CS109

Section #2
July 12th, 2023

Section 2: Random Variables

1 Warmups

1. Definitions: Cite Bayes' Theorem. Can you explain to your partner why $P(A|B)$ is different than $P(B|A)$?

2. True or False. Note that true means true for ALL cases.

   (a) In general, $P(AB|C) = P(B|C)P(A|BC)$

   (b) If A and B are independent, so are A and B^C.

2 Conditional Independence vs Unconditional Independence

Here are two examples demonstrating that conditional dependence relationships don't imply unconditional dependence relationships, and vice versa. It is also a good exercise for getting used to manipulating probability rules in the conditional paradigm.

1. There is 10% chance that the traffic is heavy, and 20% chance that Alice wakes up late. If Alice wakes up early and there is no traffic, she is never late to work. When Alice wakes up late and the traffic is heavy, she is definitely late to work. Even if Alice wakes up early, she's late to work 80% of the time if the traffic is heavy. When there is no traffic but Alice wakes up late, she's late to work 70% of the time.

   Assume that whether Alice wakes up late and whether the traffic is heavy are independent of one another. Conditioned on Alice being late to work, are these two events conditionally independent?

2. When the weather is nice, Bob goes outside 80% of the time and Charlie goes outside 90% the time, conditionally independent of each other. When the weather is bad, Bob goes outside 20% time and Charlie goes outside 30% of the time, also conditionally independent of each other. The weather is nice 40% of the time. Are whether Bob and Charlie go outside independent of one another?

3 Taking Expectation: Breaking Vegas

Preamble: When a random variable fits neatly into a family we've seen before (e.g. Binomial), we get its expectation for free. When it does not, we have to use the definition of expectation.

Problem: If you bet on "Red" in Roulette, there is $p = 18/38$ that you with win \$Y and a $(1 - p)$ probability that you lose \$Y. Consider this algorithm for a series of bets:

Let Y = \$1. First you bet Y. If you win, then stop. If you lose, then set Y to be 2Y and repeat.

What are your expected winnings when you stop? It will help to recall that the sum of a geometric series $a^0 + a^1 + a^2 + \cdots = \frac{1}{1-a}$ if $0 < a < 1$. Vegas breaks you: Why doesn't everyone do this?

4 Conditional Probabilities: Missing Not at Random

Preamble: We have three big tools for manipulating conditional probabilities:

- Definition of conditional probability: $P(EF) = P(E|F)P(F)$
- Law of Total Probability: $P(E) = P(EF) + P(EF^C) = P(E|F)P(F) + P(E|F^C)P(F^C)$
- Bayes Rule: $P(E|F) = \frac{P(F|E)P(E)}{P(F)} = \frac{P(F|E)P(E)}{P(F|E)P(E)+P(F|E^C)P(E^C)}$

This is a good time to commit these three to memory and start thinking about when each of them is useful.

Problem: You collect data on whether or not people intend to vote for Ayesha, a candidate in an upcoming election. You send an electronic poll to 100 randomly chosen people. You assume all 100 responses are IID.

User Response	**Count**
Responded that they will vote for Ayesha	40
Responded that they will **not** vote for Ayesha	45
Did not respond	15

Let A be the event that a person says they will vote for Ayesha. Let M be the event that a user did not respond to the poll. We are interested in estimating $P(A)$, however that is hard given the 15 users who did not respond.

a. What is the probability that a user said they will vote for Ayesha and that they responded to the poll $P(A \text{ and } M^C)$?

b. Which formula from class would you use to calculate $P(A)$? Your formula should rely on the context that voters for Ayesha are in one of two (mutually exclusive) groups: those that missed the poll, and those that did not.

c. Calculate the $P(A)$. You estimate that the probability that a voter is missing, given that they were going to vote for Ayesha is $P(M|A) = \frac{1}{5}$.

5 Sending Bits to Space

Preamble: When sending binary data to satellites (or really over any noisy channel), the bits can be flipped with high probability. In 1947, Richard Hamming developed a system to more reliably send data. By using Error Correcting Hamming Codes, you can send a stream of 4 bits along with 3 redundant bits. If zero or one of the seven bits are corrupted, using error correcting codes, a receiver can identify the original 4 bits.

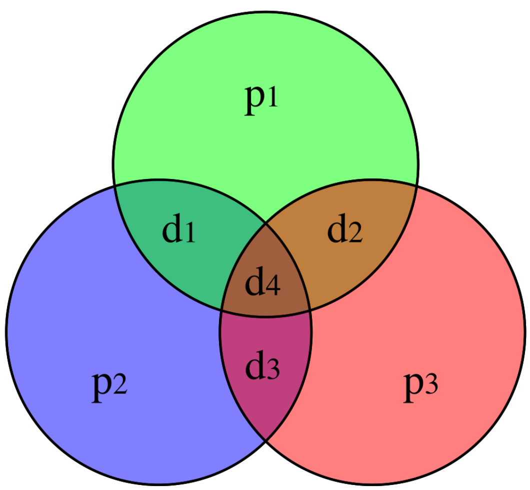
Problem: Lets consider the case of sending a signal to a satellite where each bit is independently flipped with probability $p = 0.1$.

a. If you send 4 bits, what is the probability that the correct message was received (i.e. none of the bits are flipped).

b. If you send 4 bits, with 3 Hamming error correcting bits, what is the probability that an interpretable message (i.e. a message with zero or one errors) was received?

c. Instead of using Hamming codes, you decide to send 100 copies of each of the four bits. If for every single bit, more than 50 of the copies are not flipped, the signal will be correctable. What is the probability that a correctable message was received?

Extra: Explanation of the ”`Hamming(7,4)`” technique



If we are trying to transmit 4 bits, we can send an additional 3 ”parity” bits that we can use to correct our original message if a bit gets flipped due to an error in transmission. Consider the diagram. The data bits are d_1 through d_4. The ”parity” bits are p_1 through p_3. A parity bit is set to whatever value would make it’s large circle have an even number of bits. For example, the green circle consists of p_1, d_1, d_2, and d_4. If $d_1 = 1$, $d_2 = 1$, and $d_4 = 1$, then p_1 would be set to 1 in order to ensure there are an even number of bits in that circle (in this case, 4 bits).

Convince yourself that a single error which appeared in any bit could be identified and corrected! For example, if d_2 is flipped, it would throw off the parity for the green and red circles. Therefore, flipping d_2 back is the only way to correct the parity. As another example, if p_2 is flipped, then only the blue circle would have a parity issue, and flipping p_2 back is the unique solution to fixing the parity.