lambda$

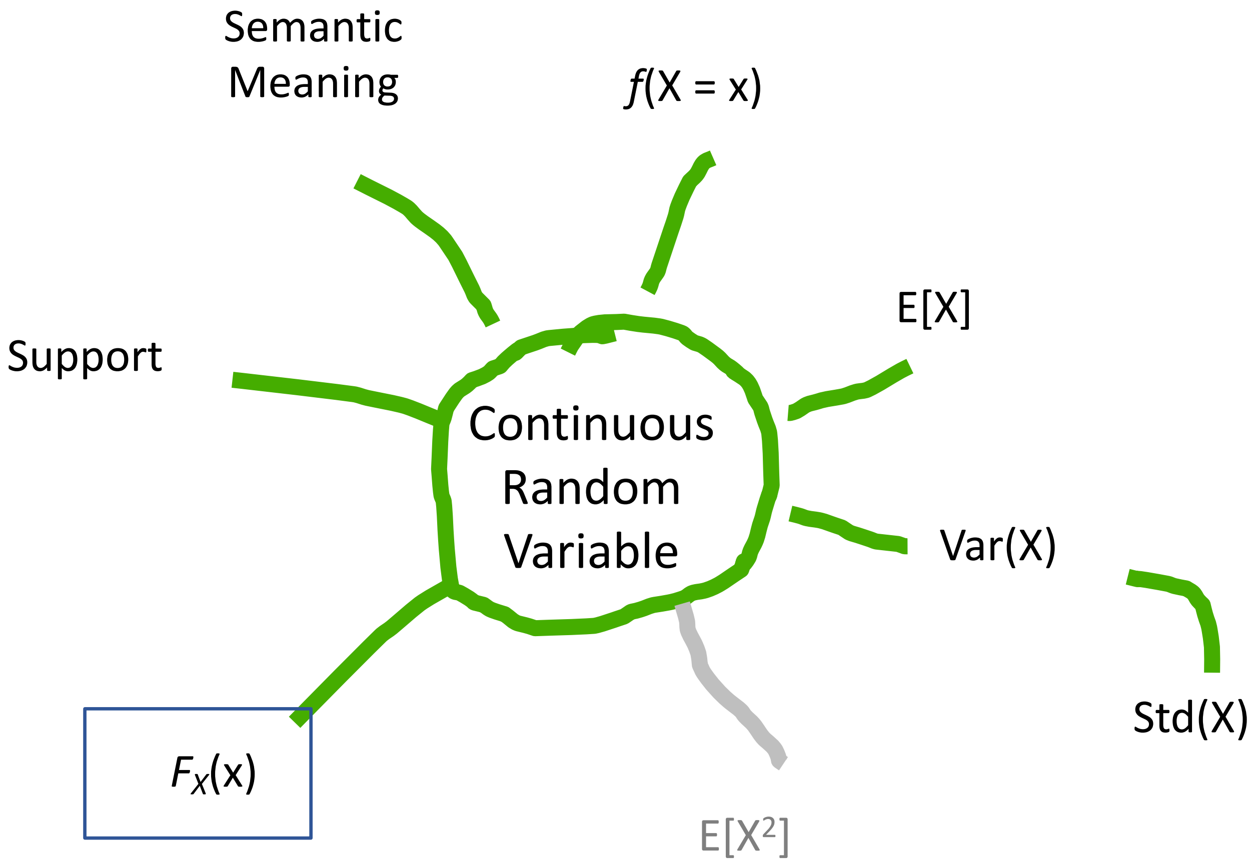
Variance: $\text{Var}(X) = 1/\lambda^2$

PDF graph:

Parameter λ: 5



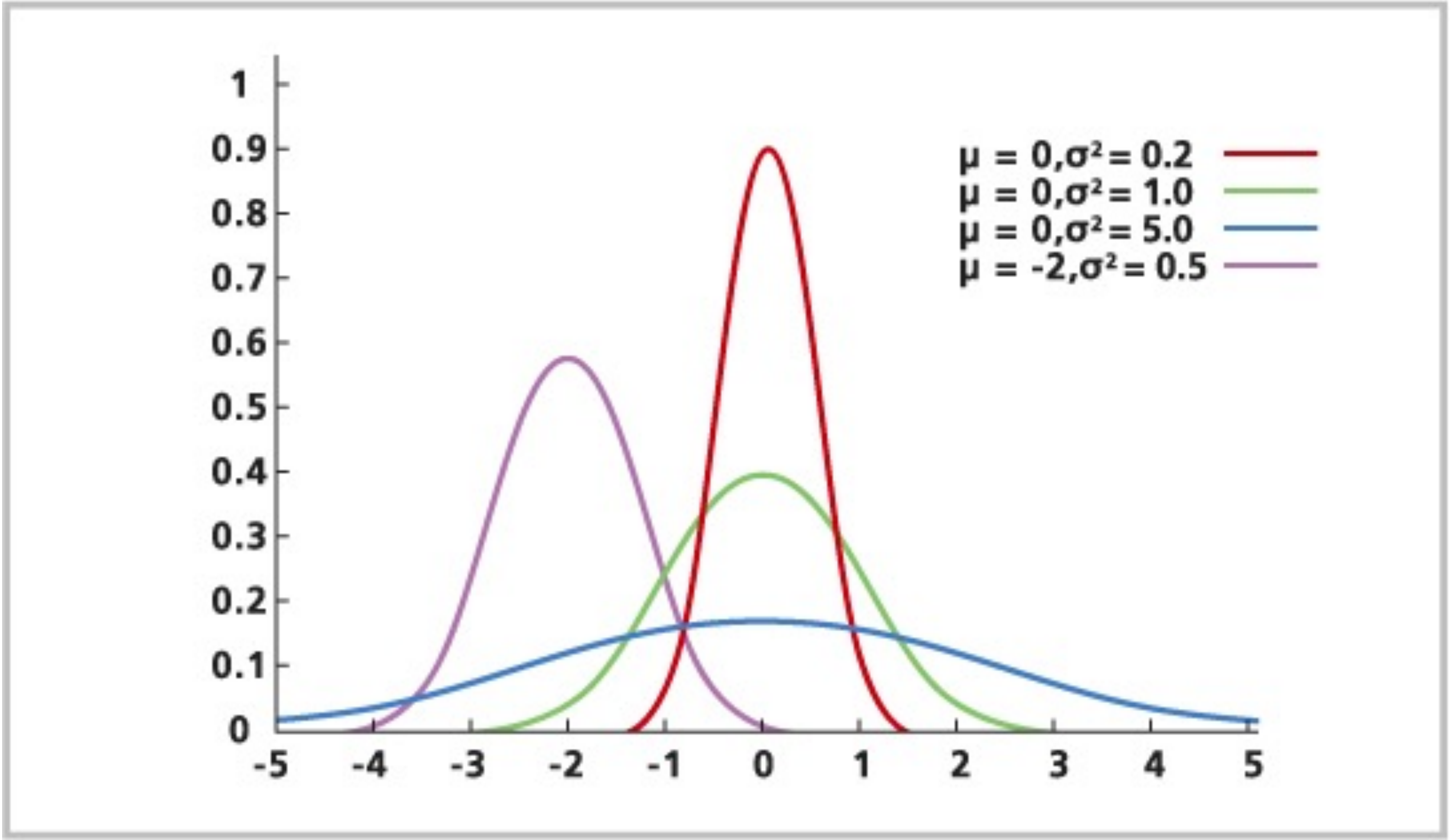
Properties for Continuous Random Variable



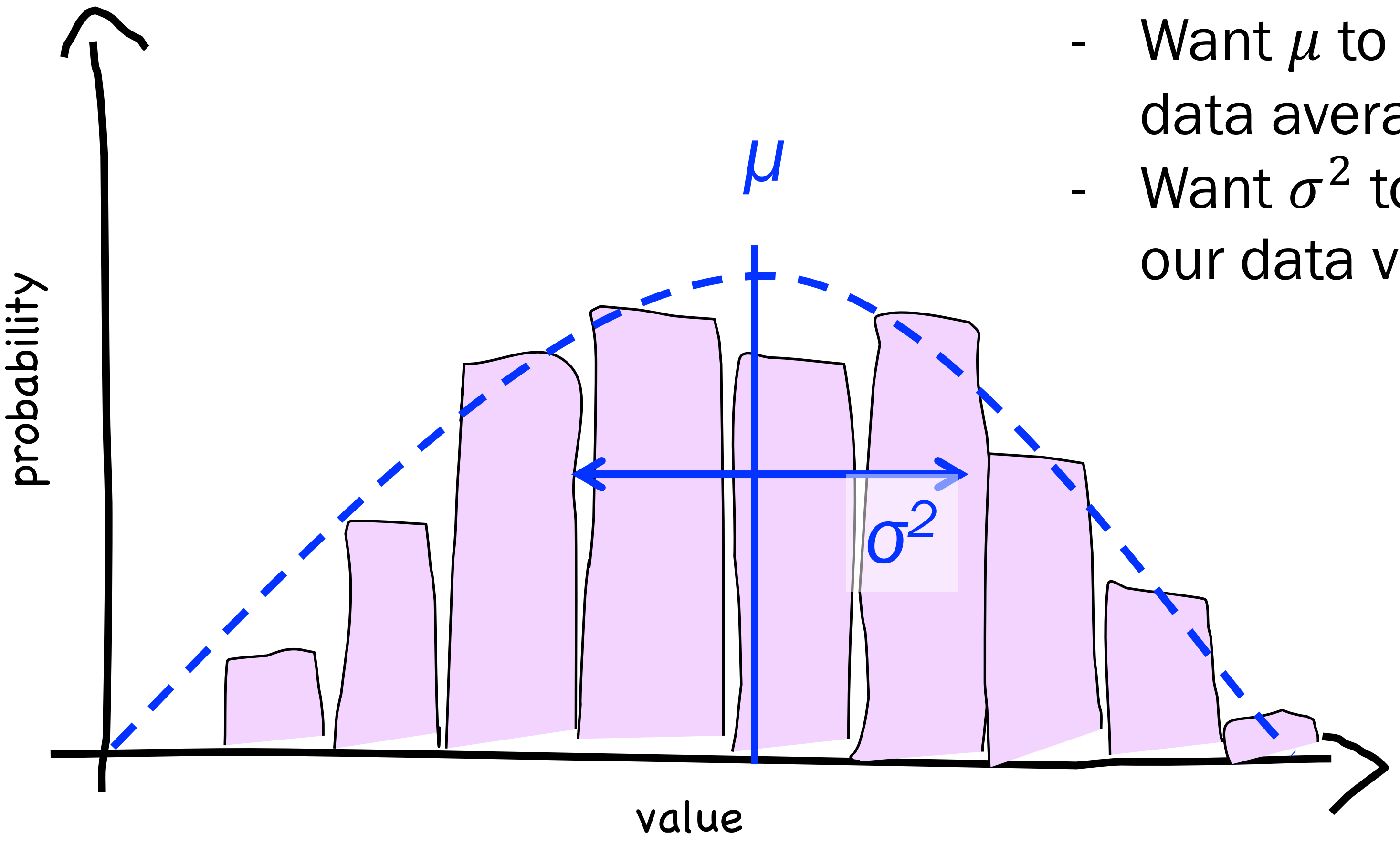
/Review

Big Day

NormCore: A Few Normal Examples



Motivation: Want to control both Expectation and Variance



Idea:

- Want μ to be equal to our data average.
- Want σ^2 to be equal to our data variance.

Want to specify mean independently of variance.

Motivation : Want to control both Expectation and Variance

Bernoulli	Binomial	Poisson	Geometric	Exponential
Param: p	Param: n, p	Param: λ	Param: p	Param: λ
↓	↓	↓	↓	↓
Bernoulli	Binomial	Poisson	Geometric	Exponential
↓	↓	↓	↓	↓
$E[X] = p$	$E[X] = np$	$E[X] = \lambda$	$E[X] = p$	$E[X] = 1/\lambda$
$Var(X) = p(1-p)$	$Var(X) = np(1-p)$	$Var(X) = \lambda$	$Var(X) = \frac{(1-p)^2}{p}$	$Var(X) = 1/\lambda^2$

So far, we have no way to really change the mean and variance independently using just the parameter.

Normal Random Variable

<u>def</u> An **Normal** random variable X is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

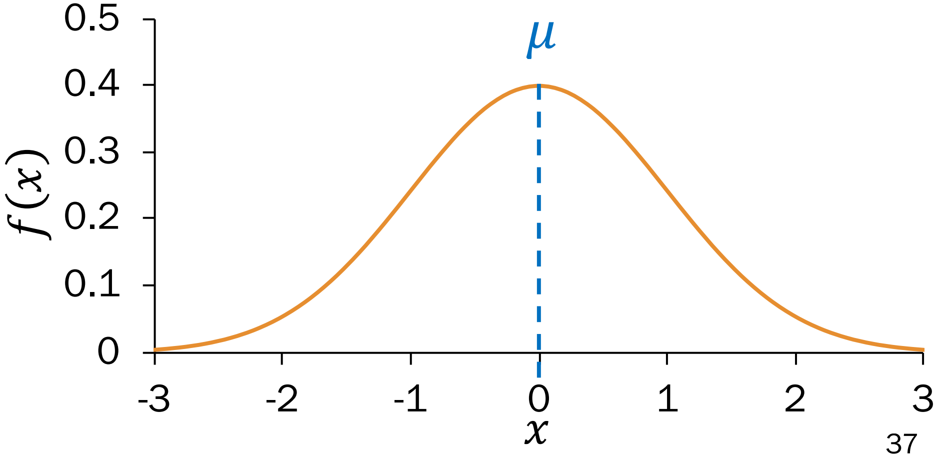
PDF	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
Expectation	$E[X] = \mu$
Variance	$\text{Var}(X) = \sigma^2$

Other names: **Gaussian** random variable

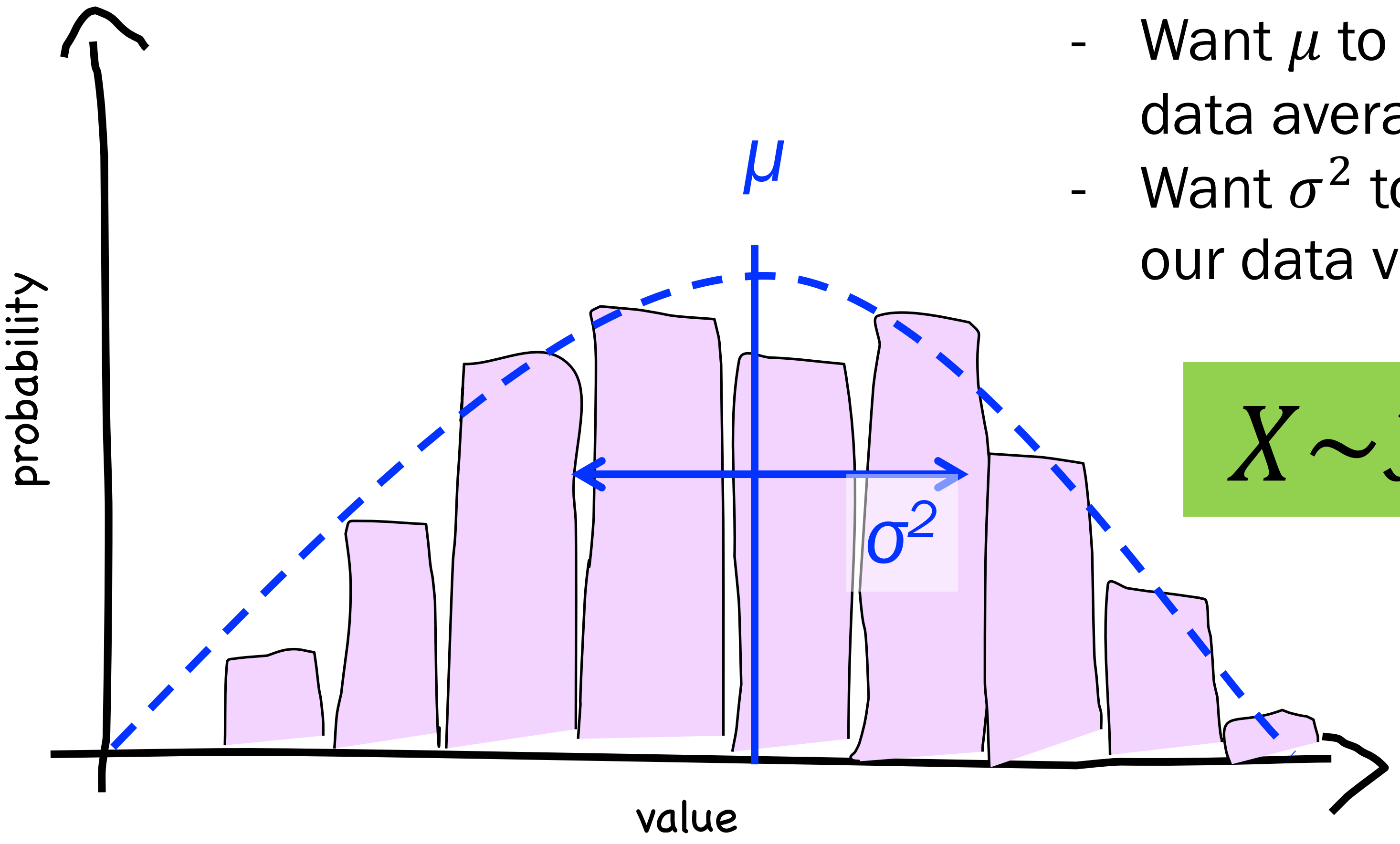
mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



Normal!



Idea:

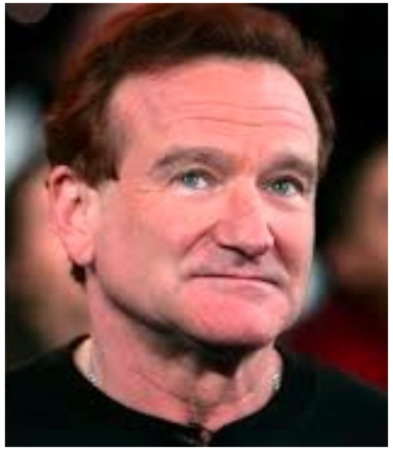
- Want μ to be equal to our data average.
- Want σ^2 to be equal to our data variance.

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Want to specify mean independently of variance.

Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician.



Johann Carl Friedrich Gauss (/ɡaʊs/; German: *Gauß* [ɡaʊs] (listen); Latin: *Carolus Fridericus Gauss*; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields, including algebra, analysis, astronomy, differential geometry, electrostatics, geodesy, geophysics, magnetic fields, matrix theory, mechanics, number theory, optics and statistics.

Sometimes referred to as the *Princeps mathematicorum*[1] (Latin for "the foremost of mathematicians") and "the greatest mathematician since antiquity", Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.[2]

Did not invent Normal distribution but rather popularized it

Why the Normal?

These are log-normal

- Common for natural phenomena: height, weight, etc.

Most noise is assumed normal

- Most noise in the world is Normal

Only if they are equally weighted and independent

- Often results from the sum of many random variables

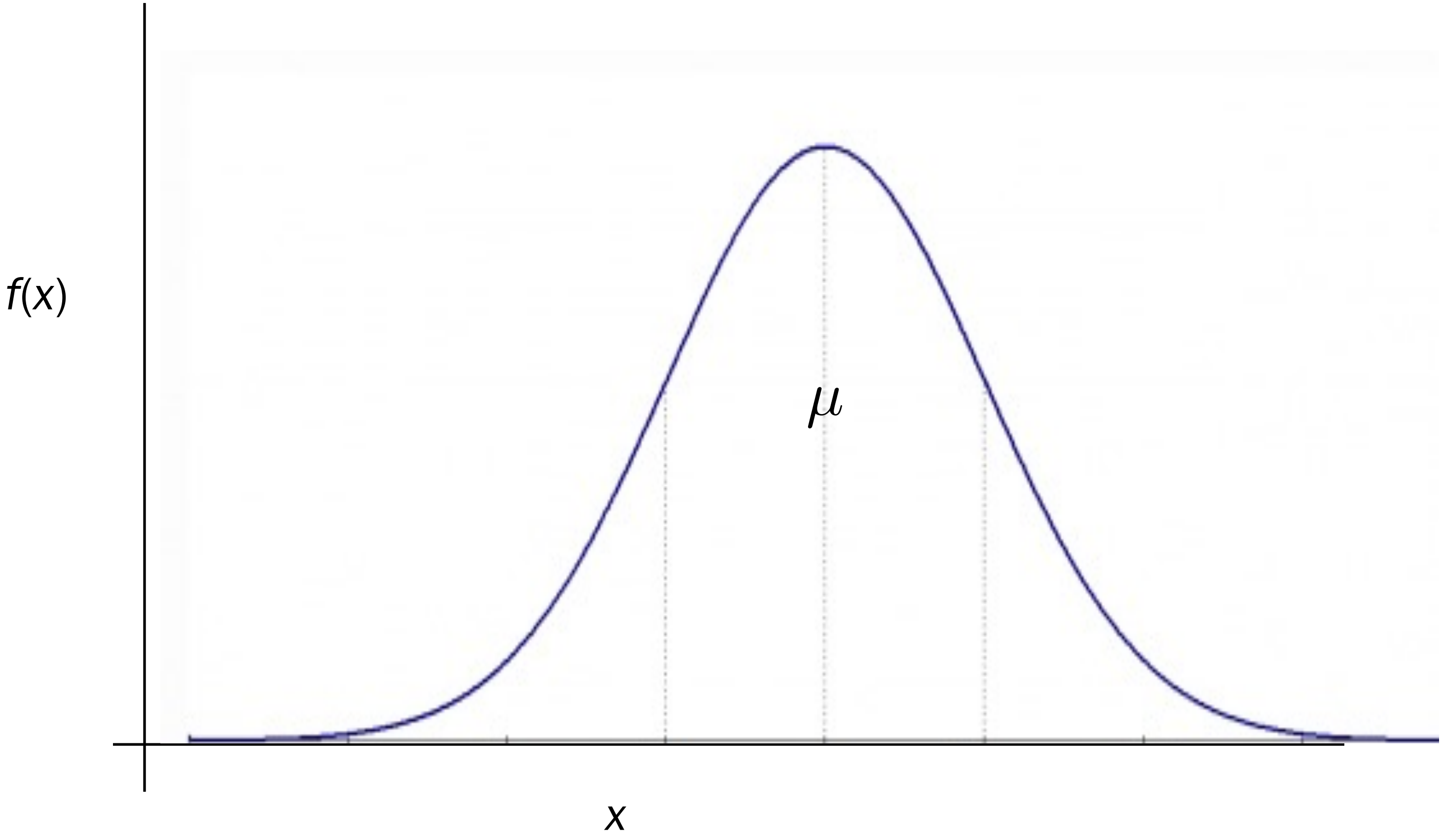
- Sample means are distributed normally

It is easy to use

Normal Probability Density Function

$$\mathcal{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



Anatomy of a Normal PDF

$$\mathcal{N}(\mu, \sigma^2)$$

"exponential"

the distance to the mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at x

a constant

sigma shows up twice

Campus bikes

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = \int_6^\infty f(x)dx = \int_6^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

(call me if you analytically solve this)

Loving, not scary
...except this time

No closed form for the integral

No closed form for $F(x)$

Spoiler: Numerically Solved CDF

$$\mathcal{N}(\mu, \sigma^2)$$

A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function of any normal

* We are going to spend the next few slides getting here

Linear Transform of Normal is Normal

$$\text{Let } X \sim \mathcal{N}(\mu, \sigma^2)$$

If $Y = aX + b$ then *Y* is also Normal

$$E[Y] = E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(Y) = Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Special Linear Transform

If $Y = aX + b$ then *Y* is also Normal

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

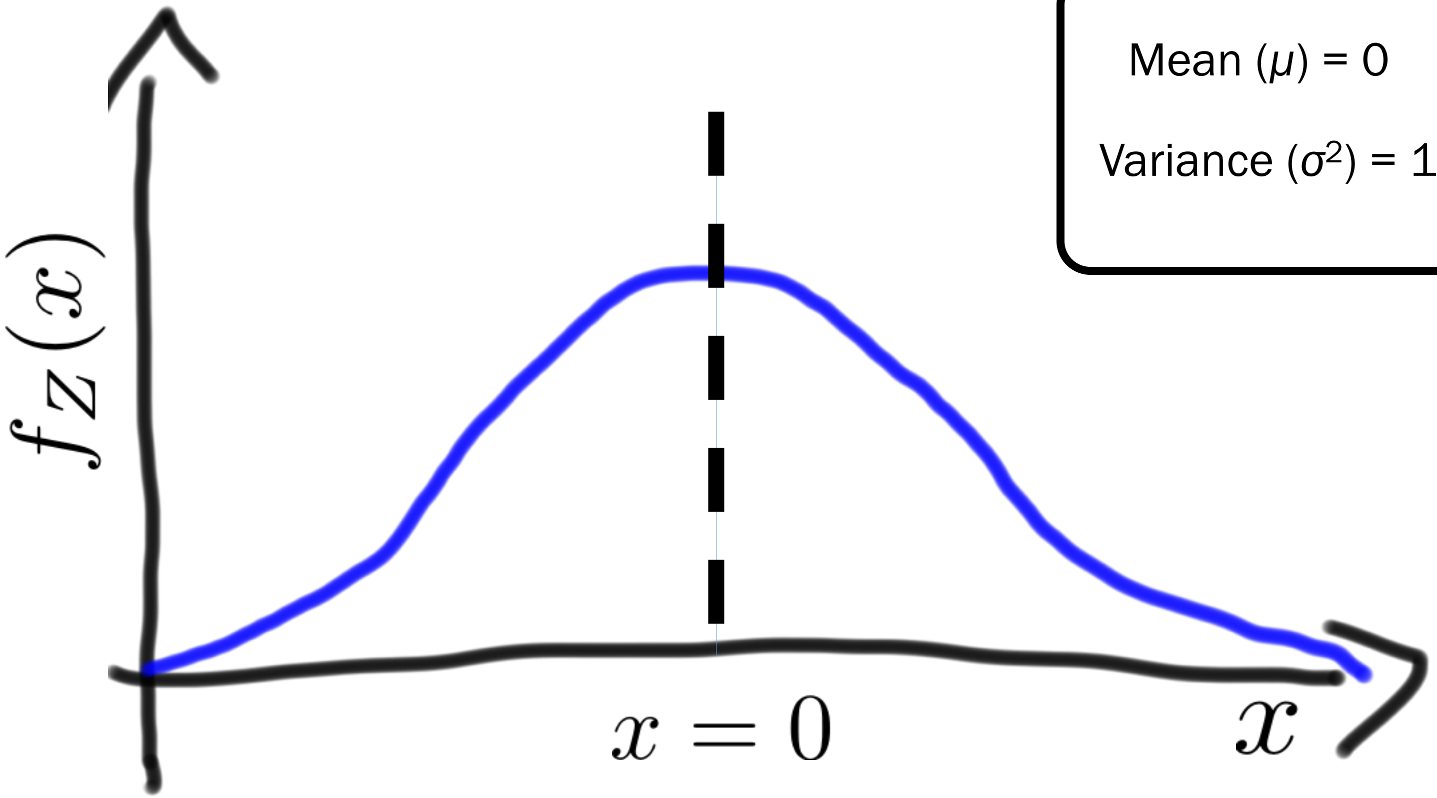
There is a special case of linear transform for any *X*:

$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \qquad a = \frac{1}{\sigma} \quad b = -\frac{\mu}{\sigma}$$

$$\begin{aligned} Z &\sim \mathcal{N}(a\mu + b, a^2\sigma^2) \\ &\sim \mathcal{N}(\frac{\mu}{\sigma} - \frac{\mu}{\sigma}, \frac{\sigma^2}{\sigma^2}) \\ &\sim \mathcal{N}(0, 1) \end{aligned}$$

The Standard Normal

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$



Mean (μ) = 0

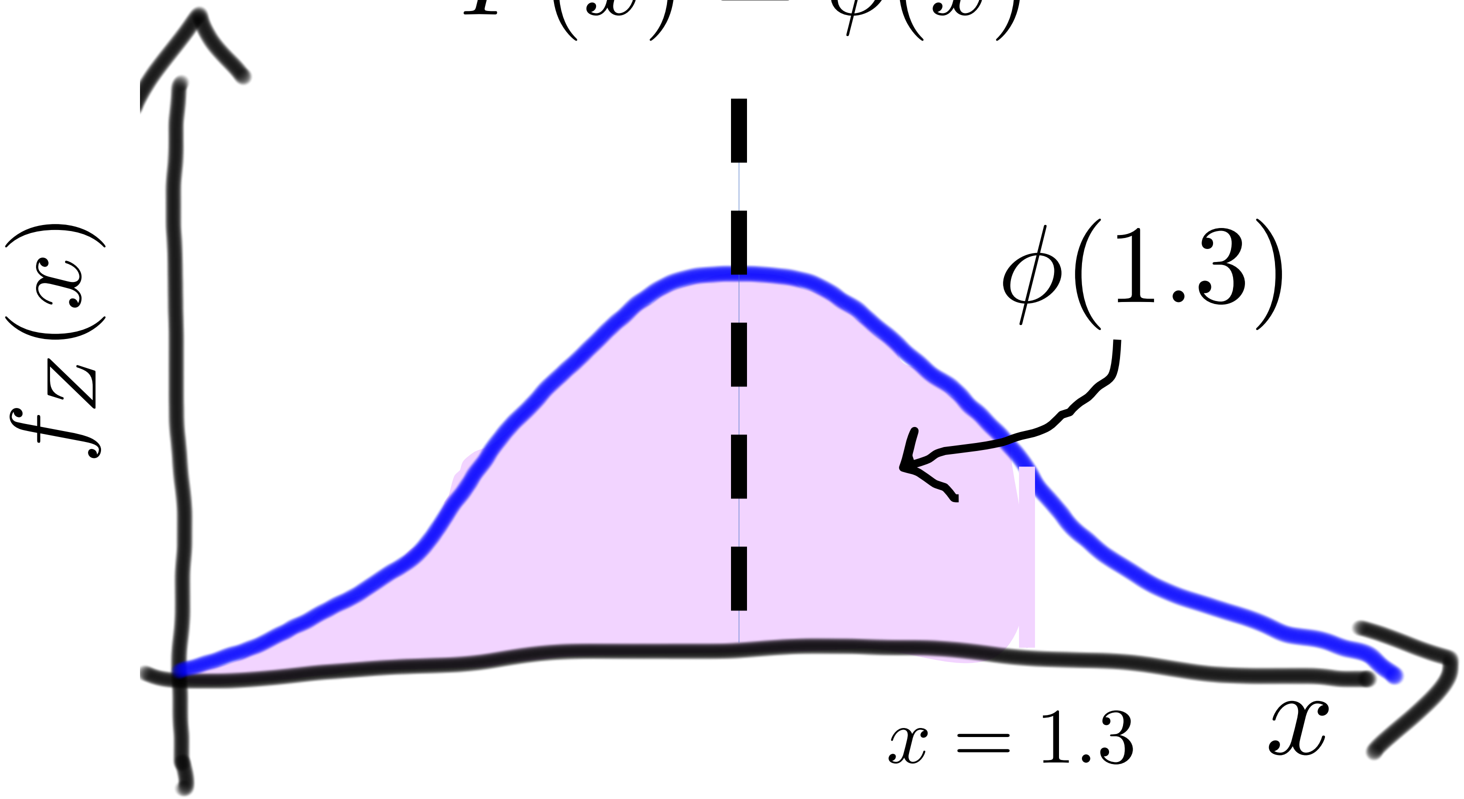
Variance (σ^2) = 1

*This is the probability density function for the standard normal

Phi

$$Z \sim N(\mu = 0, \sigma^2 = 1)$$

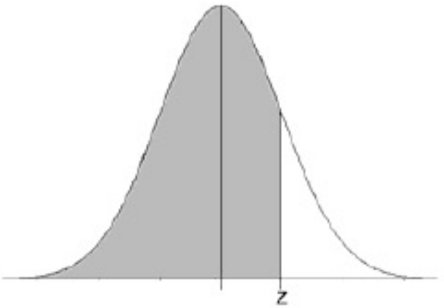
$$F(x) = \phi(x)$$



*This is the probability density function for the standard normal

Using Table of ϕ

Standard Normal Cumulative Probability Table



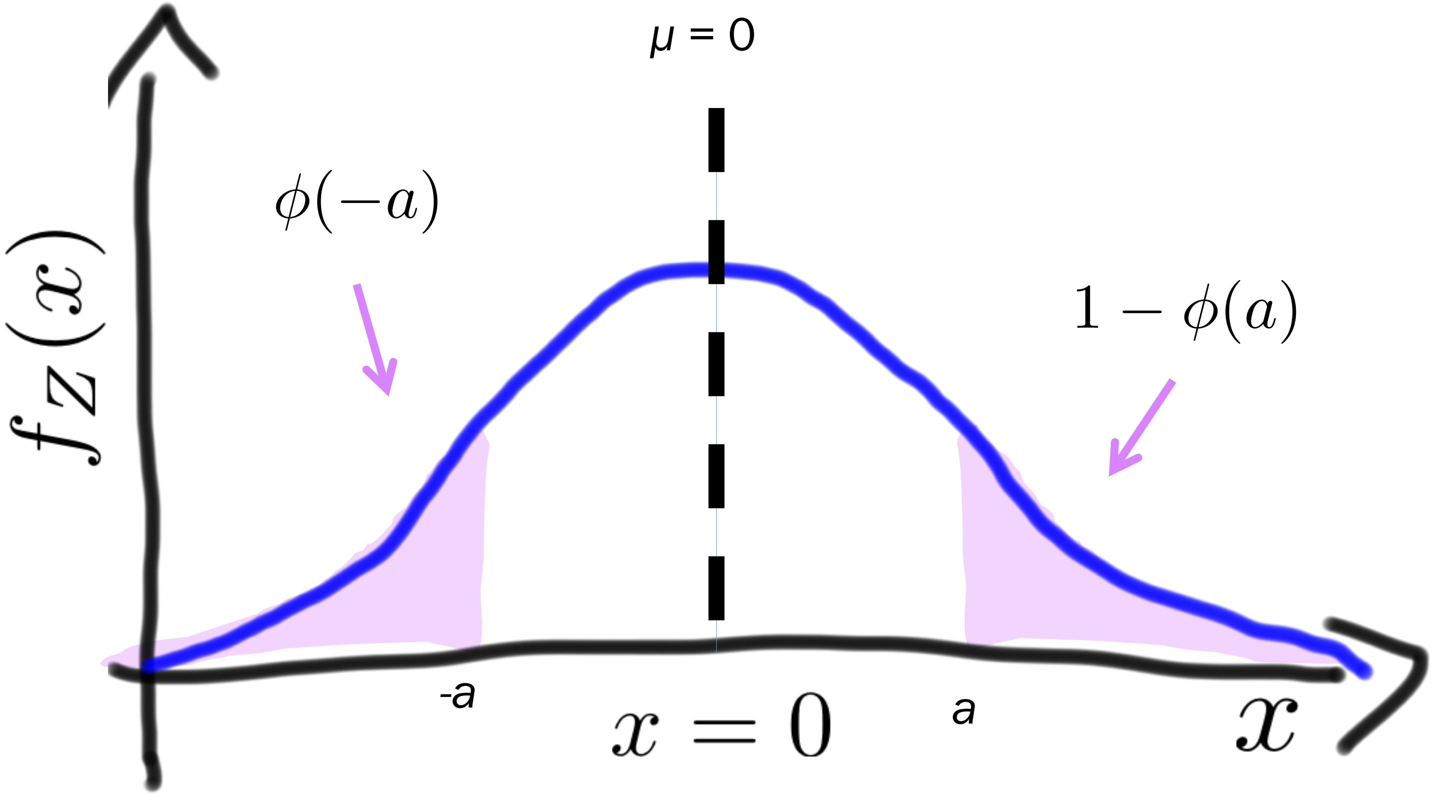
$\Phi(1.31) = 0.9049$

Cumulative probabilities for **POSITIVE** z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Symmetry of Phi

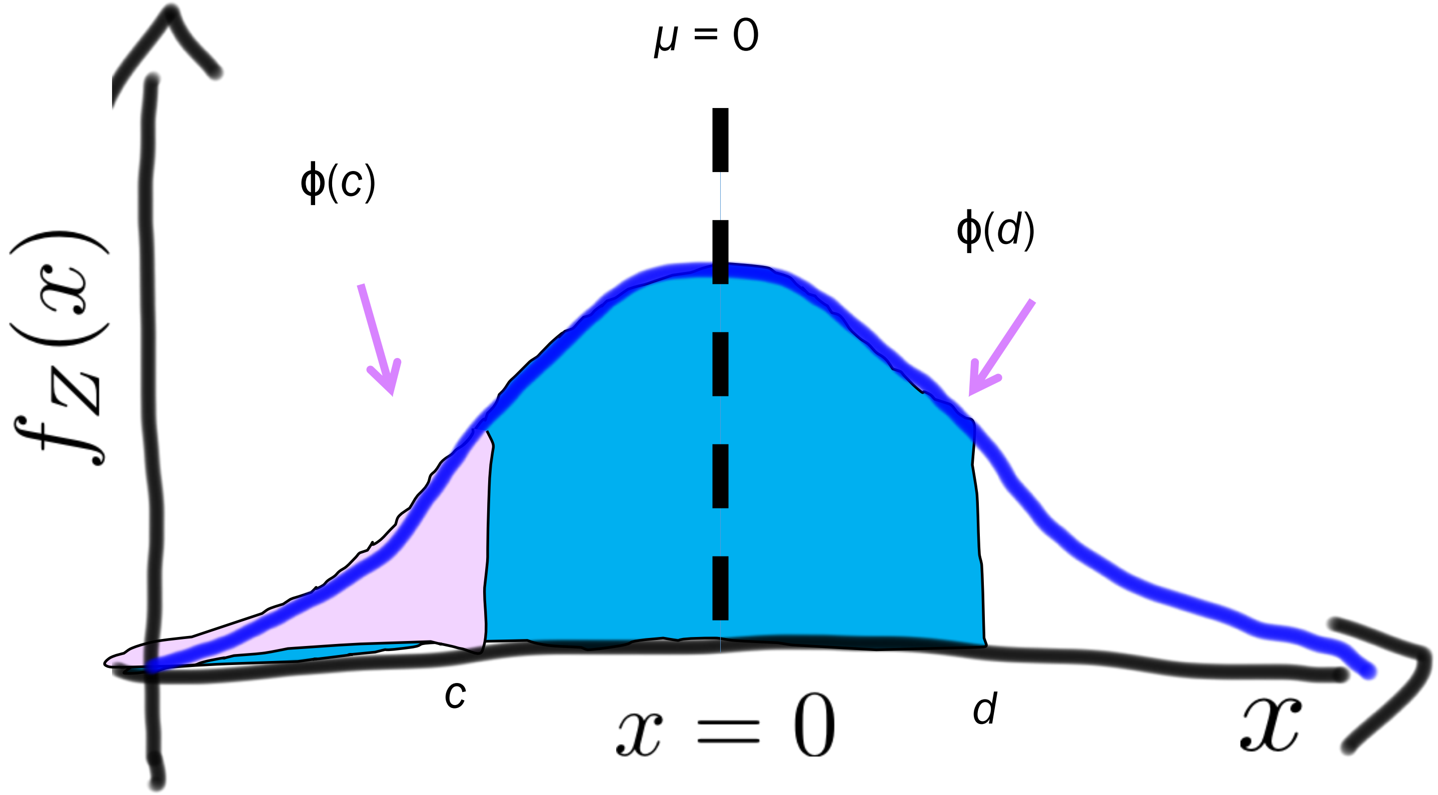
$$\phi(-a) = 1 - \phi(a)$$



*This is the probability density function for the standard normal

Interval of Phi

$$P(c < Z < d) = \phi(d) - \phi(c)$$



Compute F(x) via Transform

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ $\qquad Z = \frac{X - \mu}{\sigma}$

Use *Z* to compute *F*(*x*)

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(X - \mu \leq x - \mu) \\
&= P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\
&= P\left(Z \leq \frac{x - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{x - \mu}{\sigma}\right)
\end{aligned}
$$

For normal distribution,
F(x) is computed using
the phi transform.

And here we are

$$\mathcal{N}(\mu, \sigma^2)$$

CDF of Standard Normal: A function that has been solved for numerically

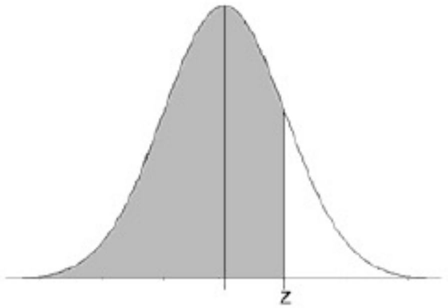
$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

Using the Phi Table

Standard Normal Cumulative Probability Table

$\Phi(0.54) = 0.7054$



Cumulative probabilities for **POSITIVE** z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Table is kinda old school

Using Programming Library

Every modern programming language has a normal library

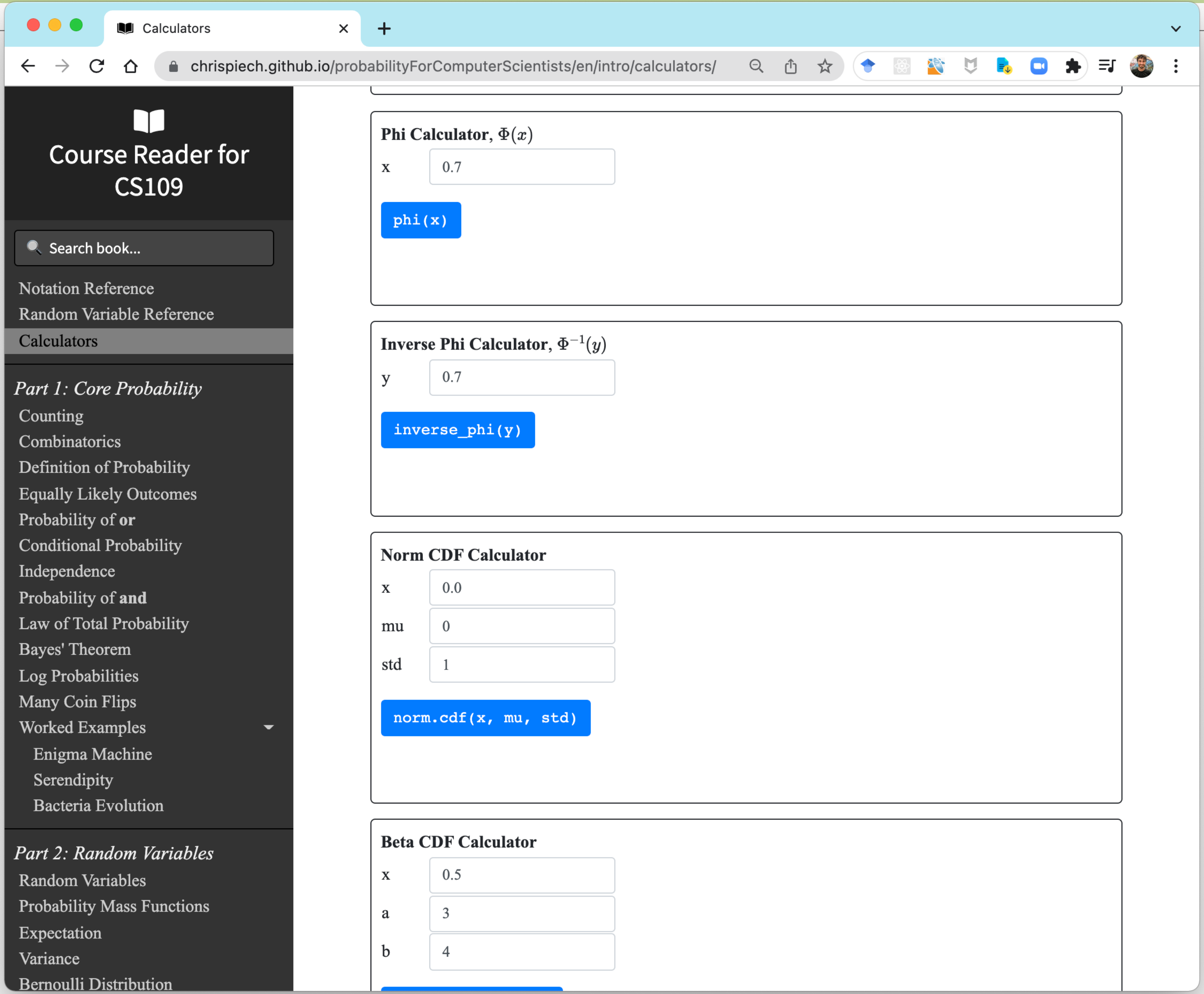
```
stats.norm.cdf(x, mean, std)
```

$$= P(X < x) \text{ where } X \sim \mathcal{N}(\mu, \sigma^2)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

* This is from Python's scipy library

We have one for you



Campus bikes

You spend some minutes, X, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X is normally distributed. What is the probability you spend ≥ 6 minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

✕ $P(X \geq 6) = \int_6^\infty f(x)dx$ (no analytic solution)

1. Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X \geq 6) = 1 - F_X(6)$$
$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right)$$
$$\approx 1 - \Phi(1.41)$$

2. Look up $\Phi(z)$ in table

$$1 - \Phi(1.41)$$
$$\approx 1 - 0.9207$$
$$= 0.0793$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$.
Note standard deviation $\sigma = 4$.

How would you write each of the below probabilities as a function of the standard normal CDF, Φ?

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

1. $P(X > 0)$ (we just did this)
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

Look up $\Phi(z)$ in table

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + \left(1 - F(9)\right)$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + (1 - F(9))$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Look up $\Phi(z)$ in table

$$= \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$= 2\left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$\approx 0.1337$$

Normal Approximation

Imagine you are taking a quiz…
With no computer!!!

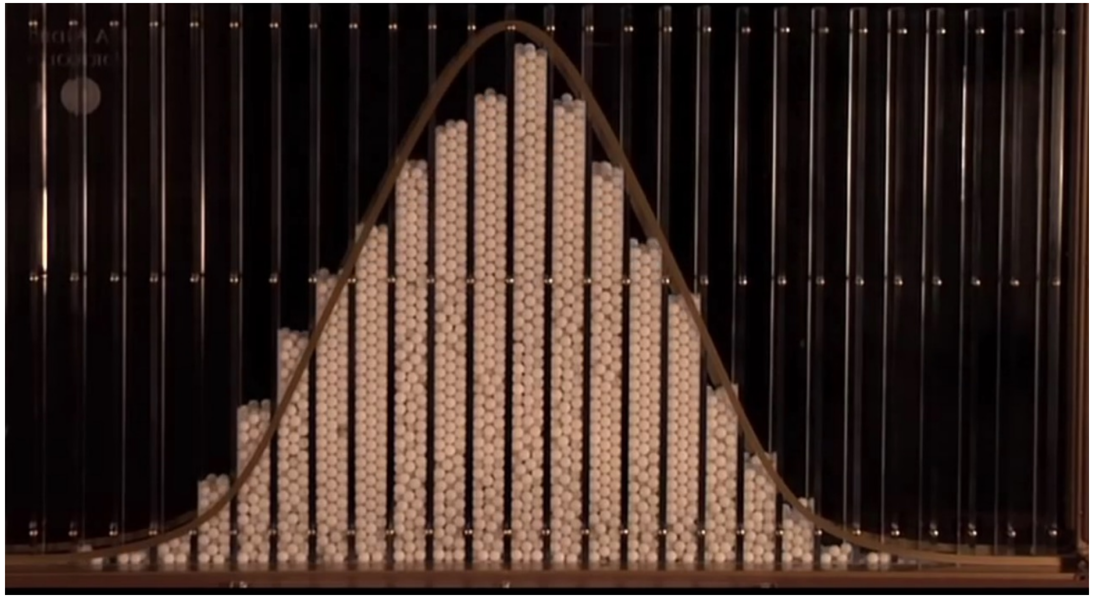
Website Testing

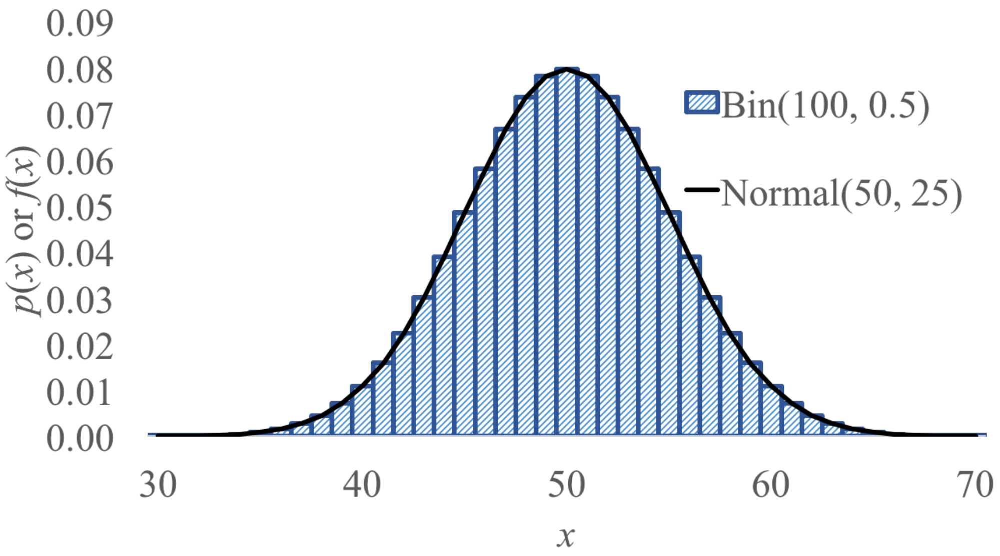
100 people are given a new website design

- X = # people whose time on site increases
- CEO will endorse new design if X ≥ 65 What is P(CEO endorses change| it has no effect)?
- X ~ Bin(100, 0.5). Want to calculate P(X ≥ 65)
- Give a numerical answer…

$$P(X \geq 65) = \sum_{i=65}^{100} \binom{100}{i} (0.5)^i (1-0.5)^{100-i}$$

Don’t worry, Normal approximates Binomial





Galton Board

(We’ll explain ***why*** in 2 weeks’ time)

Website testing

- 100 people are given a new website design.
- X = # people whose time on site increases
- The design actually has no effect, so P(time on site increases) = 0.5 independently.
- CEO will endorse the new design if $X \geq 65$.

What is P(CEO endorses change)? *Give a numerical approximation*.

Approach 1: Binomial

Define

$X \sim \text{Bin}(n = 100, p = 0.5)$

Want: $P(X \geq 65)$

Solve

$P(X \geq 65) \approx 0.0018$

Approach 2: approximate with Normal

Define

$Y \sim \mathcal{N}(\mu, \sigma^2)$

$\mu = np = 50$

$\sigma^2 = np(1-p) = 25$

$\sigma = \sqrt{25} = 5$

Solve

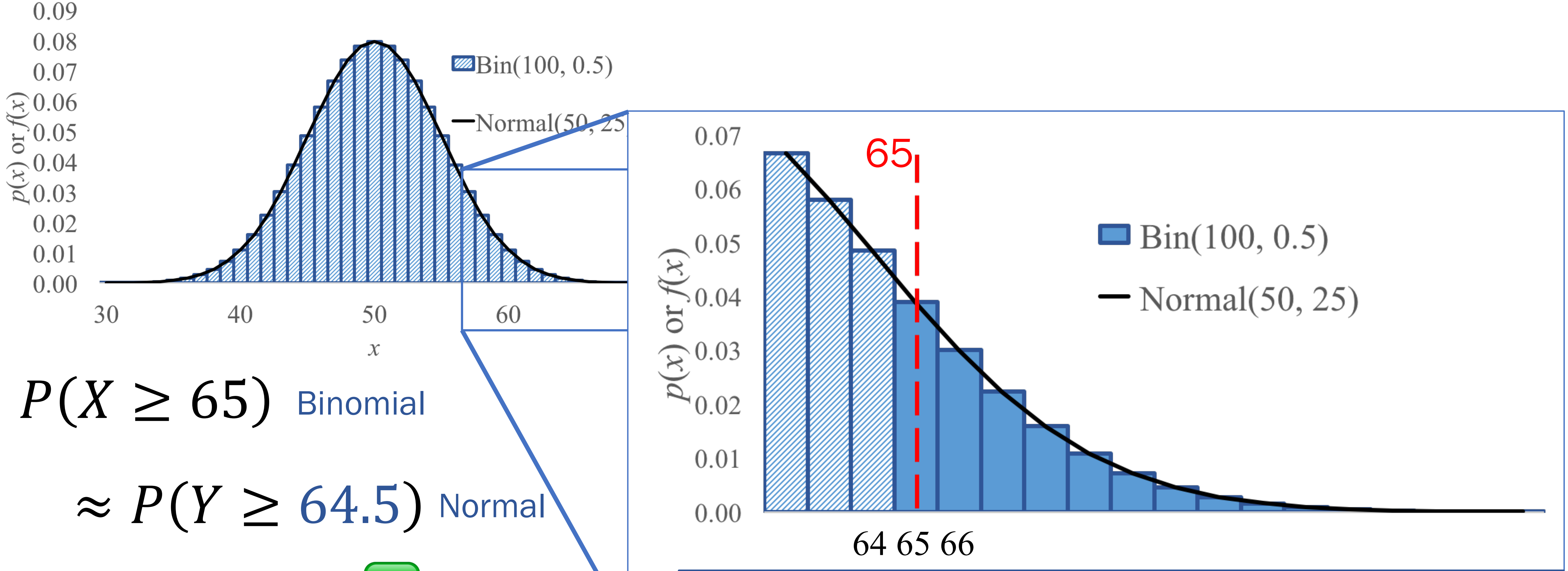
$$P(X \geq 65) \approx P(Y \geq 65) = 1 - F_Y(65)$$

$$= 1 - \Phi\left(\frac{65-50}{5}\right) = 1 - \Phi(3) \approx 0.0013 \text{ ?}$$

⚠️ ⚠️ (this approach is missing something important)

Website testing (with continuity correction)

In our website testing, $Y \sim \mathcal{N}(50, 25)$ approximates $X \sim \text{Bin}(100, 0.5)$.



$P(X \geq 65)$ Binomial

$\approx P(Y \geq 64.5)$ Normal

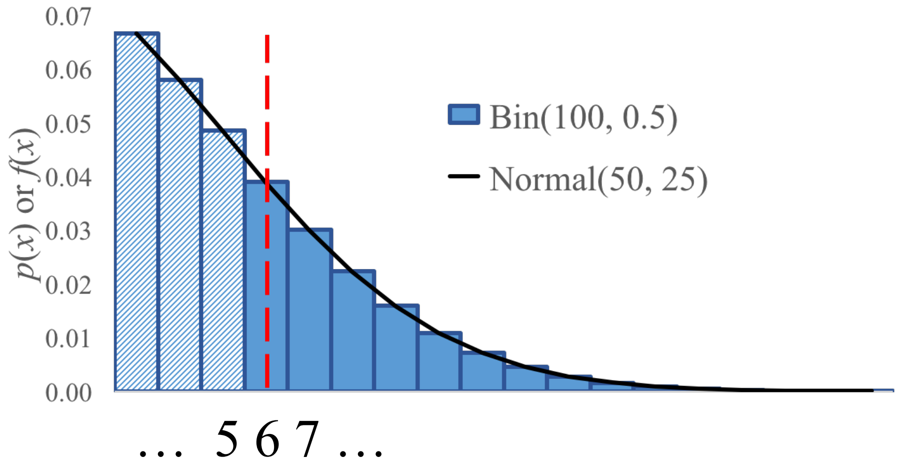
≈ 0.0018 ✅ the better Approach 2

You must perform a **continuity correction** when approximating a Binomial RV with a Normal RV.

Continuity correction

If $Y \sim \mathcal{N}(np, np(1-p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

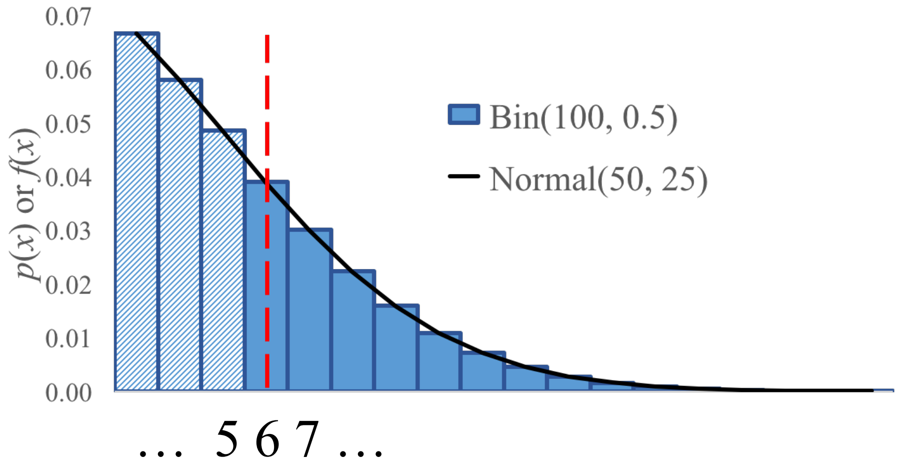
Discrete (e.g., Binomial) probability question	Continuous (Normal) probability question
$P(X = 6)$	
$P(X \geq 6)$	
$P(X > 6)$	
$P(X < 6)$	
$P(X \leq 6)$	



Continuity correction

If $Y \sim \mathcal{N}(np, np(1-p))$ approximates $X \sim \text{Bin}(n, p)$, how do we approximate the following probabilities?

Discrete (e.g., Binomial) probability question	Continuous (Normal) probability question
$P(X = 6)$	$P(5.5 \le Y \le 6.5)$
$P(X \ge 6)$	$P(Y \ge 5.5)$
$P(X > 6)$	$P(Y \ge 6.5)$
$P(X < 6)$	$P(Y \le 5.5)$
$P(X \le 6)$	$P(Y \le 6.5)$



Who gets to approximate?

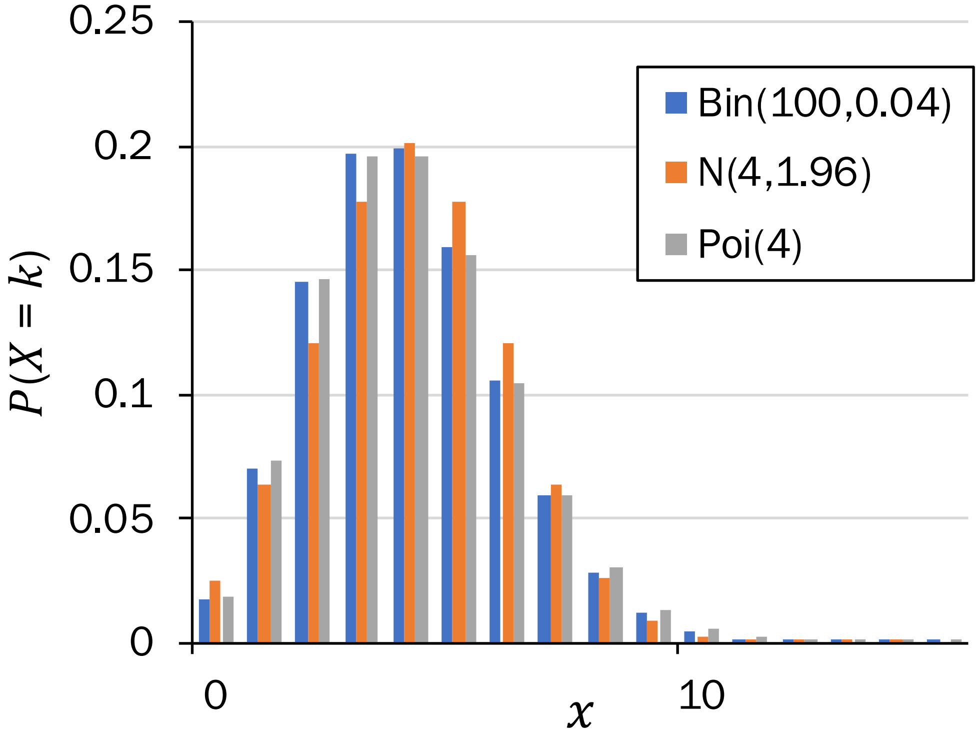
$$X \sim \text{Bin}(n, p)$$
$$E[X] = np$$
$$\text{Var}(X) = np(1-p)$$

$$Y \sim \text{Poi}(\lambda)$$
$$\lambda = np$$

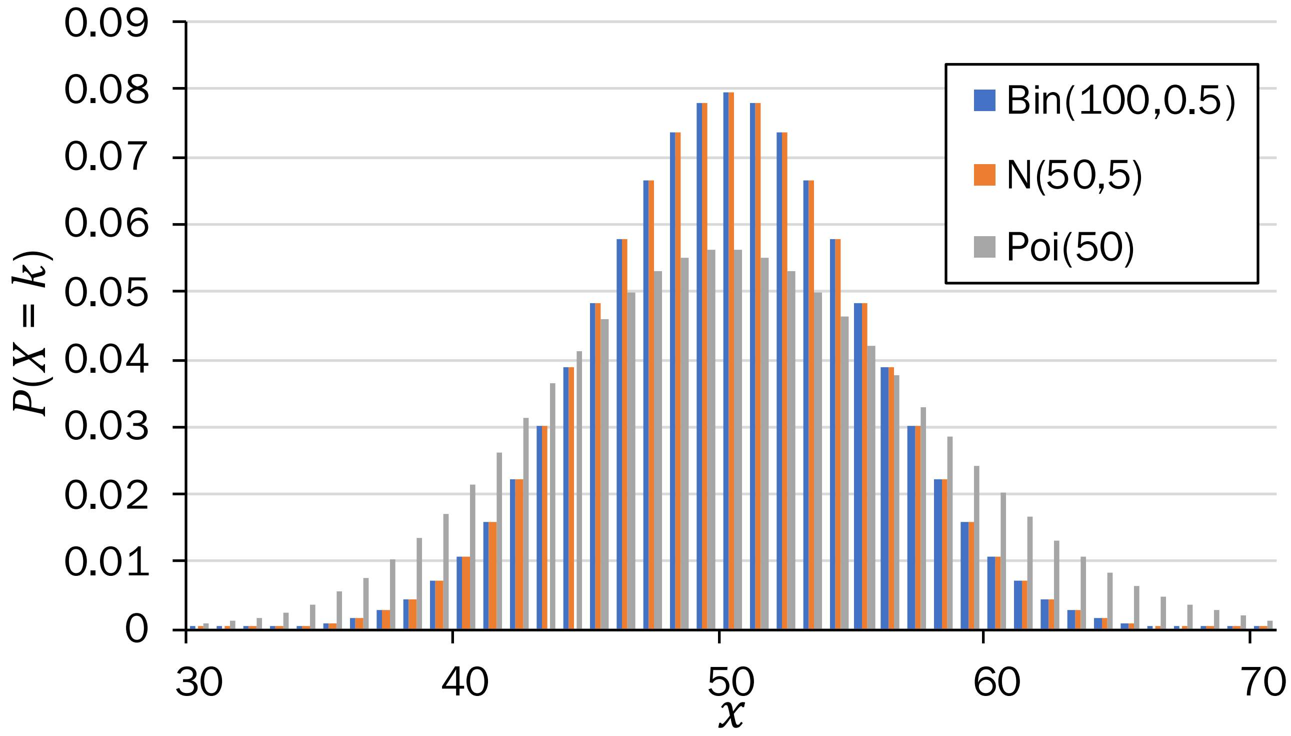
?

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mu = np$$
$$\sigma^2 = np(1-p)$$

Who gets to approximate?



Poisson approximation
n large (> 20), p small (< 0.05)
slight dependence okay



Normal approximation
n large (> 20), variance large ($np(1-p) > 10$)
independence

1. If there is a choice, either is fine.
2. When using Normal to approximate a discrete RV, use a continuity correction.

Stanford Admissions (a while back)

Stanford accepts 2480 students.

- Each admitted student matriculates w.p. 0.68 (independent trials)
- Let X = # of students who will attend

What is $P(X > 1745)$? *Give a numerical approximation.*

Strategy:

A. Just Binomial
B. Poisson
C. Normal
D. None/other

(by yourself)

Stanford Admissions

Stanford accepts 2480 students.

- Each admitted student matriculates w.p. 0.68 (independent trials)
- Let X = # of students who will attend

What is $P(X > 1745)$? *Give a numerical approximation.*

Strategy:

A. Just Binomial — not an approximation (also computationally expensive)
B. Poisson — $p = 0.68$, not small enough
C. Normal — ✅ Variance $np(1-p) = 540 > 10$
D. None/other

Define an approximation

Let $Y \sim \mathcal{N}(E[X], \text{Var}(X))$

$E[X] = np = 1686$

$\text{Var}(X) = np(1-p) \approx 540 \rightarrow \sigma = 23.3$

$P(X > 1745) \approx P(Y \geq 1745.5)$ ⚠️ Continuity correction

Solve

SciPy can do this

$$P(Y \geq 1745.5) = 1 - F(1745.5)$$
$$= 1 - \Phi\left(\frac{1745.5 - 1686}{23.3}\right)$$
$$= 1 - \Phi(2.54) \approx 0.0055$$

How many students should Stanford admit?

The Stanford Daily

NEWS · SPORTS · OPINIONS · ARTS & LIFE · THE GRIND · MULTIMEDIA · FEATURES · ARCHIVES

Class of 2018 admit rates lowest in University history

March 28, 2014 16 Comments Tweet Like 901

Alex Zivkovic
Desk Editor

Stanford admitted 2,138 students to the Class of 2018 in this year's admissions cycle, producing – at 5.07 percent – the lowest admit rate in University history.

The University received a total of 42,167 applications this year, a record total and a 8.6 percent increase over last year's figure of 38,828. Stanford accepted 748 students

Admit rate: 4.3%

Yield rate: 81.9%

68%

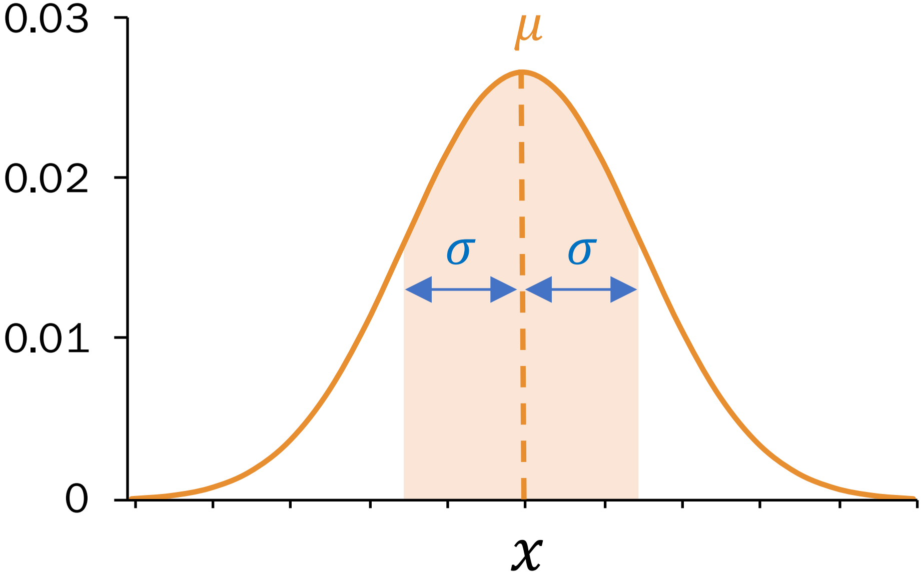
Why Be Normal? 68% rule

You may have heard the statement:

"68% of the class will fall within 1 standard deviation of the exam average."

In general, this is only true of normal distributions:

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF F.



$$P(|X-\mu| < \sigma) = P(\mu - \sigma < X < \mu + \sigma)$$
$$= F(\mu+\sigma) - F(\mu-\sigma)$$
$$= \Phi\left(\frac{(\mu+\sigma)-\mu}{\sigma}\right) - \Phi\left(\frac{(\mu-\sigma)-\mu}{\sigma}\right)$$
$$= \Phi(1) - \Phi(-1) = \Phi(1) - \big(1 - \Phi(1)\big)$$
$$= 2\Phi(1) - 1 \quad \approx 2(0.8413) - 1 = 0.6826$$

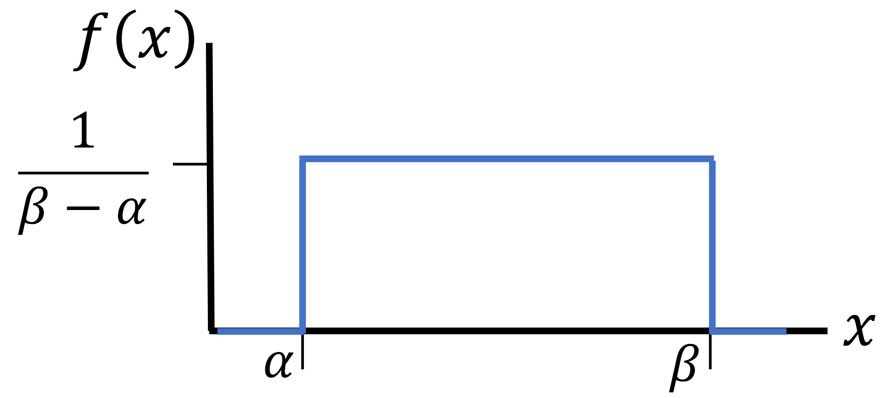
Why Be Normal? 68% rule

You may have heard the statement:

"68% of the class will fall within 1 standard deviation of the exam average."

In general, this is only true of normal distributions:

Counterexample: Let $X \sim \text{Uni}(\alpha, \beta)$.



$$\mu = E[X] = \frac{\alpha + \beta}{2}$$

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12} \Rightarrow \sigma = \text{SD}(X) = \frac{\beta - \alpha}{\sqrt{12}}$$

$$P(|X - \mu| < \sigma) = P(\mu - \sigma < X < \mu + \sigma)$$

$$= \frac{1}{\beta - \alpha} \cdot [(\mu + \sigma) - (\mu - \sigma)]$$

$$= \frac{1}{\beta - \alpha}[2\sigma] = \frac{1}{\beta - \alpha} \cdot \left[2 \cdot \frac{\beta - \alpha}{\sqrt{12}}\right]$$

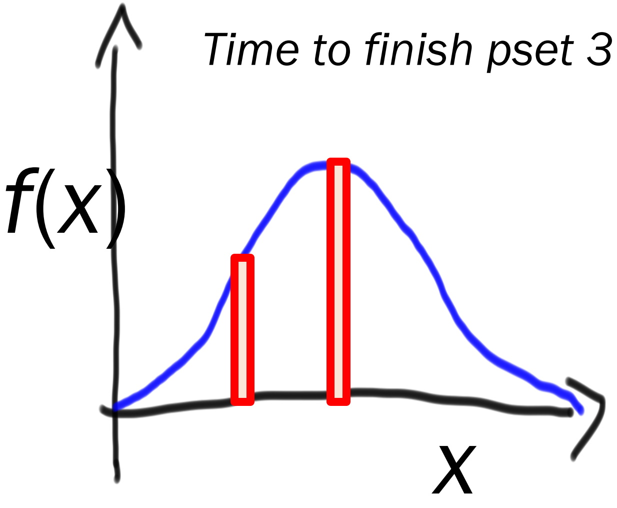
$$= 2/\sqrt{12} \approx 0.58$$

Relative values of a PDF

Relative Probability of Continuous Variables

X = time to finish pset 3

X ~ N(μ = 10, σ^2 = 2)



How much more likely are you to complete in 10 hours than in 5?

$$\frac{P(X=10)}{P(X=5)} = \frac{\varepsilon f(X=10)}{\varepsilon f(X=5)}$$

$$= \frac{f(X=10)}{f(X=5)}$$

$$= \frac{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}}$$

$$= \frac{\frac{1}{\sqrt{4\pi}} e^{-\frac{(10-10)^2}{4}}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{(5-10)^2}{4}}}$$

$$= \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

Fin!