Logistic Regression

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Slides by Chris Piech
CS109, Stanford University

The Last CS 109 Lecture!

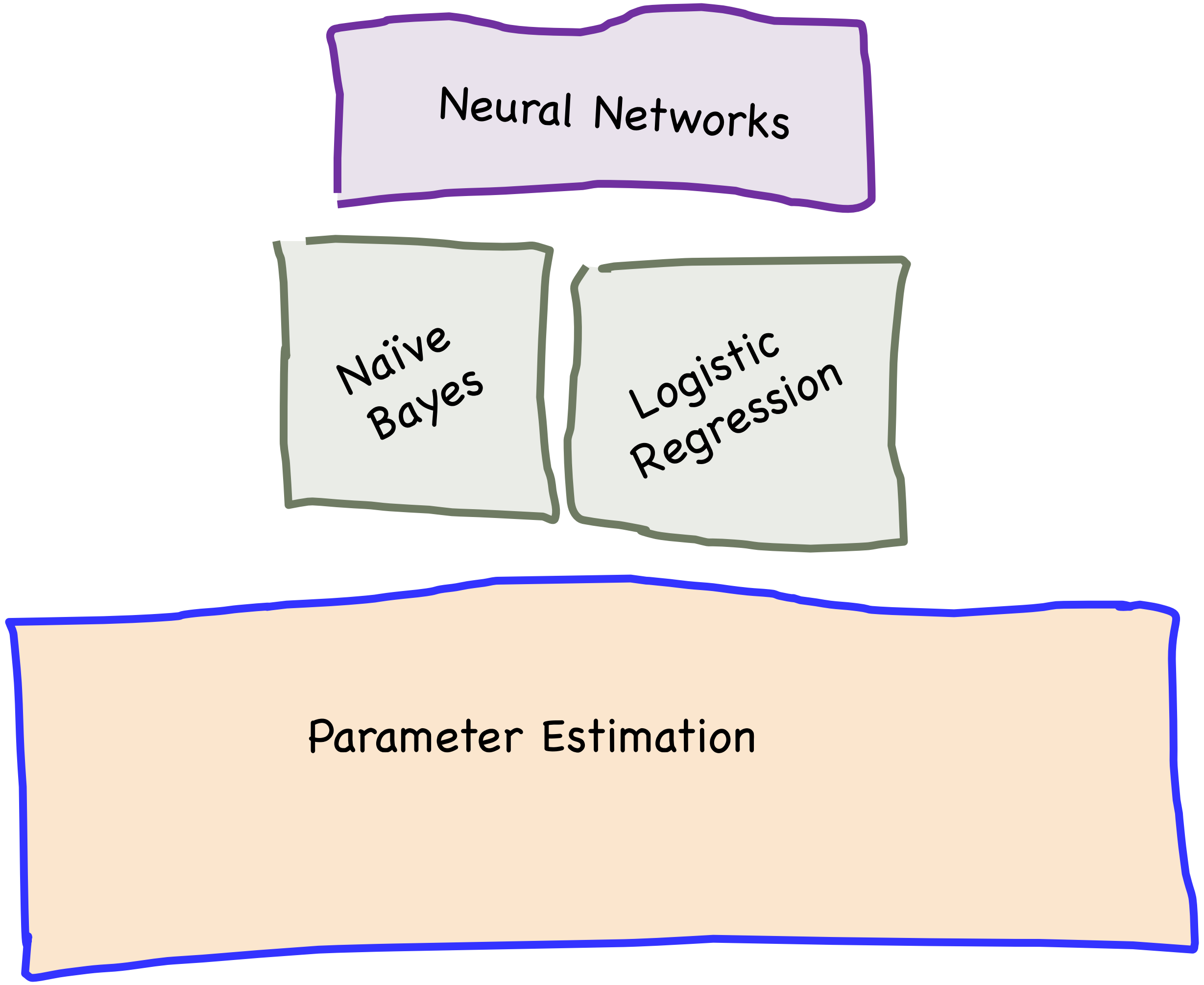
Review

Machine Learning

Great Idea

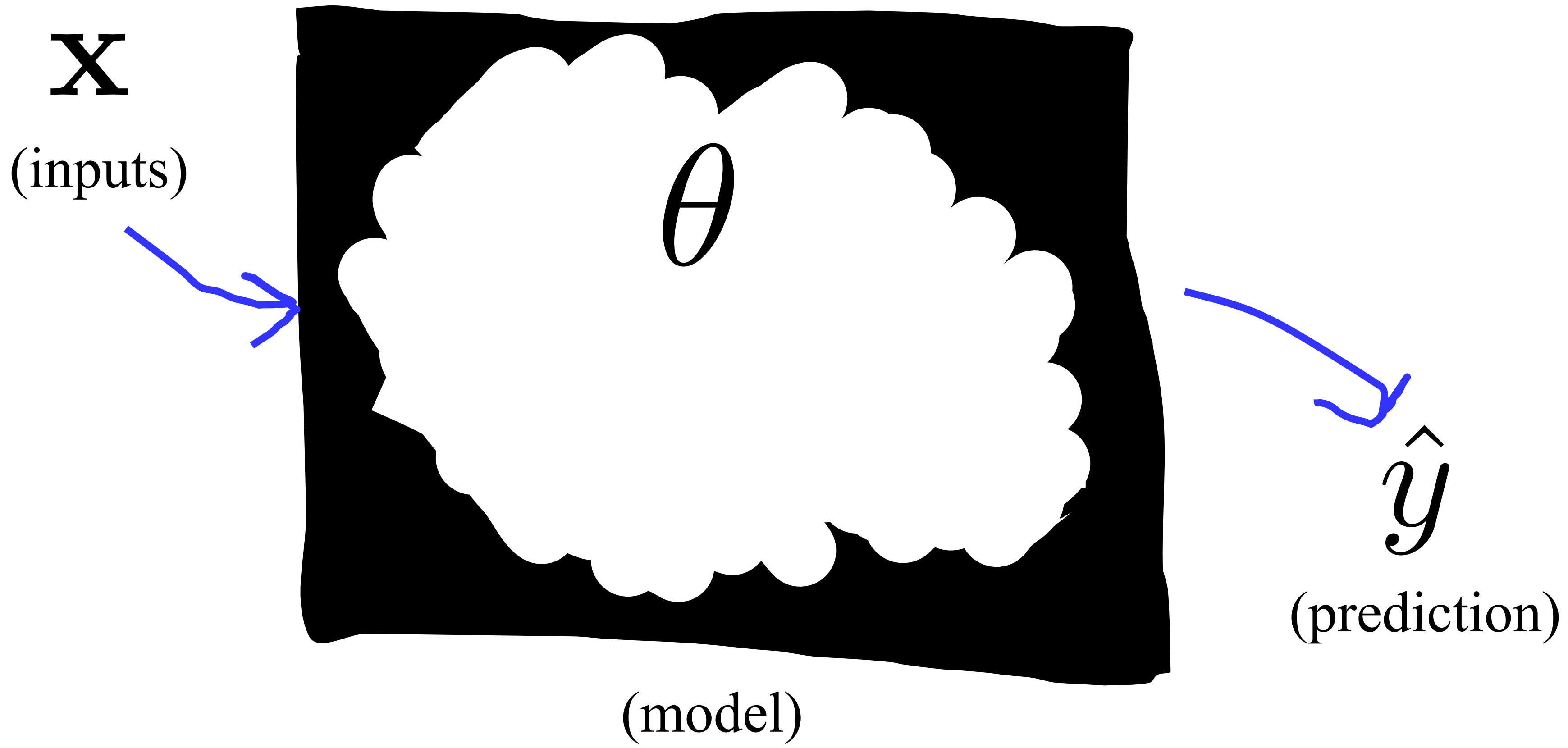
Classification
Algorithms

Theory

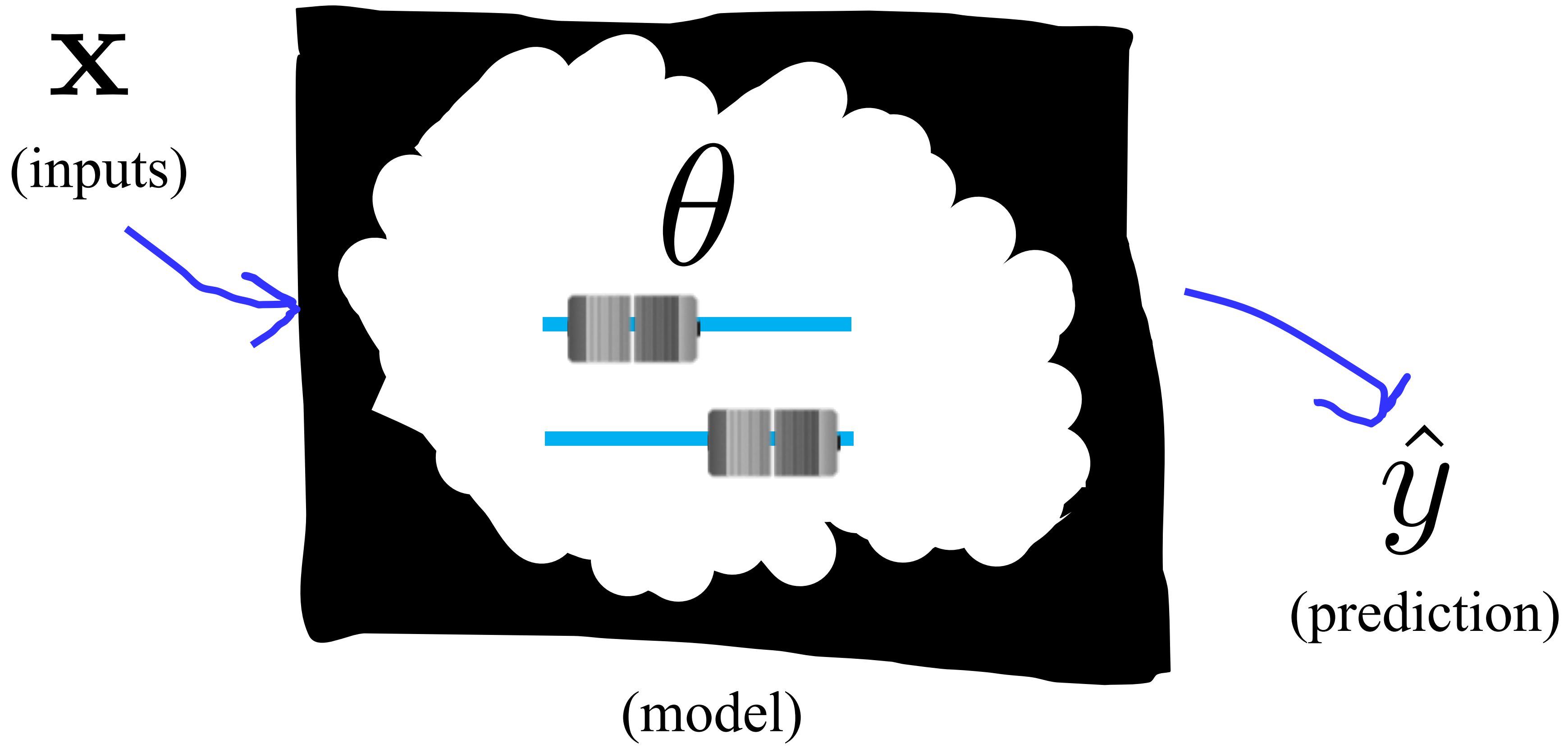


Machine Learning
(aka Applied Probability)

Machine Learning



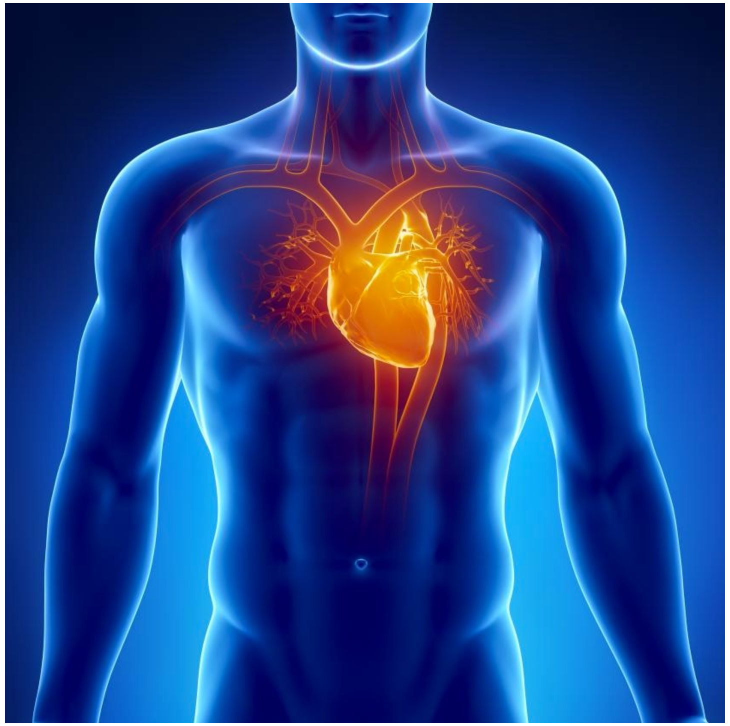
Machine Learning



Classification

Classification Task

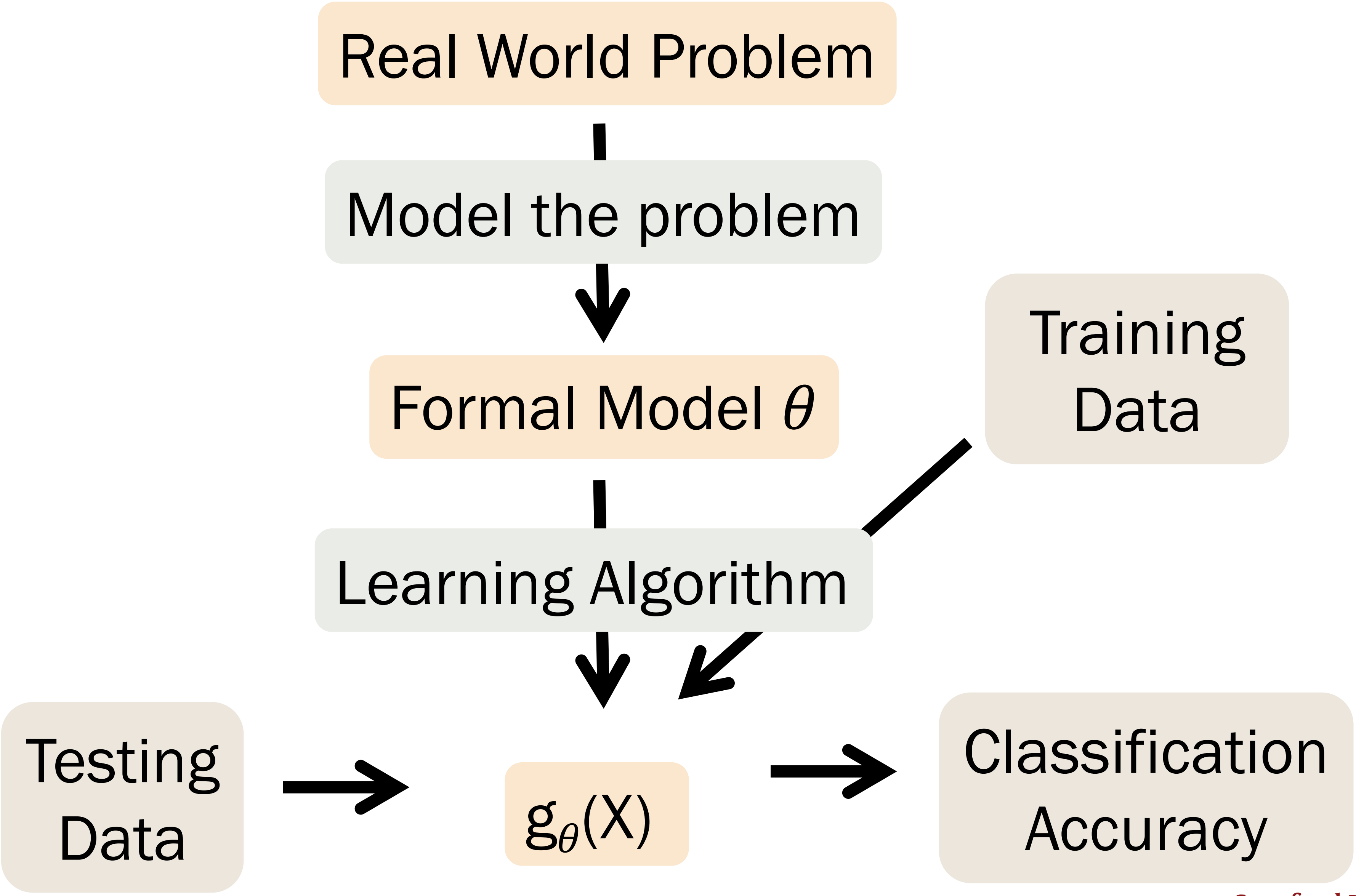
Heart



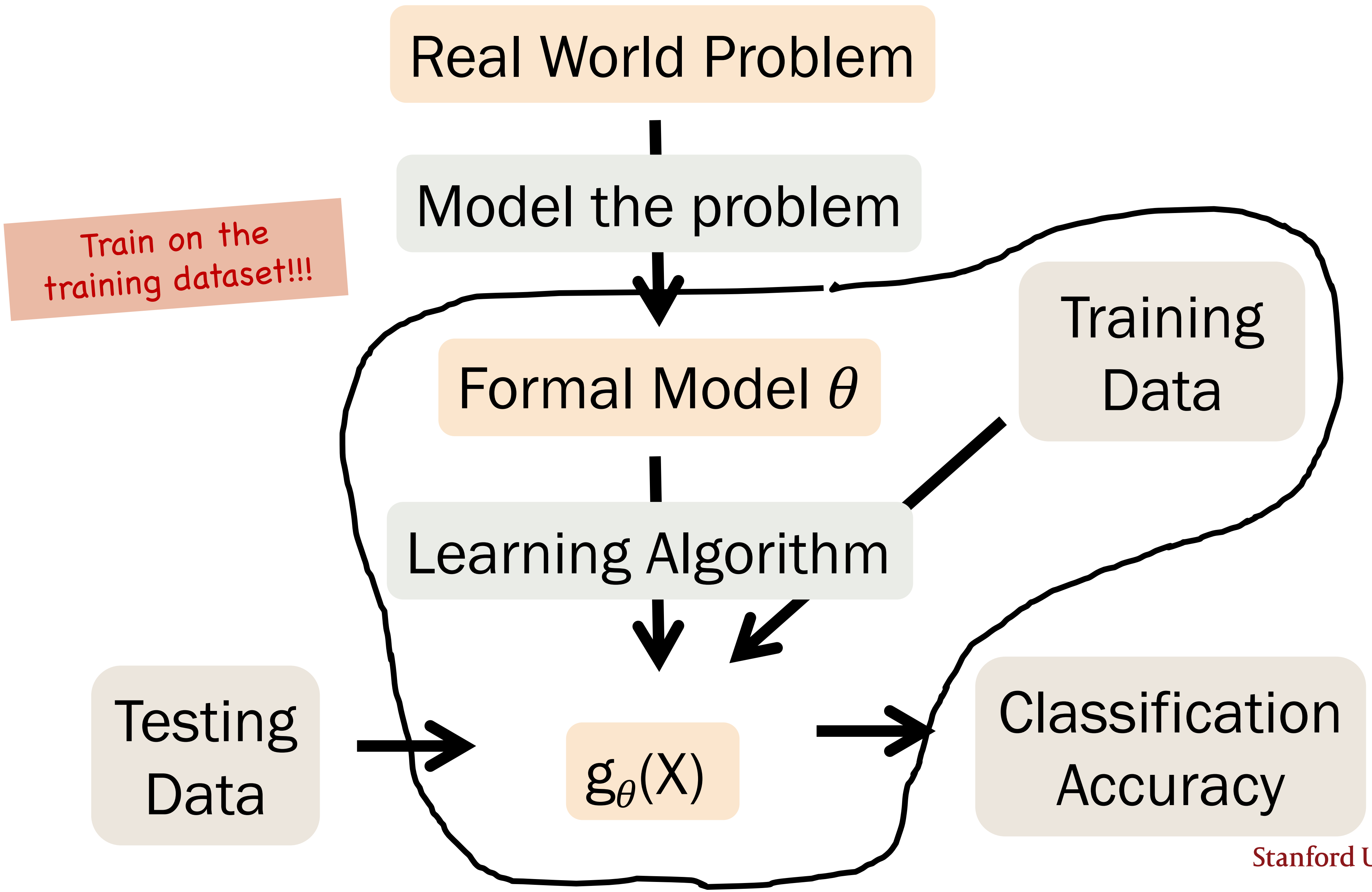
Ancestry

Netflix

Probability in Life!

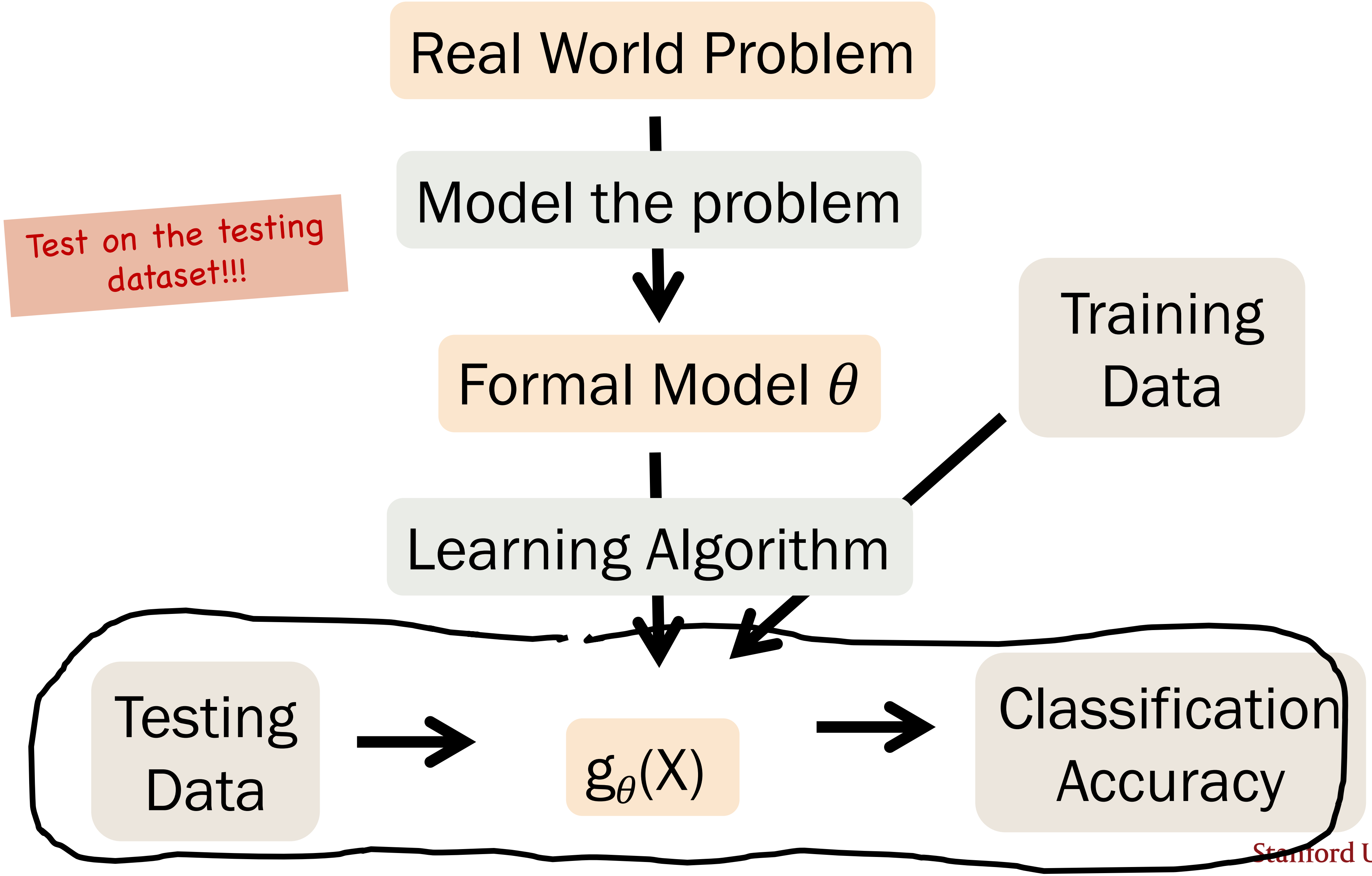


Training



Real World Problem
Model the problem
Train on the training dataset!!!
Formal Model θ
Training Data
Learning Algorithm
Testing Data
$g_\theta(X)$
Classification Accuracy

Testing



Real World Problem
Model the problem
Test on the testing dataset!!!
Formal Model θ
Training Data
Learning Algorithm
Testing Data
$g_\theta(X)$
Classification Accuracy

Training Data

Assume IID data:

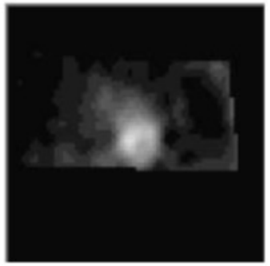
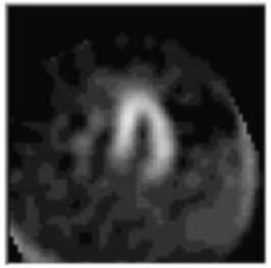
n training datapoints

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \dots (\mathbf{x}^{(n)}, y^{(n)})$$

$$m = |\mathbf{x}^{(i)}|$$

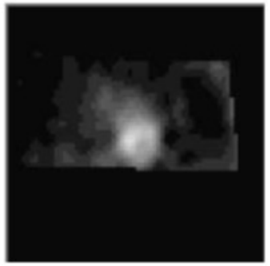
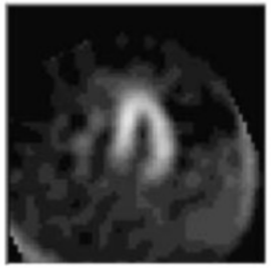
Each datapoint has m features and a single output

Training: Heart Disease Classifier

	ROI 1	ROI 2		ROI m	Output
			...		
Heart 1	0	1		1	0
Heart 2	1	1		1	0
			⋮		⋮
Heart n	0	0		0	1

$g_\theta(X)$

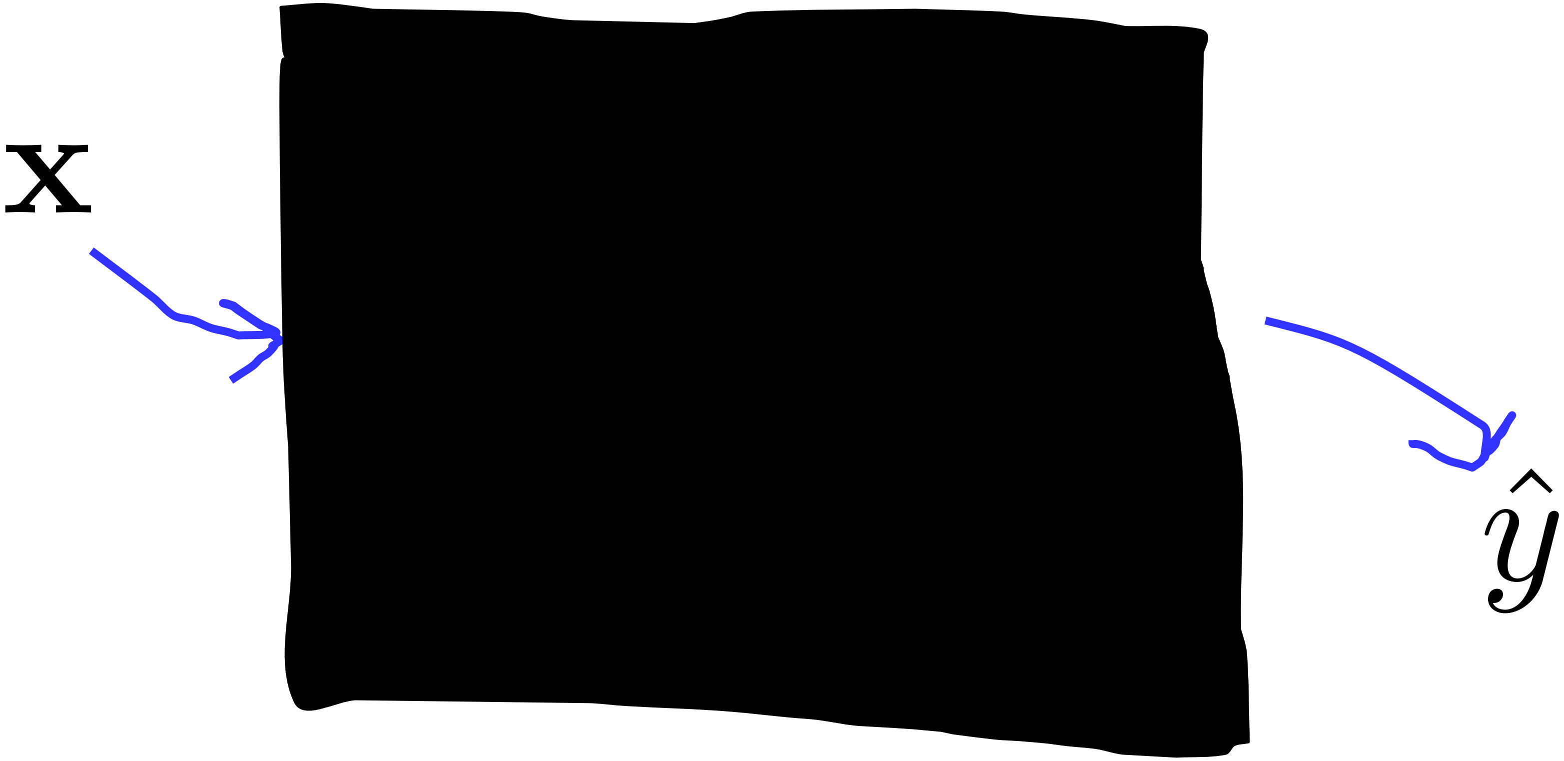
Testing: Heart Disease Classifier

	ROI 1	ROI 2		ROI m	Output
			…		
New Heart	1	0		1	$\boxed{1}$

$g_\theta(X)$

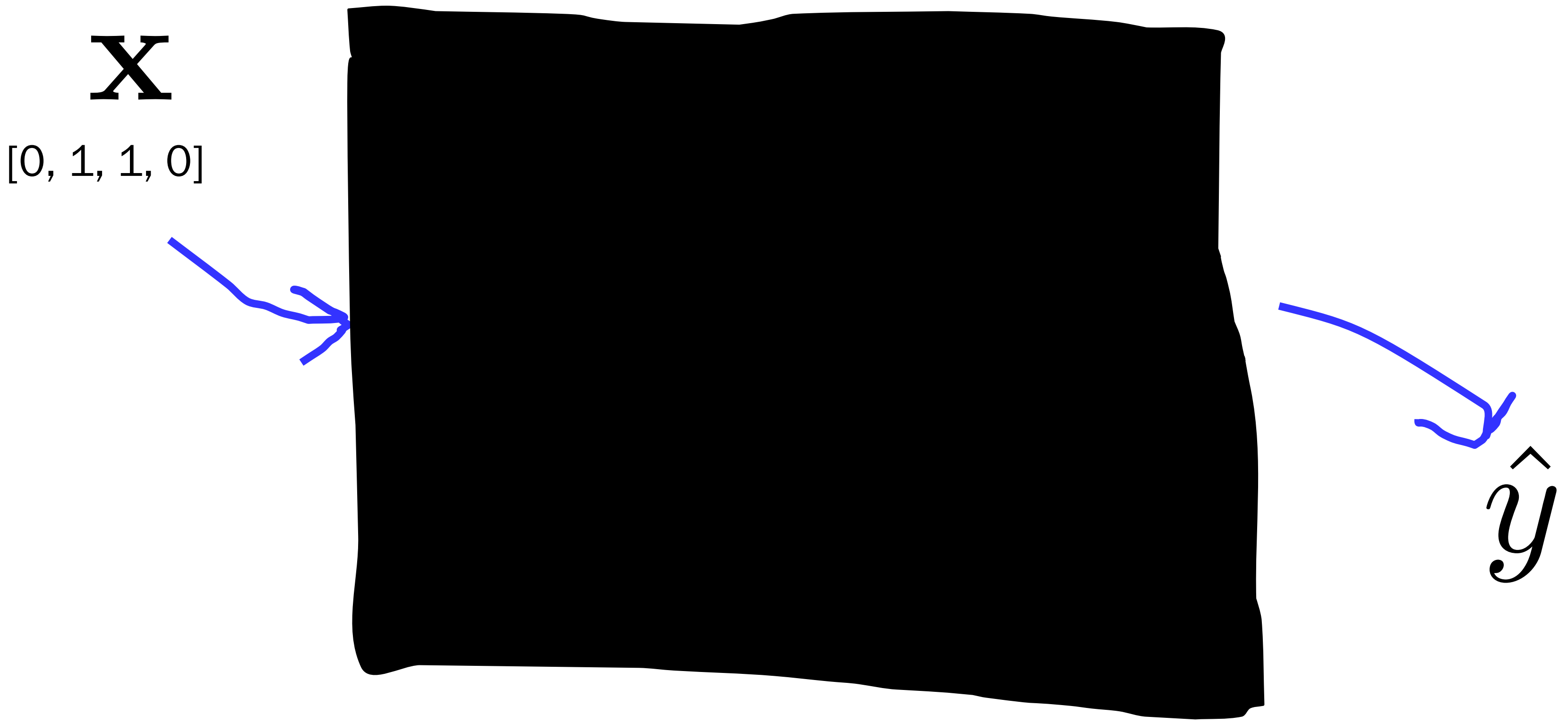
Naïve Bayes Classification

$g_\theta(\mathbf{x})$?



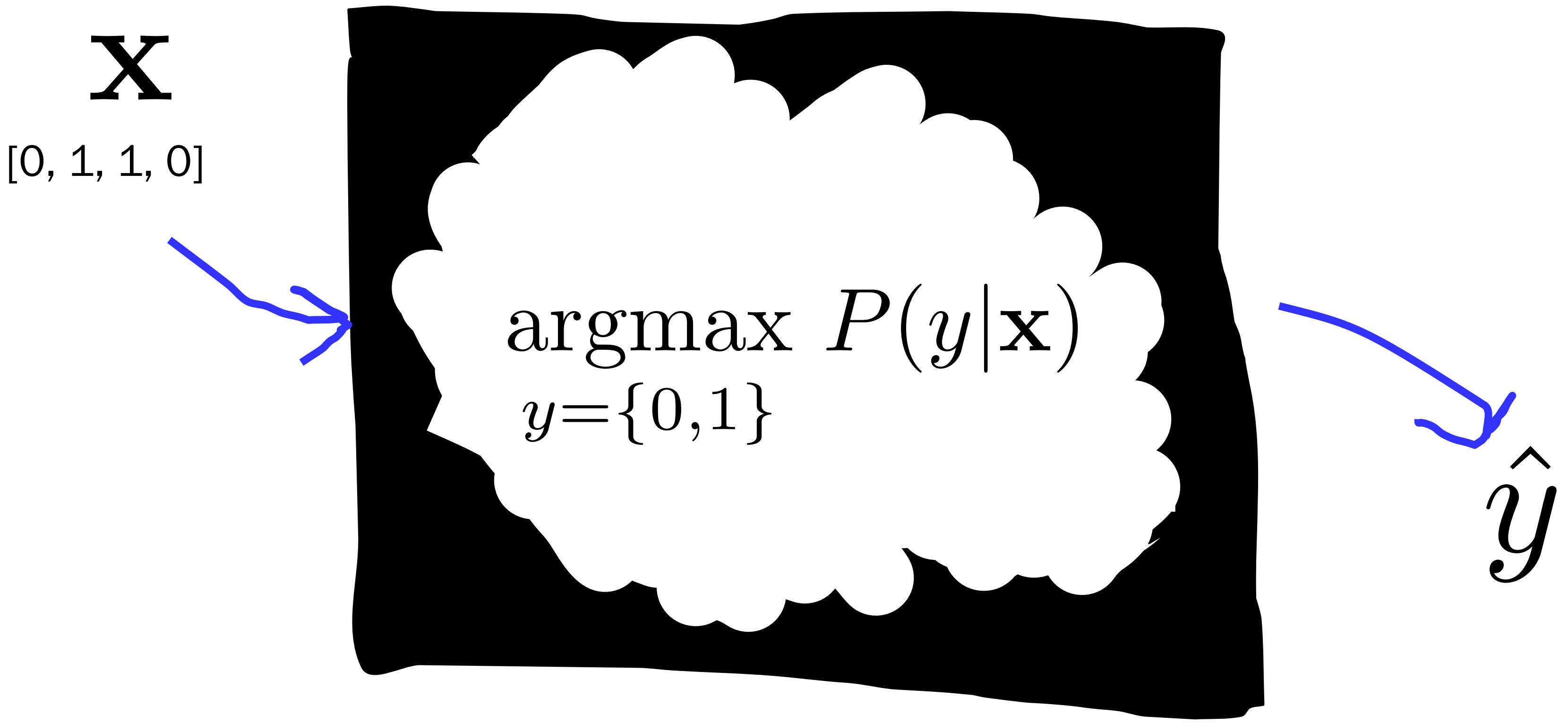
Making a prediction…

$g_\theta(\mathbf{x})$?



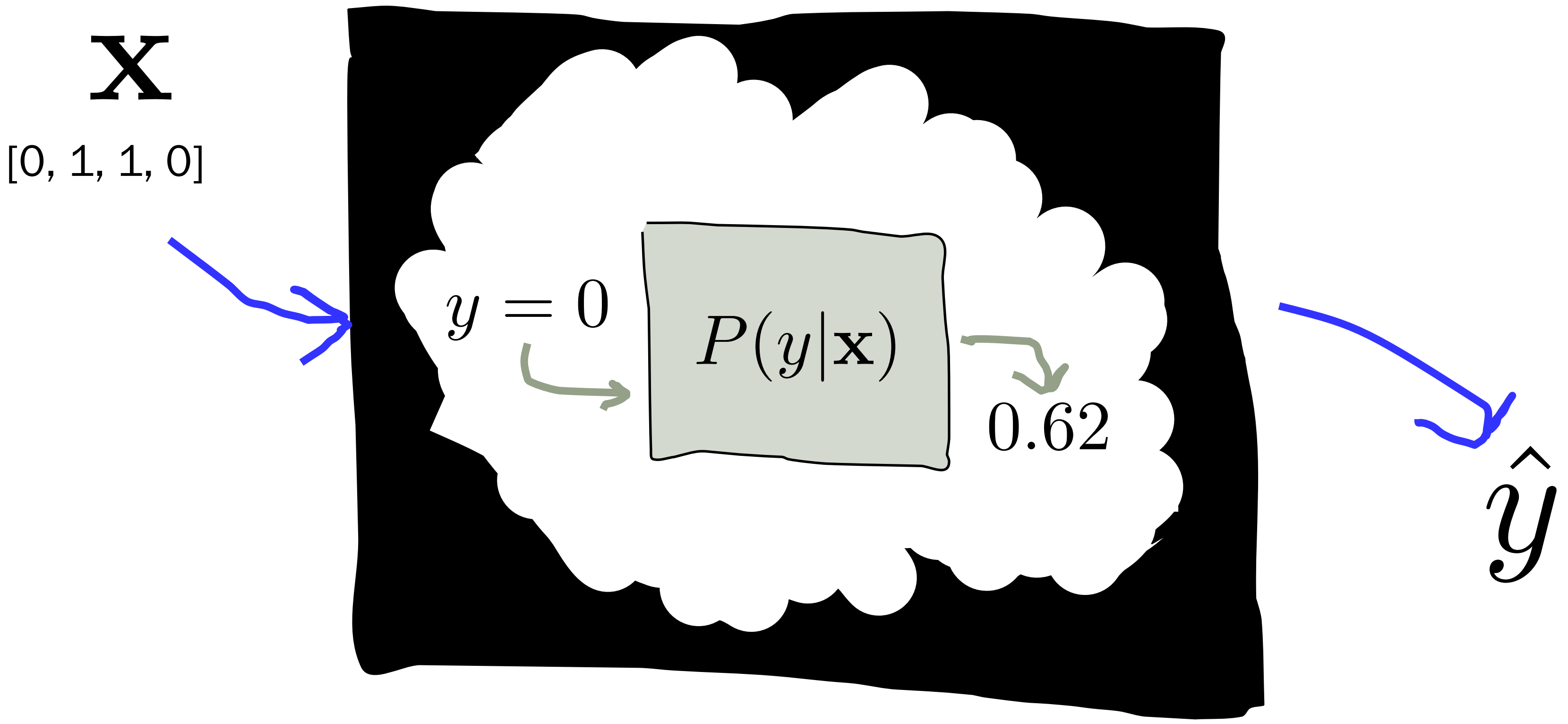
Making a prediction…

$g_\theta(\mathbf{x})$?



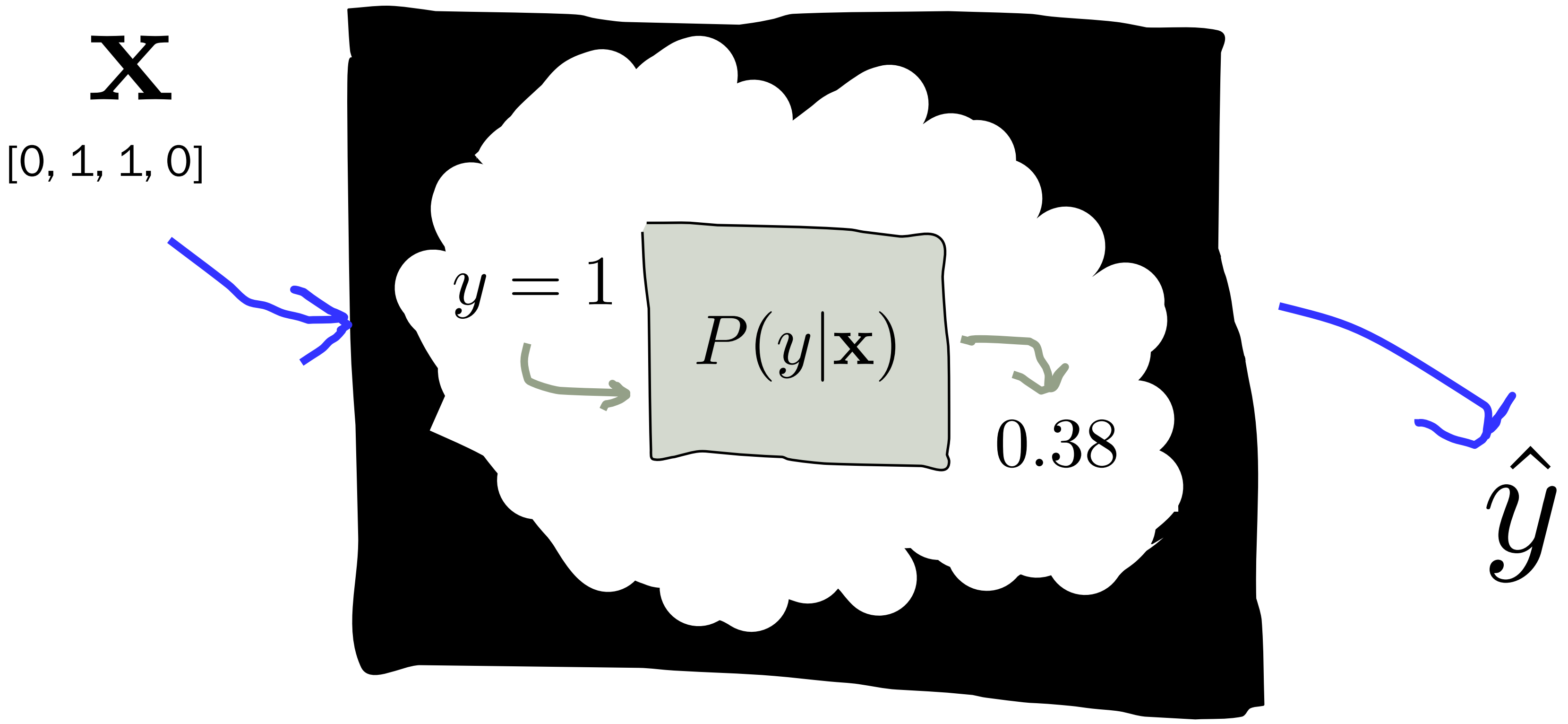
Making a prediction…

$g_\theta(\mathbf{x})$?



Making a prediction…

$g_\theta(\mathbf{x})$?

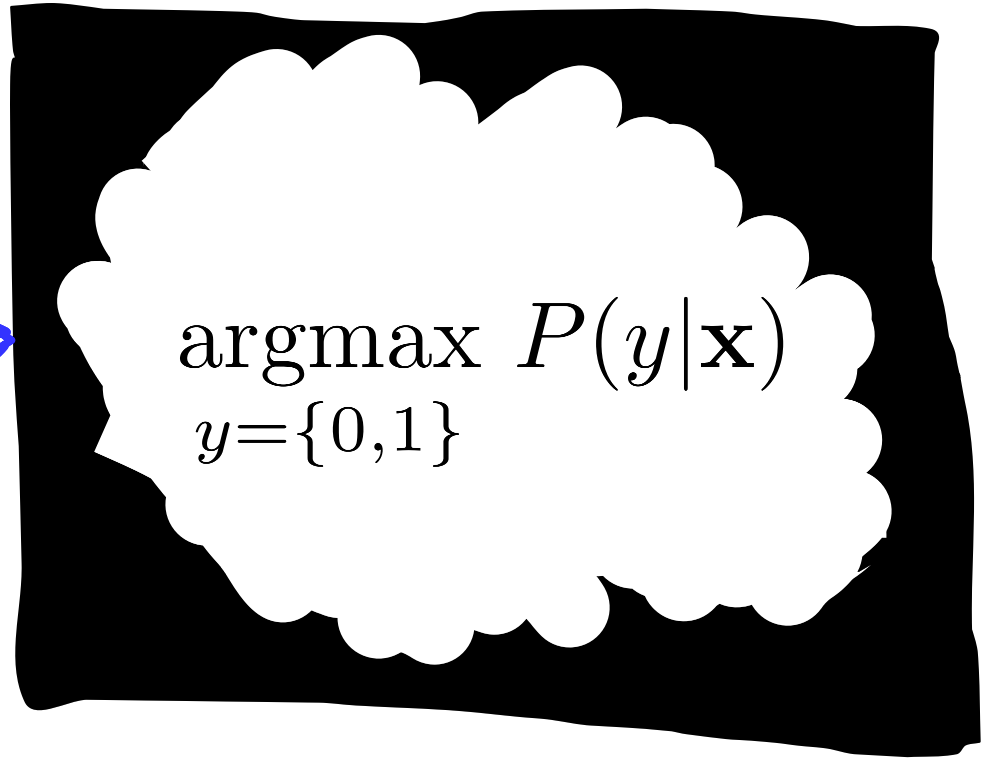


Making a prediction…

$g_\theta(\mathbf{x})$?

$\mathbf{x}$

[0, 1, 1, 0]



$$\underset{y=\{0,1\}}{\text{argmax}}\; P(y|\mathbf{x})$$

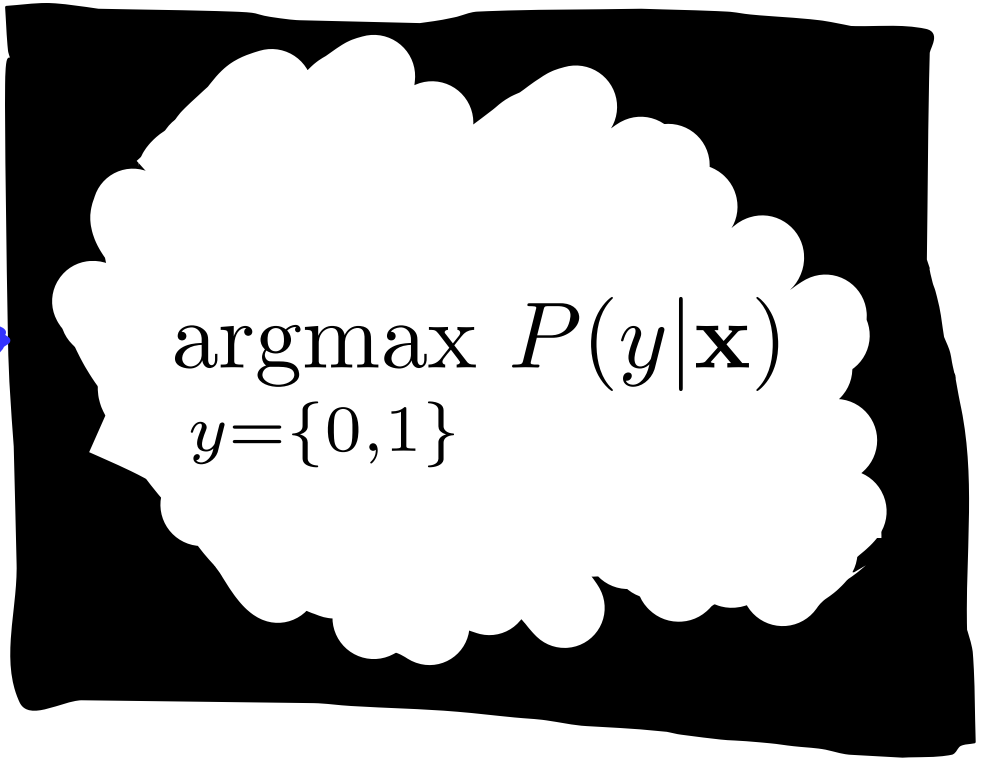
$\hat{y}$

Making a prediction…

$g_\theta(\mathbf{x})$?

$\mathbf{x}$

[0, 1, 1, 0]



$\hat{y} = 0$

Making a prediction…

Big Assumption

Naïve Bayes Assumption:

$$P(\mathbf{x}|y) = \prod_i P(x_i|y)$$

Prediction

Simply chose the class label that is the most likely given the data. Make Naïve Bayes assumption

$$\hat{y} = \arg\max_{y=\{0,1\}} P(Y = y | \mathbf{X} = \mathbf{x})$$

$$= \arg\max_{y=\{0,1\}} \frac{P(Y = y)P(\mathbf{X} = \mathbf{x} | Y = y)}{P(\mathbf{X} = \mathbf{x})}$$

$$= \arg\max_{y=\{0,1\}} P(Y = y)P(\mathbf{X} = \mathbf{x} | Y = y)$$

$$= \arg\max_{y=\{0,1\}} P(Y = y) \prod_i P(X_i = x_i | Y = y)$$

$$= \arg\max_{y=\{0,1\}} \log P(Y = y) + \sum_i \log P(X_i = x_i | Y = y)$$

Must learn params for this

Must learn params for this

Learning Probabilities from Data

Various probabilities you will need to compute for Naive Bayesian Classifier (using **MLE** here):

$$\hat{p}(X_i = 1 | Y = 0) = \frac{(\#\ \text{training examples where } X_i = 1 \text{ and } Y = 0)}{(\#\ \text{training examples where } Y = 0)}$$

$$\hat{p}(Y = 1) = \frac{(\#\ \text{training examples where } Y = 1)}{(\#\ \text{training examples})}$$

Learning Probabilities from Data

Various probabilities you will need to compute for Naive Bayesian Classifier (using **MAP with Laplace prior** here):

$$\hat{p}(X_i = 1 | Y = 0) = \frac{(\#\text{ training examples where } X_i = 1 \text{ and } Y = 0) + 1}{(\#\text{ training examples where } Y = 0) + 2}$$

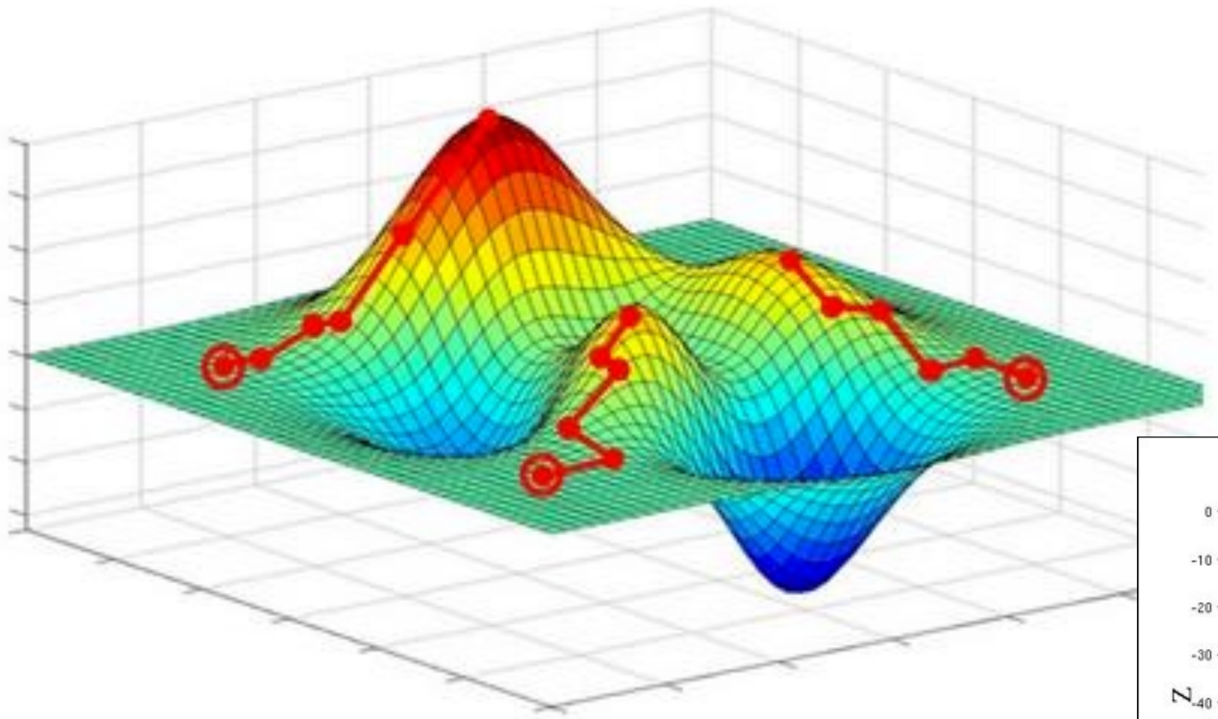
$$\hat{p}(Y = 1) = \frac{(\#\text{ training examples where } Y = 1) + 1}{(\#\text{ training examples}) + 2}$$

Training Naïve Bayes, is estimating parameters for a multinomial (or bernoulli).

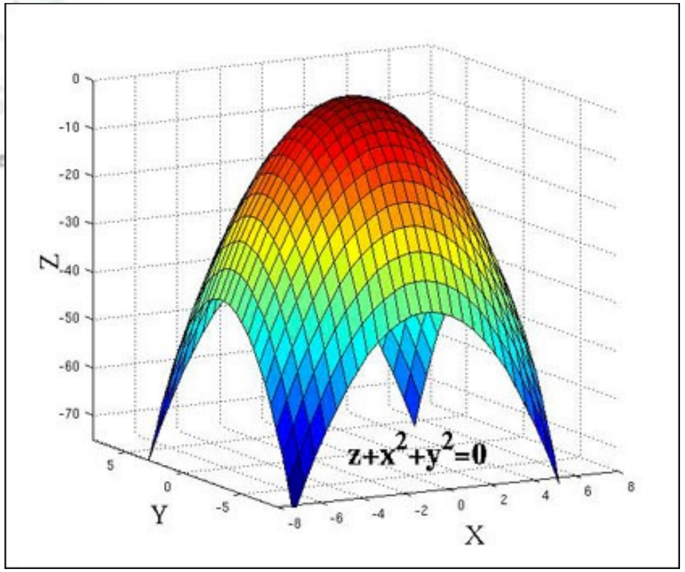
Thus training is just counting.

Optimization

Gradient Ascent



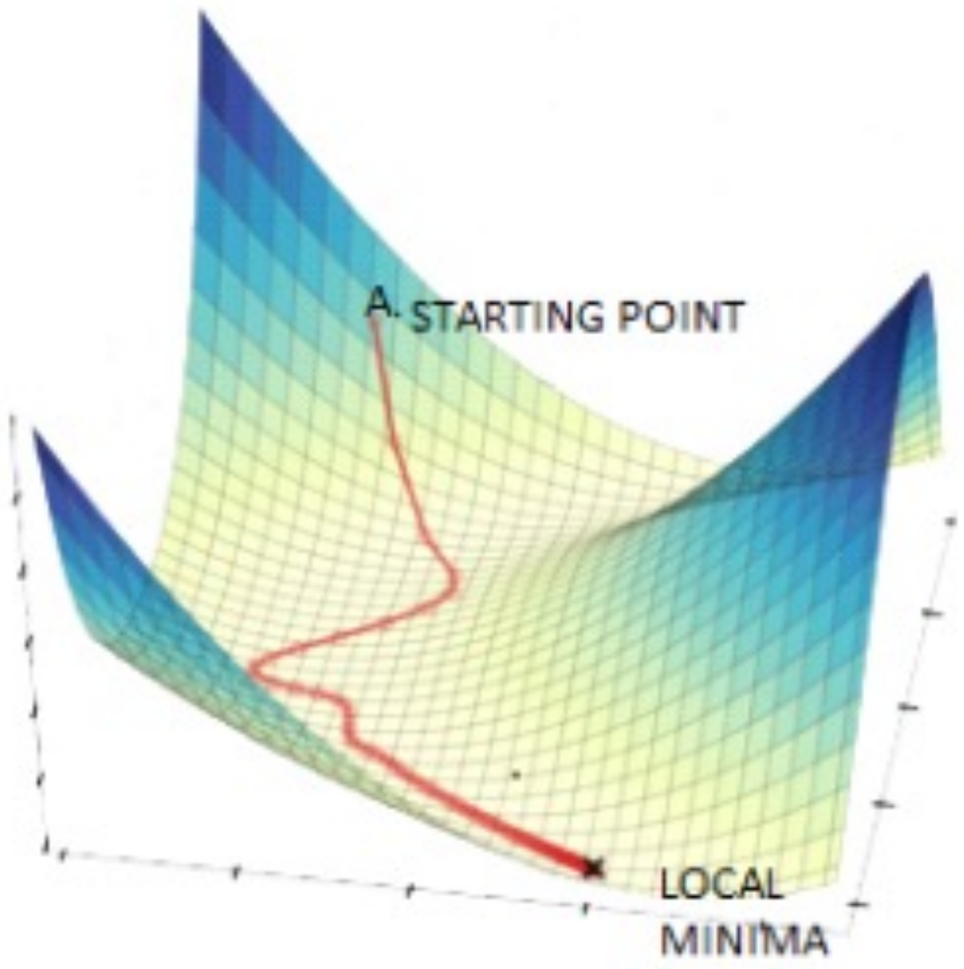
Logistic regression
LL function is
convex



Walk uphill and you will find a local maxima
(if your step size is small enough)

Gradient descent is your bread and butter algorithm for optimization (eg argmax)

Gradient Decent



Walk downhill and you will find a local maxima
(if your step size is small enough)

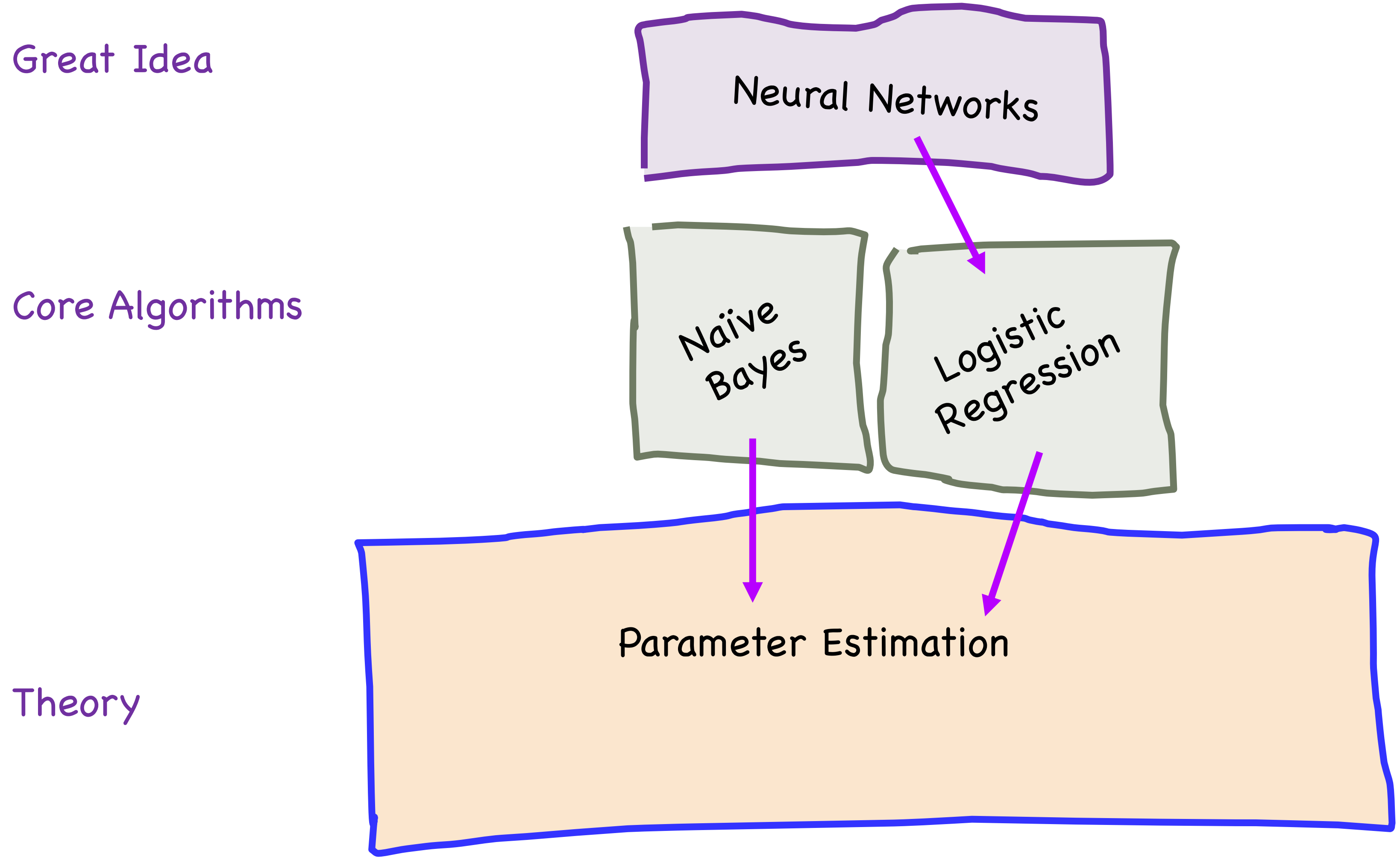
If someone gives you a gradient descent package, you should minimize **negative** log likelihood.

If you are writing optimization yourself, feel free to gradient **ascent** on log likelihood :-)

End Review

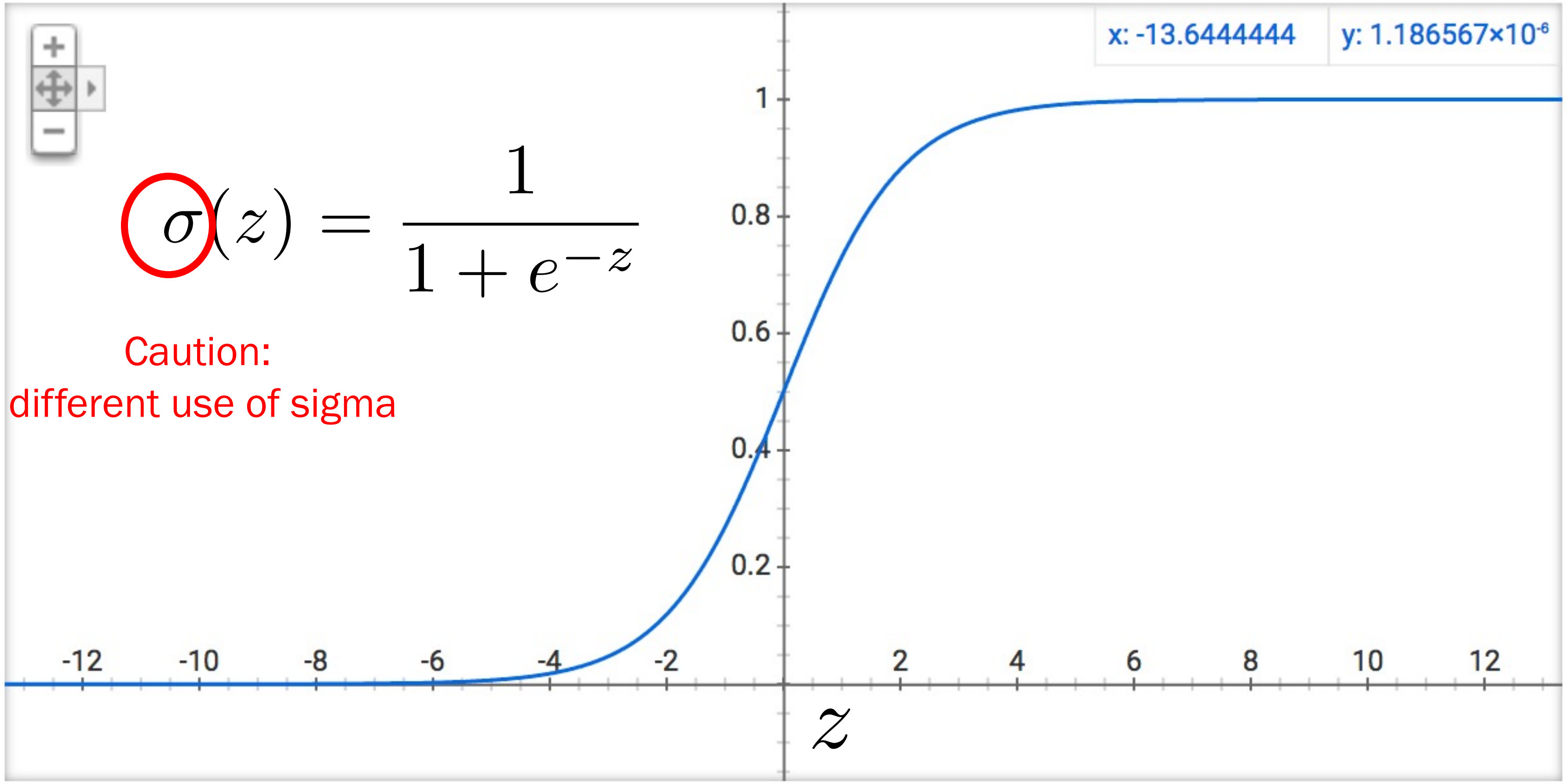
Logistic Regression

Machine Learning *Dependencies*



Chapter 0: Background

Background: Sigmoid Function



$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Caution:
different use of sigma

The sigmoid function squashes z to be a number between 0 and 1

Bigger and Bigger Questions

I want something in [0,1]

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

Sigmoid function

I want params for my data

$$\theta^T \mathbf{x} = \sum_{i=1}^{n} \theta_i x_i$$
$$= \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$$

Weighted sum
(aka dot product)

I want the probability of my data

$$\sigma(\theta^T \mathbf{x}) = \frac{1}{1+e^{-\theta^T \mathbf{x}}}$$

Sigmoid function of weighted sum

Background: Chain Rule

Who knew calculus would be so useful?

$$\frac{\partial f(x)}{\partial x} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x}$$

Aka decomposition of composed functions

$$f(x) = f(z(x))$$

Chapter 1: Big Picture

From Naïve Bayes to Logistic Regression

In classification we care about P(Y | **X**)

Recall the Naive Bayes Classifier

- Predict P(Y | **X**)
- Use assumption that $P(\mathbf{X} \mid Y) = P(X_1, X_2, \ldots X_m \mid Y) = \prod_{i=1}^{m} P(X_i \mid Y)$
- That is a pretty big assumption…

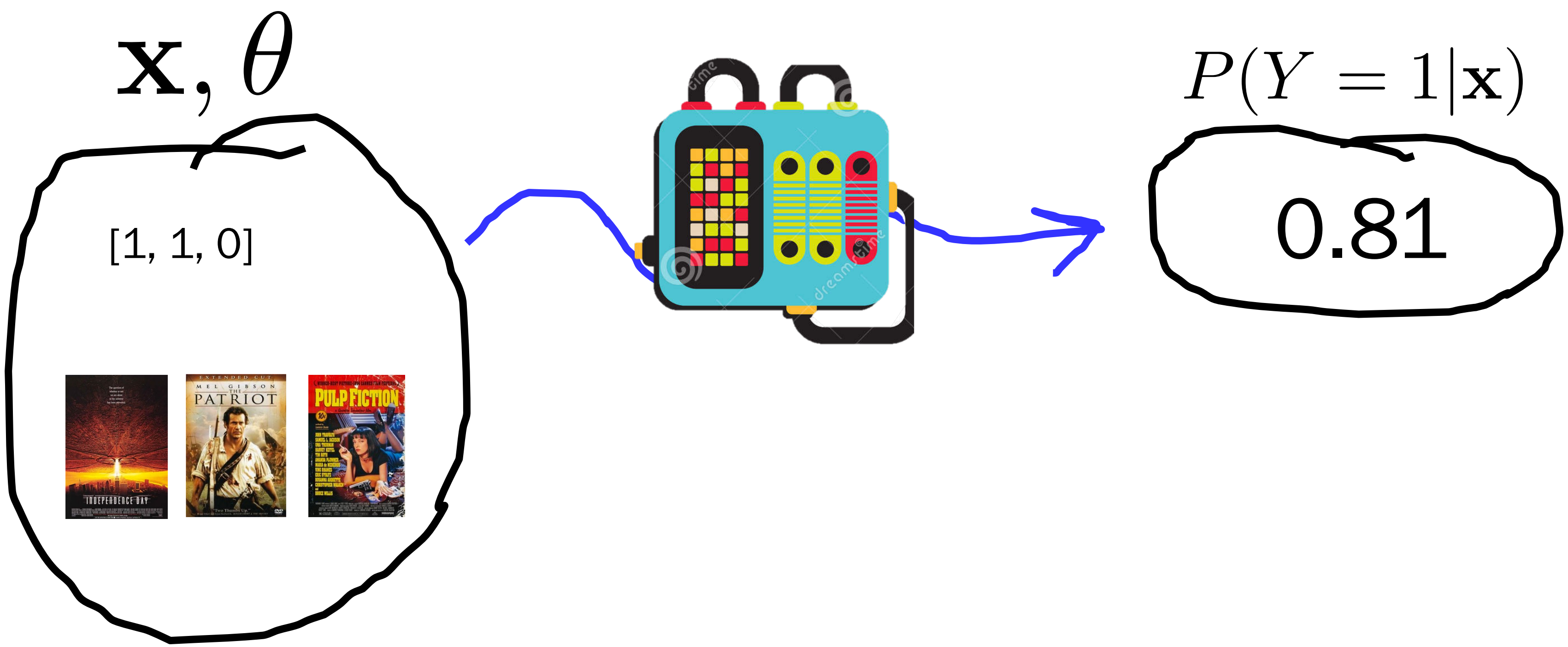
Could we model P(Y | **X**) directly?

- Welcome our friend: logistic regression!

Logistic Regression Assumption

Could we model P(Y | **X**) directly?

- Welcome our friend: logistic regression!



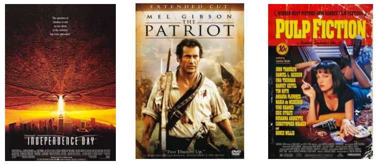
Logistic Regression Assumption

Could we model P(Y | **X**) directly?

- Welcome our friend: logistic regression!

$\mathbf{x}, \theta$

[1, 1, 0]



$P(Y = 1|\mathbf{x})$

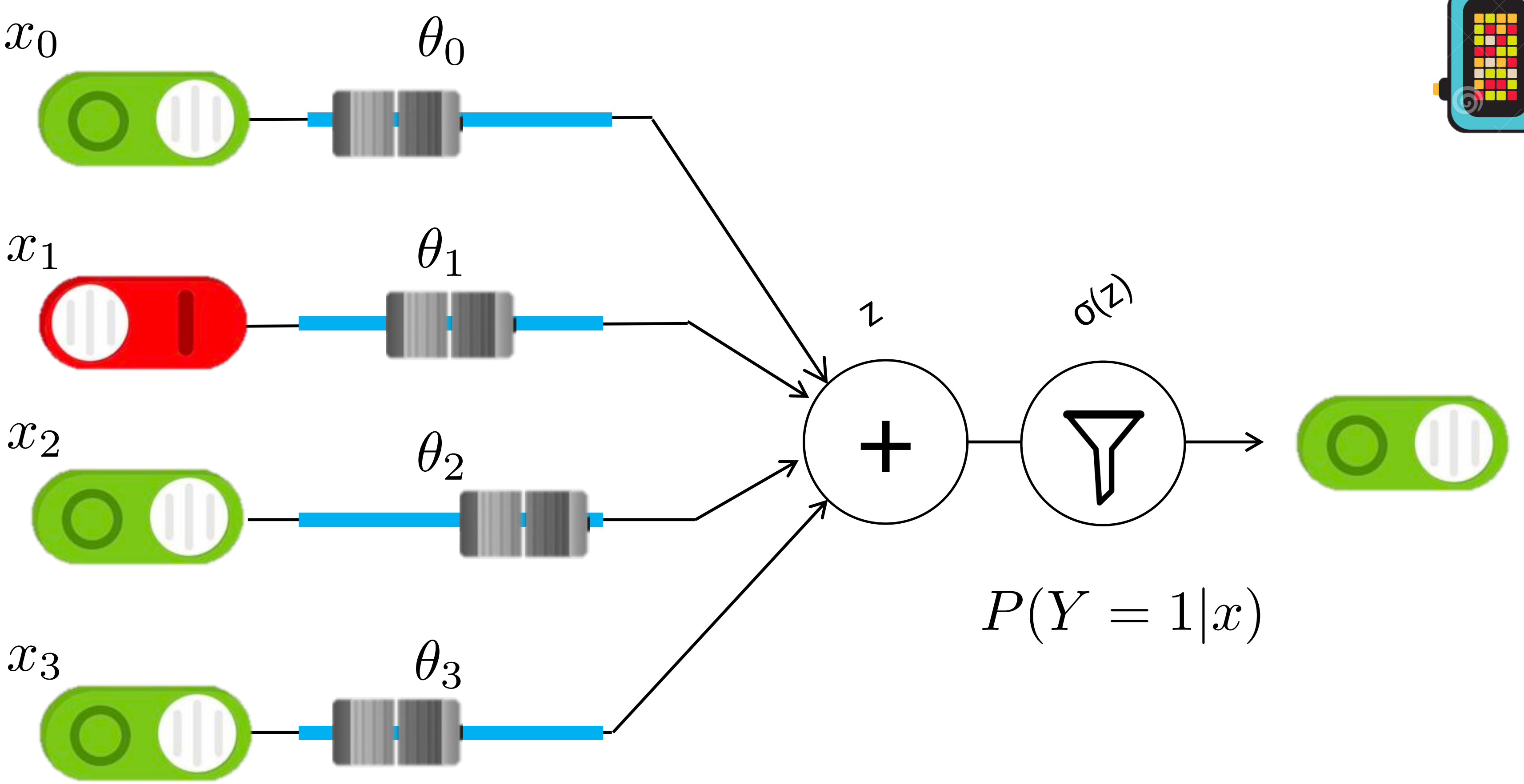
0.81

$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression Assumption

$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression

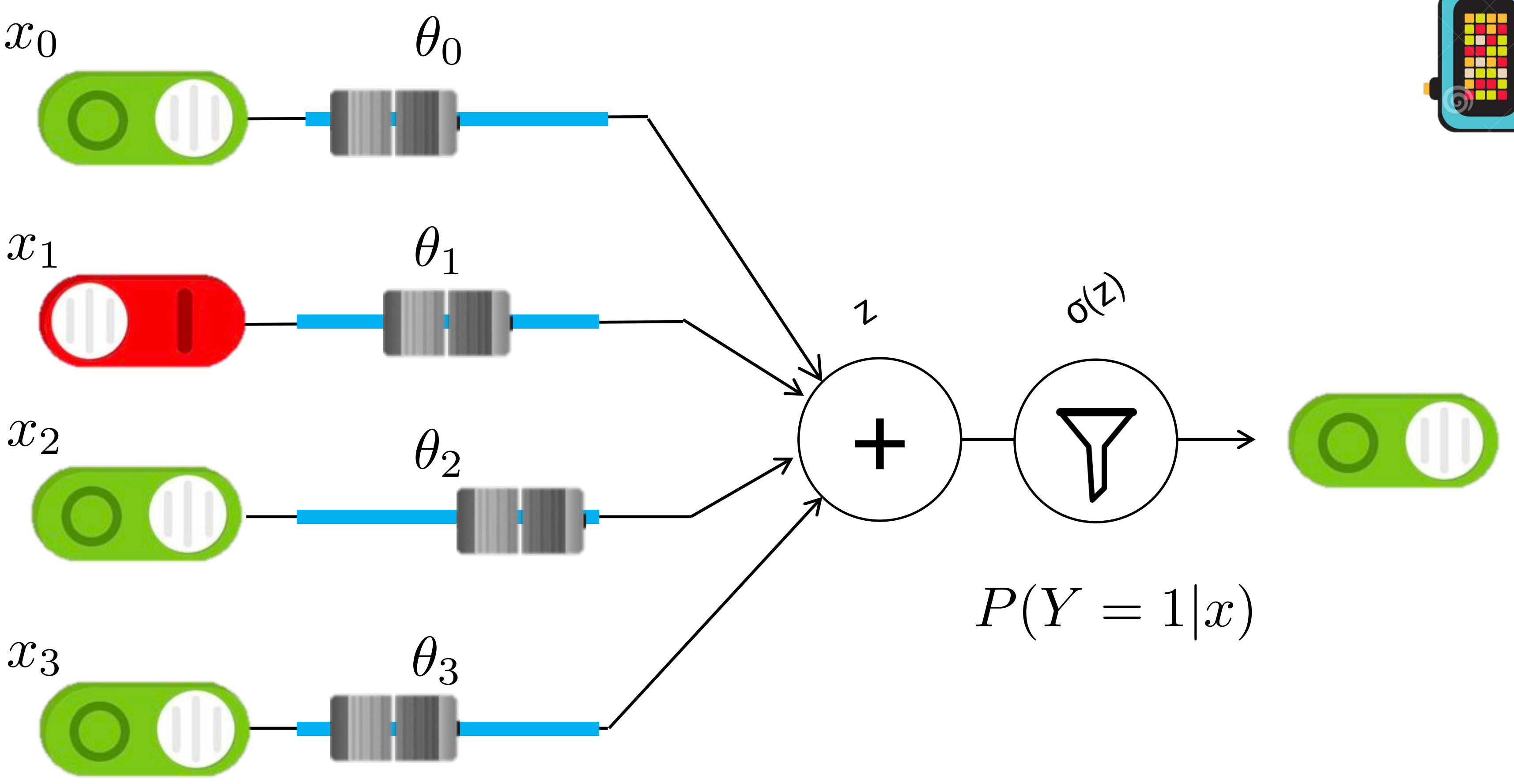


$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression

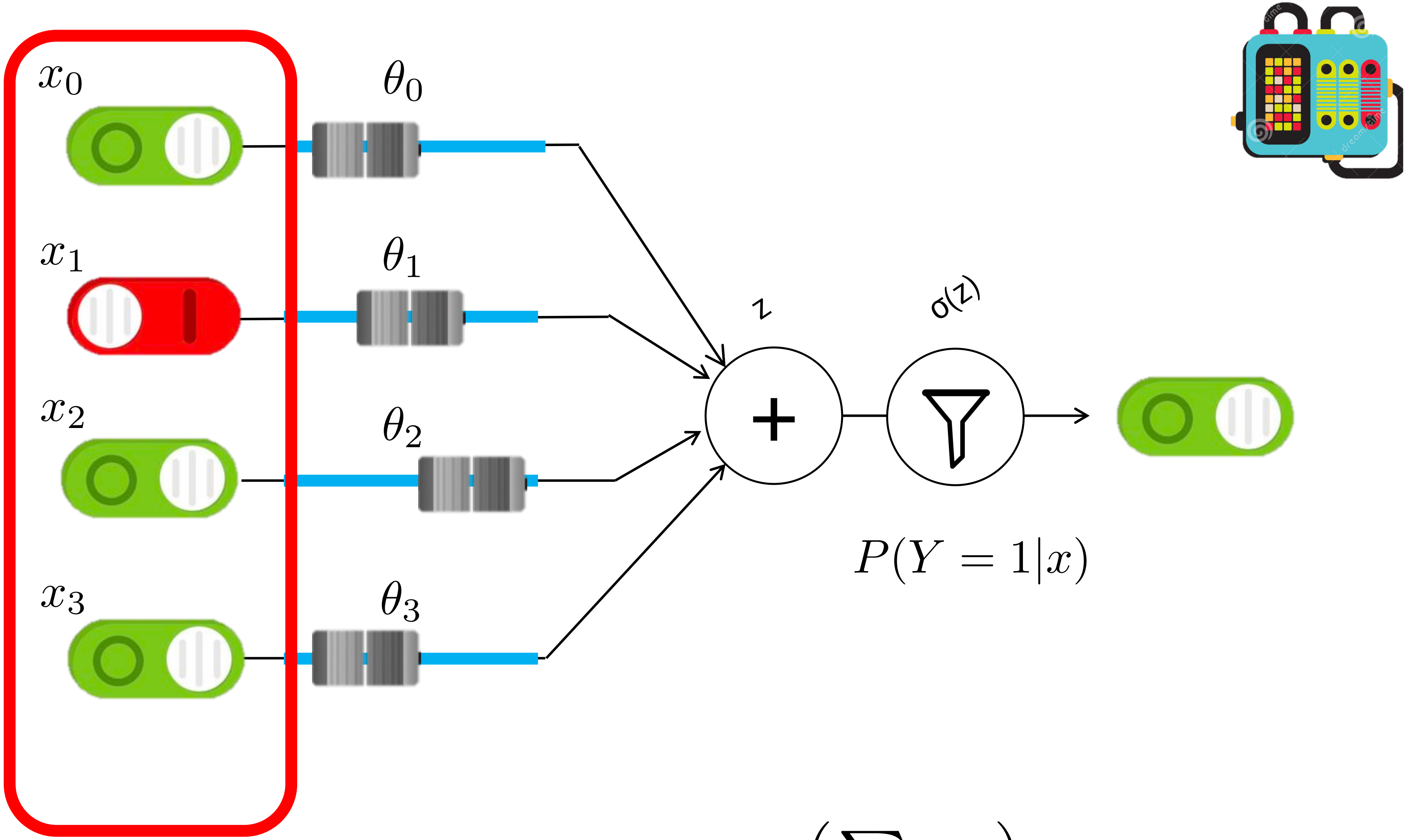
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Logistic Regression Cartoon



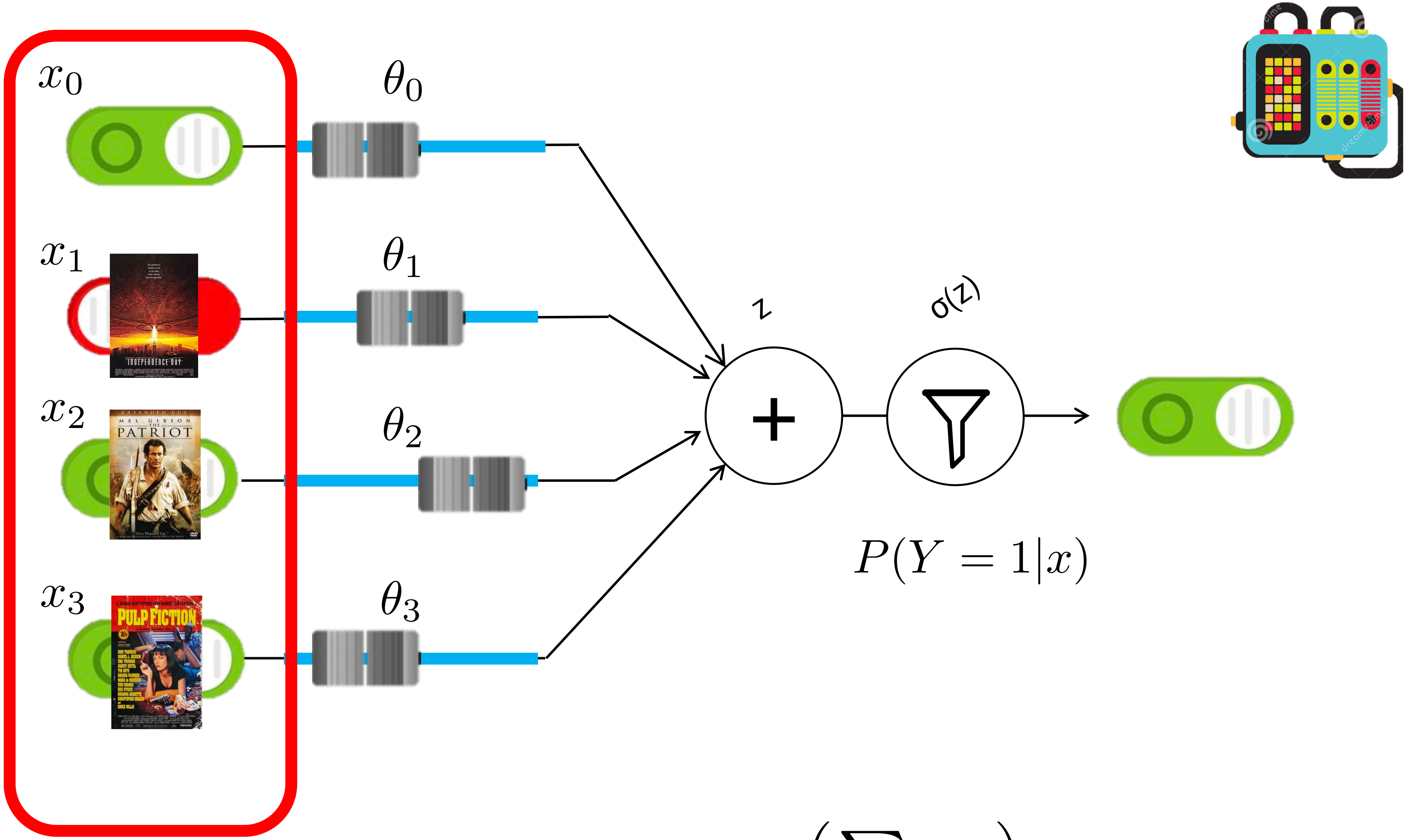
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs $x = [0, 1, 1]$



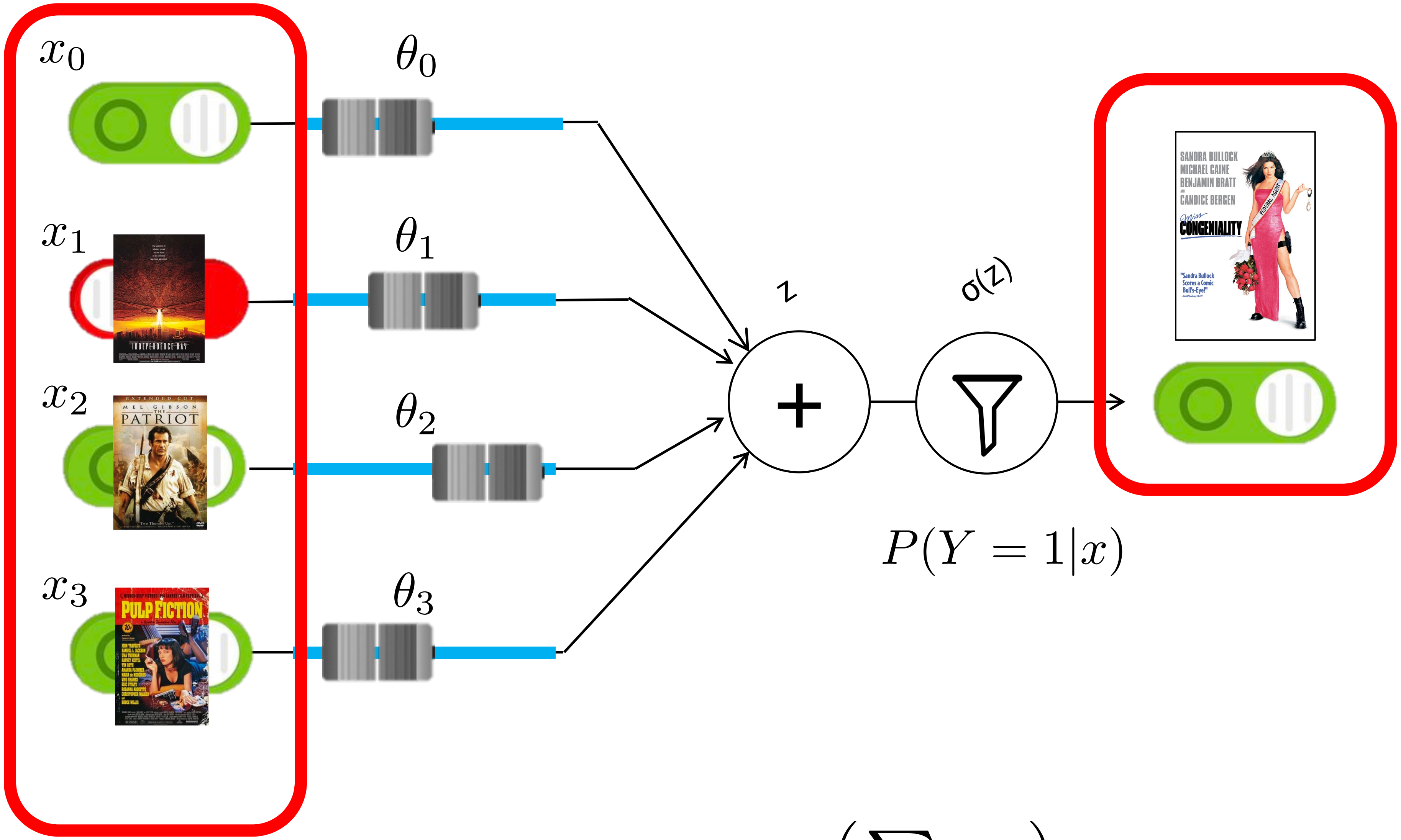
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs



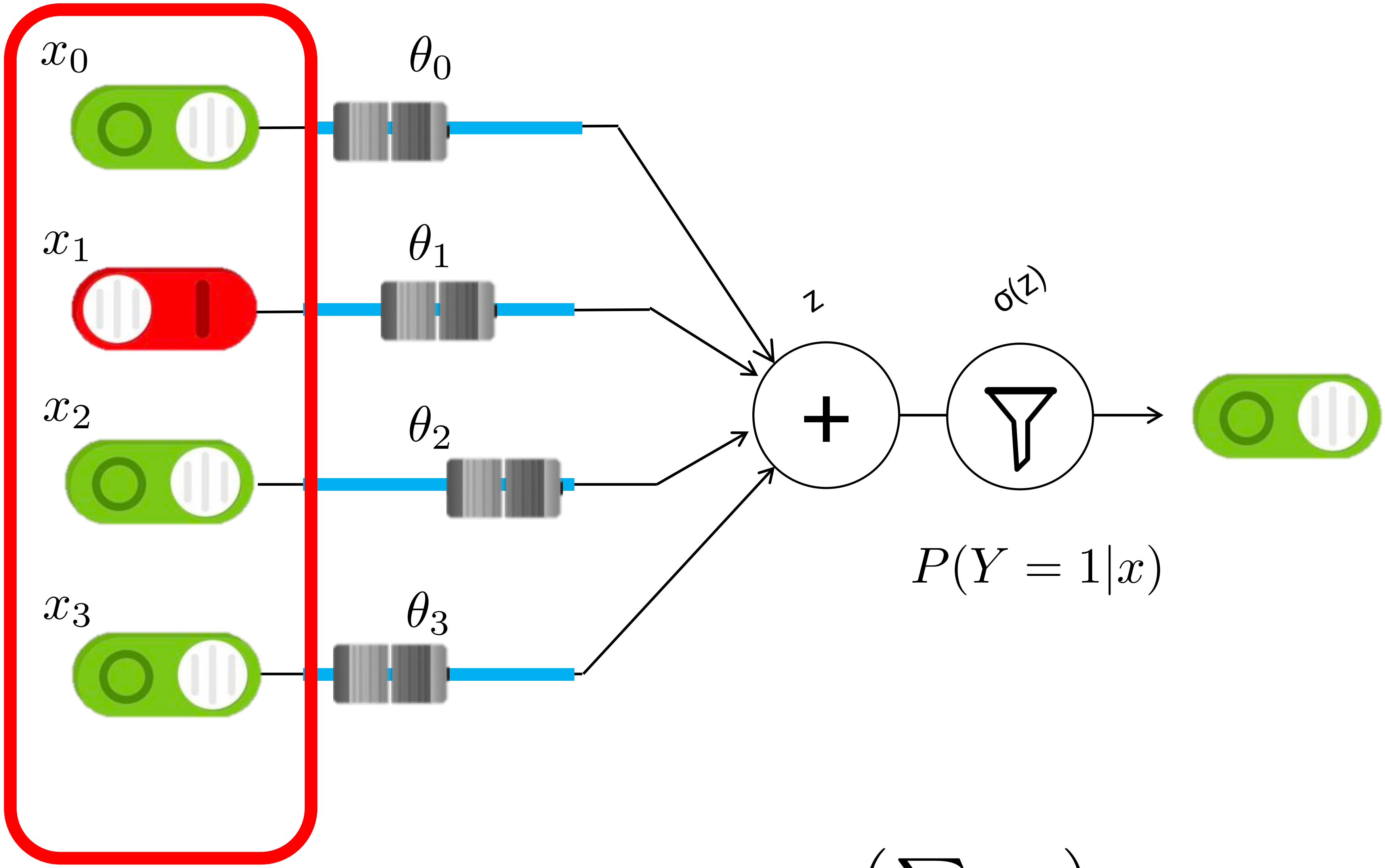
$$P(Y=1|\mathbf{X}=\mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs + Output



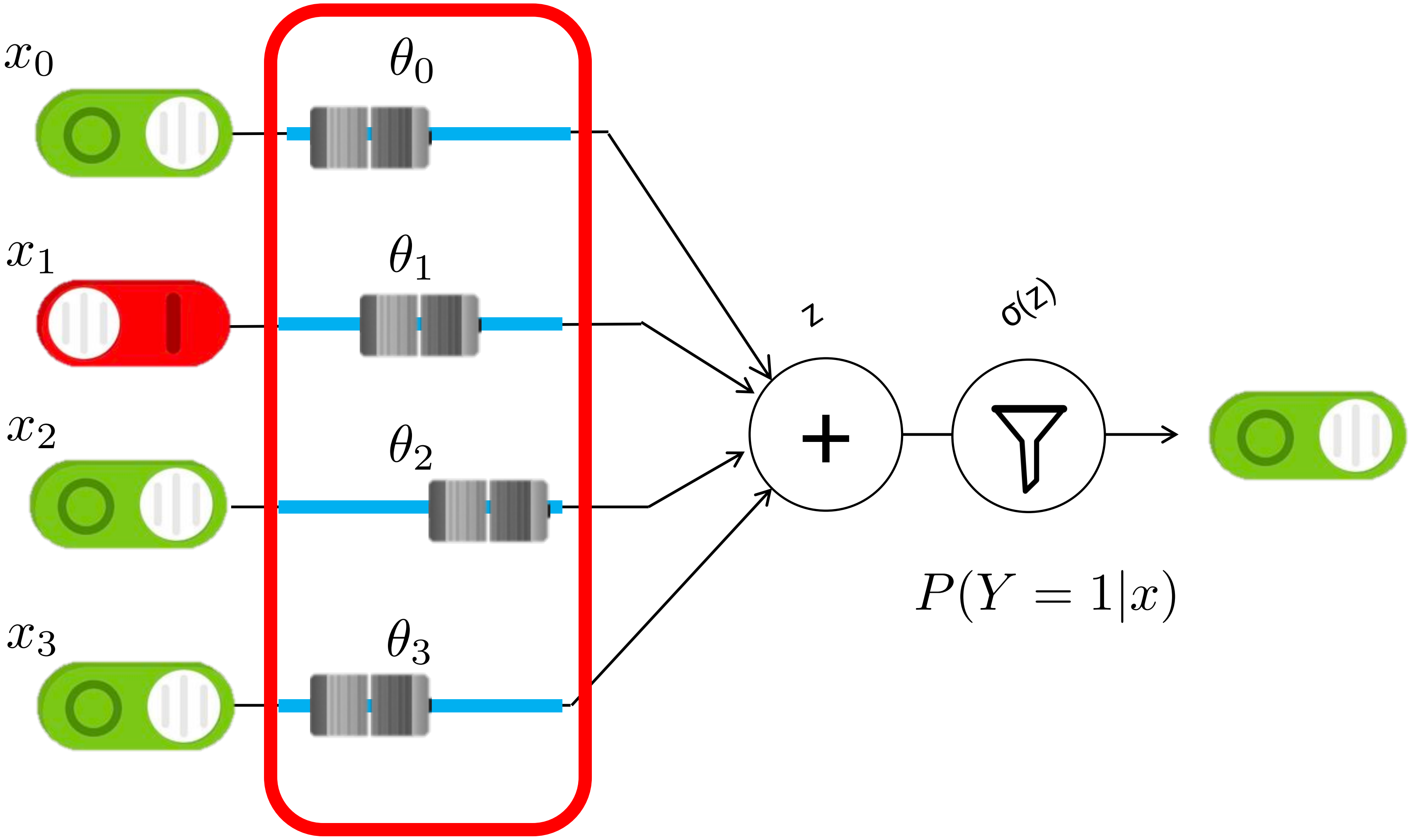
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Inputs



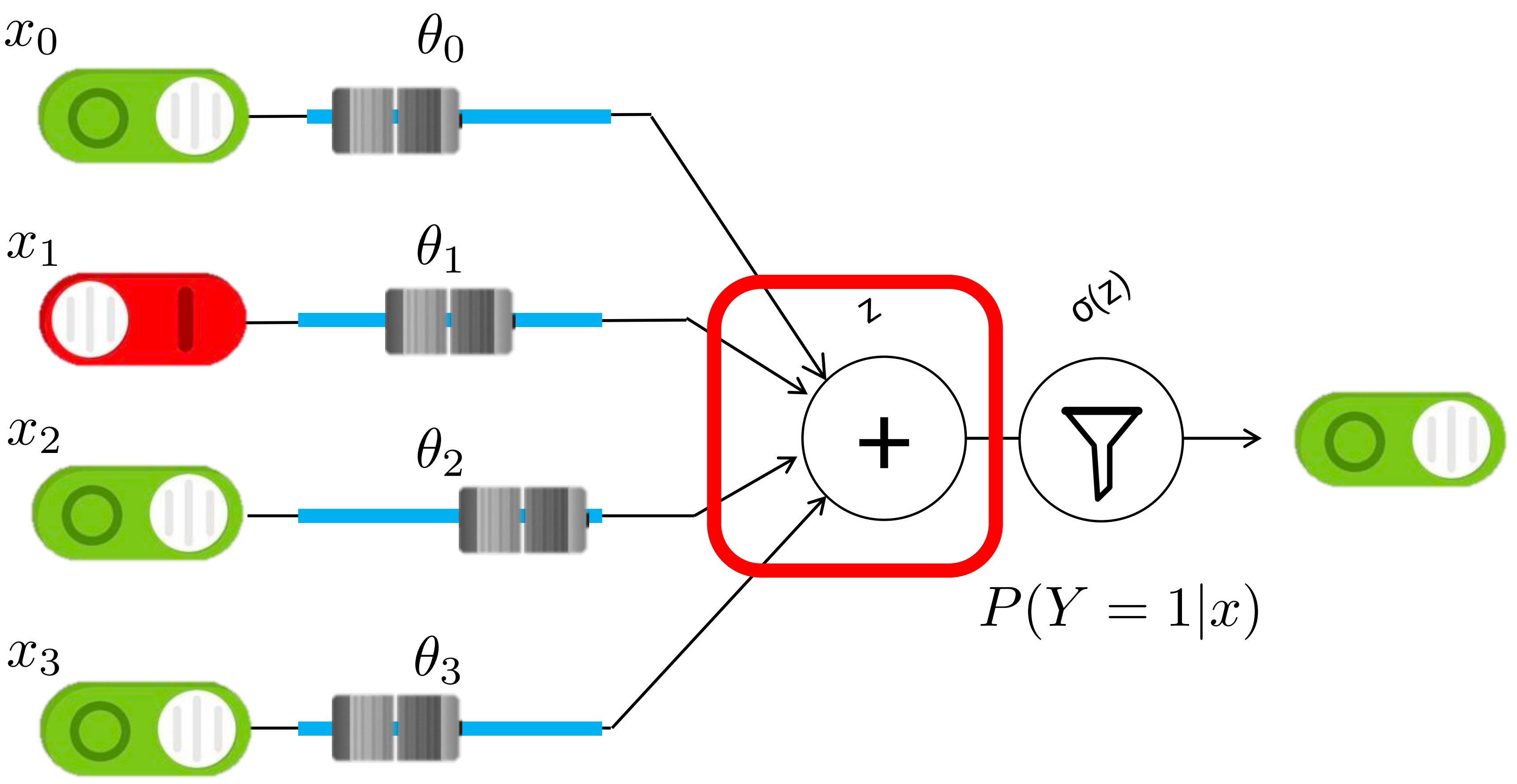
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Weights



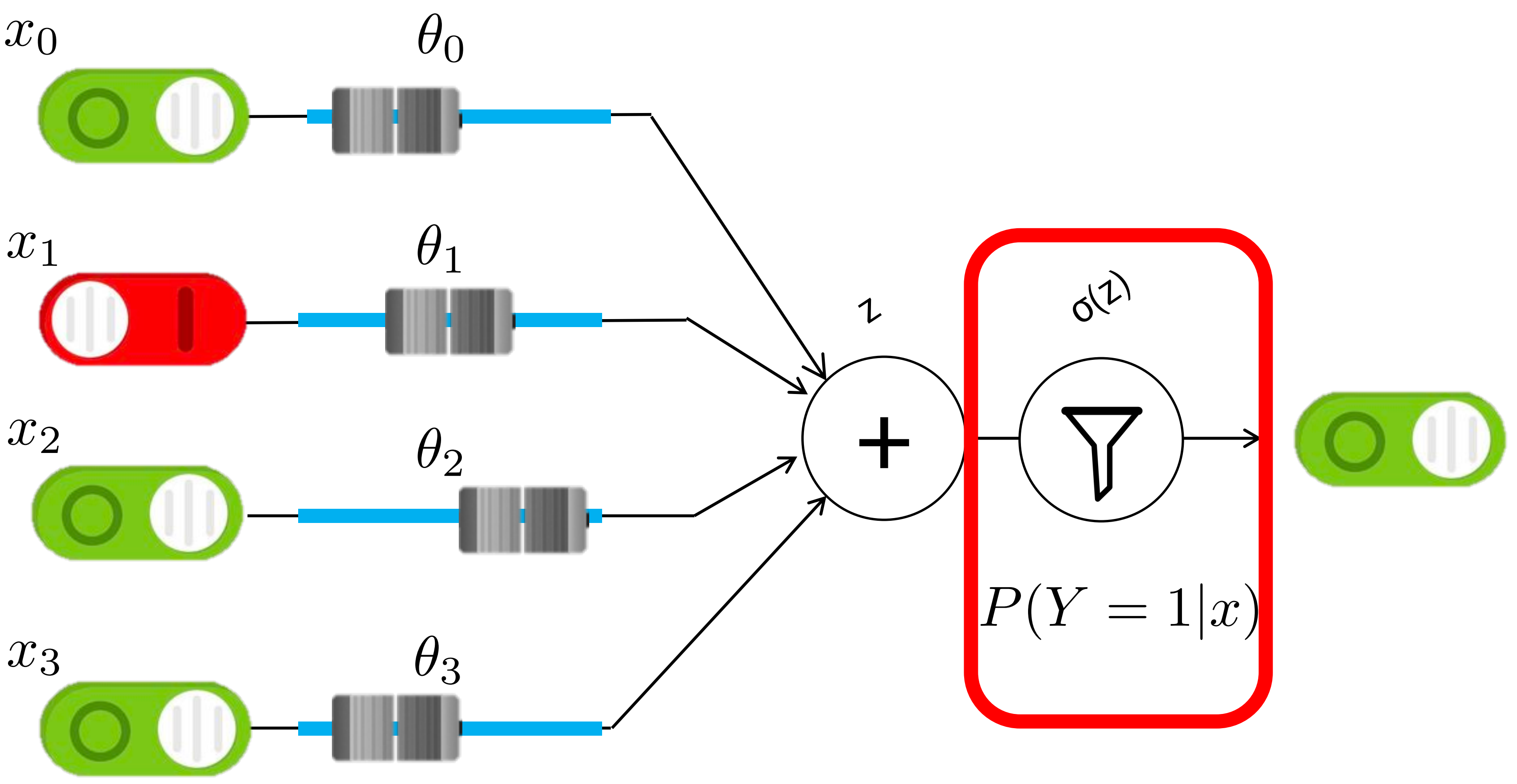
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Weighed Sum



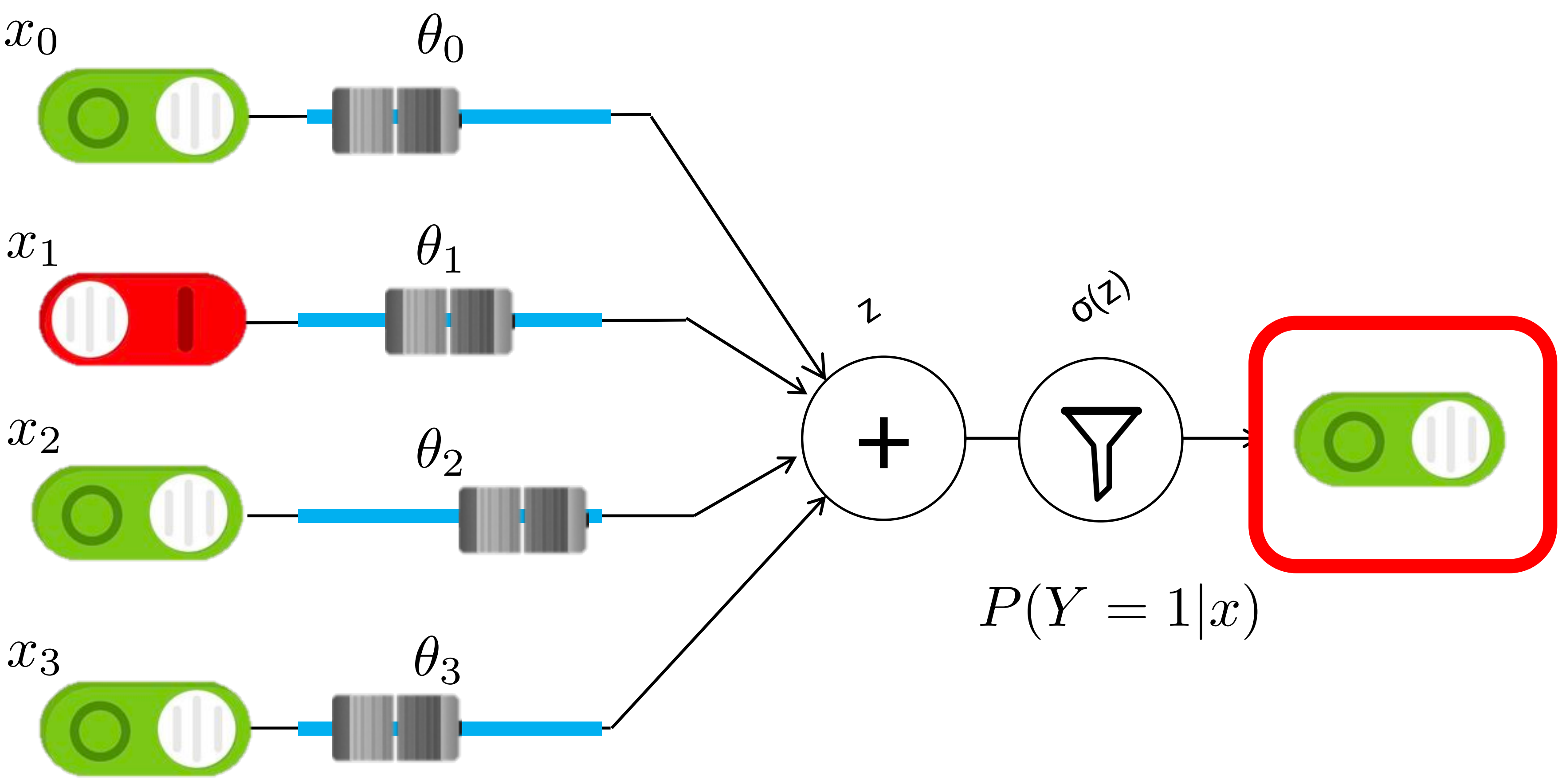
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Squashing Function



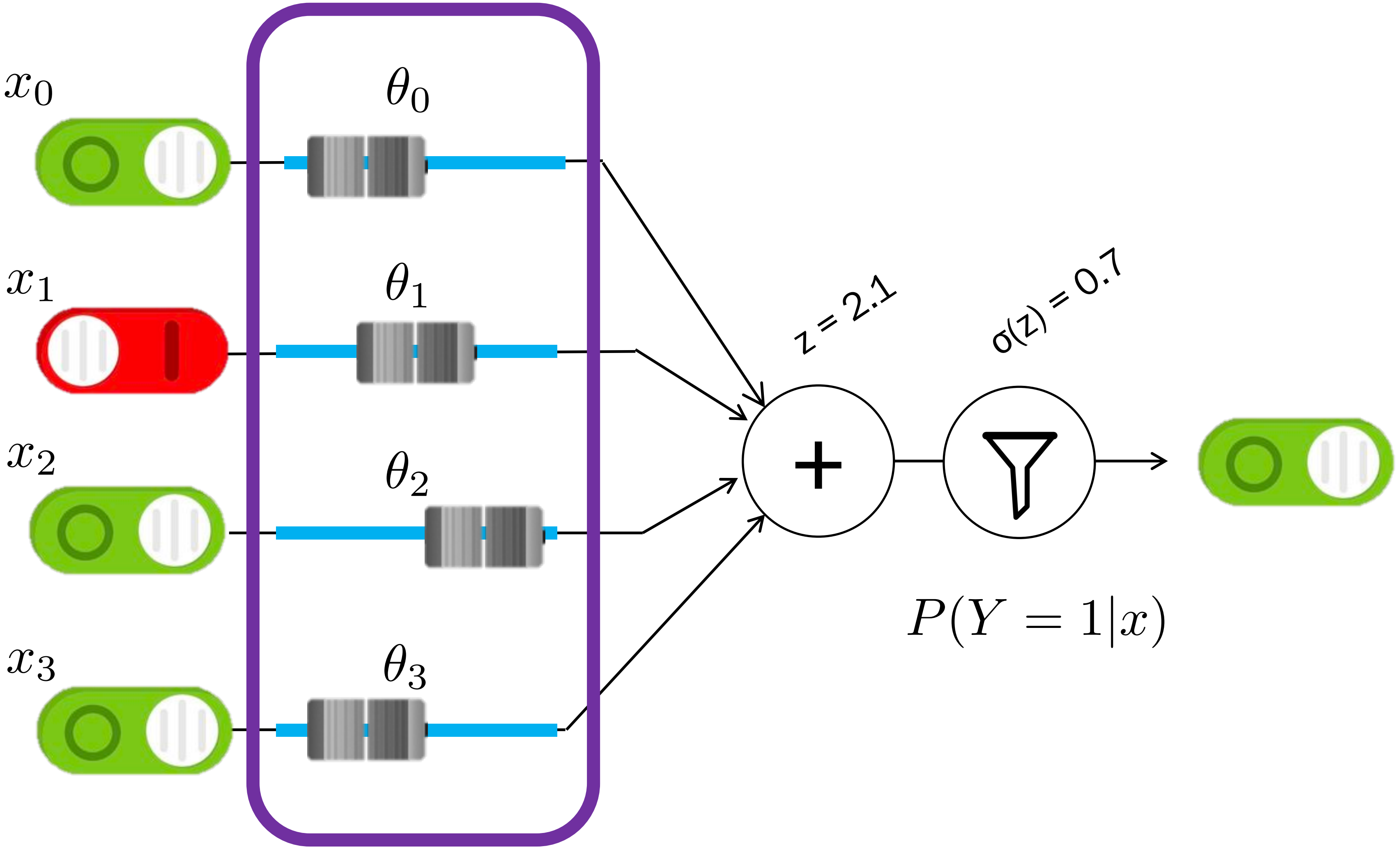
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Prediction



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Parameters Affect Prediction



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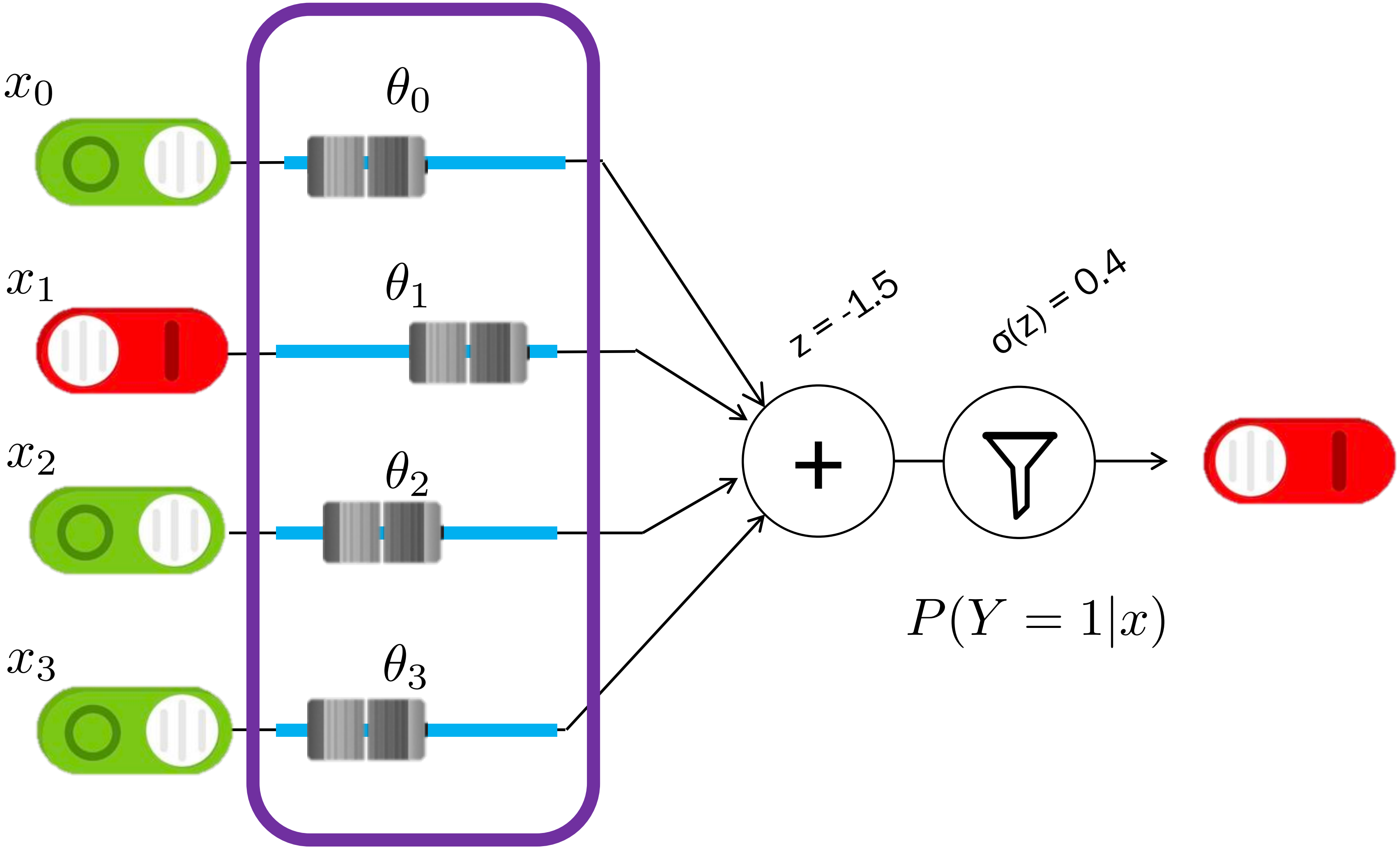
Parameters Affect Prediction



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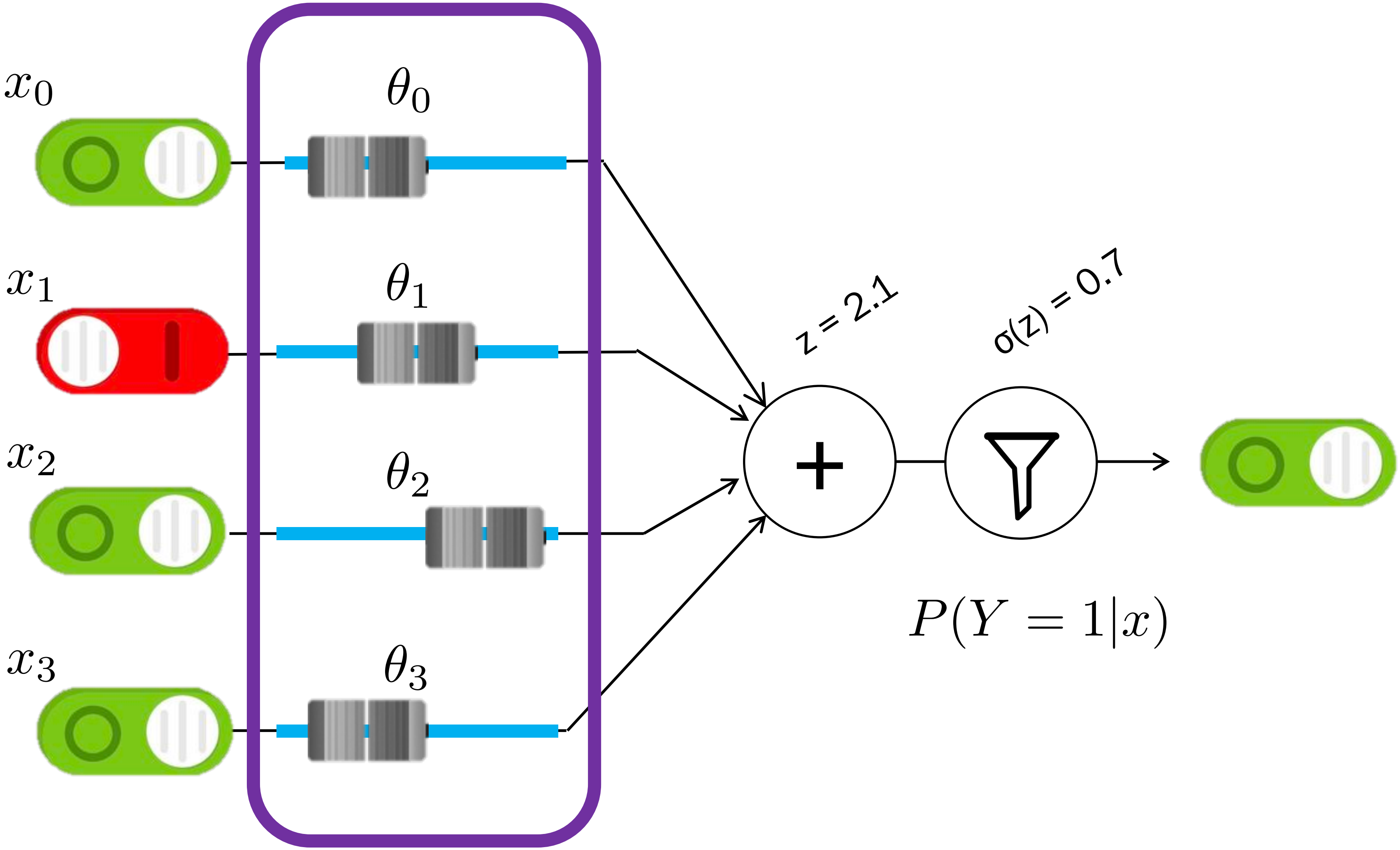
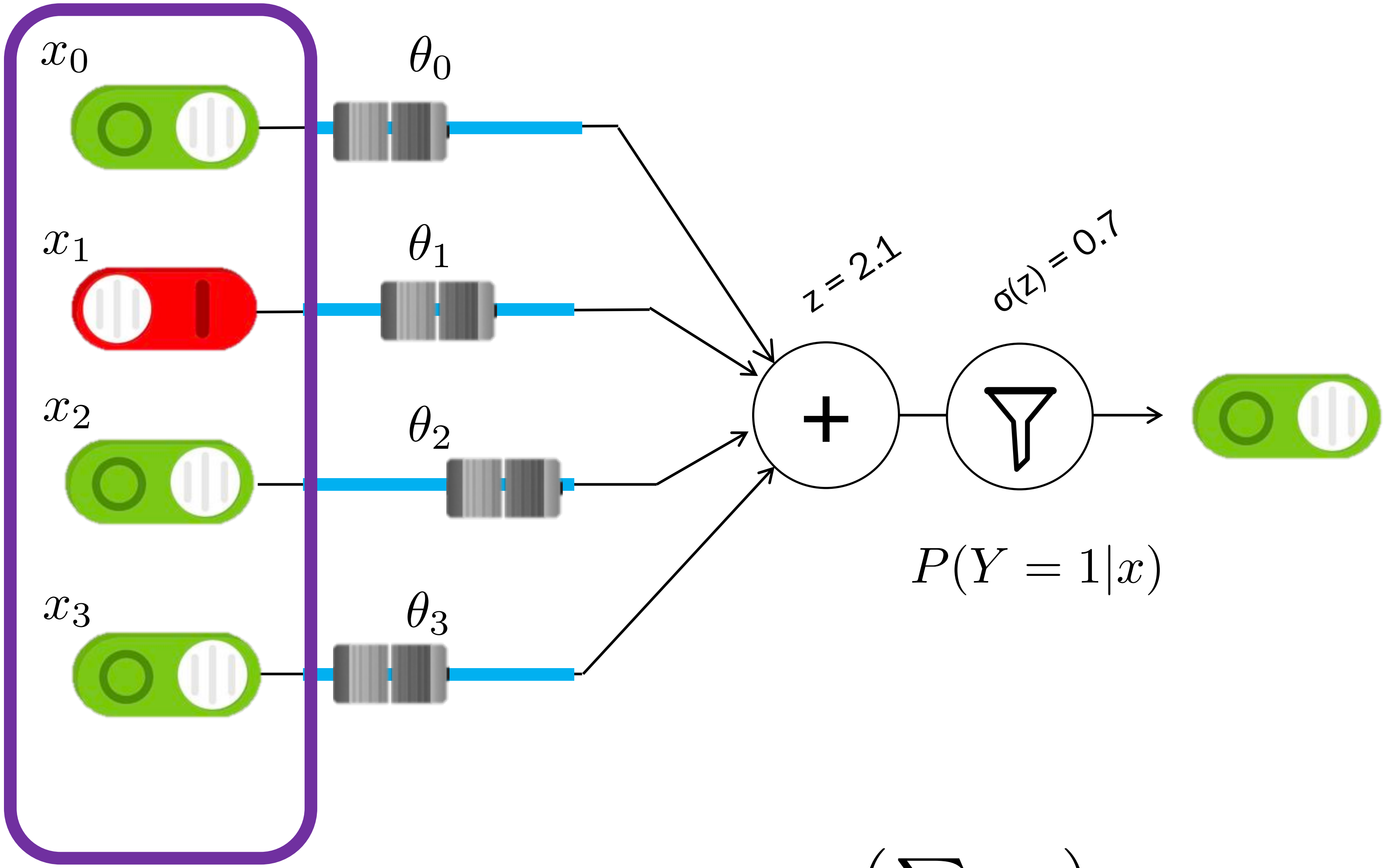
Parameters Affect Prediction



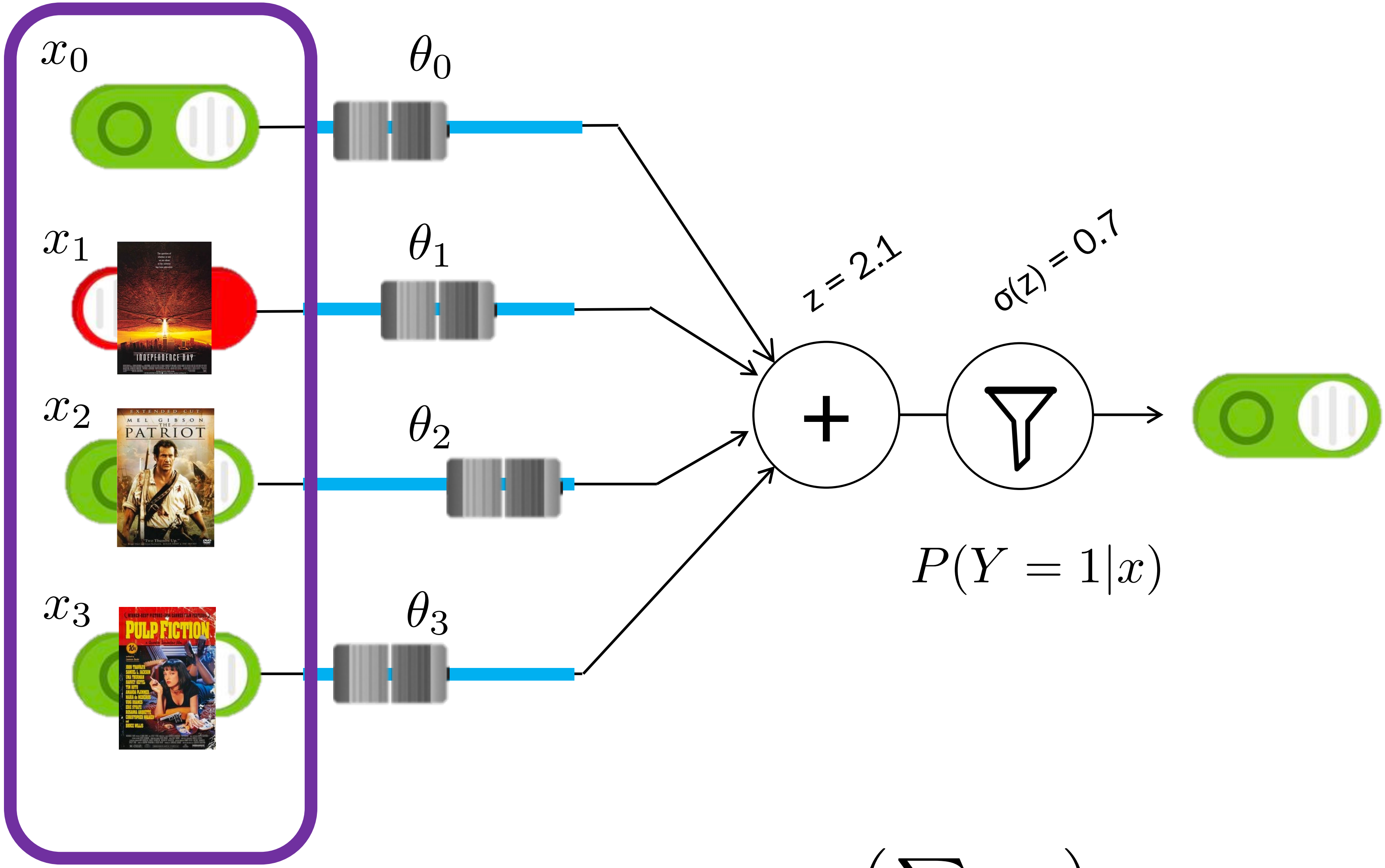
$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Different Predictions for Different Inputs



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

Different Predictions for Different Inputs



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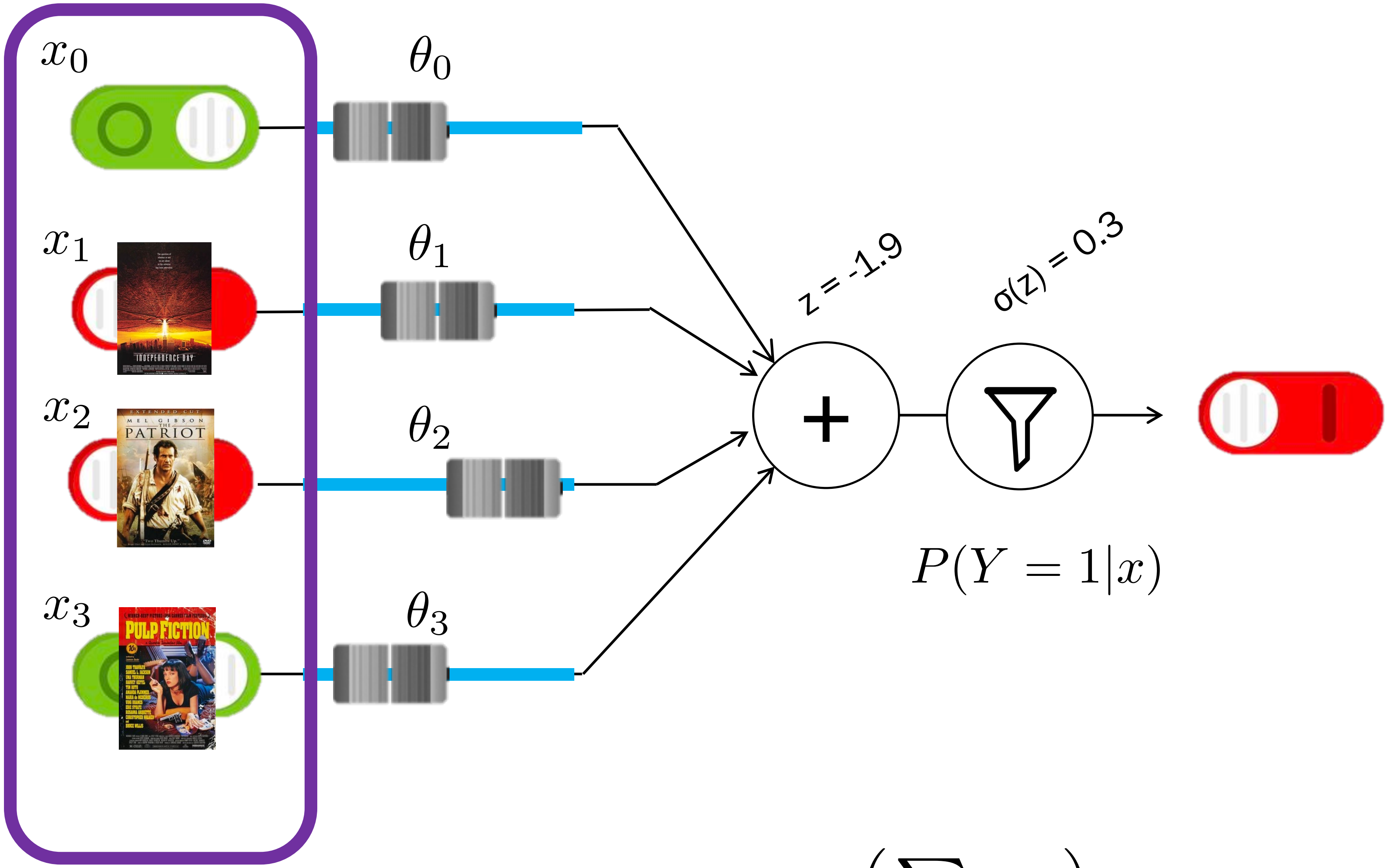
Different Predictions for Different Inputs



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right)$$

What are we trading?

Logistic Regression Assumption

Model *conditional* likelihood P(Y | **X**) directly

- Assume we can Model this probability with *logistic* function:

$$P(Y = 1|\mathbf{X}) = \sigma(z) \text{ where } z = \theta_0 + \sum_{i=1}^{m} \theta_i x_i$$

- For simplicity define $x_0 = 1$ so $z = \theta^T \mathbf{x}$

- Since $\mathrm{P}(Y = 0 \mid \mathbf{X}) + \mathrm{P}(Y = 1 \mid \mathbf{X}) = 1$:

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

Recall:
Sigmoid function

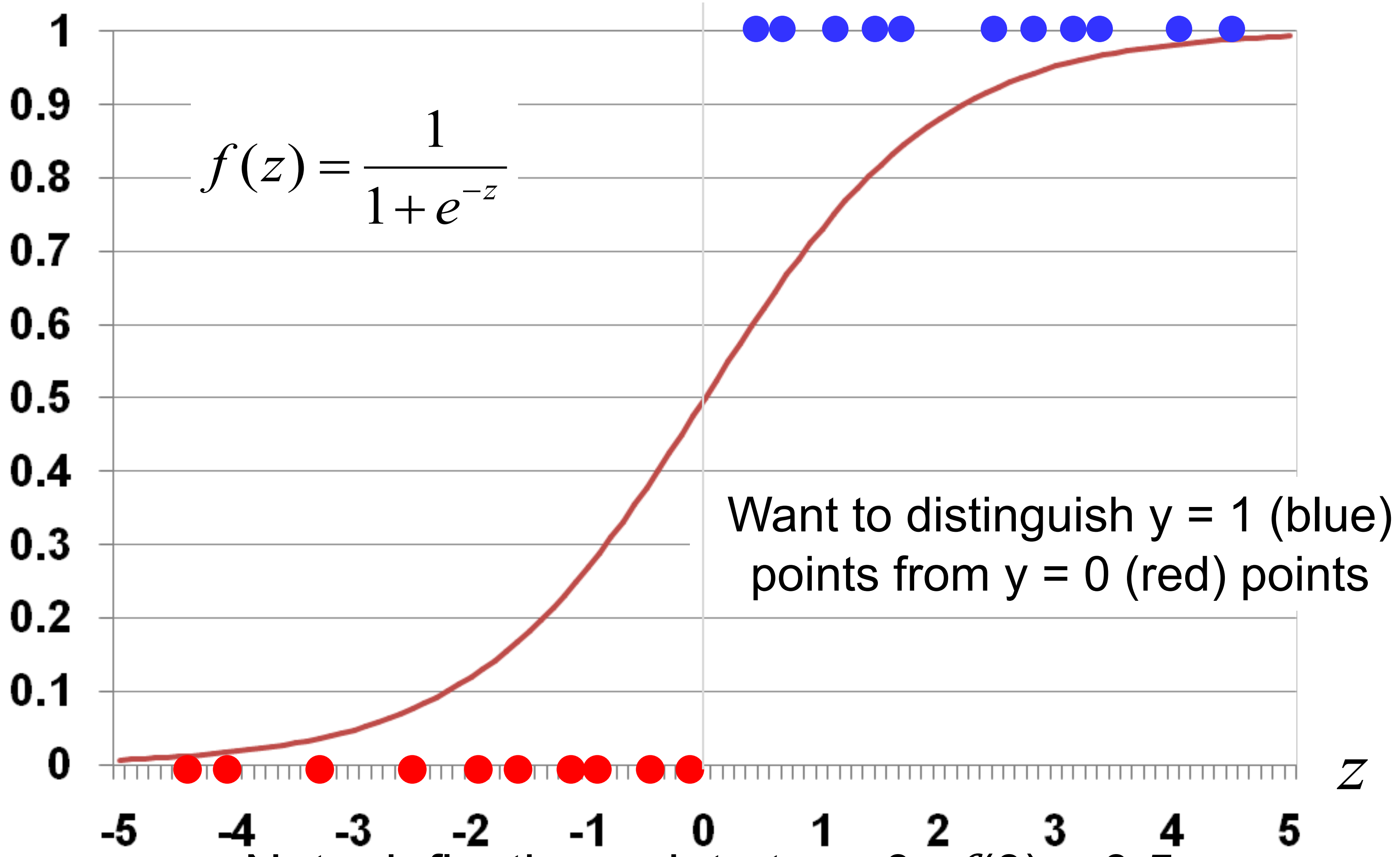
$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Big Assumption

Logistic Regression Assumption:

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

The Sigmoid Function



Note: inflection point at z = 0. $f(0) = 0.5$

What is in a Name

Regression Algorithms

Linear Regression 

Classification Algorithms

Naïve Bayes 

Logistic Regression 

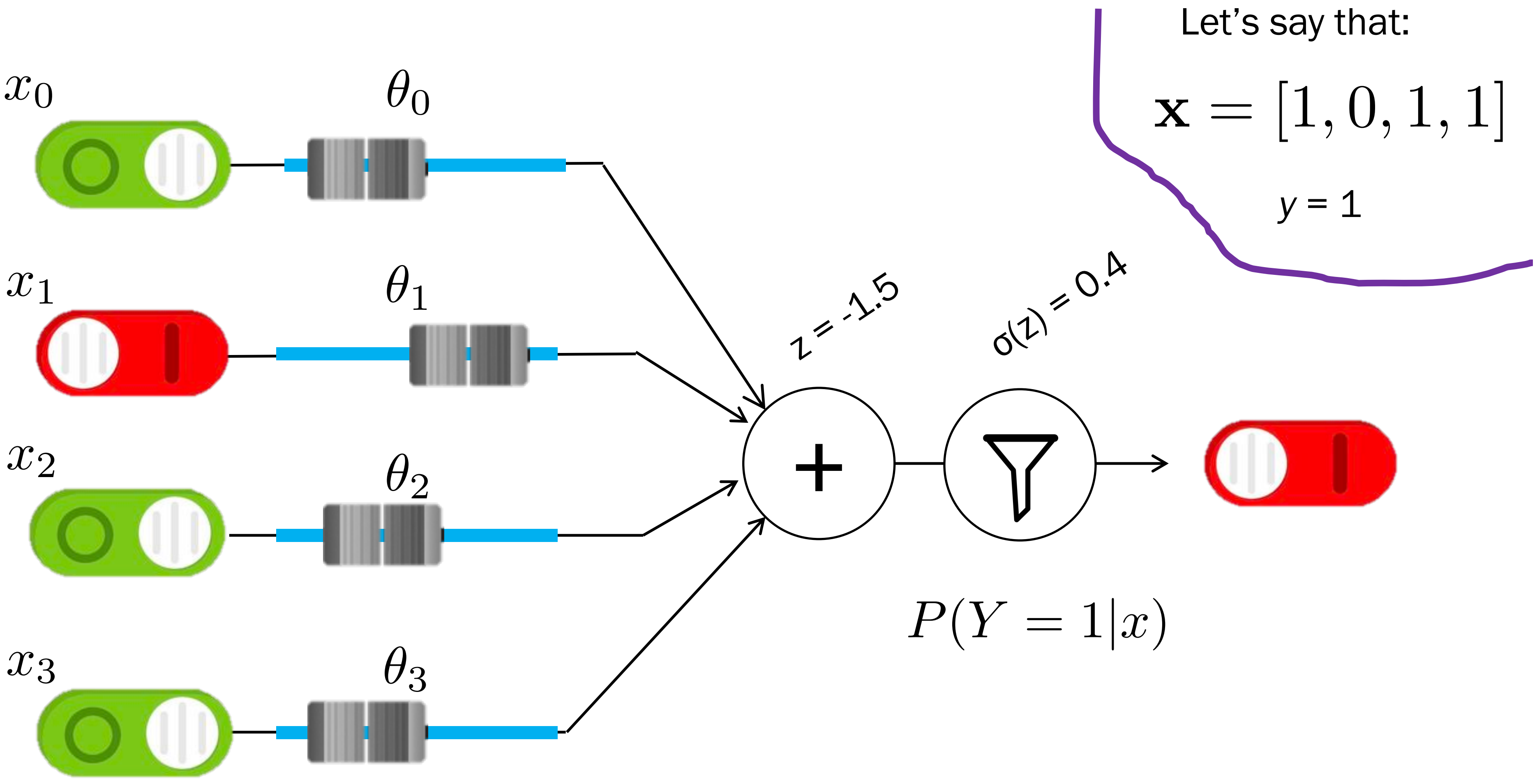
Awesome classifier,
terrible name

If our advisor (Chris) could rename it he would call it: Sigmoidal Classification

What makes for a “smart” logistic regression algorithm?

Logistic regression gets its ***intelligence*** from its thetas (aka its parameters)

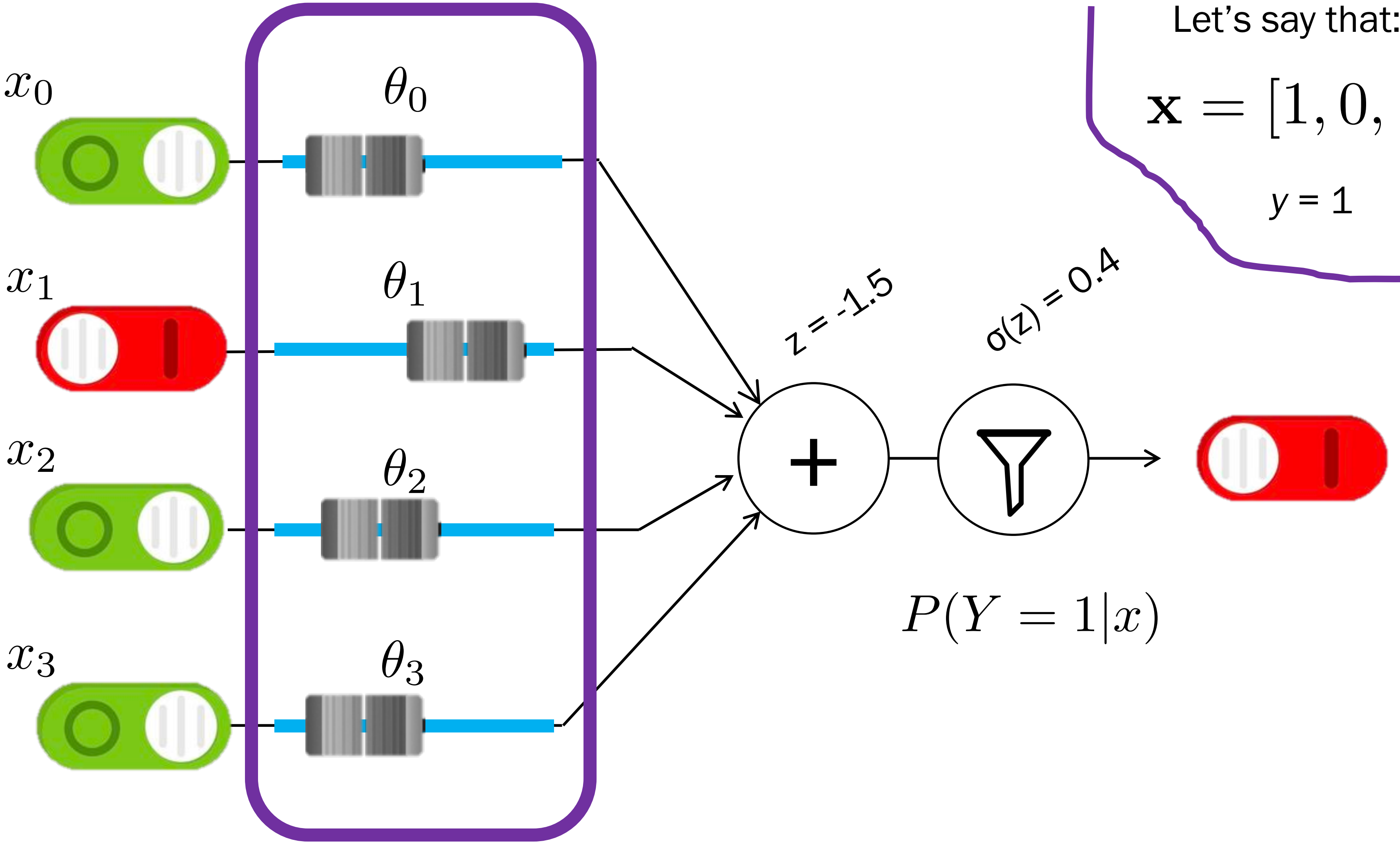
How Do We Learn Parameters?



$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right) = 0.4$$

Data looks unlikely

How Do We Learn Parameters?



Let's say that:

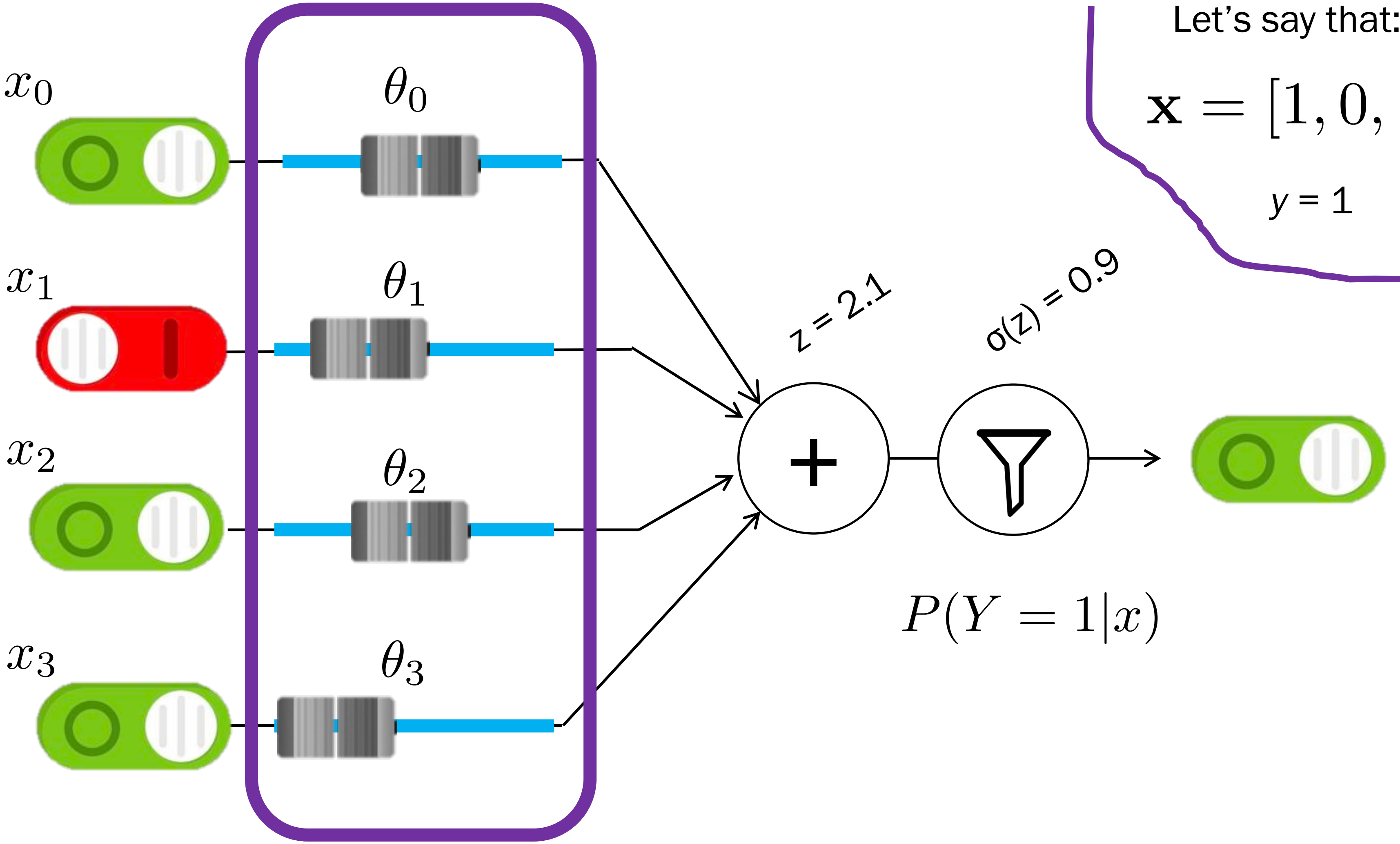
$$\mathbf{x} = [1, 0, 1, 1]$$

$y = 1$

$$P(Y = 1|\mathbf{X} = \mathbf{x}) = \sigma\Big(\sum_i \theta_i x_i\Big) = 0.4$$

Data looks unlikely

How Do We Learn Parameters?



Let's say that:

$$\mathbf{x} = [1, 0, 1, 1]$$

$y = 1$

$$P(Y=1|\mathbf{X}=\mathbf{x}) = \sigma\left(\sum_i \theta_i x_i\right) = 0.9$$

Data is much more likely!

Maximum Likelihood Estimation

Chose your parameter estimates

Parameter μ: 5 Parameter σ: 1.1

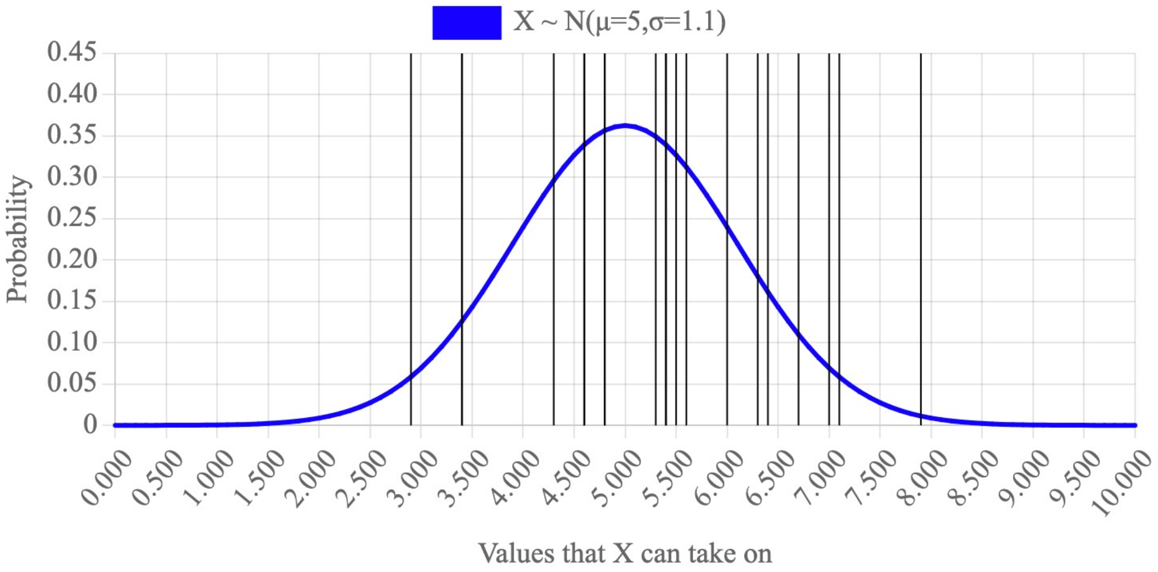
Likelihood of the data given your params

Likelihood: 5.204152095194613e-16

Log Likelihood: -314.1

Best Seen: -311.2

Your Gaussian



Math for Logistic Regression

1. Make logistic regression assumption

$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T\mathbf{x})$$
$$P(Y=0|X=\mathbf{x}) = 1-\sigma(\theta^T\mathbf{x})$$

Often call this $\hat{y}$

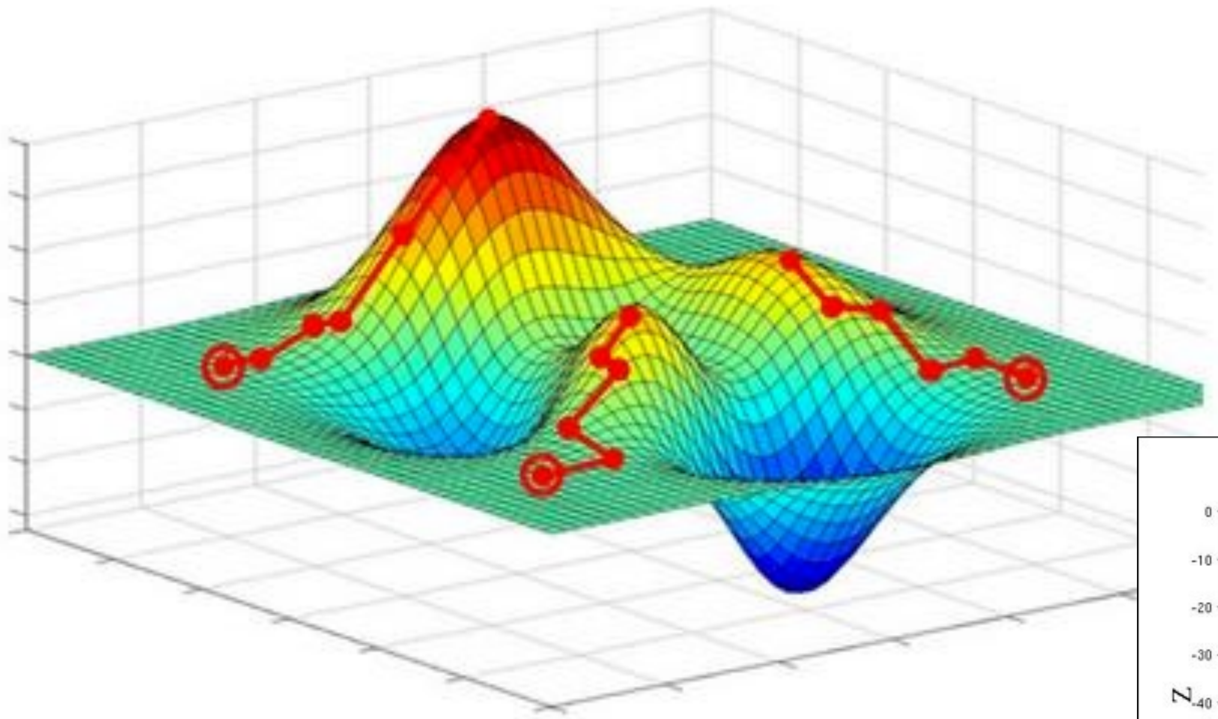
2. Calculate the log likelihood for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T\mathbf{x}^{(i)}) + (1-y^{(i)}) \log[1-\sigma(\theta^T\mathbf{x}^{(i)})]$$

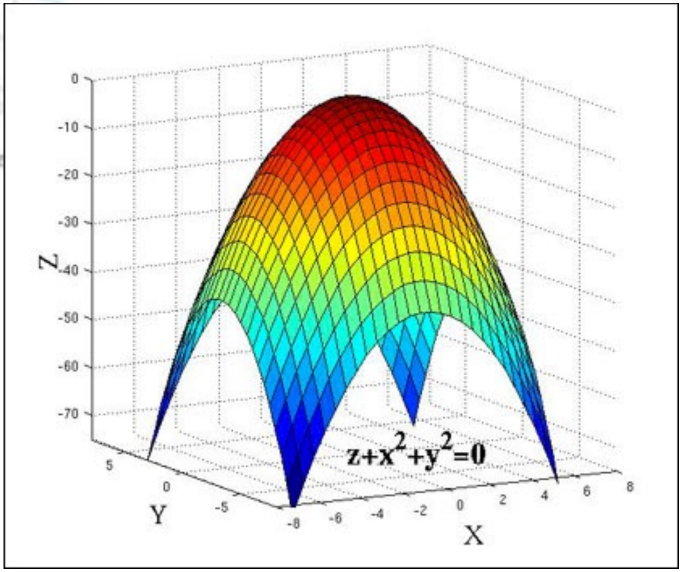
3. Get derivative of log likelihood with respect to thetas

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma(\theta^T\mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Gradient Ascent



Logistic regression
LL function is
convex



Walk uphill and you will find a local maxima
(if your step size is small enough)

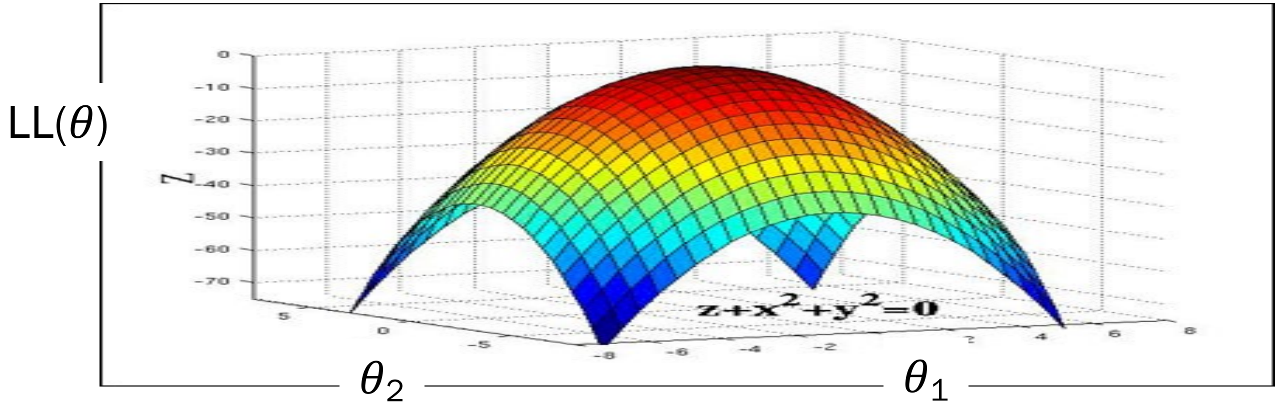
Gradient Ascent Step

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

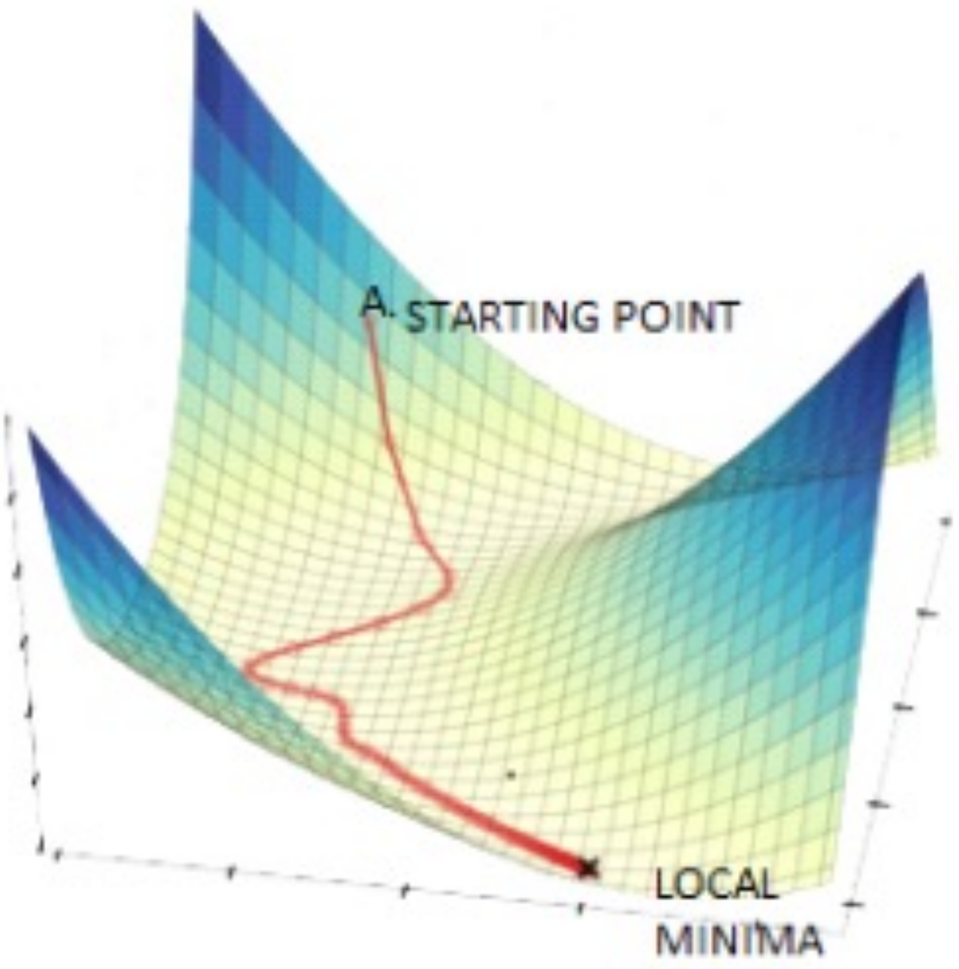
$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \frac{\partial LL(\theta^{\text{ old}})}{\partial \theta_j^{\text{old}}}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Do this for all thetas!



Gradient Decent



Walk downhill and you will find a local minima
(if your step size is small enough)

Gradient Descent with Negative LL

Assume some loss function with known derivative $\dfrac{\partial \text{ Loss}}{\partial \theta_j}$

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Loss}}{\partial \theta_j}$$

Gradient Descent with Negative LL

Assume some loss function with known derivative $\dfrac{\partial \text{ Loss}}{\partial \theta_j}$

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Loss}}{\partial \theta_j}$$

$$= \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Negative}LL}{\partial \theta_j}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Gradient Descent with Negative LL

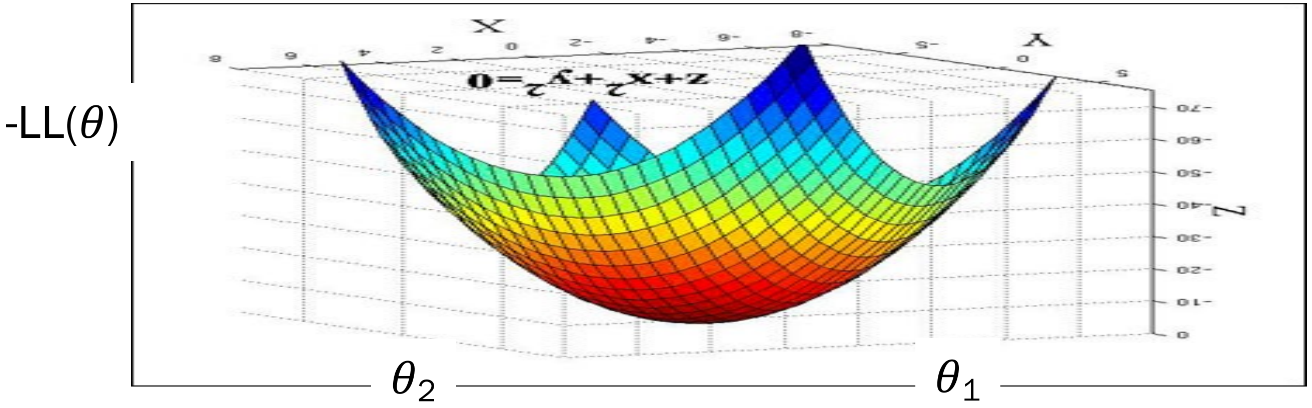
Assume some loss function with known derivative $\dfrac{\partial \text{ Loss}}{\partial \theta_j}$

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Loss}}{\partial \theta_j}$$

...exactly the same

$$= \theta_j^{\text{old}} - \eta \cdot \frac{\partial \text{ Negative}LL}{\partial \theta_j}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$



What does this look like in code?

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \frac{\partial LL(\theta^{\text{ old}})}{\partial \theta_j^{\text{old}}}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Logistic Regression Training

Initialize: $\theta_j = 0$ **for all** $0 \leq j \leq m$

Calculate all θ_j

Logistic Regression Training

```
Initialize: θ_j = 0 for all 0 ≤ j ≤ m
```

Initialize: $\theta_j = 0$ for all $0 \le j \le m$

Repeat many times:

gradient[j] = 0 for all $0 \le j \le m$

Calculate all gradient[j]*'s based on data*

θ_j += η * gradient[j] for all $0 \le j \le m$

Logistic Regression Training

`Initialize:` θ_j `= 0 for all 0` $\leq j \leq m$

`Repeat many times:`

`gradient[j] = 0 for all 0` $\leq j \leq m$

`For each training example (x, y):`

`For each parameter j:`

Update `gradient[j]` *for current training example*

θ_j `+=` η `* gradient[j] for all 0` $\leq j \leq m$

Logistic Regression Training

Initialize: $\theta_j = 0$ **for all** $0 \leq j \leq m$

Repeat many times:

`gradient[j] = 0` **for all** $0 \leq j \leq m$

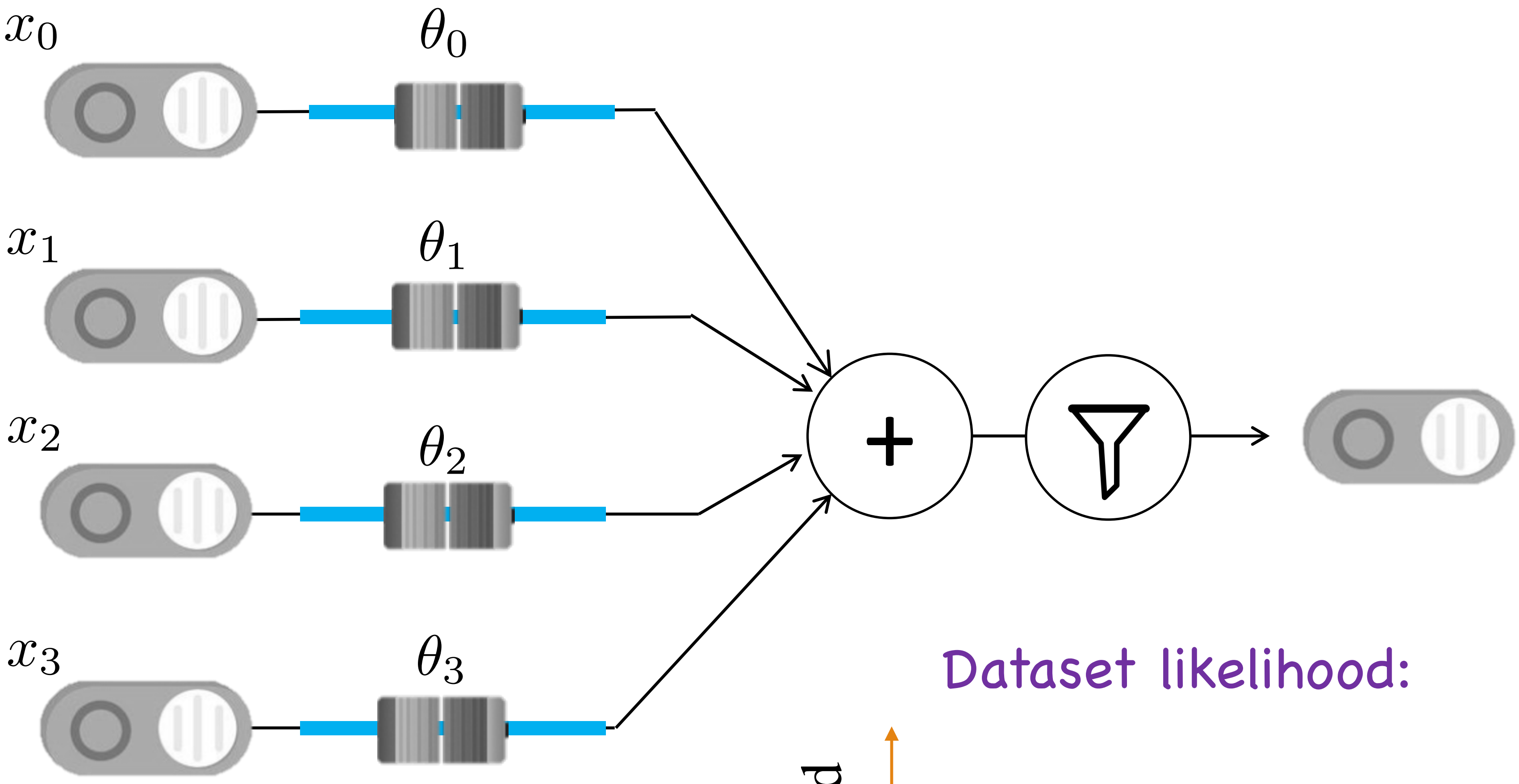
For each training example (x, y):

For each parameter j:

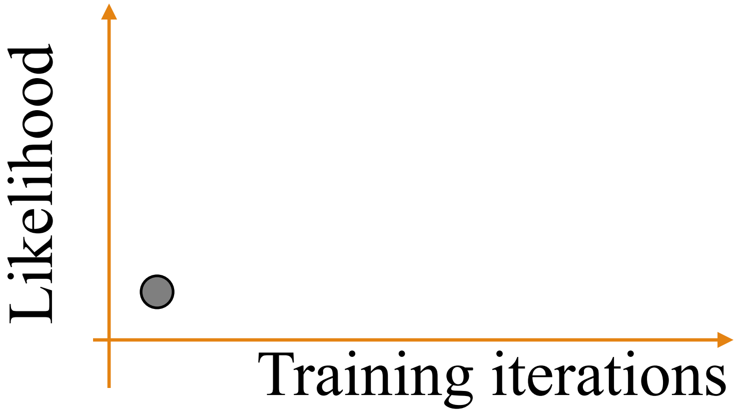
$$\texttt{gradient[j]} \mathrel{+}= x_j \left(y - \frac{1}{1 + e^{-\theta^{\mathrm{T}} \mathbf{x}}} \right)$$

θ_j += η * `gradient[j]` **for all** $0 \leq j \leq m$

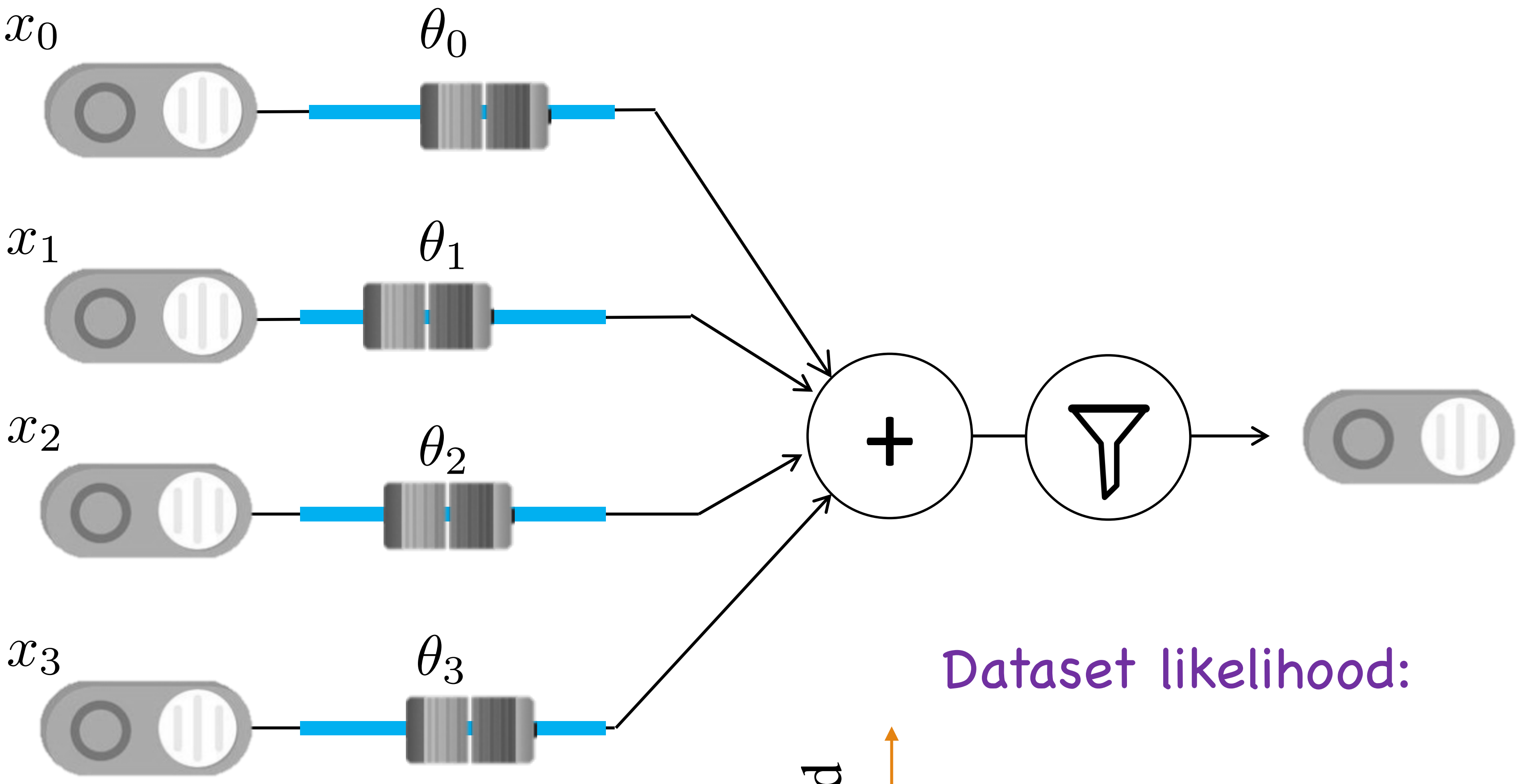
Training



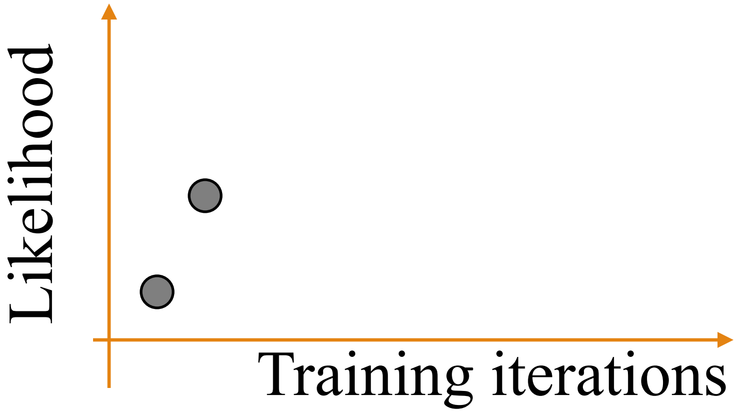
Dataset likelihood:



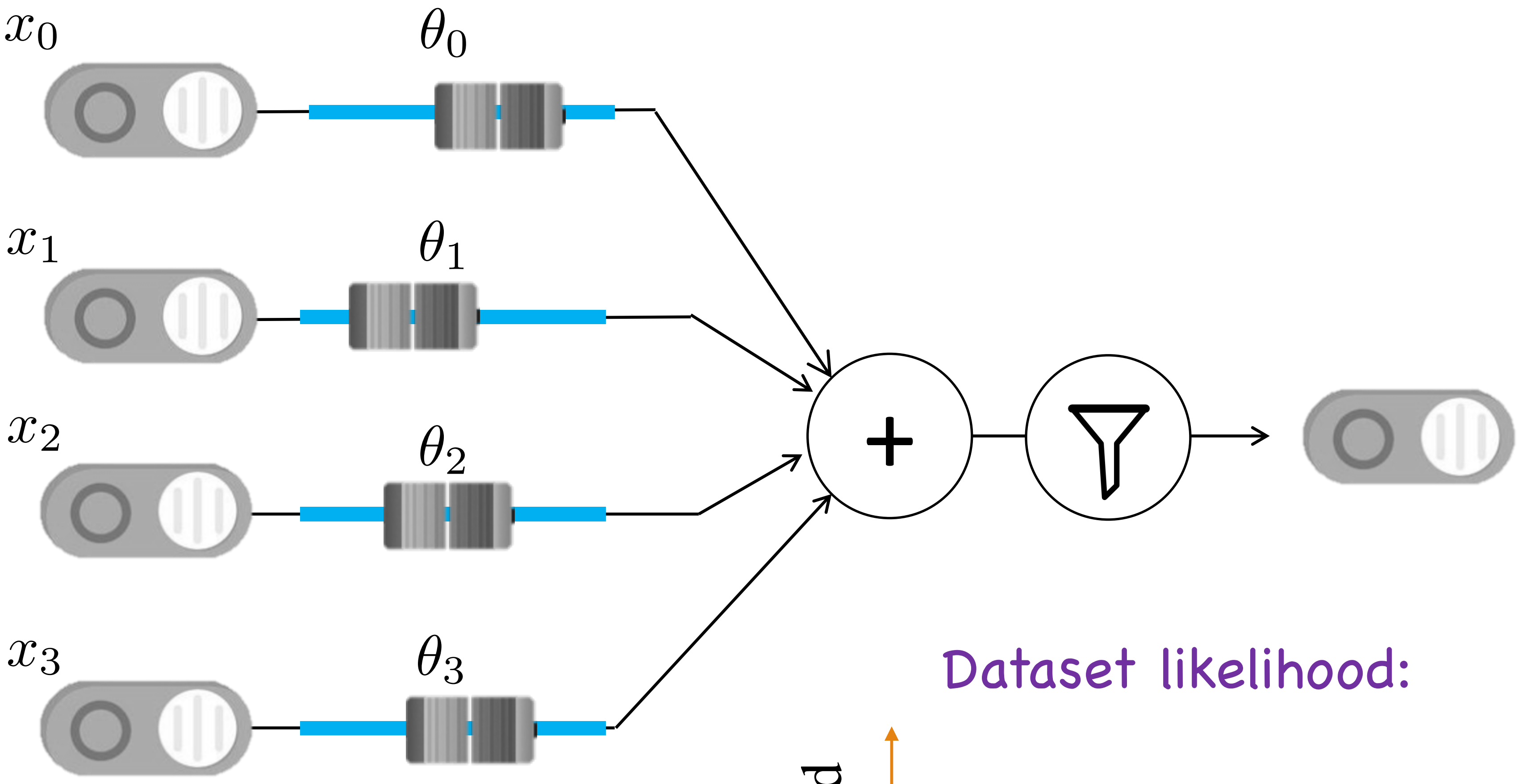
Training



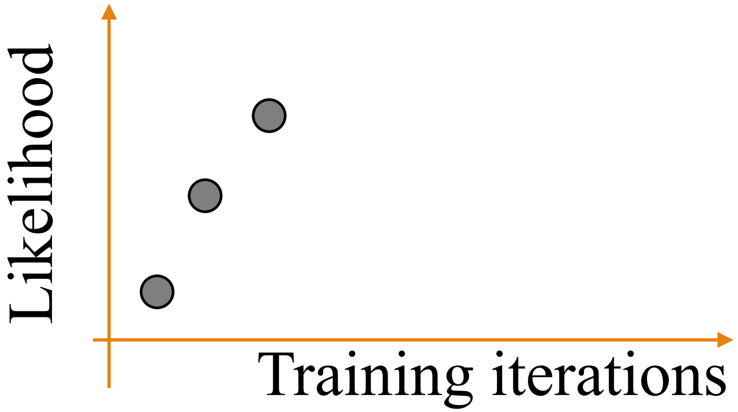
Dataset likelihood:



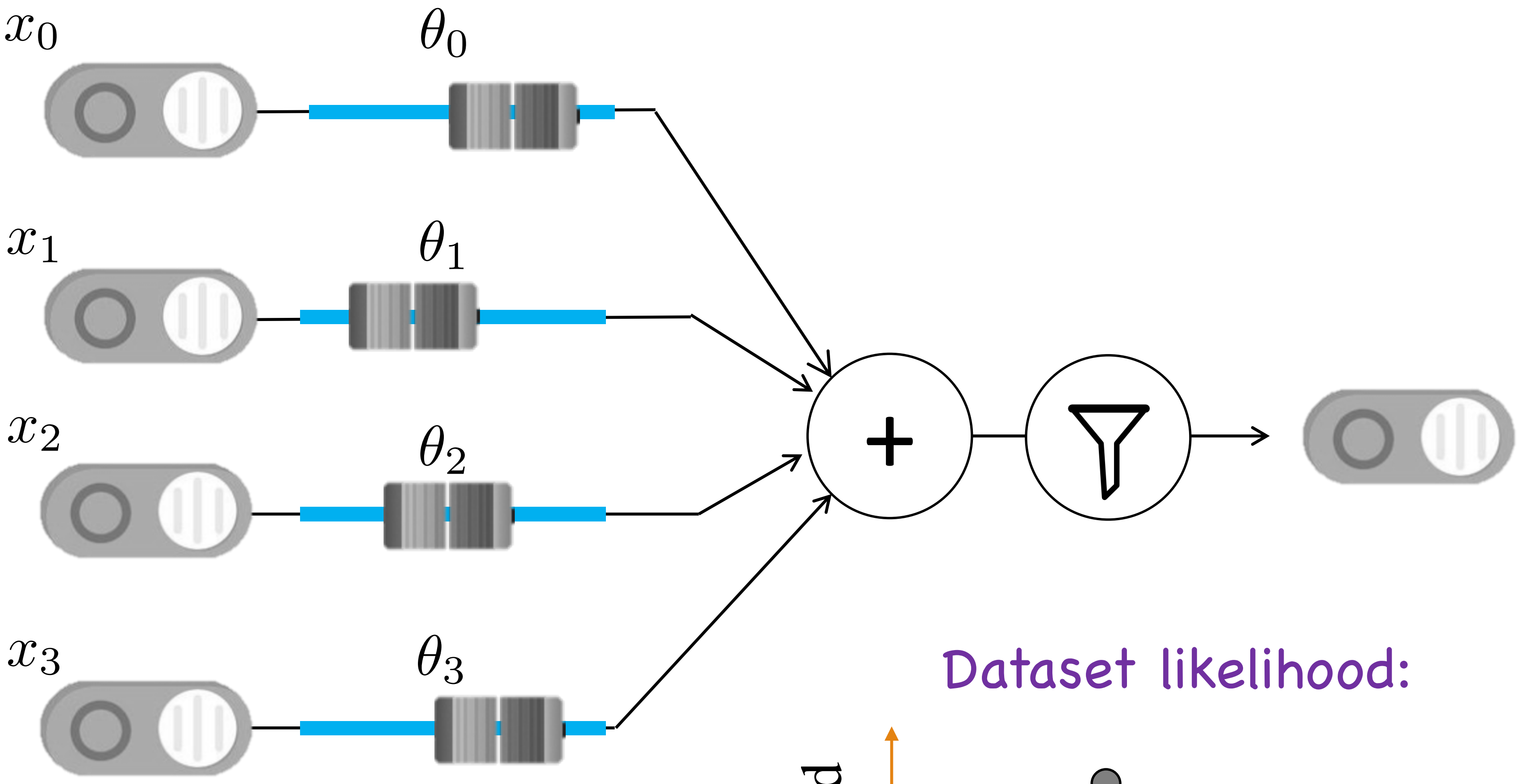
Training



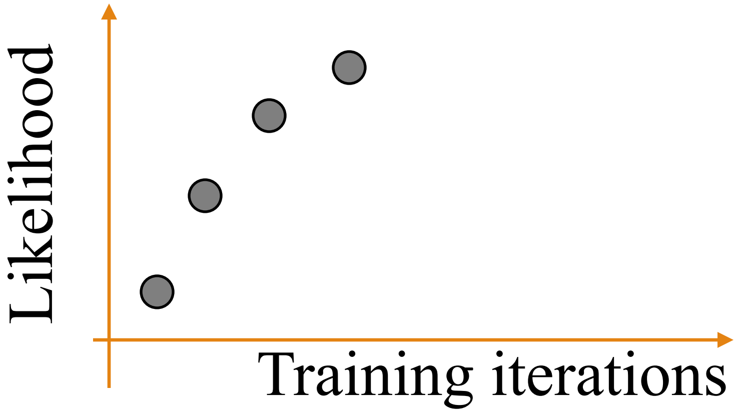
Dataset likelihood:



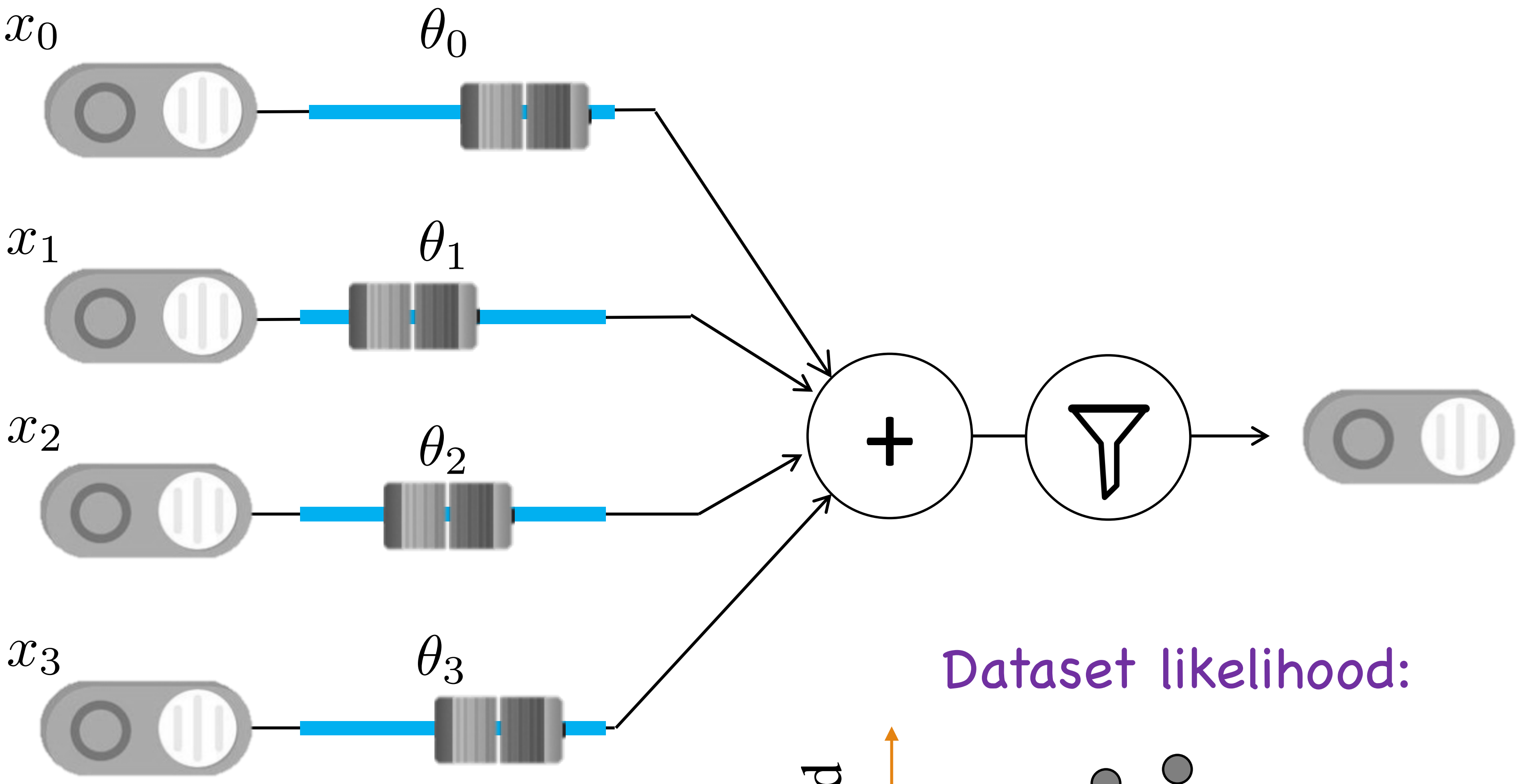
Training



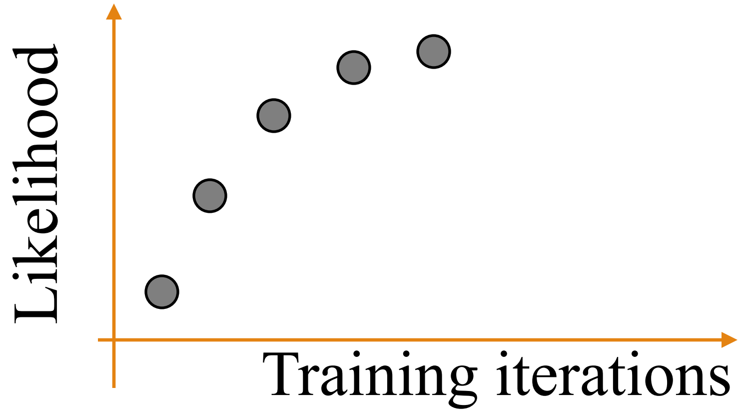
Dataset likelihood:



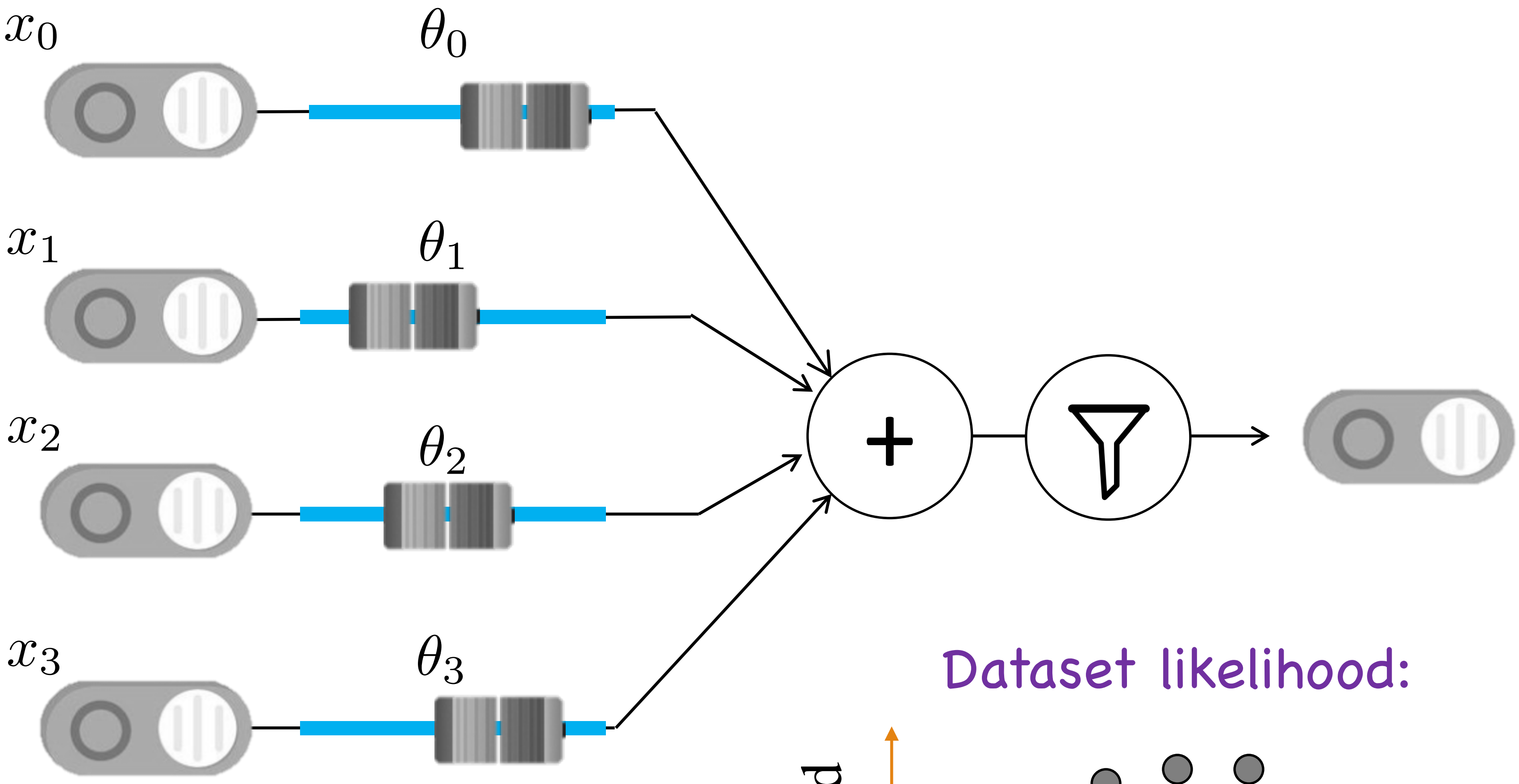
Training



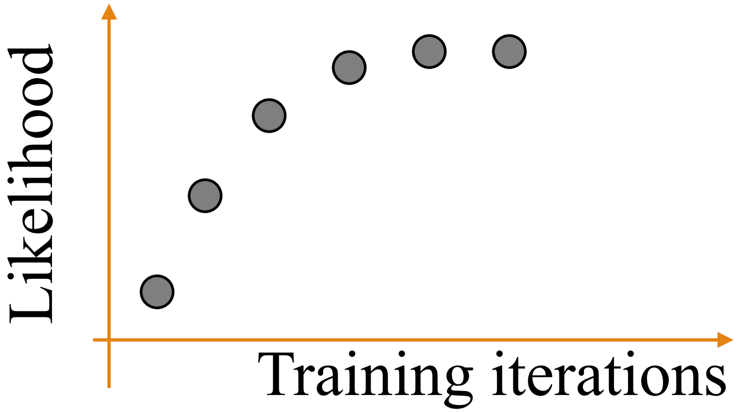
Dataset likelihood:



Training



Dataset likelihood:



Don’t forget:

x_j is j-th input variable and $x_0 = 1$.

Allows for θ_0 to be an intercept.

Classification with Logistic Regression

Training: determine parameters θ_j (for all $0 \leq j \leq m$)

- After parameters θ_j have been learned, test classifier

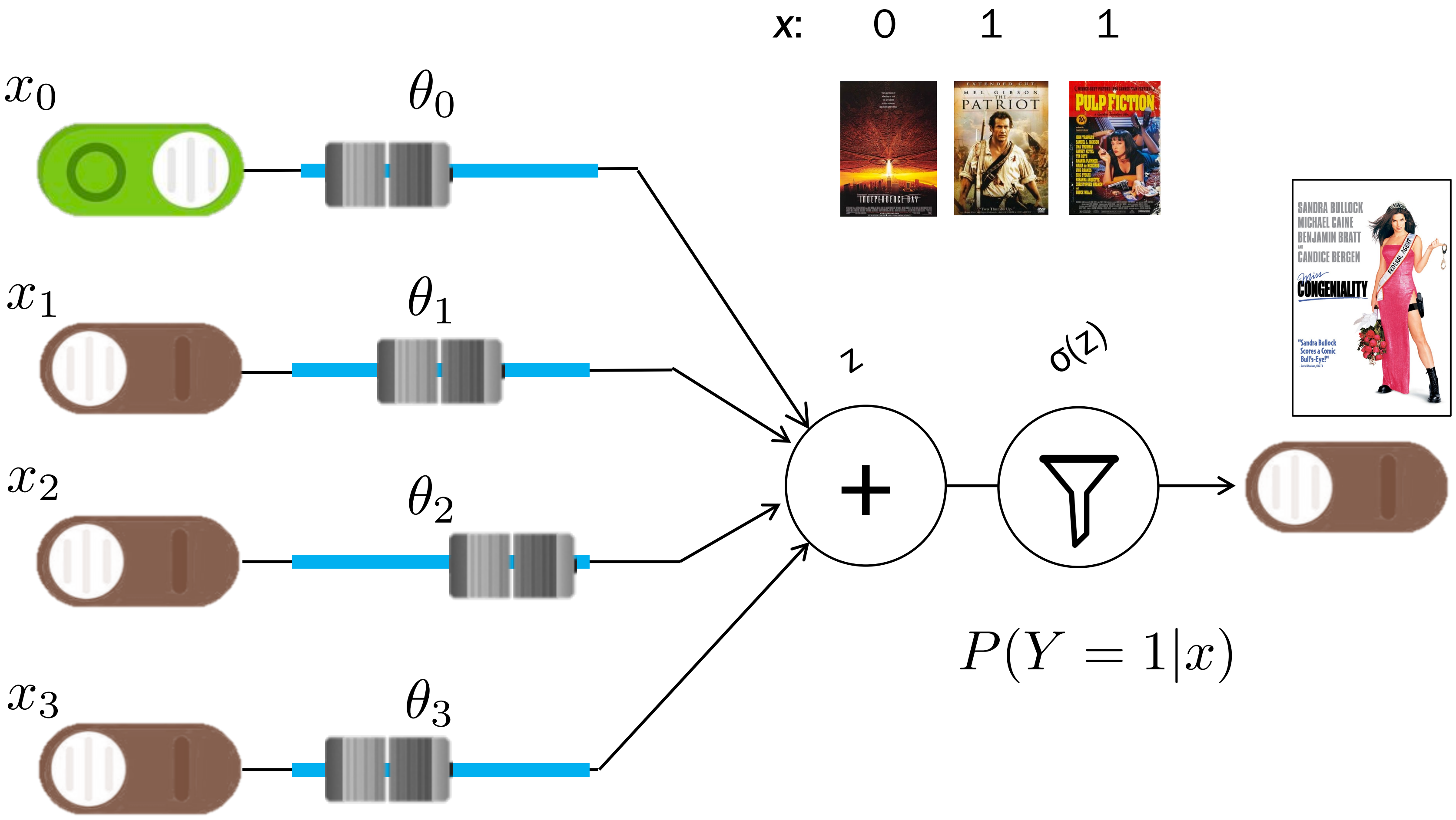
To test classifier, for each new (test) instance ***X***:

- Compute: $p = P(Y = 1 \mid \boldsymbol{X}) = \dfrac{1}{1 + e^{-z}}$, where $z = \theta^T \mathbf{x}$

- Classify instance as: $\hat{y} = \begin{cases} 1 & p > 0.5 \\ 0 & \text{otherwise} \end{cases}$

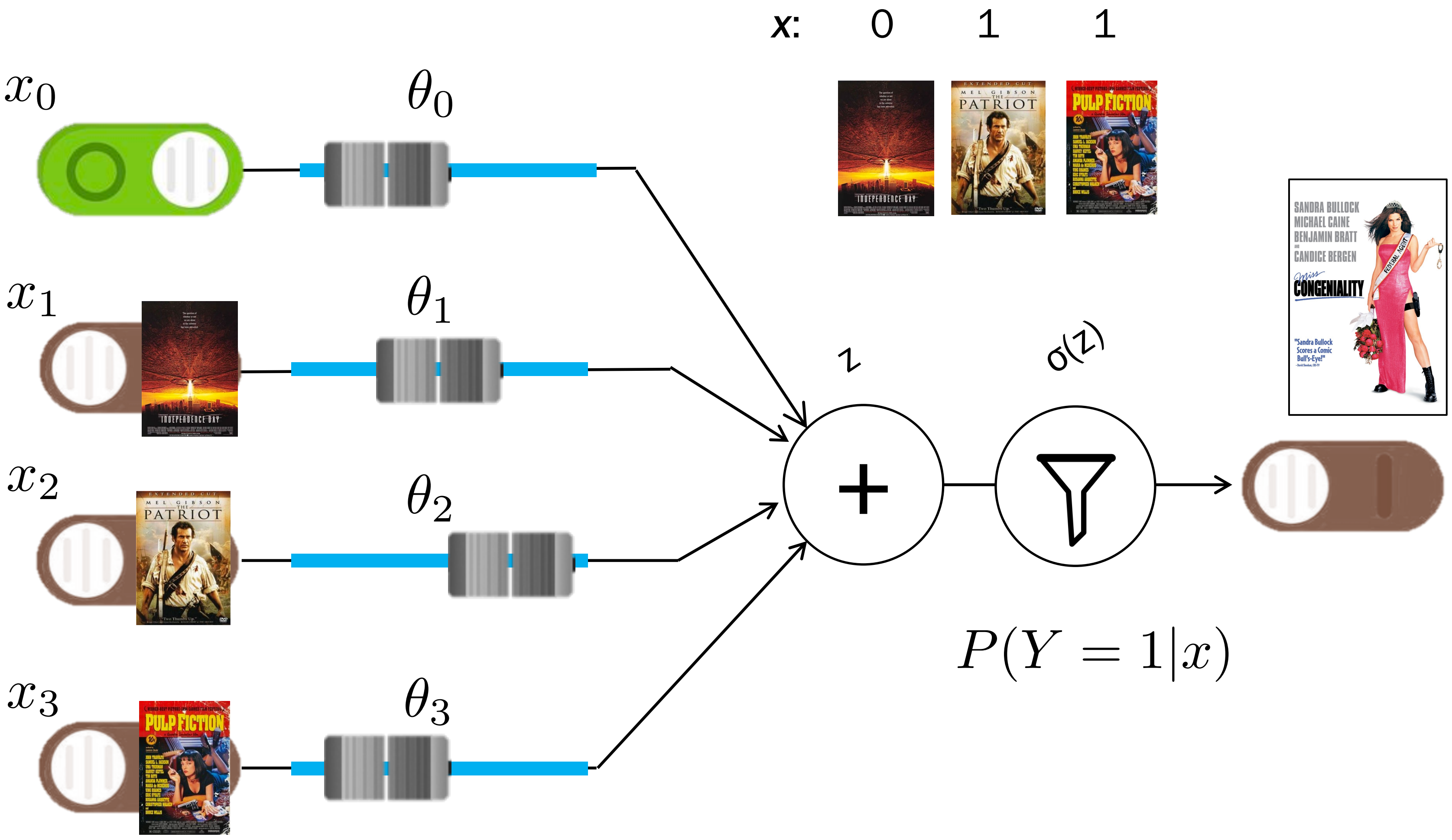
- Note about evaluation set-up: parameters θ_j are **not** updated during "testing" phase

Prediction



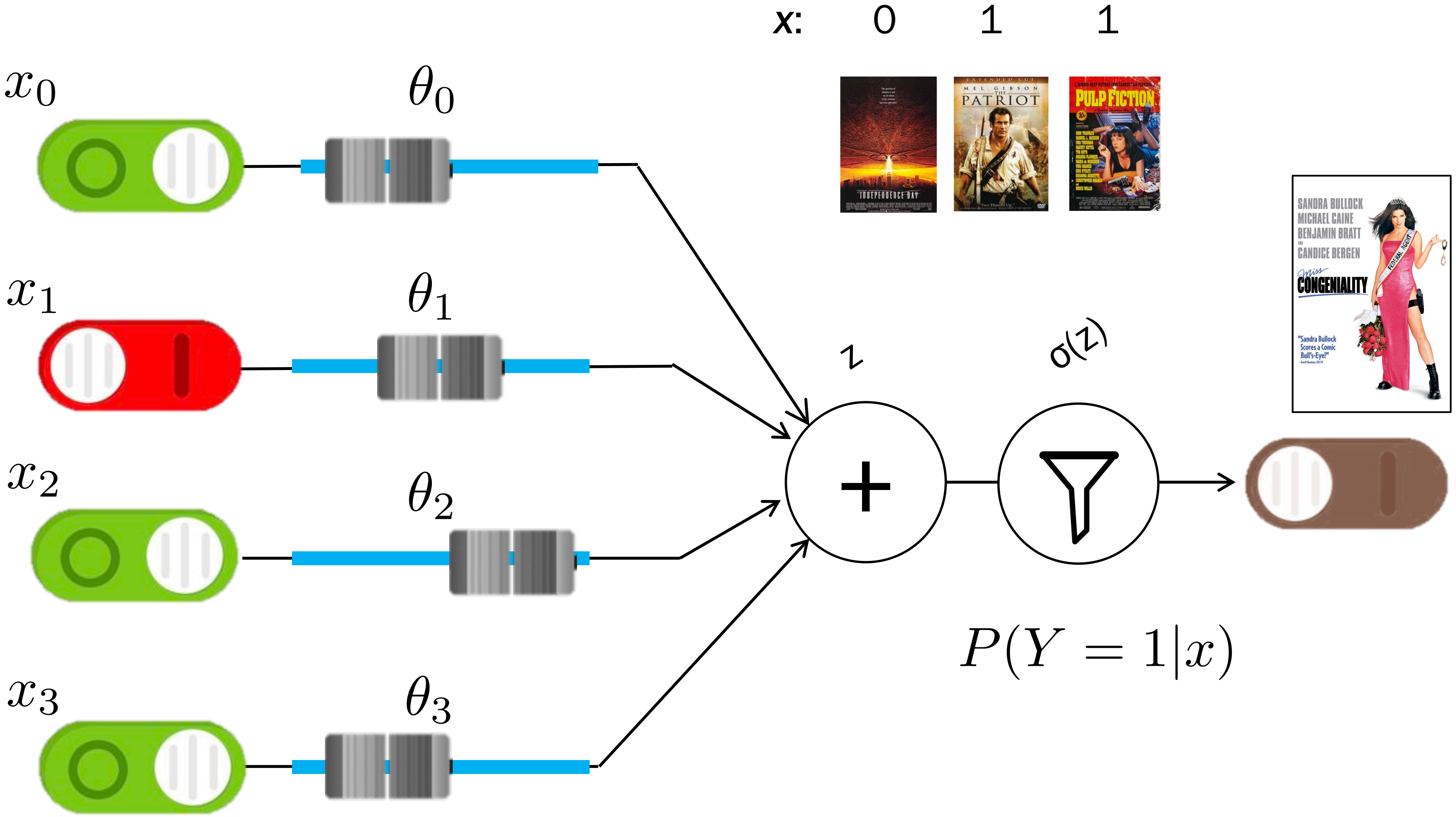
$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Prediction



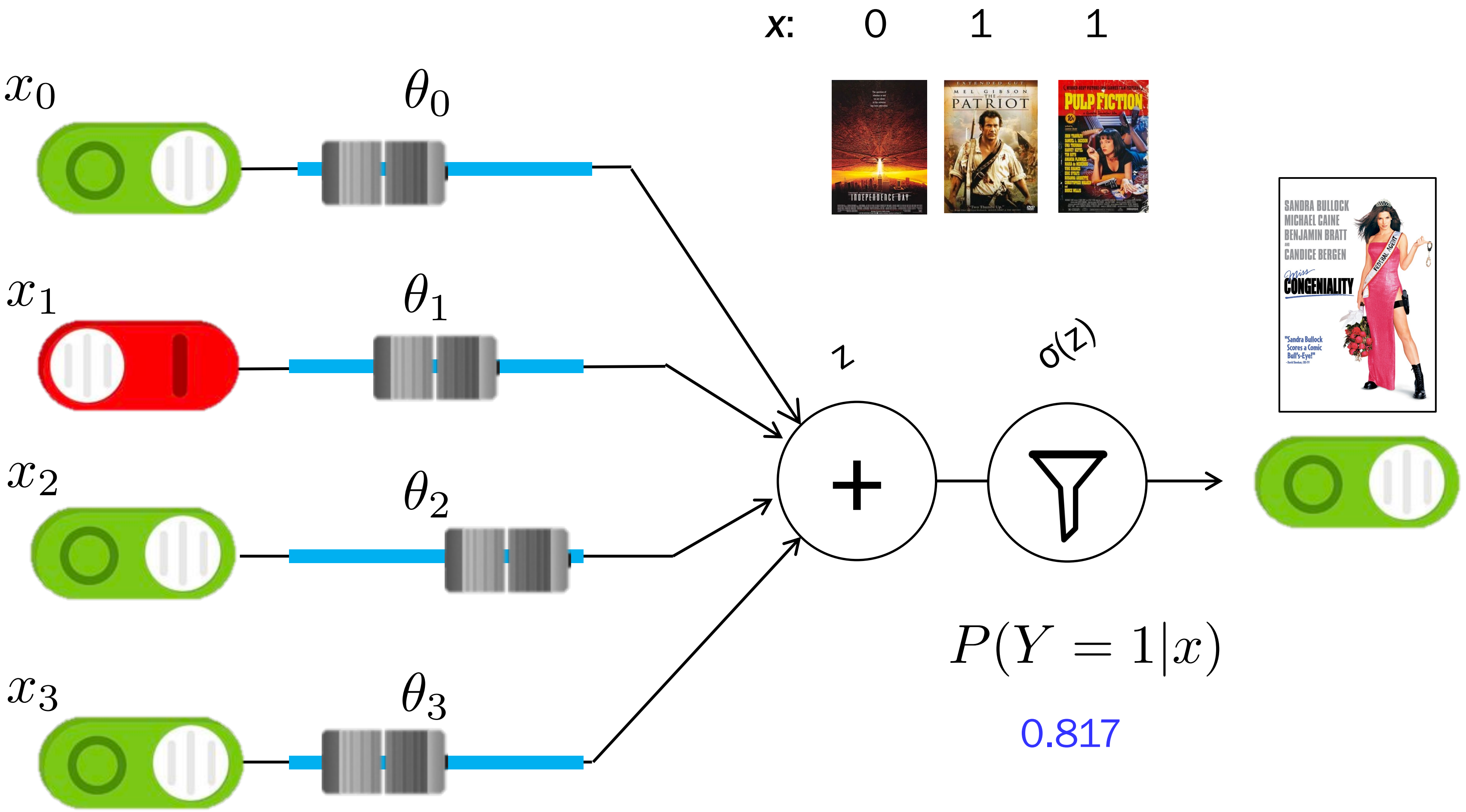
$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Prediction



$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Prediction



$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Chapter 2: How Come?

Logistic Regression

1 Make logistic regression assumption

$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T\mathbf{x})$$
$$P(Y=0|X=\mathbf{x}) = 1-\sigma(\theta^T\mathbf{x})$$

Often call this $\hat{y}$

2 Calculate the log probability for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T\mathbf{x}^{(i)}) + (1-y^{(i)}) \log[1-\sigma(\theta^T\mathbf{x}^{(i)})]$$

3 Get derivative of log probability with respect to thetas

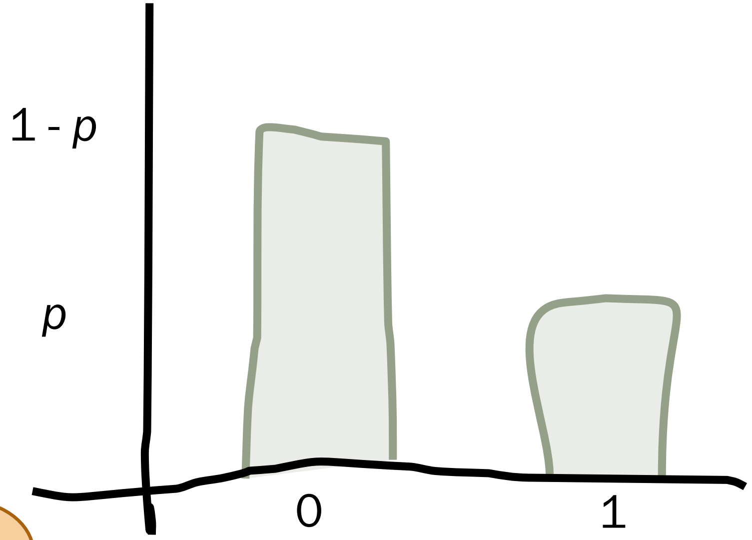
$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma(\theta^T\mathbf{x}^{(i)}) \right] x_j^{(i)}$$

How did we get that LL function?

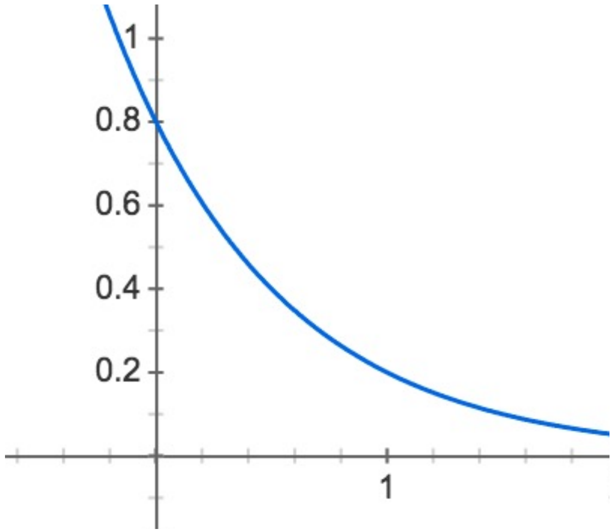
Recall: PMF of Bernoulli

- Y ~ Ber(*p*)
- Probability mass function: $P(Y = y)$

PMF of Bernoulli



PMF of Bernoulli (*p* = 0.2)



I want this smoother...

$$P(Y = y) = p^y (1 - p)^{1-y}$$

$$P(Y = y) = 0.2^y (0.8)^{1-y}$$

Log Probability of Data

$$P(Y=1|X=\mathbf{x}) = \sigma(\theta^T\mathbf{x})$$

$$P(Y=0|X=\mathbf{x}) = 1-\sigma(\theta^T\mathbf{x})$$

Implies

$$P(Y=y|X=\mathbf{x}) = \sigma(\theta^T\mathbf{x})^y \cdot \left[1-\sigma(\theta^T\mathbf{x})\right]^{(1-y)}$$

For IID data

$$L(\theta) = \prod_{i=1}^{n} P(Y=y^{(i)}|X=\mathbf{x}^{(i)})$$

$$= \prod_{i=1}^{n} \sigma(\theta^T\mathbf{x}^{(i)})^{y^{(i)}} \cdot \left[1-\sigma(\theta^T\mathbf{x}^{(i)})\right]^{(1-y^{(i)})}$$

Take the log

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)}\log\sigma(\theta^T\mathbf{x}^{(i)}) + (1-y^{(i)})\log[1-\sigma(\theta^T\mathbf{x}^{(i)})]$$

How did we get that gradient?

Sigmoid has a Beautiful Slope

True fact about
sigmoid functions

$$\frac{\partial}{\partial z}\sigma(z) = \sigma(z)[1 - \sigma(z)]$$

Sigmoid has a Beautiful Slope

$$\frac{\partial}{\partial \theta_j}\sigma(\theta^T x)?$$

$$\frac{\partial}{\partial z}\sigma(z) = \sigma(z)[1-\sigma(z)]$$

---

where $z = \theta^T x$

$$\frac{\partial}{\partial \theta_j}\sigma(\theta^T x) = \frac{\partial}{\partial z}\sigma(z) \cdot \frac{\partial z}{\partial \theta_j}$$

Chain rule!

$$\frac{\partial}{\partial \theta_j}\sigma(\theta^T x) = \sigma(\theta^T x)[1-\sigma(\theta^T x)]x_j$$

Plug and chug

Sigmoid, you should be a ski hill

Sigmoid has a Beautiful Slope

$$\hat{y} = \sigma(\theta^T x)$$

$$\frac{\partial \hat{y}}{\partial \theta_j} = \sigma(\theta^T x)[1 - \sigma(\theta^T x)]x_j$$

$$= \hat{y}(1 - \hat{y})x_j$$

Putting it all together!

I think I'm Ready…

$$\frac{\partial LL(\theta)}{\partial \theta_j}$$

Where

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

Think About Only One Training Instance

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}]$$

We only need to calculate the gradient for one training example!

$$\frac{\partial}{\partial x} \sum_{i} f(x, i) = \sum_{i} \frac{\partial}{\partial x} f(x, i)$$

We will pretend we only have one example

$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

We can sum up the gradients of each example to get the correct answer

Make it Simple

$$LL(\theta) = y \log \hat{y} + (1-y) \log[1-\hat{y}]$$

Where $\hat{y} = \sigma(\theta^T \mathbf{x})$

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \frac{\partial LL(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial \theta_j}$$

CHAIN RULZ!

$$= \frac{\partial LL(\theta)}{\partial \hat{y}} \hat{y}(1-\hat{y}) x_j$$

Already did that one

$$= \left[\frac{y}{\hat{y}} - \frac{1-y}{1-\hat{y}}\right] \hat{y}(1-\hat{y}) x_j$$

Derive this one

$$= (y - \hat{y}) x_j$$

Simplify

Now, all the data

$$LL(\theta) = \sum_{i=1}^{n} y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}]$$

$$\hat{y}^{(i)} = \sigma(\theta^T \mathbf{x}^{(i)})$$

Derivative of sum…

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_j} \left[y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}] \right]$$

$$= \sum_{i=1}^{n} [y^{(i)} - \hat{y}^{(i)}] x_j^{(i)}$$

See last slide

$$= \sum_{i=1}^{n} [y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)})] x_j^{(i)}$$

Some people don't like hats…

Now, all the data

$$\frac{\partial LL(\theta)}{\partial \theta_j}$$

$$= \sum_{i=1}^{n} [y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)})] x_j^{(i)}$$

Logistic Regression

1 Make logistic regression assumption

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$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

The Hard Way

$$LL(\theta) = y \log \sigma(\theta^T \mathbf{x}) + (1-y) \log[1 - \sigma(\theta^T \mathbf{x})]$$

$$\begin{aligned}
\frac{\partial LL(\theta)}{\partial \theta_j} &= \frac{\partial}{\partial \theta_j} y \log \sigma(\theta^T \mathbf{x}) + \frac{\partial}{\partial \theta_j} (1-y) \log[1 - \sigma(\theta^T \mathbf{x}] \\
&= \left[\frac{y}{\sigma(\theta^T x)} - \frac{1-y}{1-\sigma(\theta^T x)} \right] \frac{\partial}{\partial \theta_j} \sigma(\theta^T x) \\
&= \left[\frac{y}{\sigma(\theta^T x)} - \frac{1-y}{1-\sigma(\theta^T x)} \right] \frac{\partial}{\partial \theta_j} \sigma(\theta^T x) \\
&= \left[\frac{y - \sigma(\theta^T x)}{\sigma(\theta^T x)[1-\sigma(\theta^T x)]} \right] \sigma(\theta^T x)[1-\sigma(\theta^T x)] x_j \\
&= \left[y - \sigma(\theta^T x) \right] x_j
\end{aligned}$$

Phew!

Chapter 3: Philosophy
(if time)

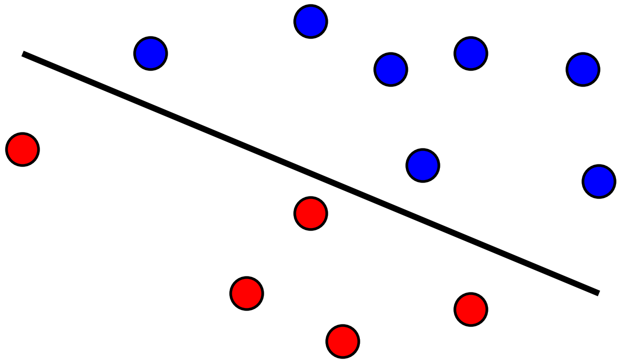
Choosing an Algorithm?

Many trade-offs in choosing learning algorithm

- Continuous input variables
  - Logistic Regression easily deals with continuous inputs
  - Naive Bayes needs to use some parametric form for continuous inputs (e.g., Gaussian) or "discretize" continuous values into ranges (e.g., temperature in range: <50, 50-60, 60-70, >70)
- Discrete input variables
  - Naive Bayes naturally handles multi-valued discrete features by using multinomial distribution for $P(X_i \mid Y)$
  - Logistic Regression requires some sort of representation of multi-valued discrete data (e.g., one hot vector)
  - Say $X_i \in \{A, B, C\}$. Not necessarily a good idea to encode X_i as taking on input values 1, 2, or 3 corresponding to A, B, or C.

Discrimination Intuition

- Logistic regression is trying to fit a **line** that separates data instances where *y* = 1 from those where *y* = 0



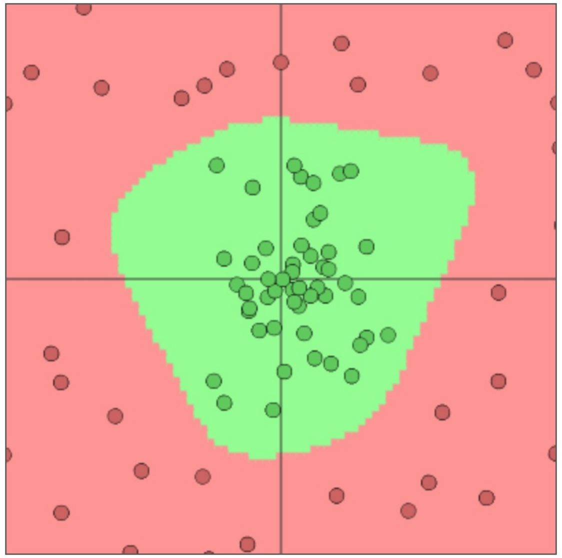
$$\theta^T \mathbf{x} = 0$$

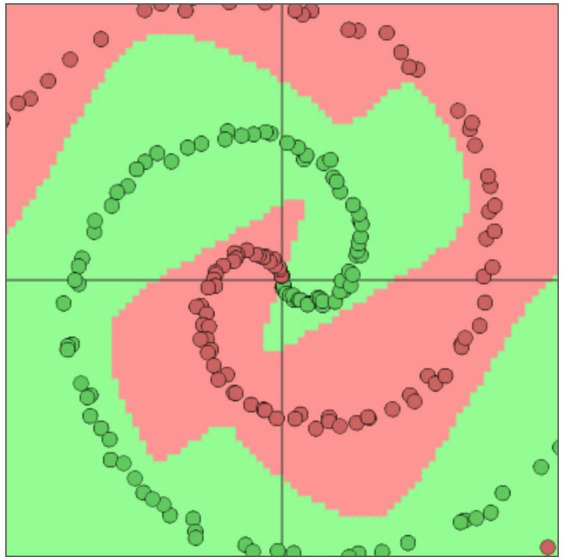
$$\theta_0 x_0 + \theta_1 x_1 + \cdots + \theta_m x_m = 0$$

- We call such data (or the functions generating the data) "**linearly separable**"
- **Naïve bayes is linear too** as there is no interaction between different features.

Some Data Not Linearly Seperable

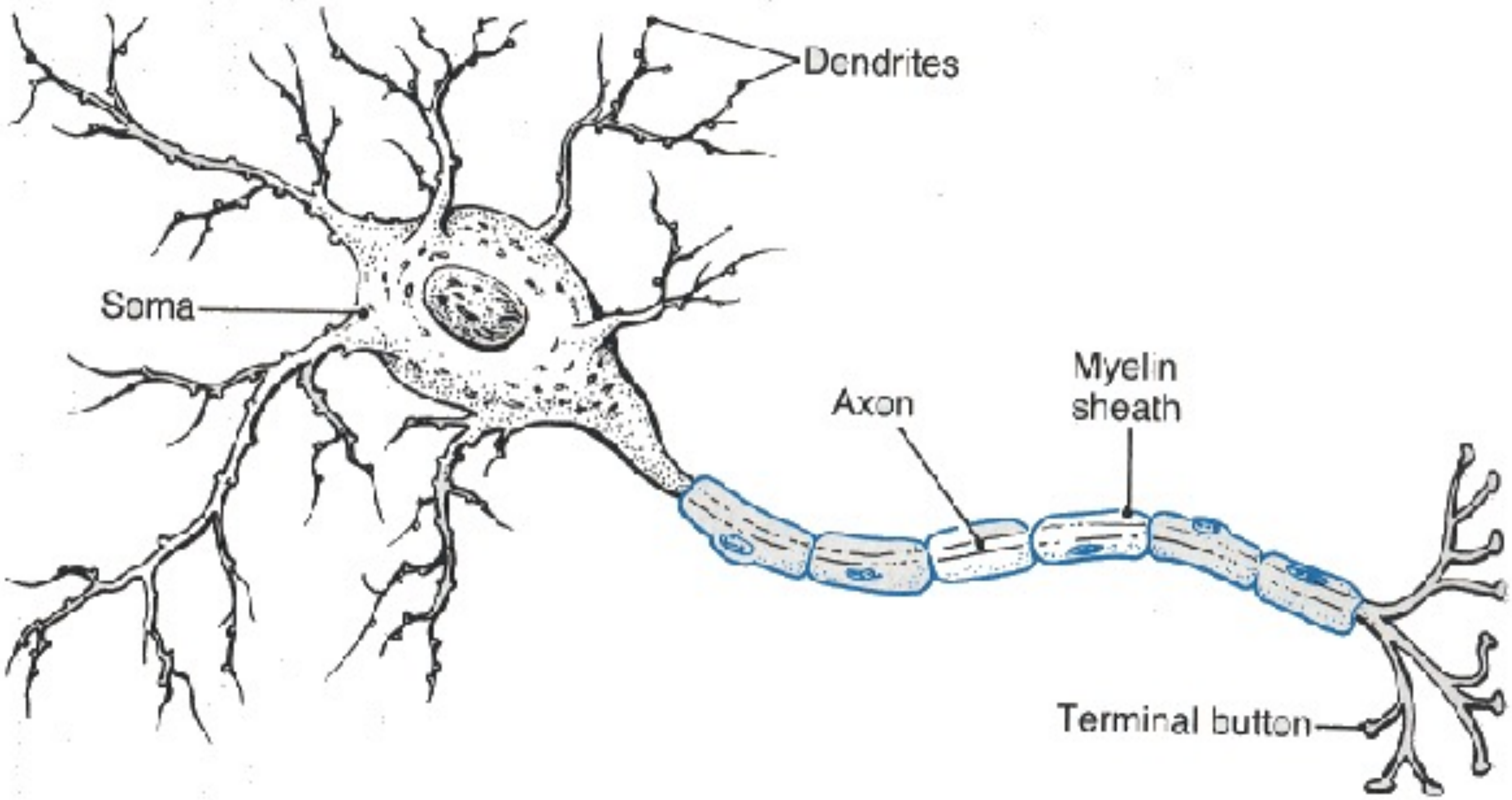
Some data sets/functions are not separable



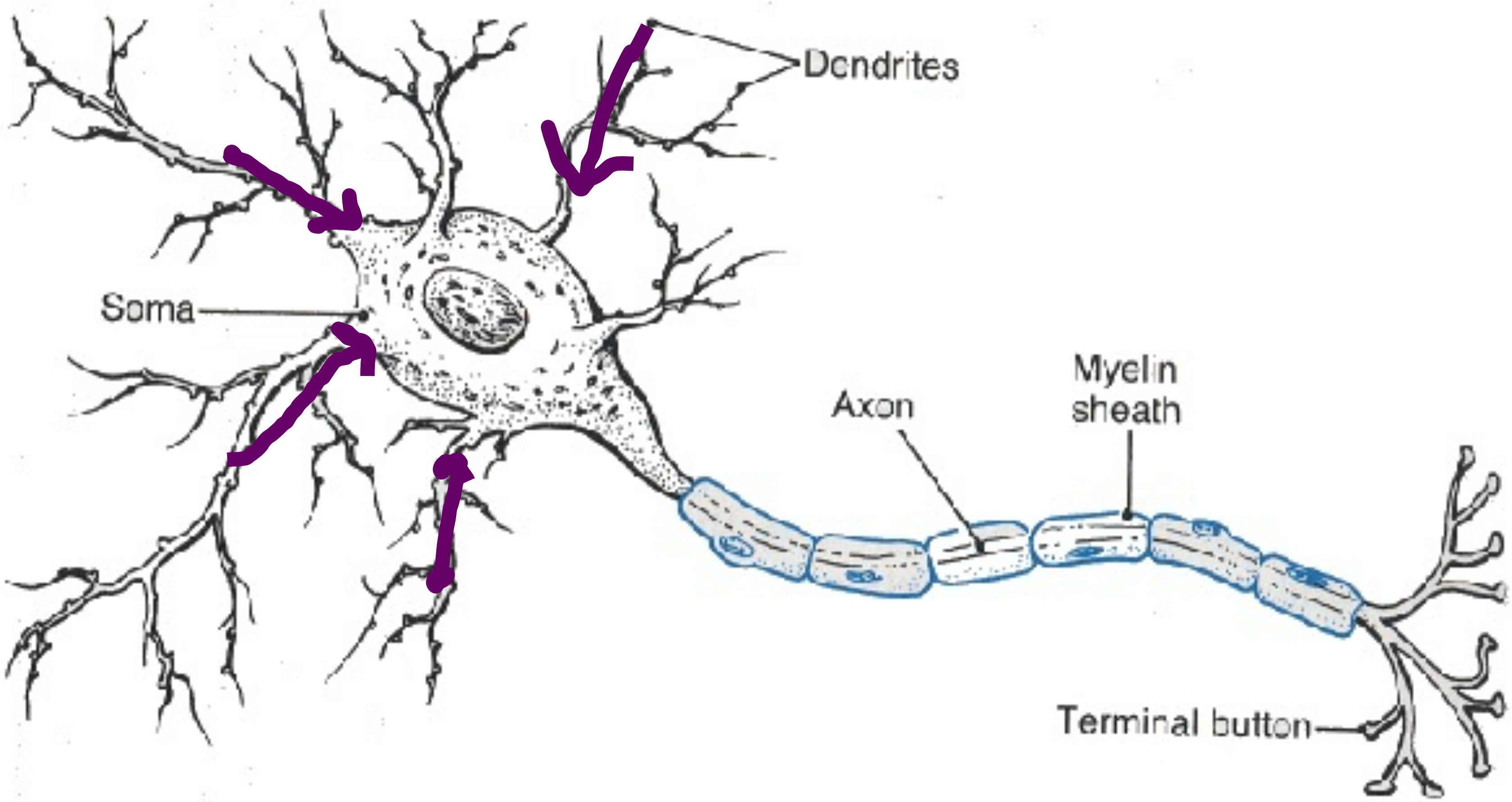


- Not possible to draw a line that successfully separates all the $y = 1$ points (green) from the $y = 0$ points (red)
- Despite this fact, logistic regression and Naive Bayes still often work well in practice

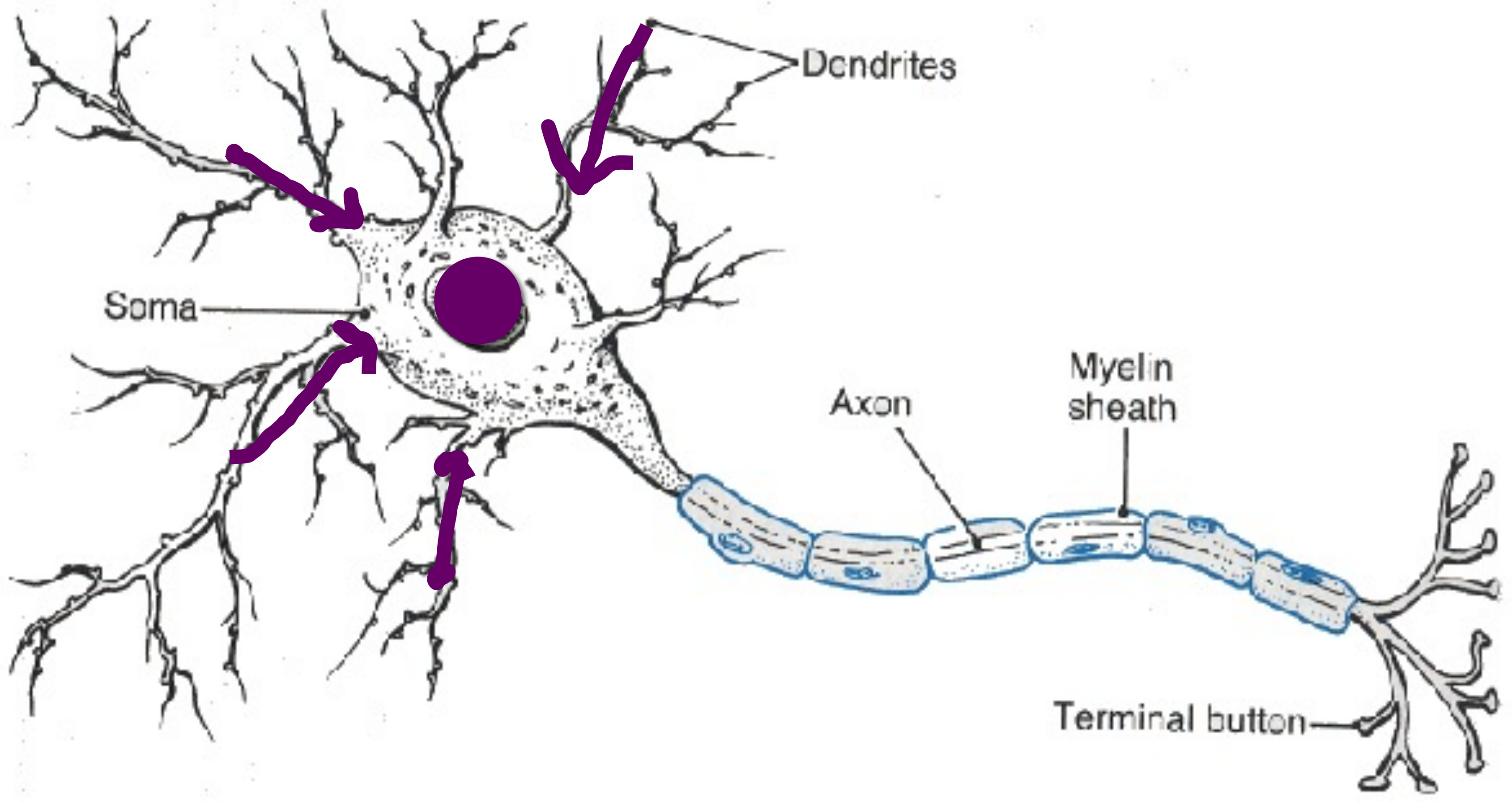
Neuron



Neuron



Neuron



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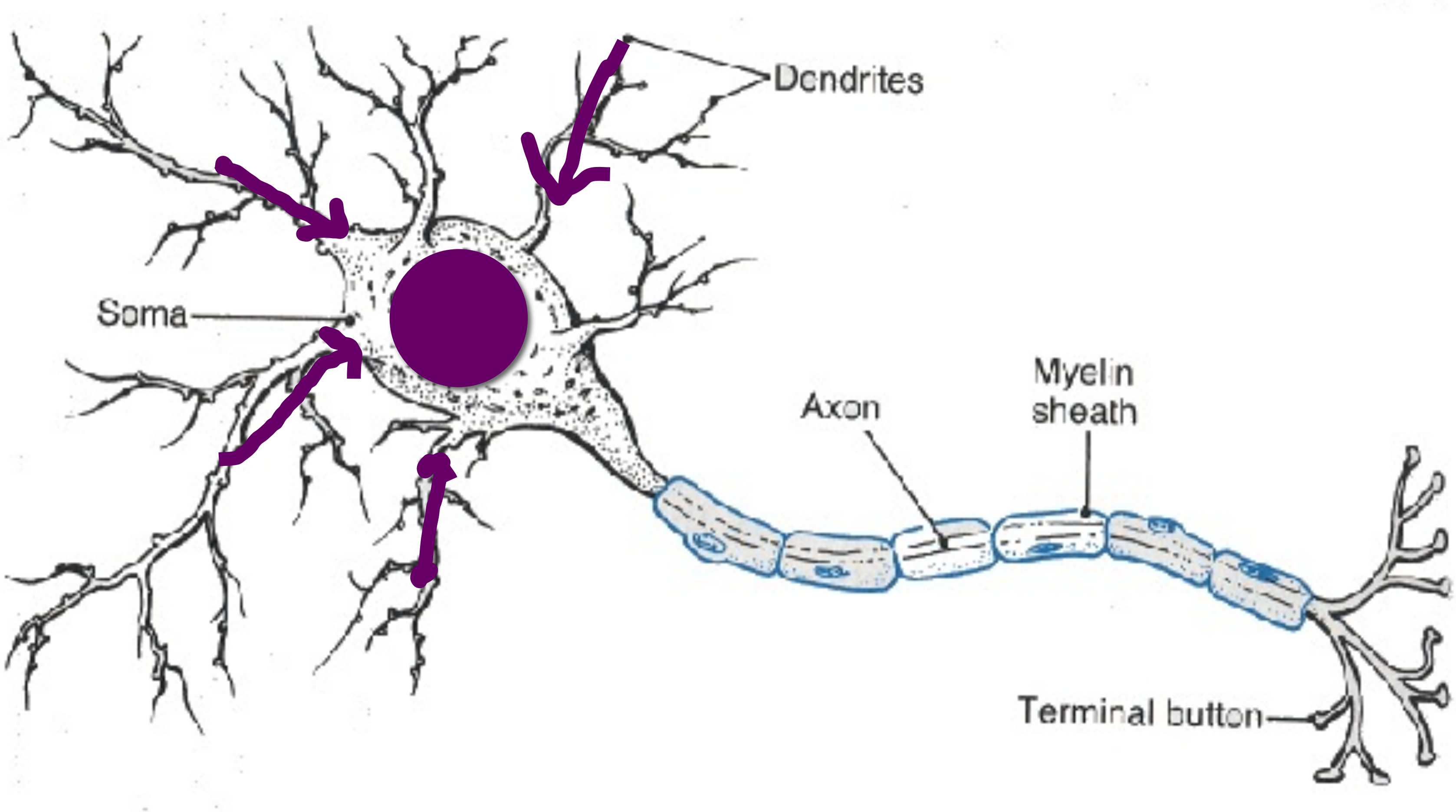
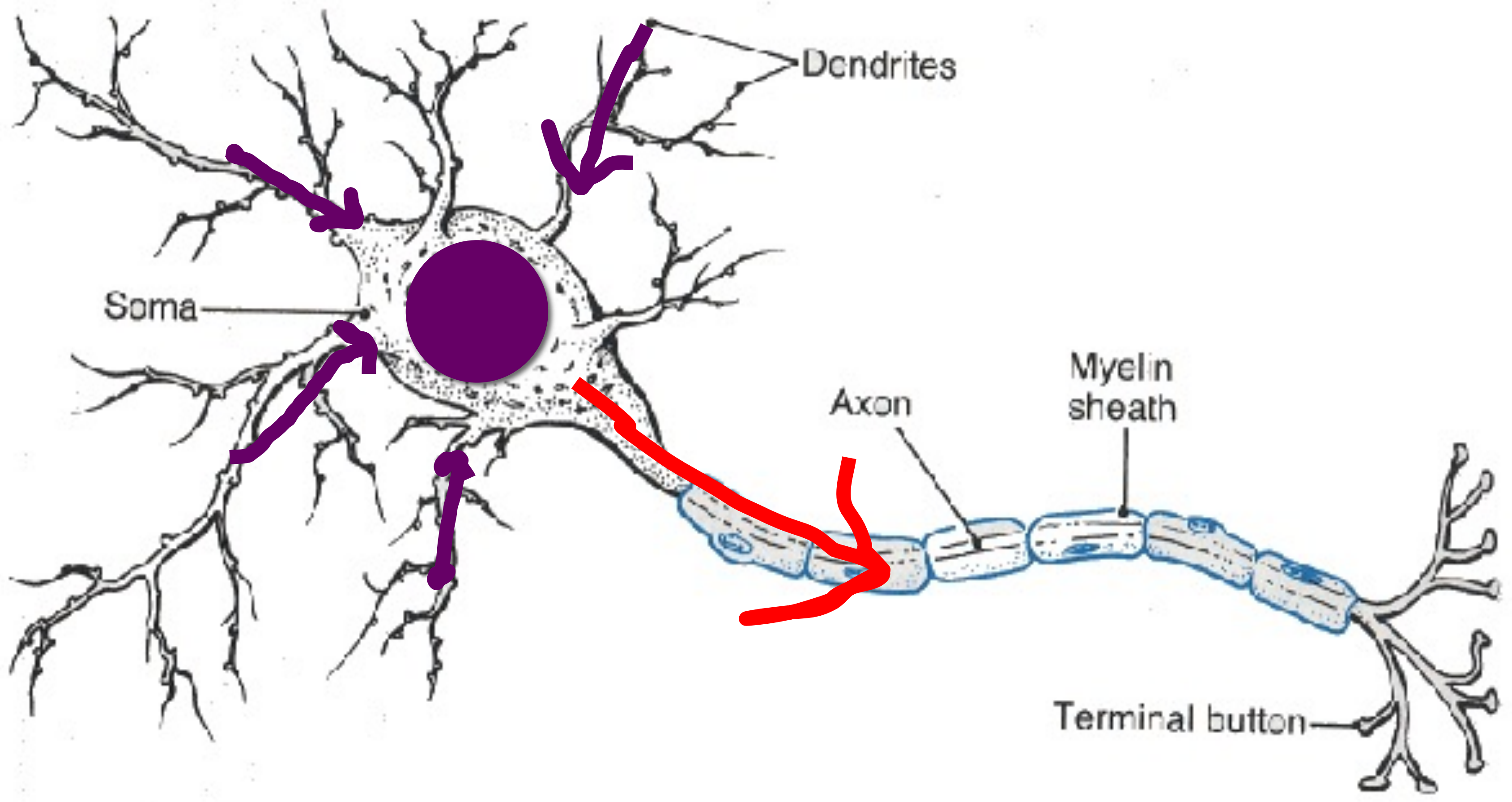
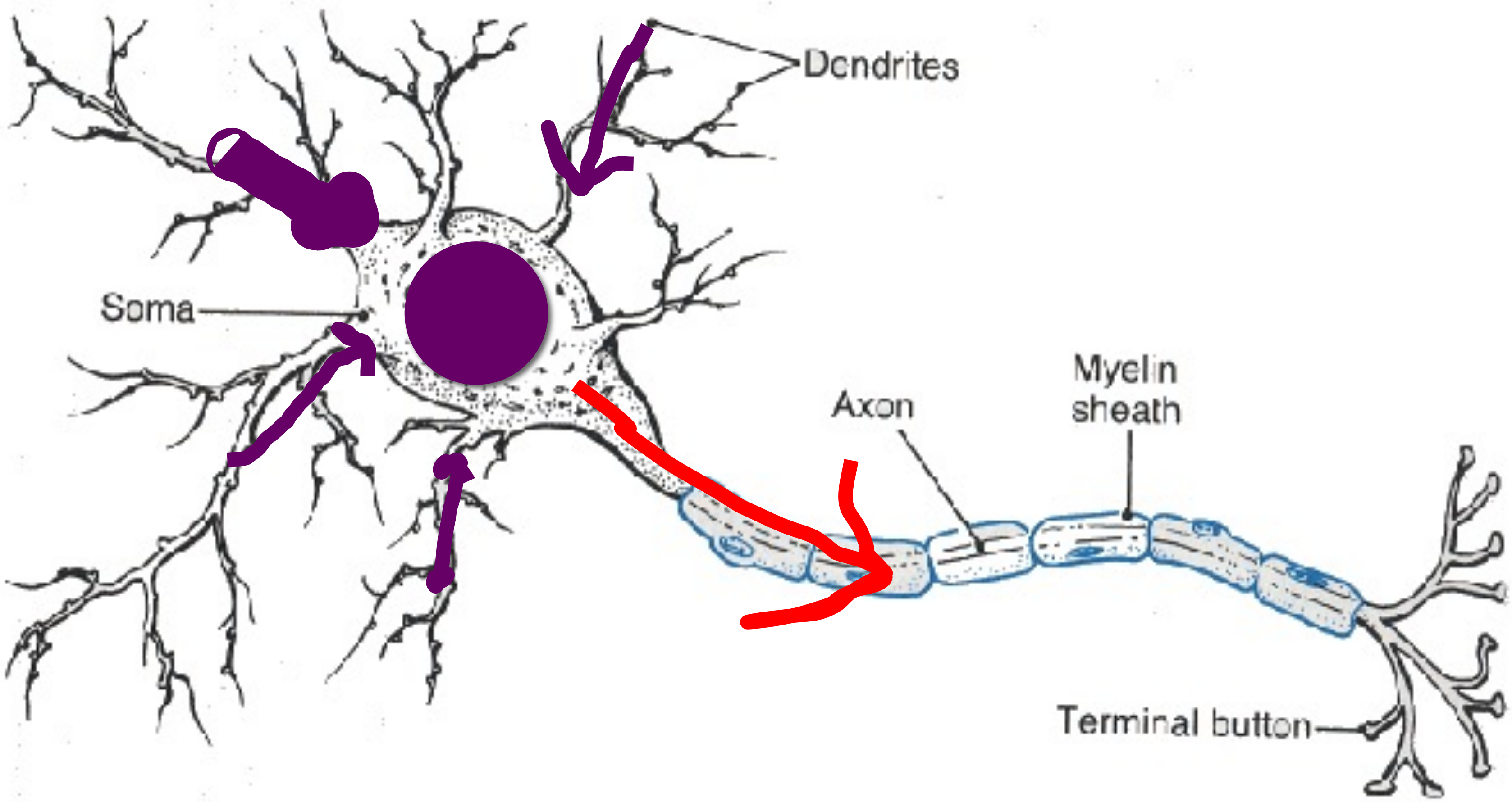


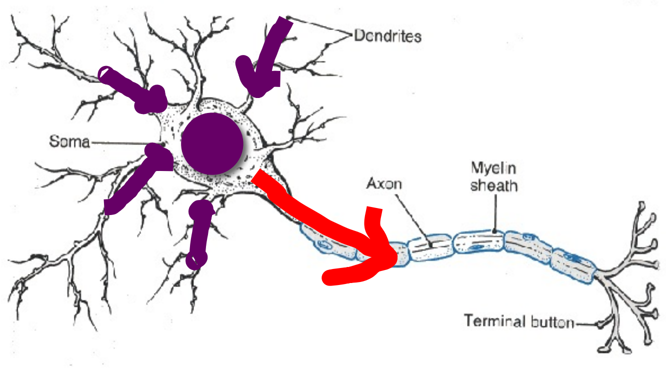
Neuron

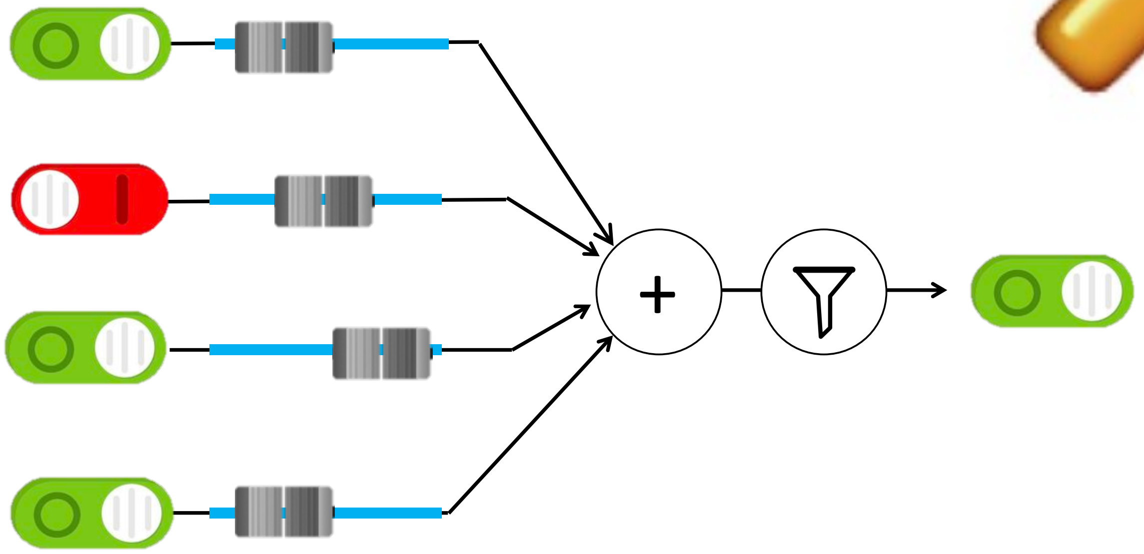


Some inputs are more important



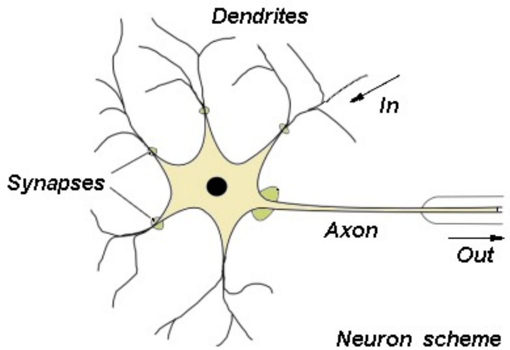
Artificial Neurons

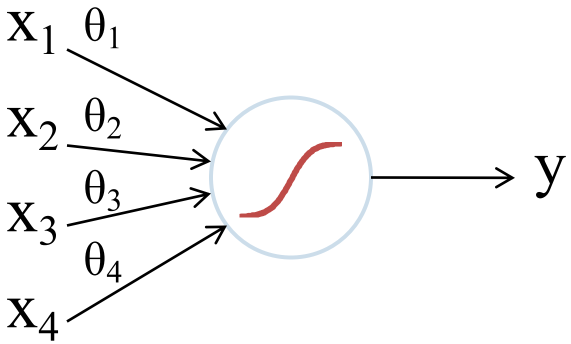




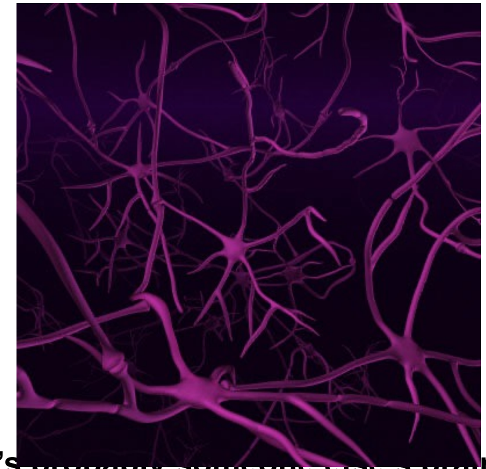
Biological Basis for Neural Networks

A neuron

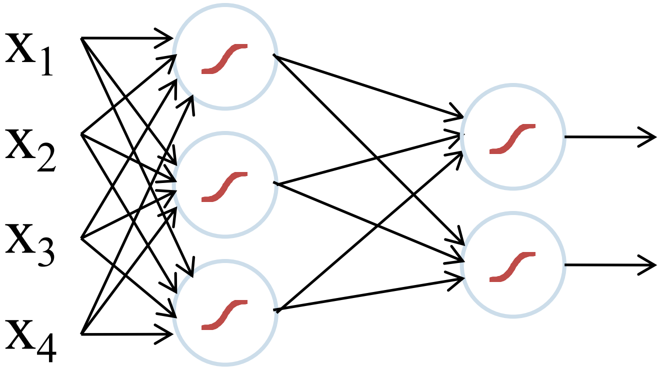




Your brain



Actually, it's probably someone else's brain

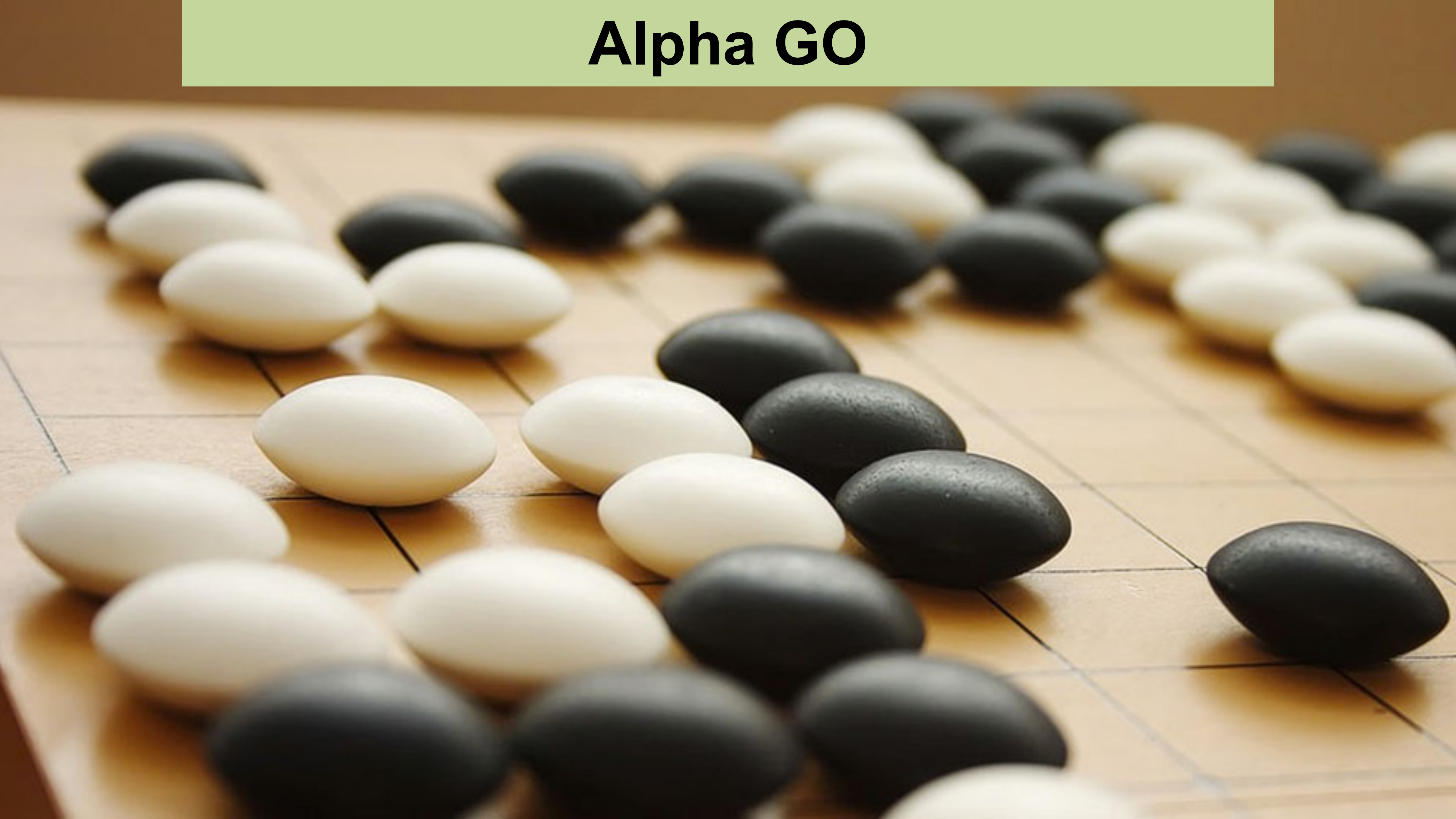


(aka Neural Networks)

Deep learning is (at its core) many logistic regression pieces stacked on top of each other.

Computer Vision

Alpha GO



Revolution in AI

Computers Making Art

Logistic Regression is the last tool we will present you!

Learning Goal: Abundance of important problems

Last Class…

CS109

AI

Uncertainty Theory

Single Random Variables

Probabilistic Models

Counting

Probability Fundamentals

Counting Rules

Counting operations on *n* objects

Sort (order matters) (perms)

- Distinct: $n!$
- Some Distinct: $\dfrac{n!}{n_1!n_2!\ldots}$

Choose *k* (combinations)

- Distinct: $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Put in *r* buckets

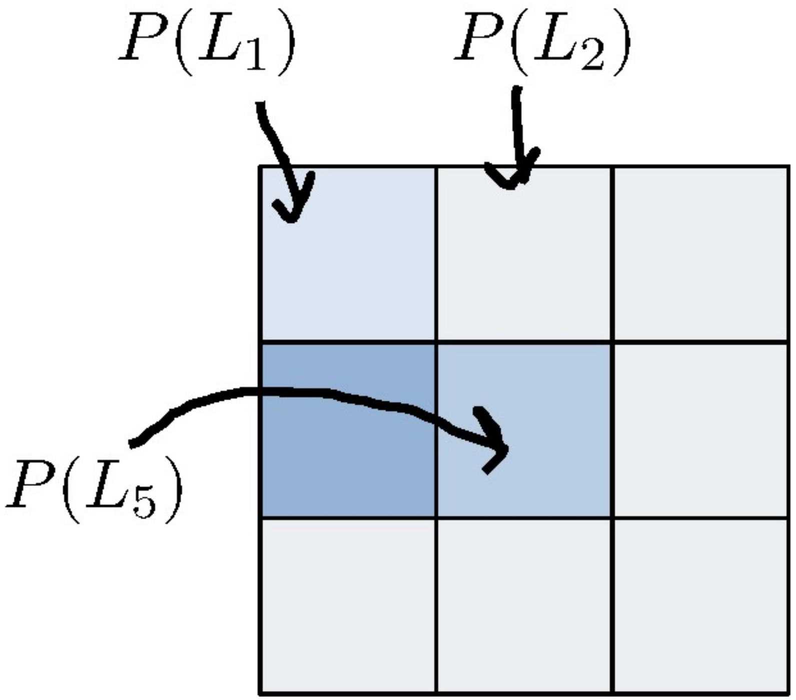
- Distinct: r^n
- None Distinct: $\dfrac{(n+r-1)!}{n!(r-1)!}$

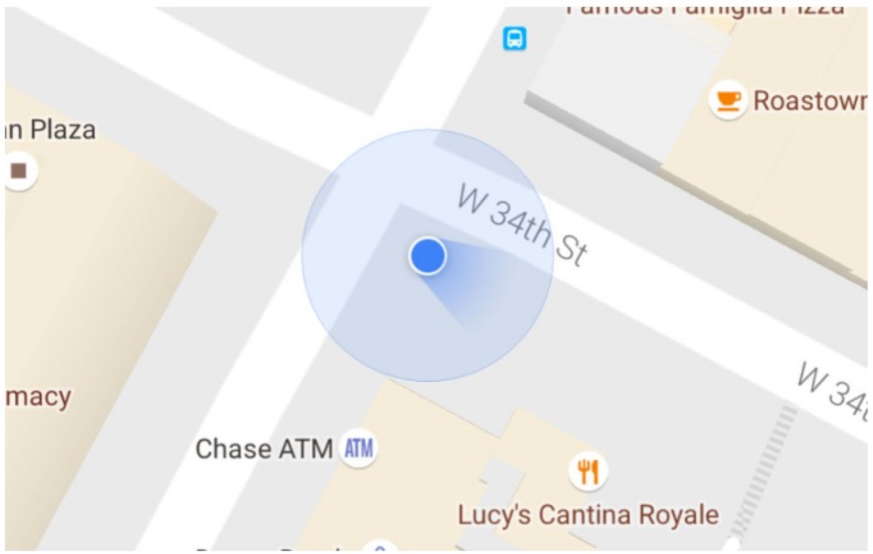
Counting

Ayesha Tim Irina Joey Waddie

Trailing the dovetail shuffle to it’s lair – Persi Diaconosis

Update Belief

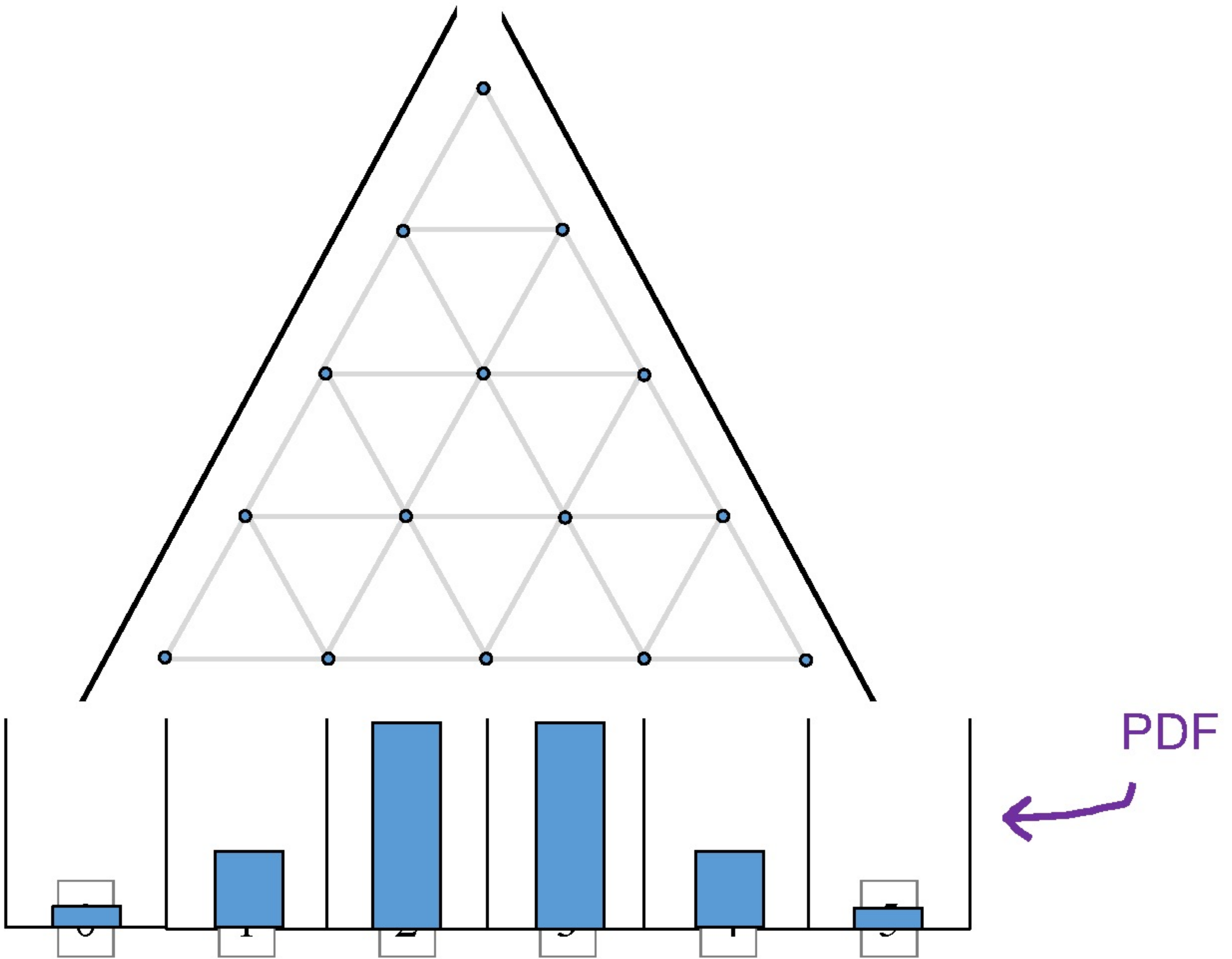




Before Observation

Random Variables

Binomial

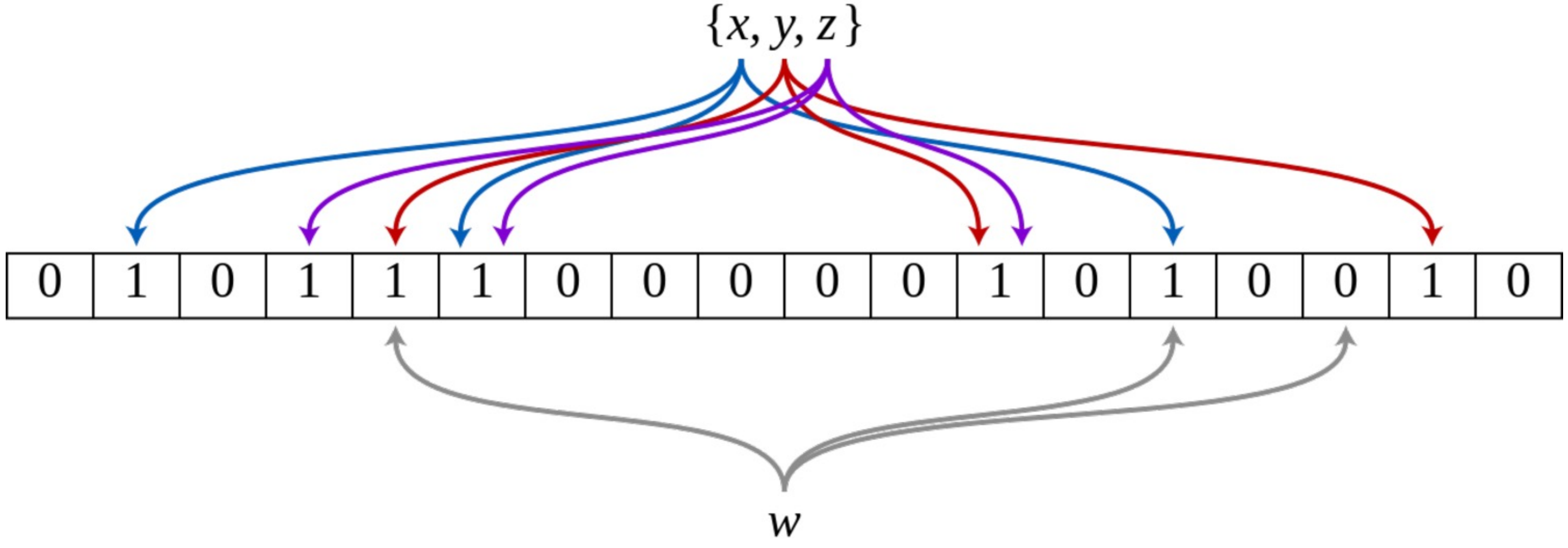


Geometric

Sequence 1:

TTHHTHTTHTTTHTTTHTTTHTTHTHHTH
HTTHHTTHHHTHHTHTTHTHTTHHTHHHT
HTHHTHHHTHTHTTHTTHHTHTHTHTTHH
TTHHTHTTHTHTHTHTHTHTHHHTHTHTH
TTHHTHTHTHTHHHTTHHTHTTTHTHHTH

Bloom Filter

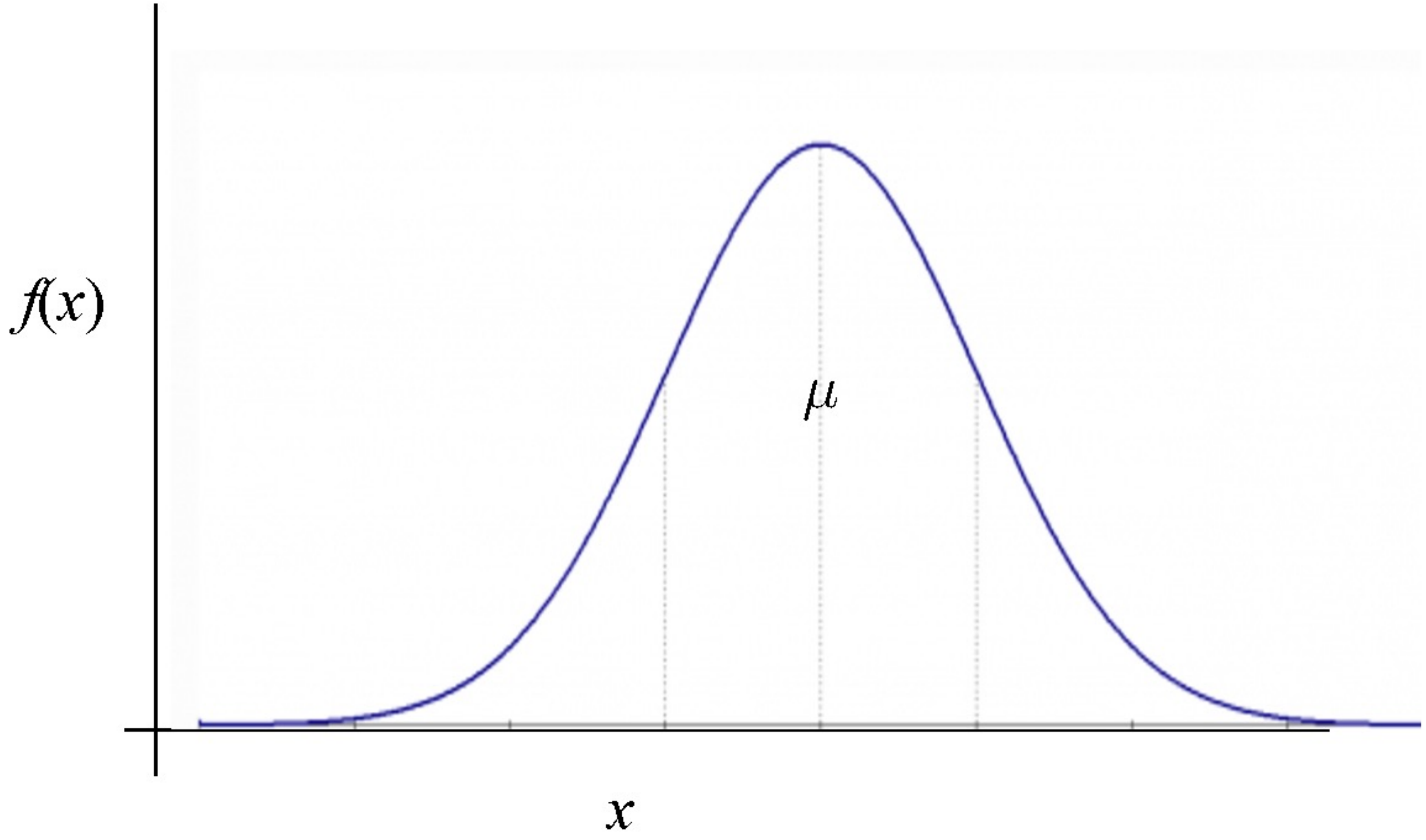


random()?

Probability Density Function

$$\mathcal{N}(\mu, \sigma^2)$$

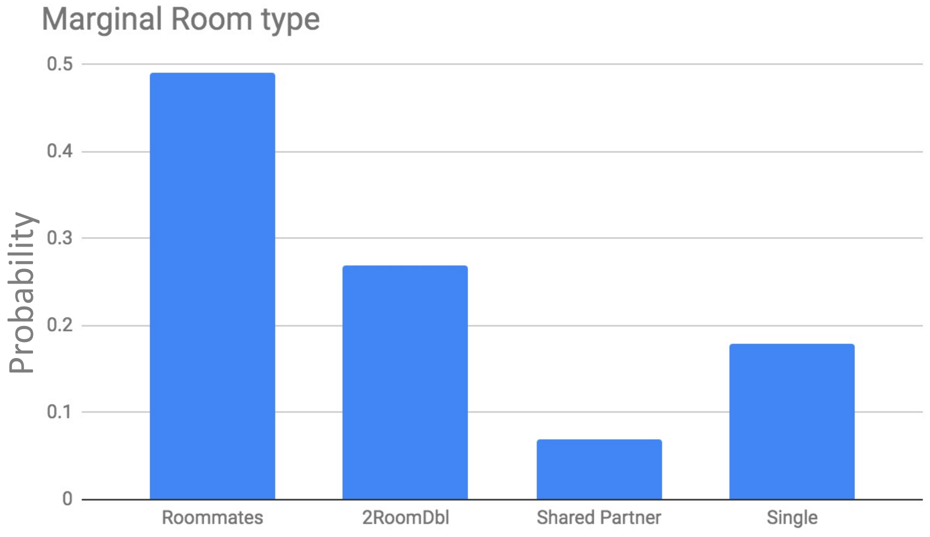
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

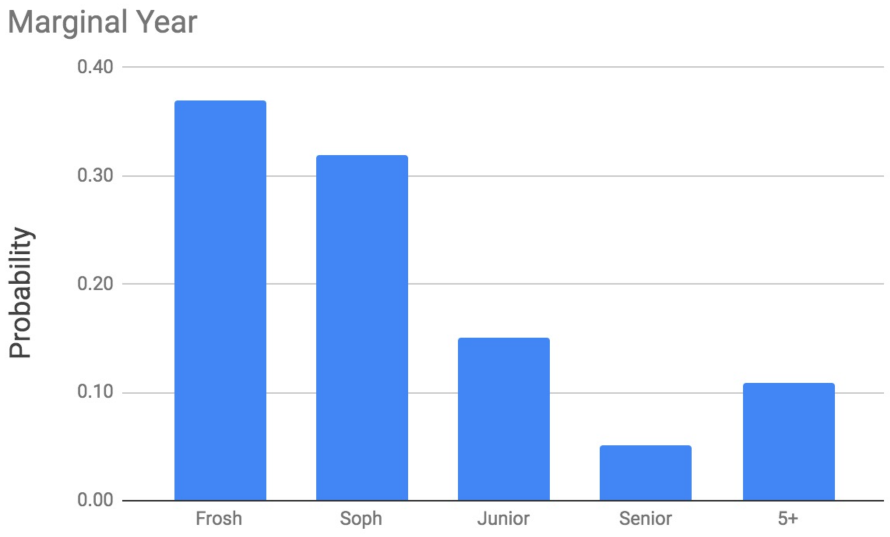


Probabilistic Models

Joint Probability Table

	Roommates	2RoomDbl	Shared Partner	Single	
Frosh	0.30	0.07	0.00	0.00	0.37
Soph	0.12	0.18	0.00	0.03	0.32
Junior	0.04	0.01	0.00	0.10	0.15
Senior	0.01	0.02	0.02	0.01	0.05
5+	0.02	0.00	0.05	0.04	0.11
	0.49	0.27	0.07	0.18	1.00





Multinomial

Example document:
"Pay for Viagra with a credit-card. Viagra is great. So are credit-cards. Risk free Viagra. Click for free."
n = 18

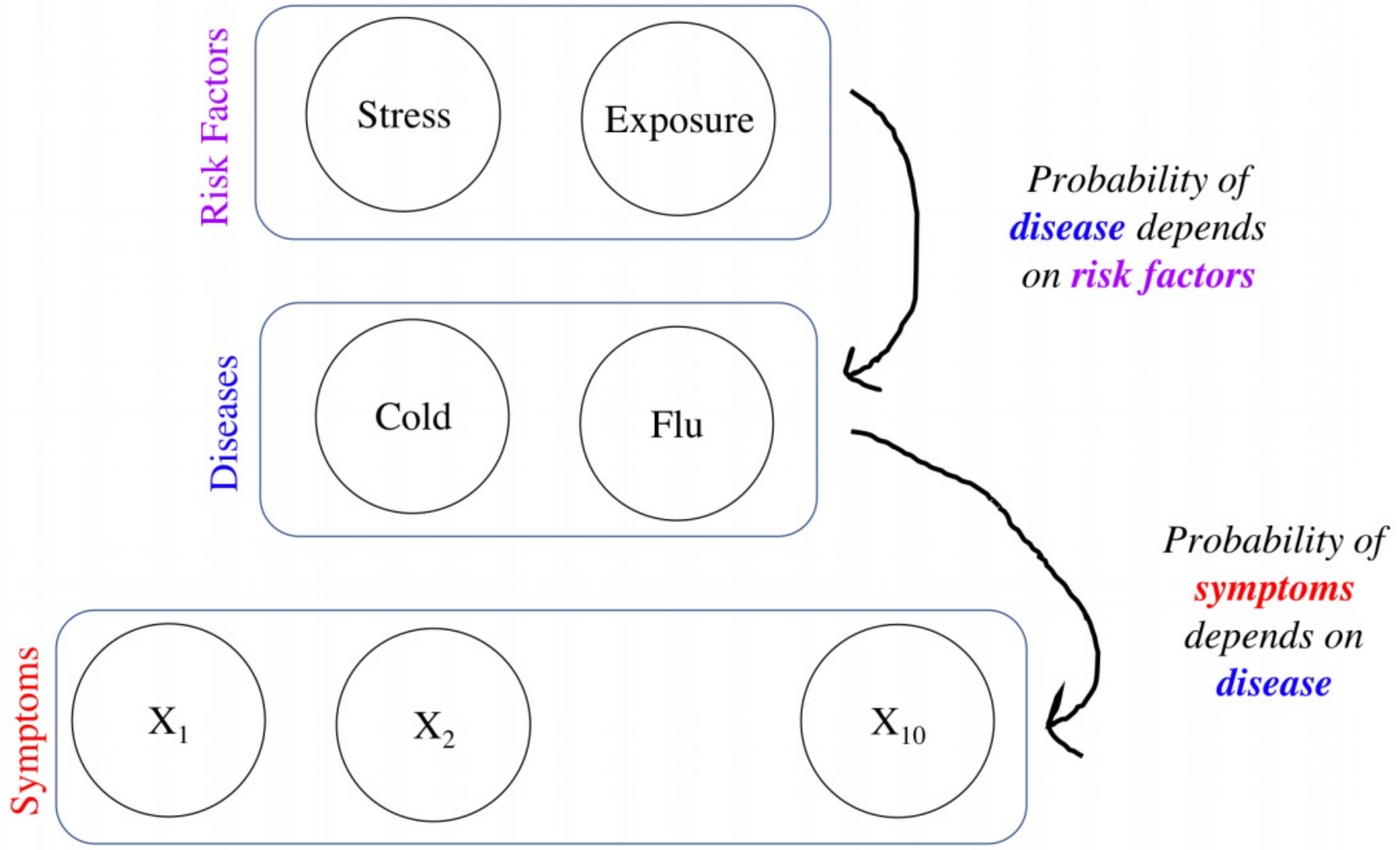
It's a Multinomial!

$$P\left(\begin{array}{l}\text{Viagra} = 2\\ \text{Free} = 2\\ \text{Risk} = 1\\ \text{Credit-card: } 2\\ \dots\\ \text{For} = 2\end{array} \middle| \text{spam}\right) = \frac{n!}{2!2!\dots2!}p^2_{\text{viagra}}p^2_{\text{free}}\cdots p^2_{\text{for}}$$

Probability of seeing this document | spam

The probability of a word in spam email being viagra

Bayes Nets!



Risk Factors
Stress
Exposure
Diseases
Cold
Flu
Symptoms
X_1
X_2
X_{10}
Probability of disease depends on risk factors
Probability of symptoms depends on disease

Alg #1: Rejection Sampling

```
N_SAMPLES = 100000

# Program: Joint Sa
# ------------------
# we can answer any
# with multivariate
# where conditioned
def main():
    obs = getObserv
    print 'Observat

    samples = sampl
    prob = probFluG
    print 'Pr(Flu)
```

webMd — -bash — 38×22

```
[0, 0, 0, 0]
[0, 1, 0, 1]
[1, 0, 1, 0]
[1, 1, 1, 1]
[0, 1, 0, 1]
[0, 1, 0, 0]
[0, 0, 0, 0]
[0, 1, 1, 1]
[0, 1, 0, 0]
[0, 1, 0, 1]
[0, 1, 0, 0]
[0, 1, 0, 1]
[0, 1, 0, 1]
[0, 0, 0, 0]
[1, 1, 1, 1]
[0, 0, 0, 0]
[0, 0, 0, 0]
[1, 1, 1, 1]
[0, 1, 0, 0]
Observation =  [None, None, None, 1]
Pr(Flu | Obs) =  0.140635888502
>
```

Each one of these is one posterior sample:

[Flu, Ugrad, Fever, Tired]

Uncertainty Theory

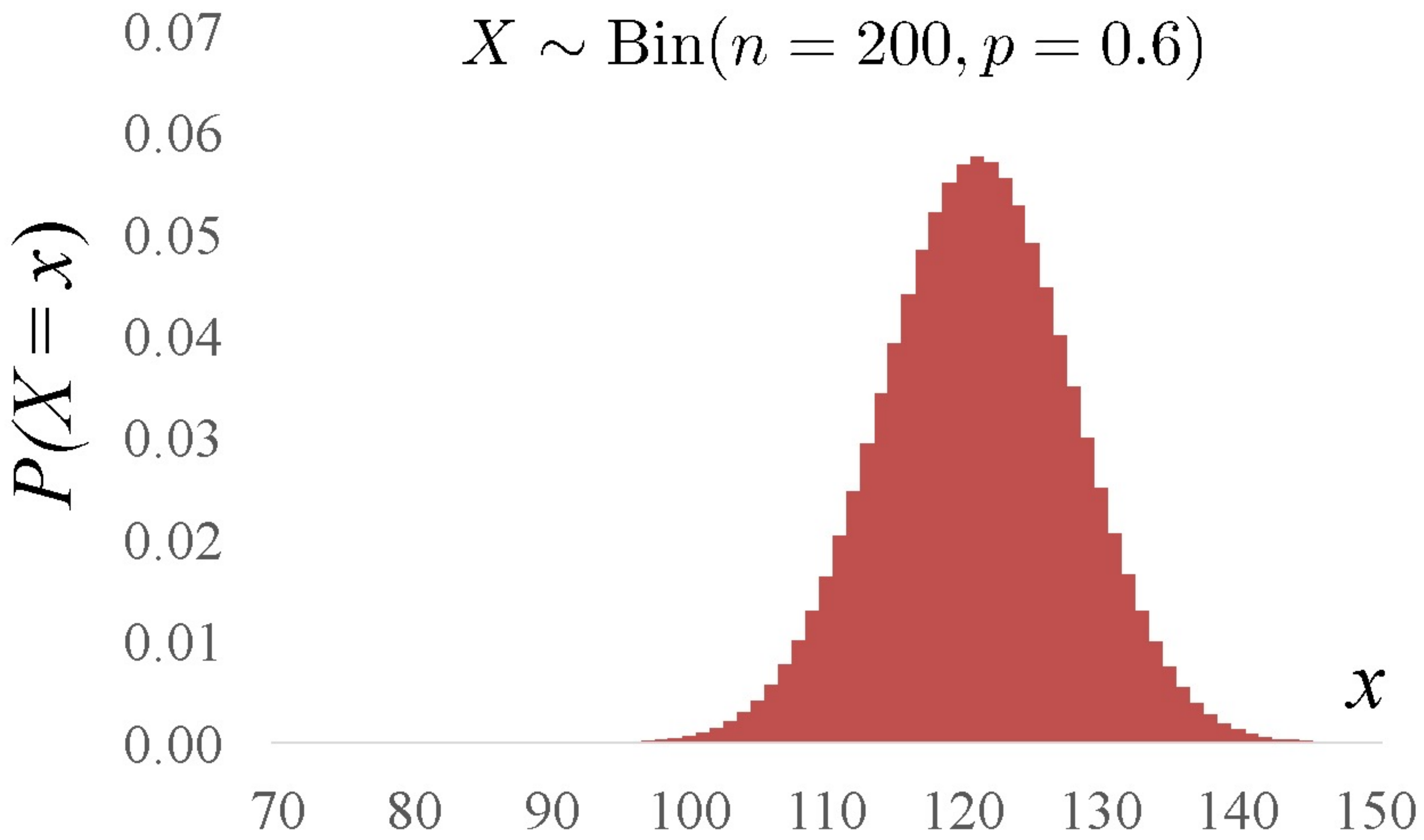
Lets Play!

Drug A

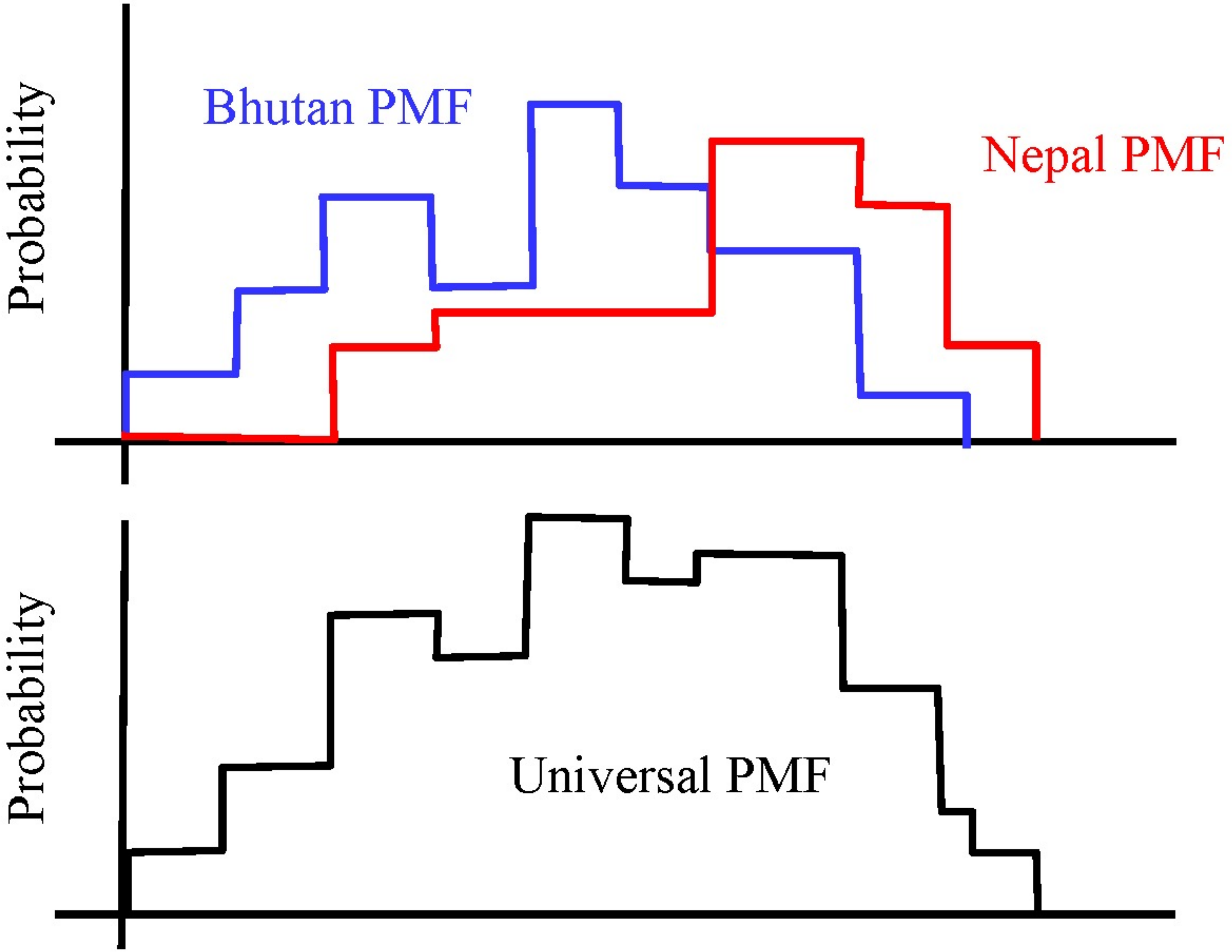
Drug B

Which one do you give to a patient?

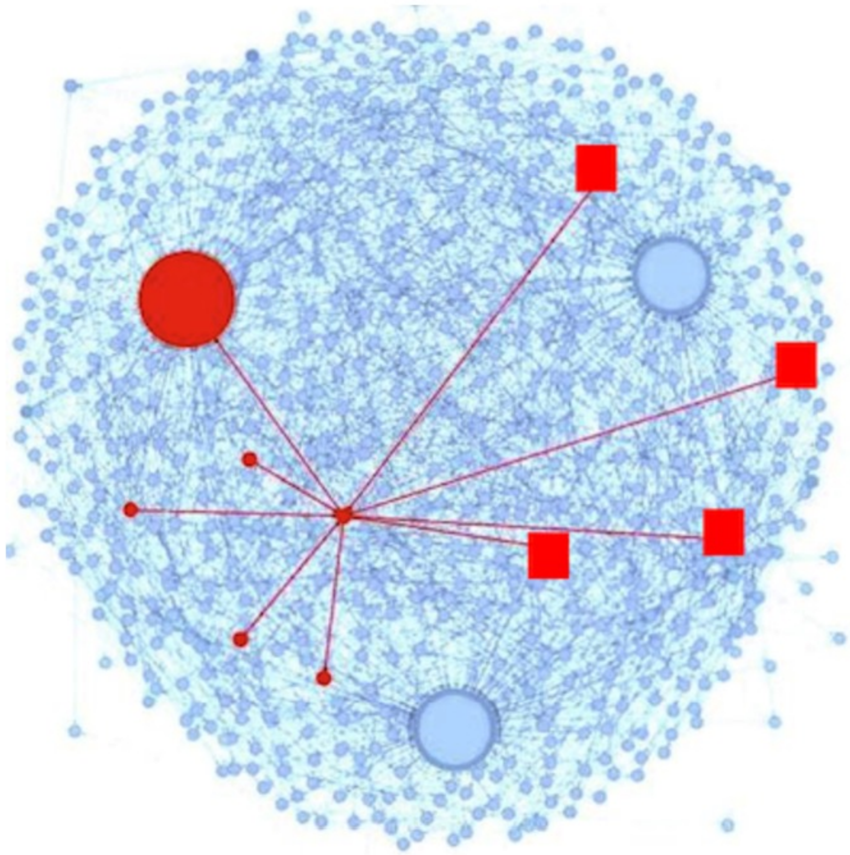
C.L.T. Explains This



Universal Sample



Peer Grading



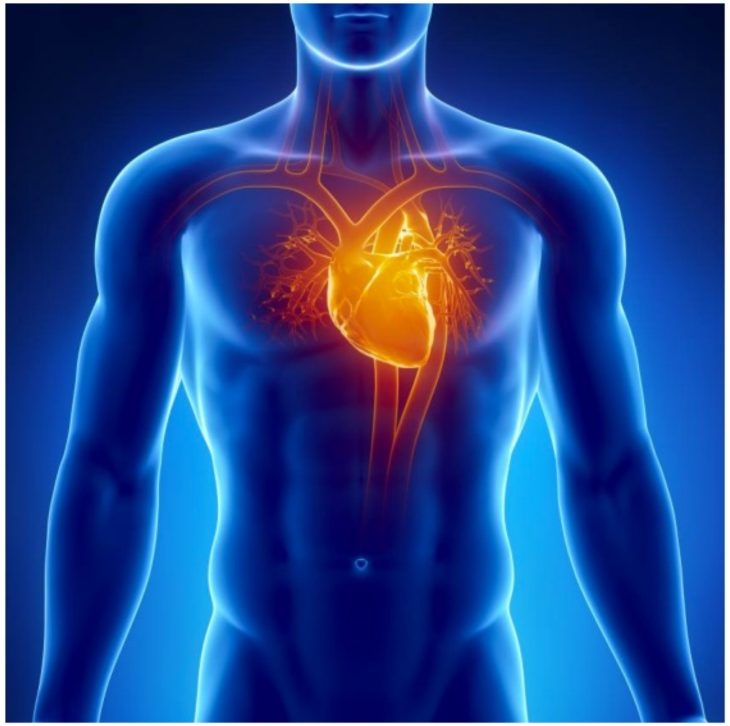
Peer Grading on Coursera HCI.

31,067 peer grades for 3,607 students.

Machine Learning

Machine Learning

Heart



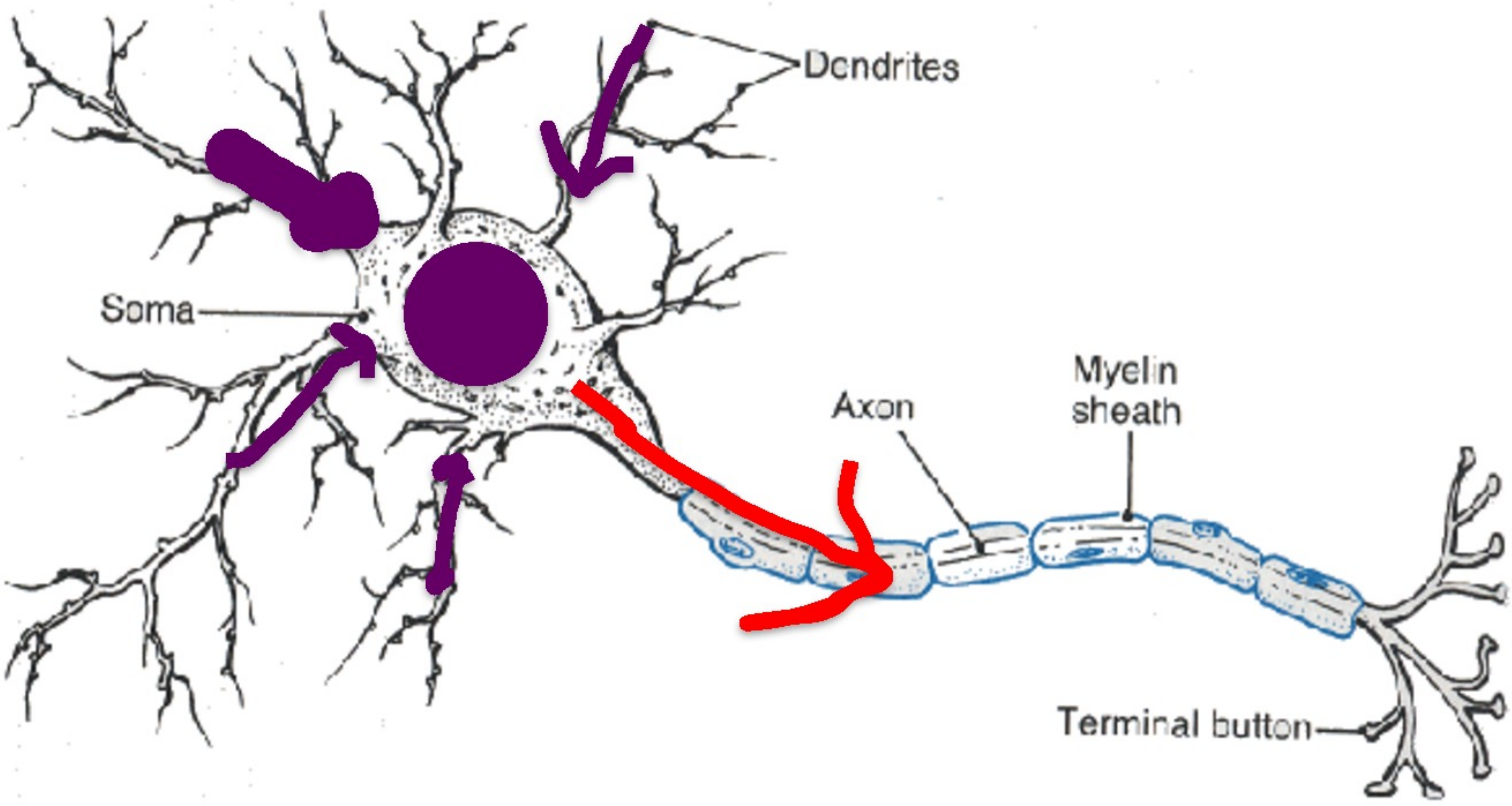
Ancestry

23andMe

Netflix

NETFLIX

Logistic Regression



CS 109 is just a
beginning in your
Journey in Probability


thank you