Adding Random Variables

CS109, Stanford University




Midterm grades…
I know I said by Monday….

New Definition

IID Random Variables

- Consider *n* random variables $X_1, X_2, \ldots X_n$
  - X_i are all independently and identically distributed (I.I.D.)
  - All have the same PMF (if discrete) or PDF (if continuous)
  - All have the same expectation
  - All have the same variance

IID iid

Quick Concept Check

Are $X_1, X_2, \dots, X_n$ i.i.d. with the following distributions?

1. $X_i \sim \text{Exp}(\lambda)$, X_i independent

2. $X_i \sim \text{Exp}(\lambda_i)$, X_i independent

3. $X_i \sim \text{Exp}(\lambda)$, $X_1 = X_2 = \cdots = X_n$

4. $X_i \sim \text{Bin}(n_i, p)$, X_i independent

Quick check

Are $X_1, X_2, \dots, X_n$ i.i.d. with the following distributions?

1. $X_i \sim \text{Exp}(\lambda),\ X_i$ independent ✅
2. $X_i \sim \text{Exp}(\lambda_i),\ X_i$ independent ❌ (unless λ_i equal)
3. $X_i \sim \text{Exp}(\lambda),\ X_1 = X_2 = \cdots = X_n$ ❌ dependent: $X_1 = X_2 = \cdots = X_n$
4. $X_i \sim \text{Bin}(n_i, p),\ X_i$ independent ❌ (unless n_i equal)
   Note underlying Bernoulli RVs are i.i.d.!

What happens when you add random variables?

Why should you care?

Zero Sum Games

What is the probability that the Warriors win?

How do you model zero sum games?

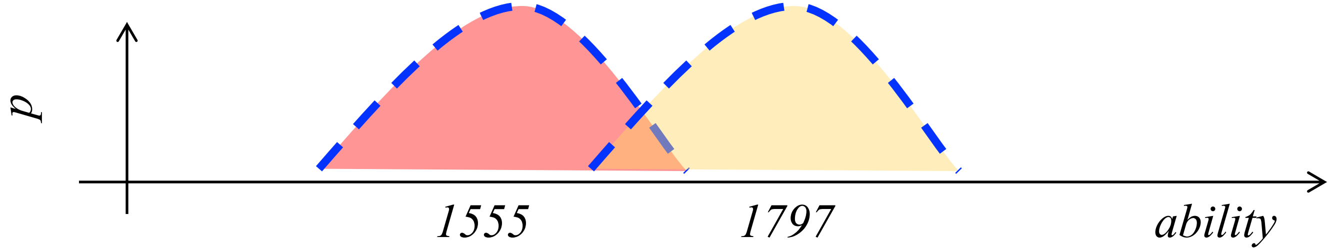
Motivating Idea: Zero Sum Games

How it works:

- Each team has an "ELO" score *S*, calculated based on their past performance.
- Each game, the team has ability $A \sim N(S, 200^2)$
- The team with the higher sampled ability wins.

Arpad Elo

$$A_B \sim \mathcal{N}(1555, 200^2) \qquad A_W \sim \mathcal{N}(1797, 200^2)$$



$$P(\text{Warriors win}) = P(A_W > A_B)$$

Motivating Idea: Zero Sum Games

$$A_W \sim \mathcal{N}(1797, 200^2)$$

$$A_B \sim \mathcal{N}(1555, 200^2)$$

$$P(\text{Warriors win}) = P(A_W > A_B)$$

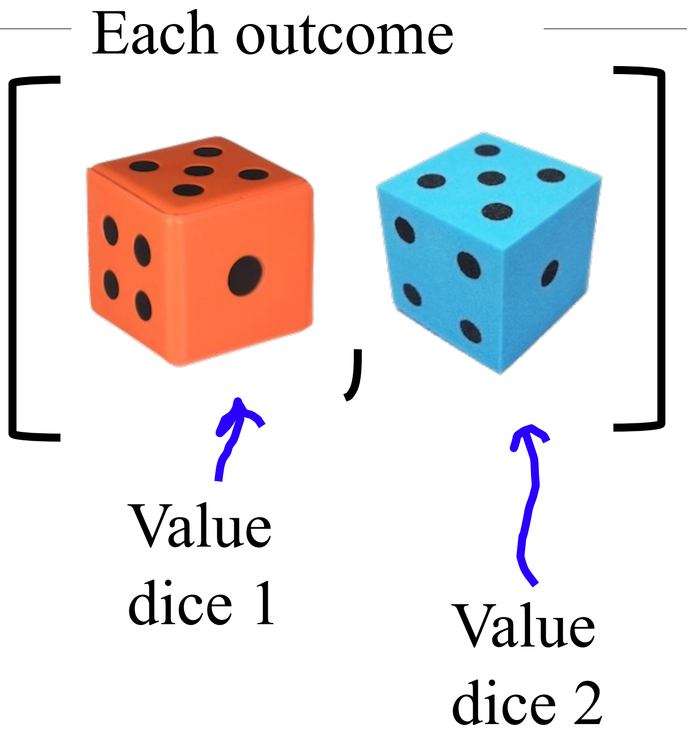
How do we do this???

Review

Sum of Two Die?

Roll two 6-sidex dice. What is P(sum = 7)?

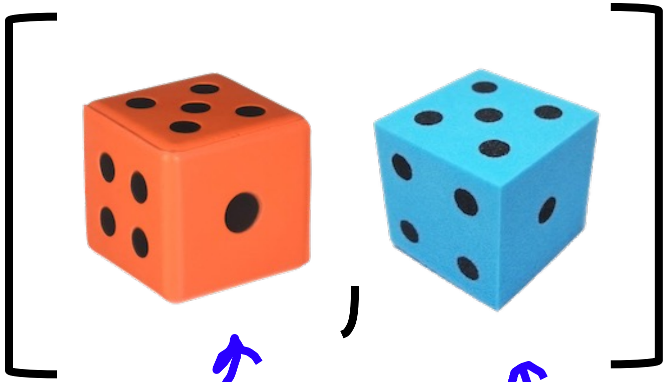
S = { [1,1] [1,2] [1,3] [1,4] [1,5] [1,6]
[2,1] [2,2] [2,3] [2,4] [2,5] [2,6]
[3,1] [3,2] [3,3] [3,4] [3,5] [3,6]
[4,1] [4,2] [4,3] [4,4] [4,5] [4,6]
[5,1] [5,2] [5,3] [5,4] [5,5] [5,6]
[6,1] [6,2] [6,3] [6,4] [6,5] [6,6] }



Sum of Two Die = 7?

Roll two 6-sidex dice. What is P(sum = 7)?

Each outcome



Value dice 1

Value dice 2

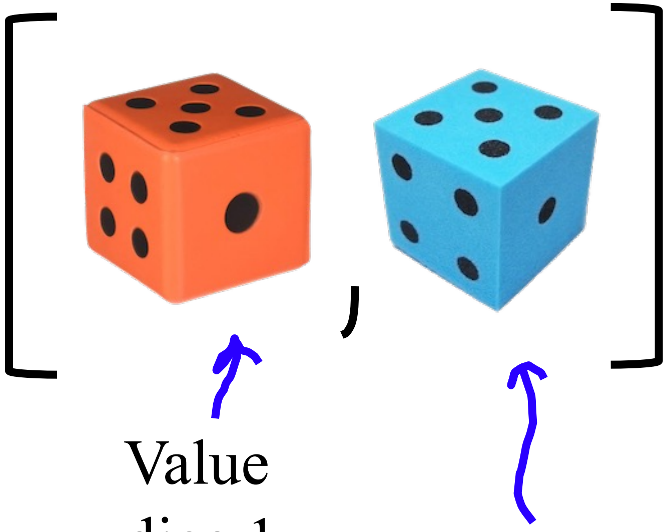
S = { [1,1] [1,2] [1,3] [1,4] [1,5] **[1,6]**
[2,1] [2,2] [2,3] [2,4] **[2,5]** [2,6]
[3,1] [3,2] [3,3] **[3,4]** [3,5] [3,6]
[4,1] [4,2] **[4,3]** [4,4] [4,5] [4,6]
[5,1] **[5,2]** [5,3] [5,4] [5,5] [5,6]
[6,1] [6,2] [6,3] [6,4] [6,5] [6,6] }

E = *in blue*

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = 0.16\bar{6}$$

Sum of Two Die = 10?

Roll two 6-sidex dice. What is P(sum = 10)?



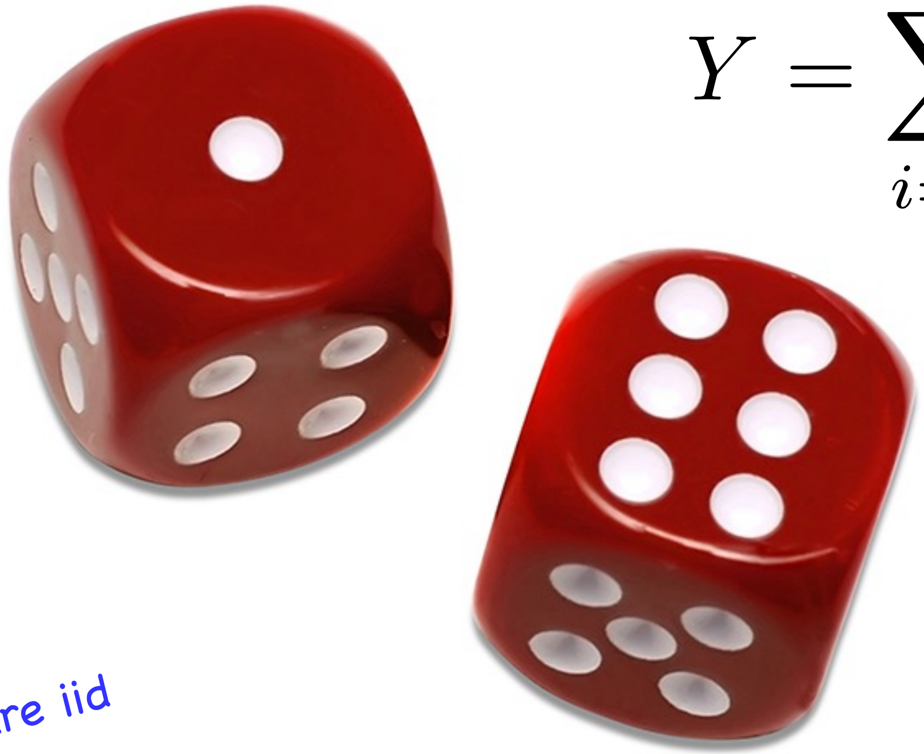
S = { [1,1] [1,2] [1,3] [1,4] [1,5] [1,6]
[2,1] [2,2] [2,3] [2,4] [2,5] [2,6]
[3,1] [3,2] [3,3] [3,4] [3,5] [3,6]
[4,1] [4,2] [4,3] [4,4] [4,5] **[4,6]**
[5,1] [5,2] [5,3] [5,4] **[5,5]** [5,6]
[6,1] [6,2] [6,3] **[6,4]** [6,5] [6,6] }

E = *in blue*

$$P(E) = \frac{|E|}{|S|} = \frac{3}{36} = 0.08\bar{3}$$

End Review

Sum of Two Dice



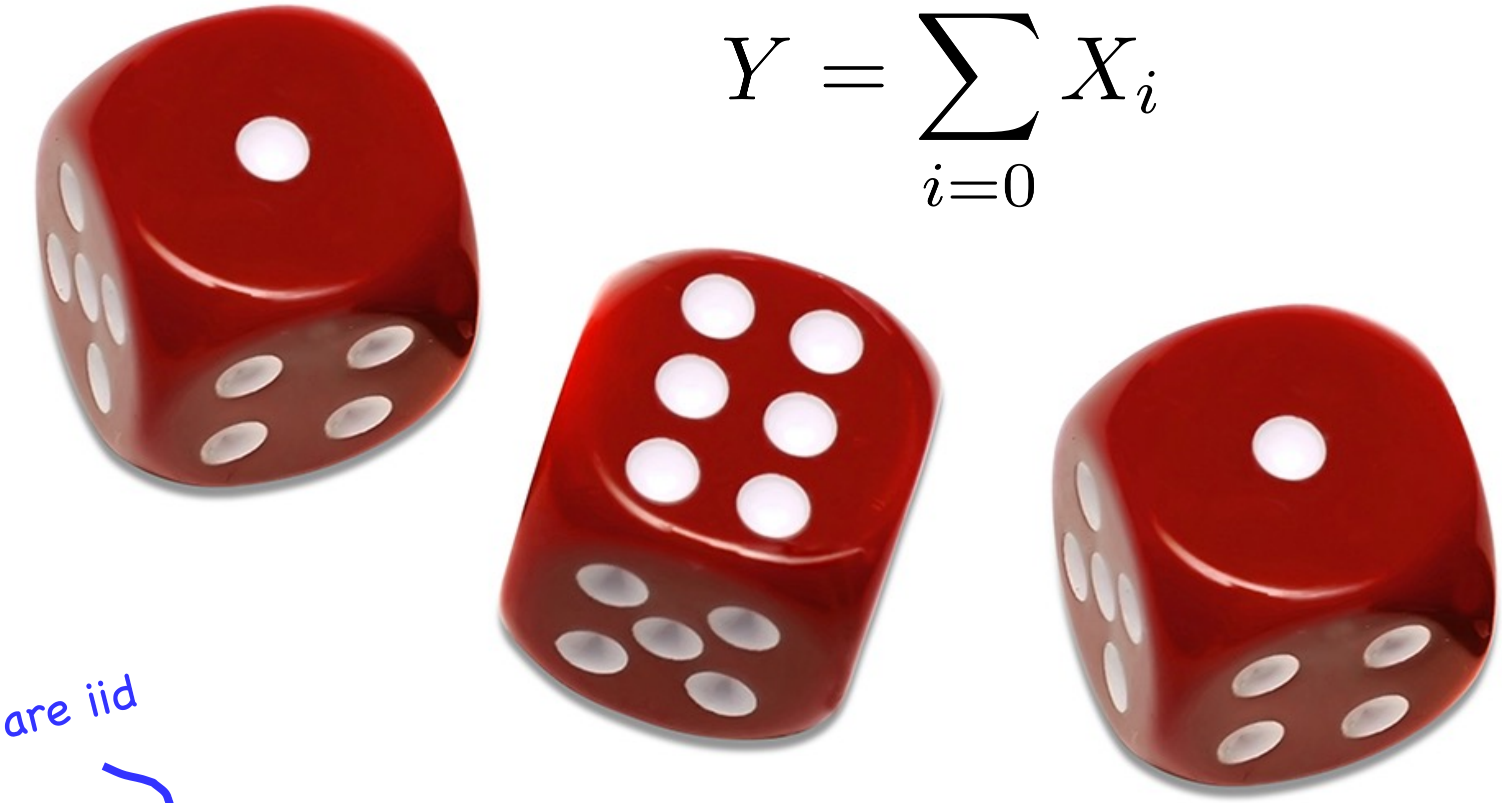
$$Y = \sum_{i=0}^{2} X_i$$

X_i s are iid

X_i is the outcome of dice roll i

Sum of Three Dice

$$Y = \sum_{i=0}^{3} X_i$$

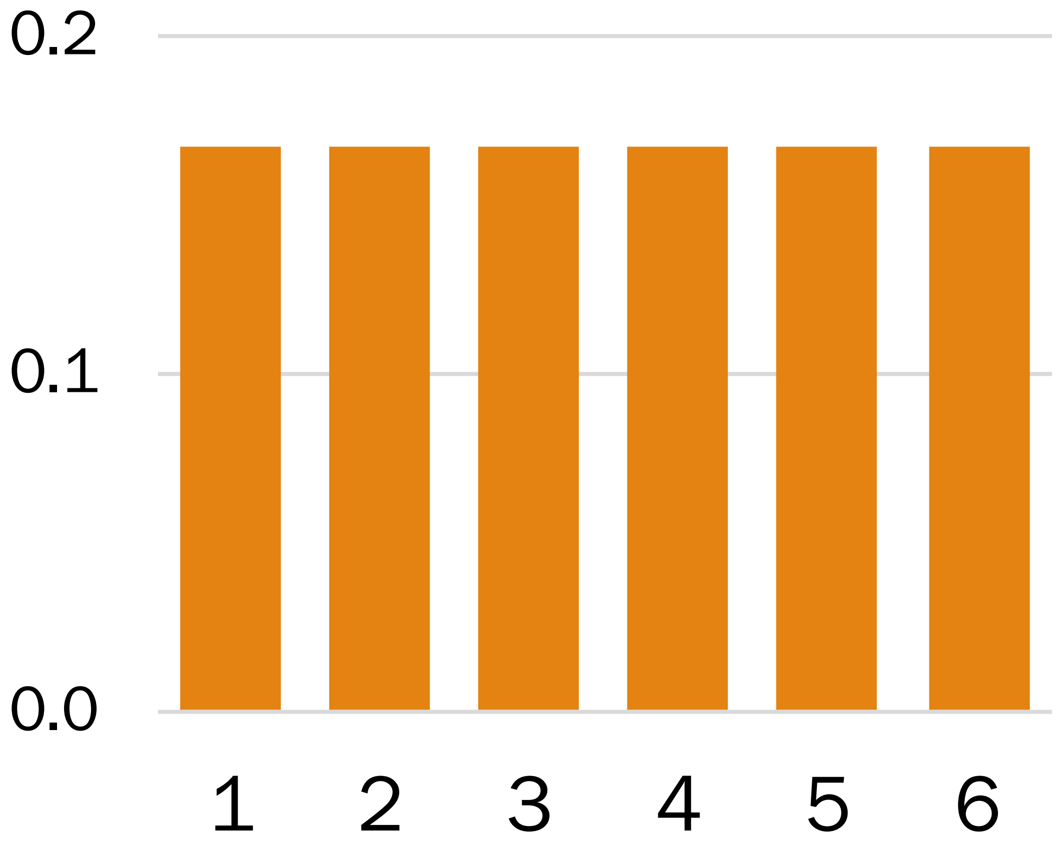


X_i s are iid

X_i is the outcome of dice roll i

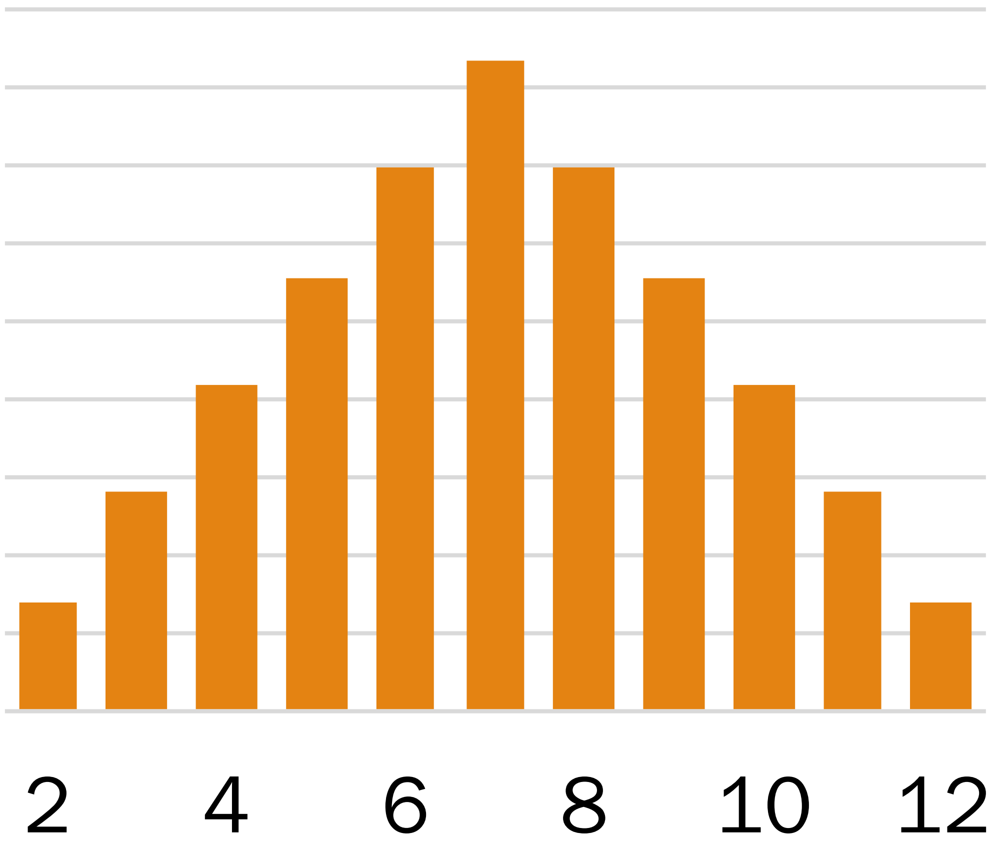
Sum of One Dice

This is the PMF of the sum of one dice



Sum of Two Dice

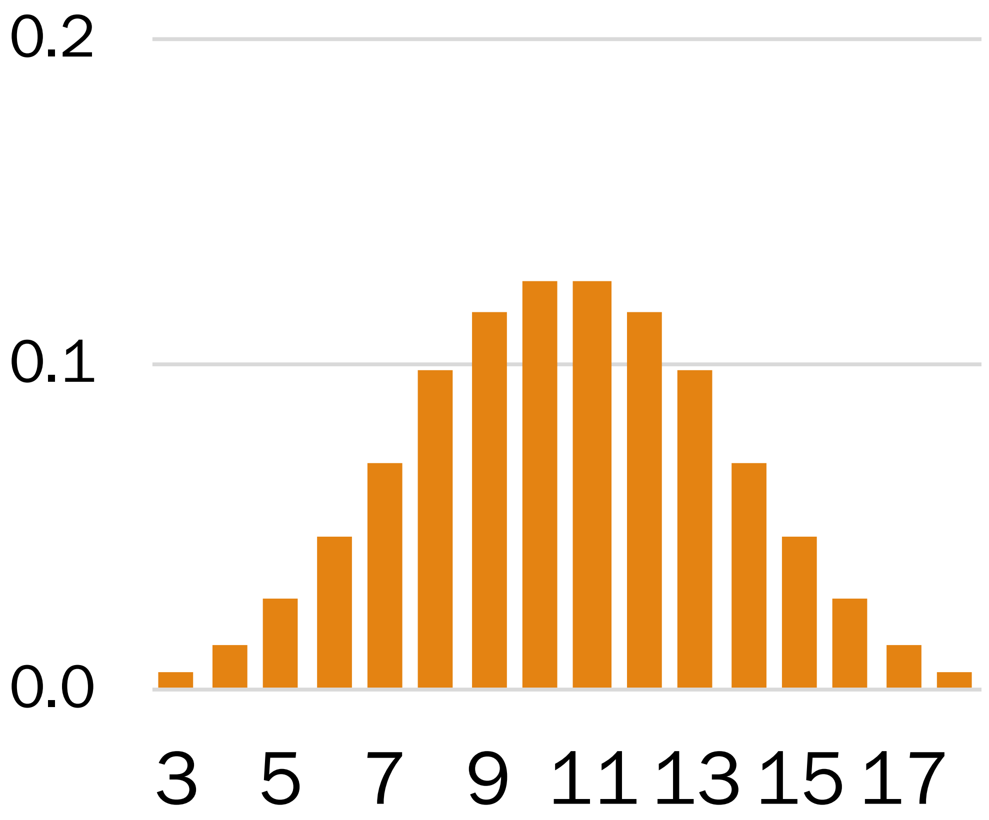
This is the PMF of the sum of two dice



Why is there more mass in the middle?

Sum of Three Dice

This is the PMF of the sum of three dice



Why is there more mass in the middle?

Sum of 50 dice?

We’ll get there soon….

The Insight to Convolution

Imagine a game
where each player *independently* scores between 0 and 100 points:

Let X be the amount of points you score.
Let Y be the amount of points your opponent scores.
Let's say you know P(X = x) and P(Y = y).

What is the probability of a tie?

$$P(\text{tie}) = \sum_{i=0}^{100} P(X = i, Y = i)$$

$$= \sum_{i=0}^{100} P(X = i)P(Y = i)$$

The Insight to Convolution Proofs

What is the probability that X + Y = n?

$$P(X+Y=n)?$$

$$P(X+Y=n)=\sum_{i=0}^{n} P(X=i, Y=n-i)$$

X	Y	i	
0	n	0	$P(X=0, Y=n)$
1	n - 1	1	$P(X=1, Y=n-1)$
2	n - 2	2	$P(X=2, Y=n-2)$
	…		
n	0	n	$P(X=n, Y=0)$

The Insight to Convolution Proofs

What is the probability that X + Y = n?

$$P(X+Y=n)?$$

Since this is the OR or mutually exclusive events

$$P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k)$$

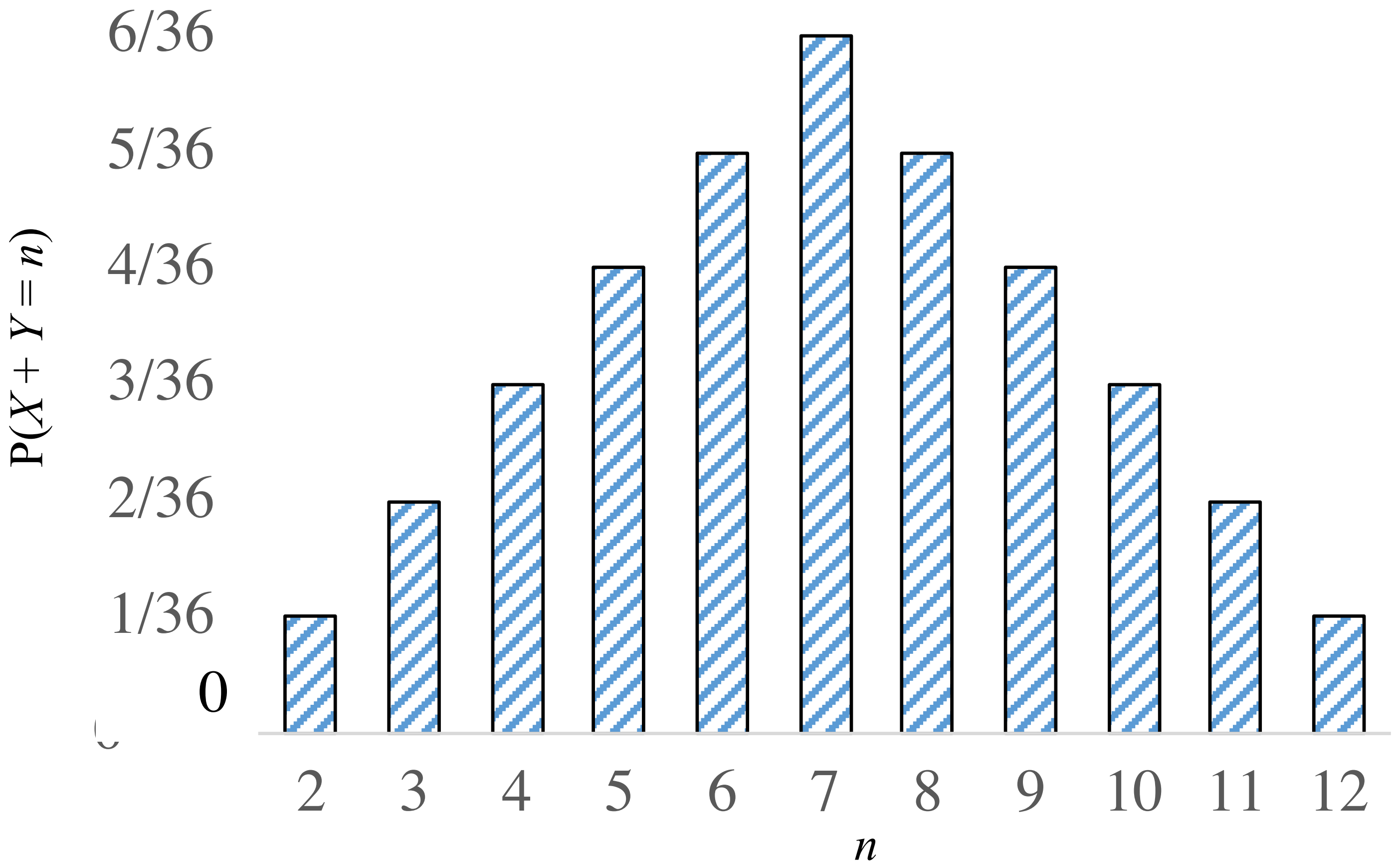
If the random variables are independent

$$= \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

Sum of Two Dice

Let $X+Y$ be the value of the sum of two dice
(aka two independent random variables)

$$P(X+Y=n) = \sum_{i=1}^{n-1} \mathrm{P}(X=i, Y=n-i)$$



Convolution: The fanciest way to say “adding random variables”

Sometimes Adding is Easy:

Sum of Independent Binomials

- Let X and Y be independent binomials with the same value for *p*:
  - $X \sim Bin(n_1, p)$ and $Y \sim Bin(n_2, p)$
  - $X + Y \sim Bin(n_1 + n_2, p)$

- Intuition:
  - X has n_1 trials and Y has n_2 trials
    - Each trial has same "success" probability p
  - Define Z to be $n_1 + n_2$ trials, each with success prob. p
  - $Z \sim Bin(n_1 + n_2, p)$, and also $Z = X + Y$

Sum of Independent Normals

- Let X and Y be independent random variables
  - $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$
  - $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

- Generally, have *n* independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, ..., n$:

$$\left(\sum_{i=1}^{n} X_i\right) \sim N\left(\sum_{i=1}^{n} \mu_i,\ \sum_{i=1}^{n} \sigma_i^2\right)$$

Sum of Independent Poissons

- Let X and Y be independent random variables
  - $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$
  - $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$

Virus Infections

- Say you are working with the WHO to plan a response to the initial conditions of a virus:
  - Two exposed groups
  - P1: 50 people, each independently infected with $p = 0.1$
  - P2: 100 people, each independently infected with $p = 0.4$
  - Question: Probability of more than 40 infections?

Sanity check: Should we use the Binomial Sum-of-RVs shortcut?

1. YES!
2. NO!
3. Other/none/more

Virus Infections

- Say you are working with the WHO to plan a response to the initial conditions of a virus:
  - Two exposed groups
  - P1: 50 people, each independently infected with p = 0.1
  - P2: 100 people, each independently infected with p = 0.4
  - A = # infected in P1 A ~ Bin(50, 0.1) ≈ X ~ N(5, 4.5)
  - B = # infected in P2 B ~ Bin(100, 0.4) ≈ Y ~ N(40, 24)
  - What is P(≥ 40 people infected)?
  - P(A + B ≥ 40) ≈ P(X + Y ≥ 39.5)
  - X + Y = W ~ N(5 + 40 = 45, 4.5 + 24 = 28.5)

$$P(W > 39.5) = 1 - P(X < 39.5)$$

$$= 1 - F_X(39.5) = 1 - \Phi(\frac{39.5 - 45}{\sqrt{28.5}}) \approx 0.8485$$

Linear Transform

Thinking of Y as a linear transform

$$X \sim N(\mu, \sigma^2)$$

$$Y = X + X = 2 \cdot X$$

$$Y \sim N(2\mu, 4\sigma^2)$$

$$Y = X + X = 2 \cdot X$$

Thinking of Y as the sum of independent normals

$$X + X \sim N(\mu + \mu, \sigma^2 + \sigma^2)$$

$$Y \sim N(2\mu, 2\sigma^2)$$

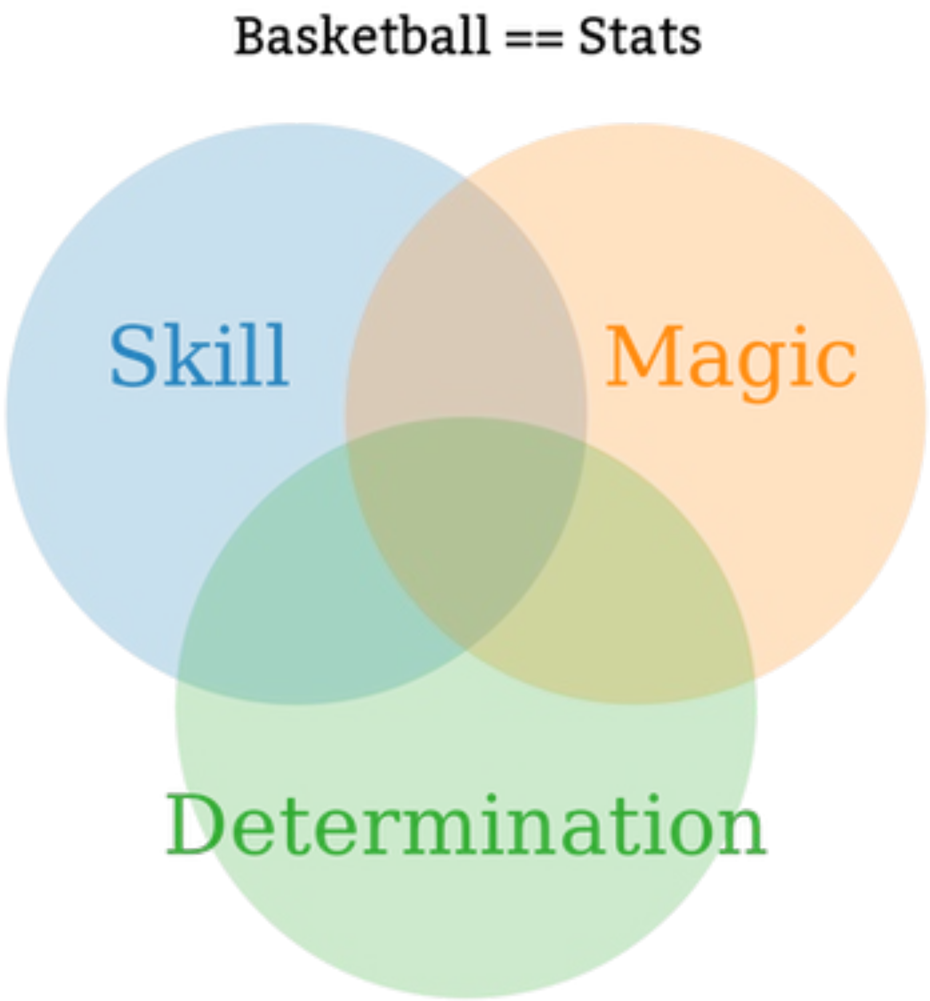
X is not independent of X

Zero Sum Games

What is the probability that the Warriors win?

How do you model zero sum games?

Gaussian Sampling and ELO ratings



What is the probability that the Warriors win?
How do you model zero-sum games?

Gaussian Sampling and ELO ratings

Each team has an ELO score S, calculated based on its past performance.

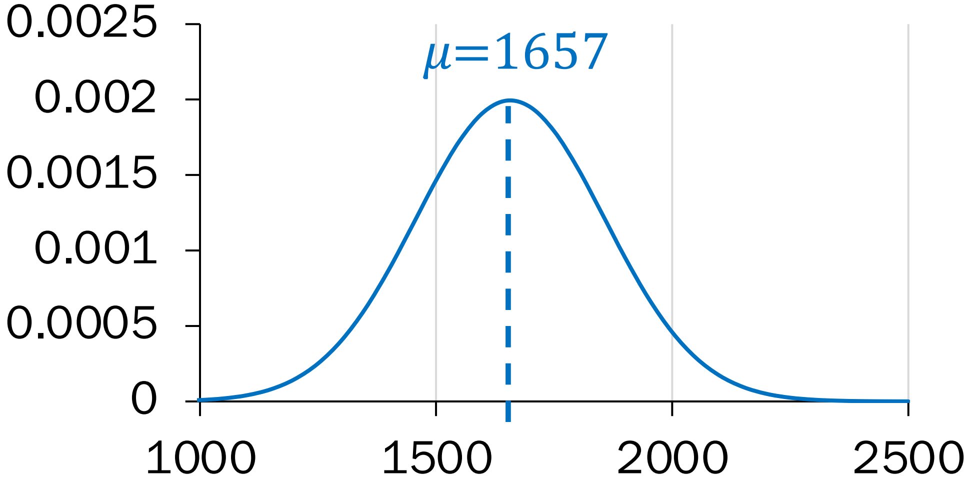
- Each game, a team has ability $A \sim \mathcal{N}(S, 200^2)$.
- The team with the higher sampled ability wins.

Arpad Elo

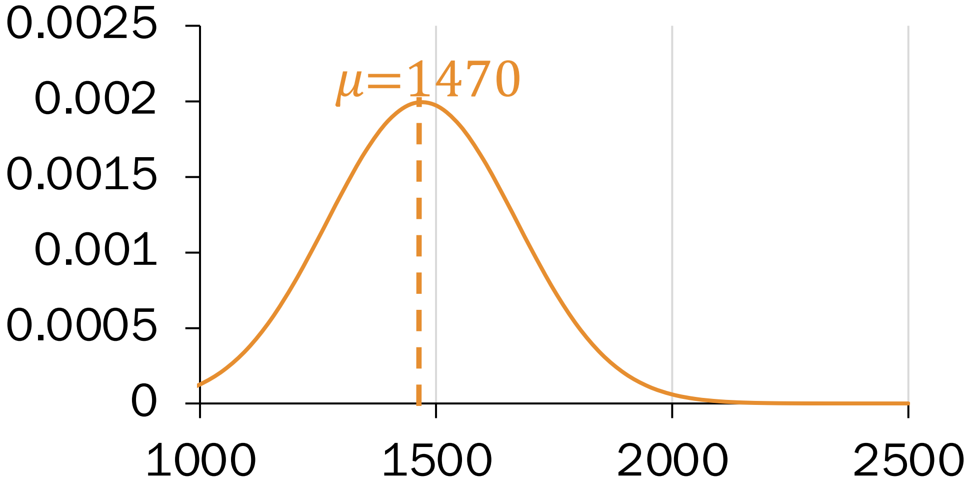
What is the probability that Warriors win this game?

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

Warriors' $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponent's $A_O \sim \mathcal{N}(S = 1470, 200^2)$



Probability of Winning a Game

$$A_W \sim N(1797, 200^2)$$

$$A_O \sim N(1555, 200^2)$$

$$P(\text{Warriors win}) = P(A_W > A_O)$$

$$P(\text{Warriors win}) = P(A_W - A_O > 0)$$

$$-A_O \sim N(-1555, 200^2)$$

$$D = A_W + (-A_O)$$

$$D \sim N(242, 2 \cdot 200^2)$$

$$P(D > 0) = 1 - F_D(0) \approx 0.804$$

We talked about sum of Binomial, Normal and Poisson…who’s missing from this party?

Uniform.

Discrete Vs Continuous

Discrete

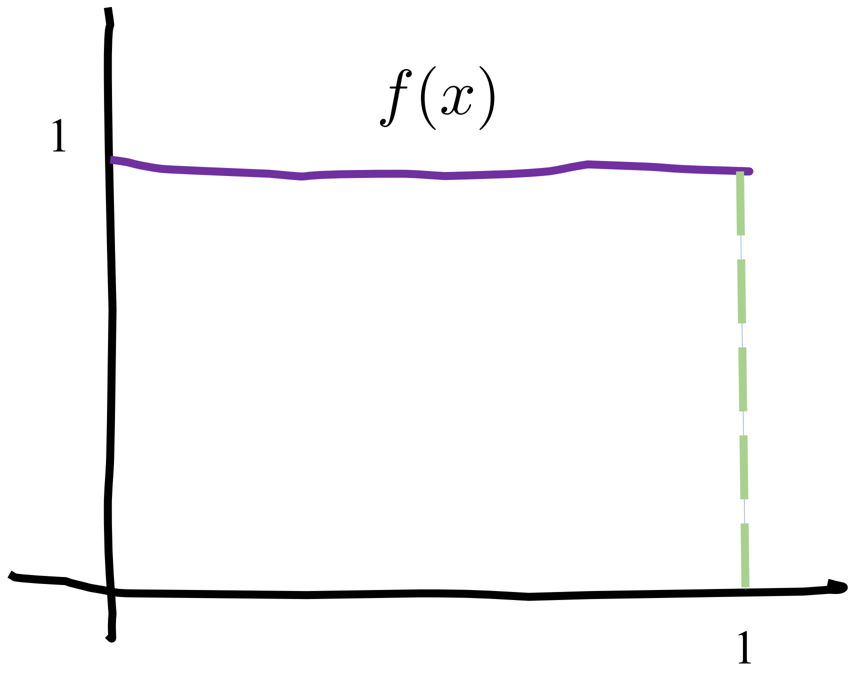
$$P(X + Y = a) = \sum_{y=-\infty}^{\infty} P(X = a - y)P(Y = y)\ dy$$

Continuous

$$f(X + Y = a) = \int_{y=-\infty}^{\infty} f(X = a - y)f(Y = y)\ dy$$

Sum of Independent Uniforms

- Let X and Y be independent random variables
  - X ~ Uni(0, 1) and Y ~ Uni(0, 1) → $f(x) = 1$ for $0 \leq x \leq 1$



For both X and Y

$1 < a < 2$

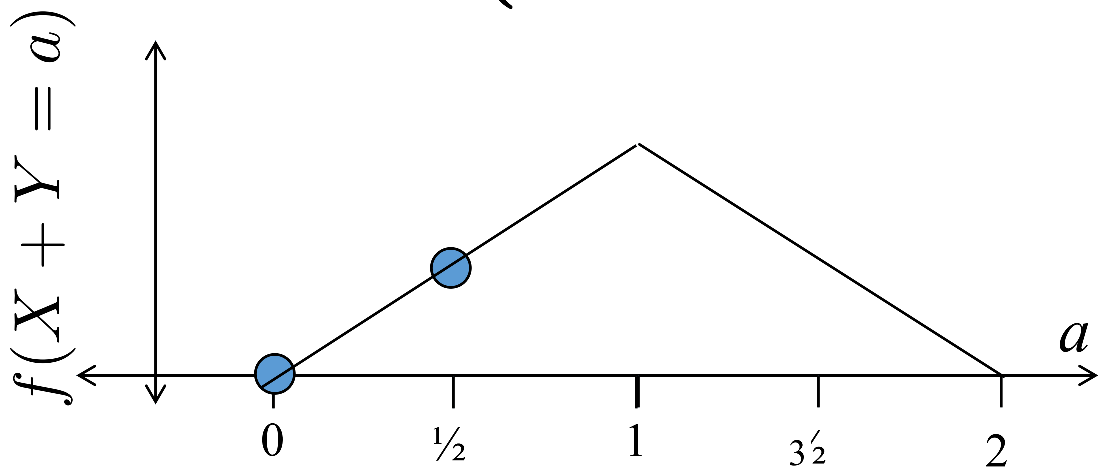
$X \sim \text{Uni}(0, 1) \quad Y \sim \text{Uni}(0, 1)$

X and *Y* are independent

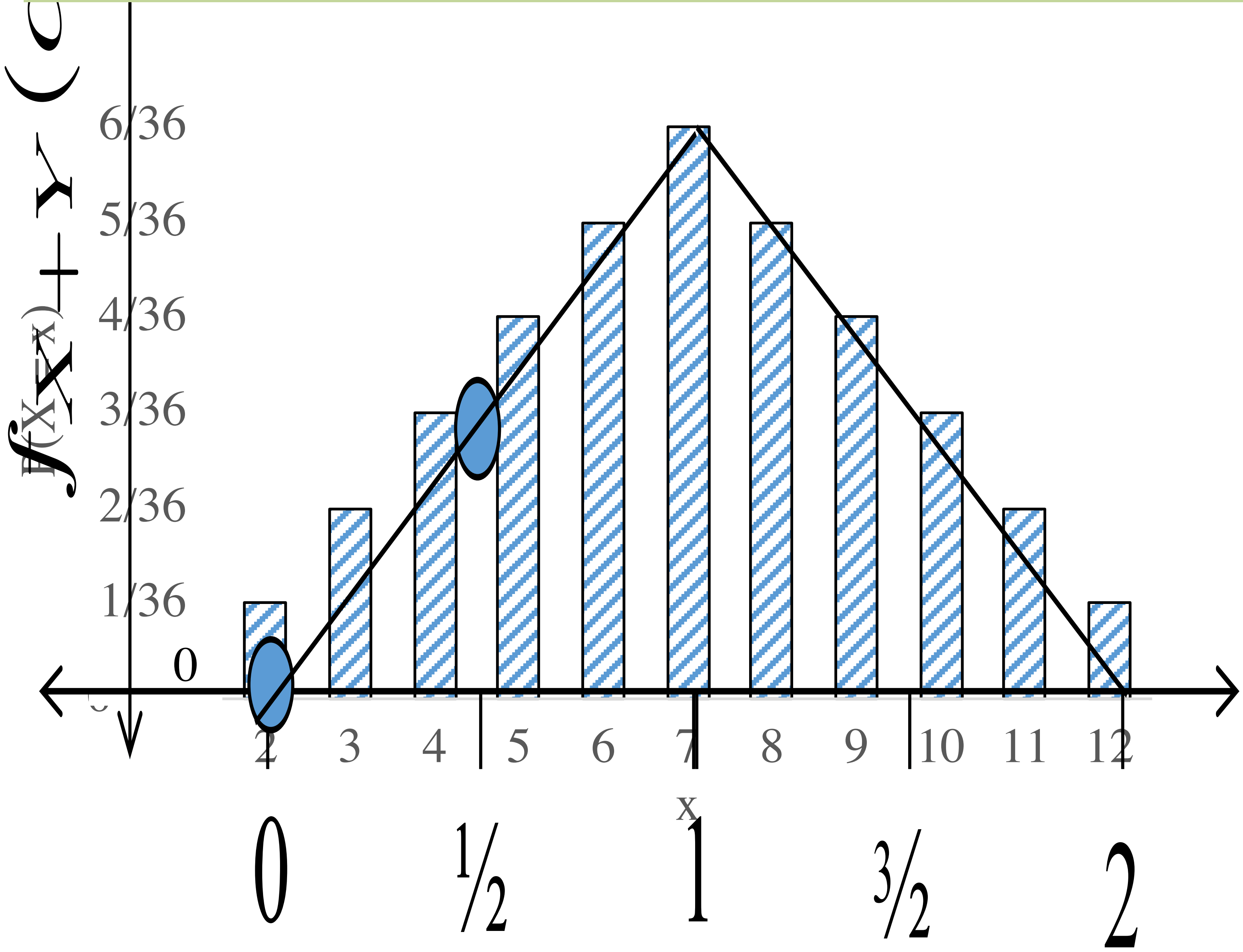
$$f(X + Y = a)?$$

$$f(X + Y = a) = \int_{y=-\infty}^{\infty} f(X = a - y) f(Y = y)\, dy$$

$$f(X + Y = a) = \begin{cases} a & 0 < a < 1 \\ 2 - a & 1 < a < 2 \\ 0 & \text{otherwise} \end{cases}$$



Sum of Uniforms and Sum of Dice



Sum of 100 uniforms???

Sum of 100 poissons???

And now a moment of silence...

...before we present...

...a beautiful result of probability theory!

(silent drumroll)

Central Limit Theorem

Consider n **independent and identically distributed (i.i.d)** variables $X_1, X_2, \dots, X_n$ with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$.

As $n \to \infty$

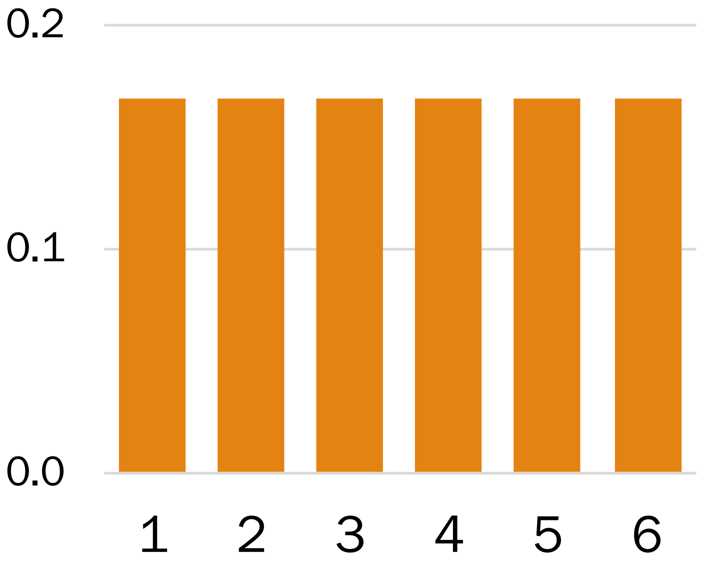
$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

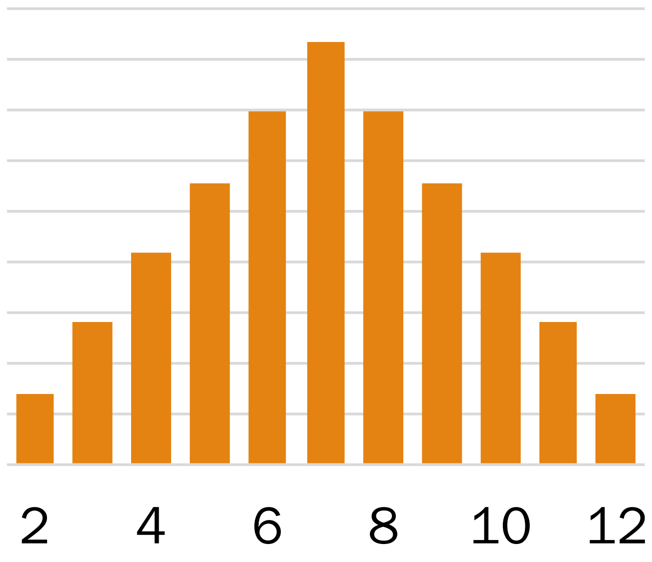
True happiness

Sum of dice rolls

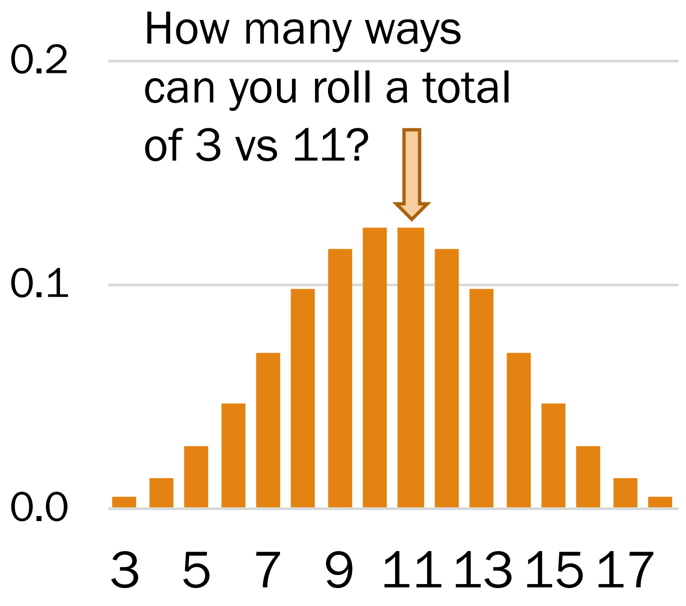
Roll n independent dice. Let X_i be the outcome of roll i. X_i are i.i.d.



$$\sum_{i=1}^{1} X_i$$ Sum of 1 die roll



$$\sum_{i=1}^{2} X_i$$ Sum of 2 dice rolls



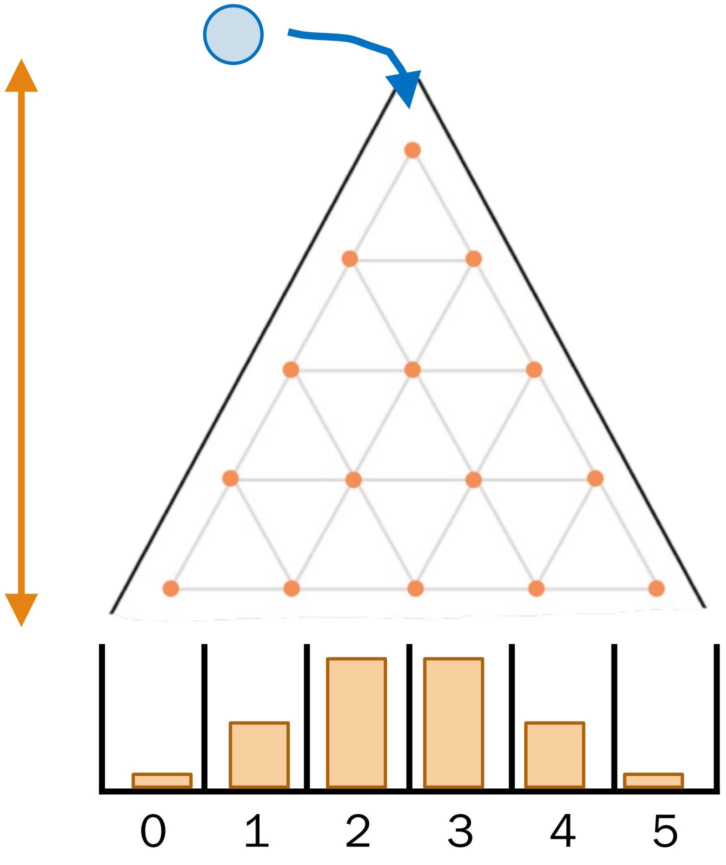
$$\sum_{i=1}^{3} X_i$$ Sum of 3 dice rolls

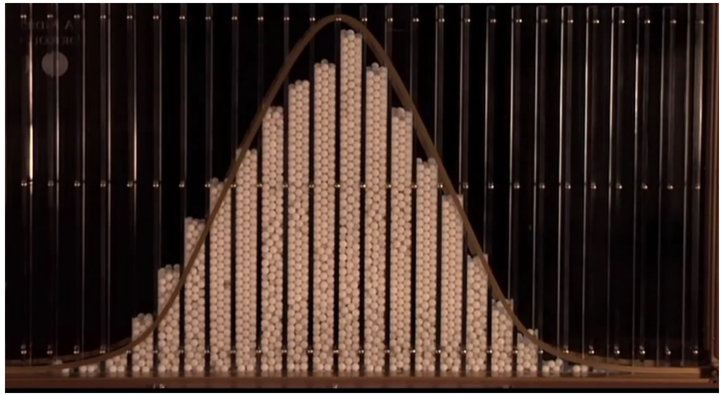
CLT explains a lot

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \quad \text{As } n \to \infty$$

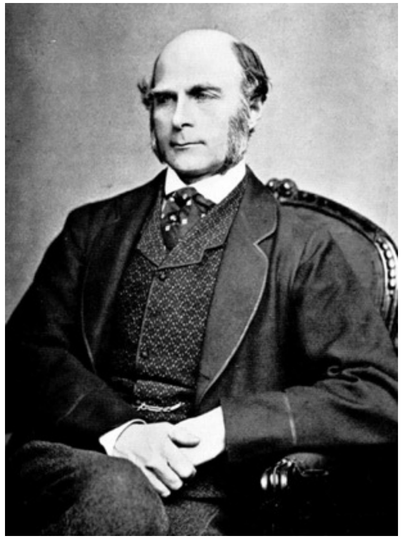
The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.

$n = 5$





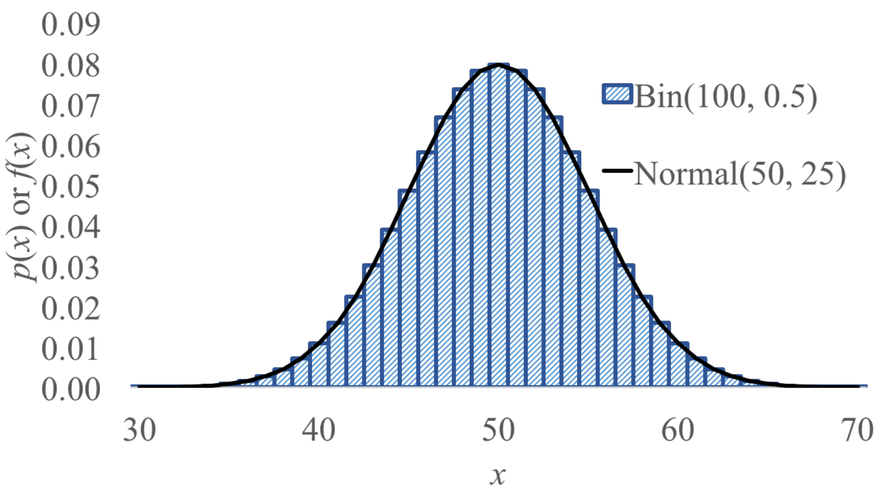
Galton Board, by Sir Francis Galton (1822-1911)



CLT explains a lot

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2) \quad \text{As } n \to \infty$$

The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.



Normal approximation of Binomial

Sum of i.i.d. Bernoulli RVs ≈ Normal

Proof:

Let $X_i \sim \text{Ber}(p)$ for $i = 1, \ldots, n$, where X_i are i.i.d.
$E[X_i] = p$, $\text{Var}(X_i) = p(1-p)$

$$X = \sum_{i=1}^{n} X_i \qquad (X \sim \text{Bin}(n, p))$$

$$X \sim \mathcal{N}(n\mu, n\sigma^2) \qquad (\text{CLT, as } n \to \infty)$$

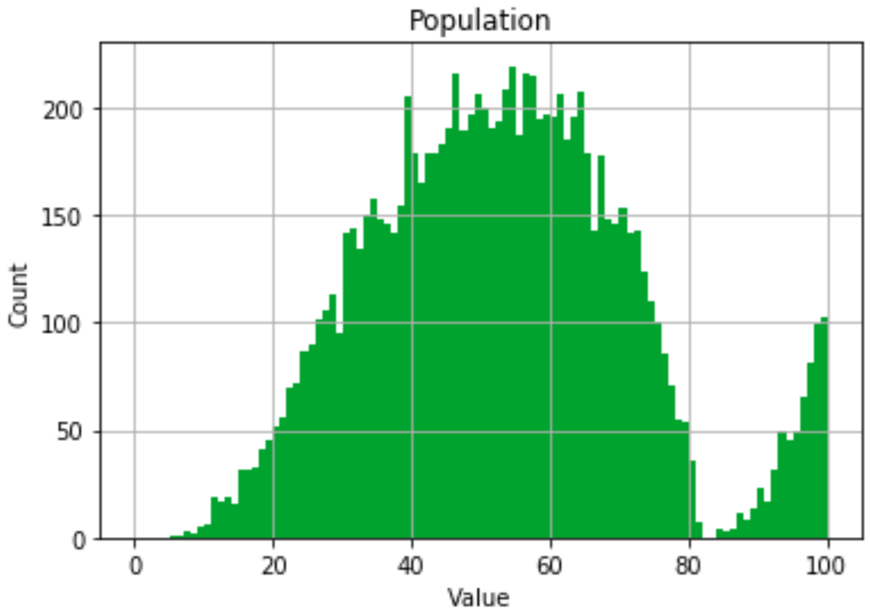
$$X \sim \mathcal{N}(np, np(1-p)) \qquad (\text{substitute mean, variance of Bernoulli})$$

CLT explains a lot

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

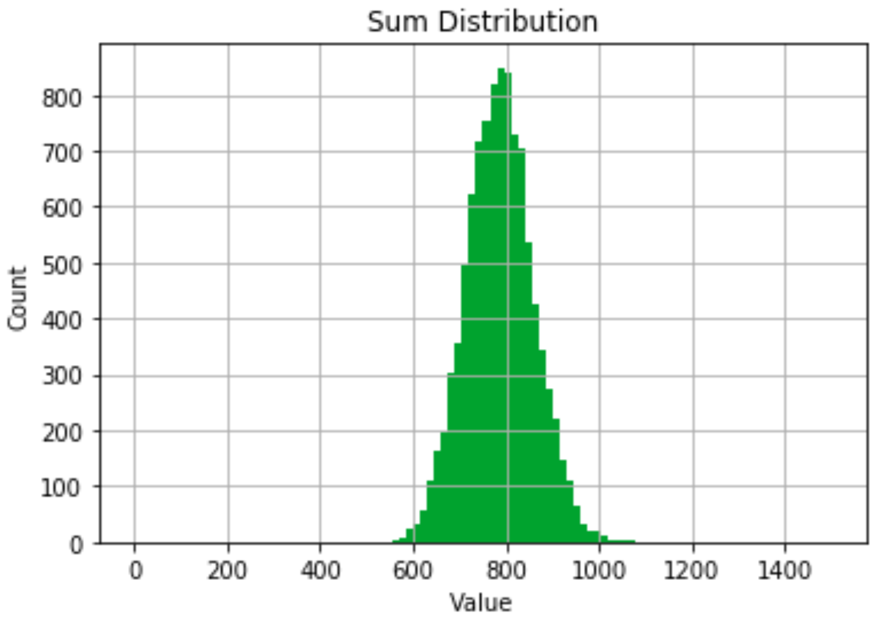
As $n \to \infty$

The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.



Distribution of X_i

Sample of size 15, **sum** values



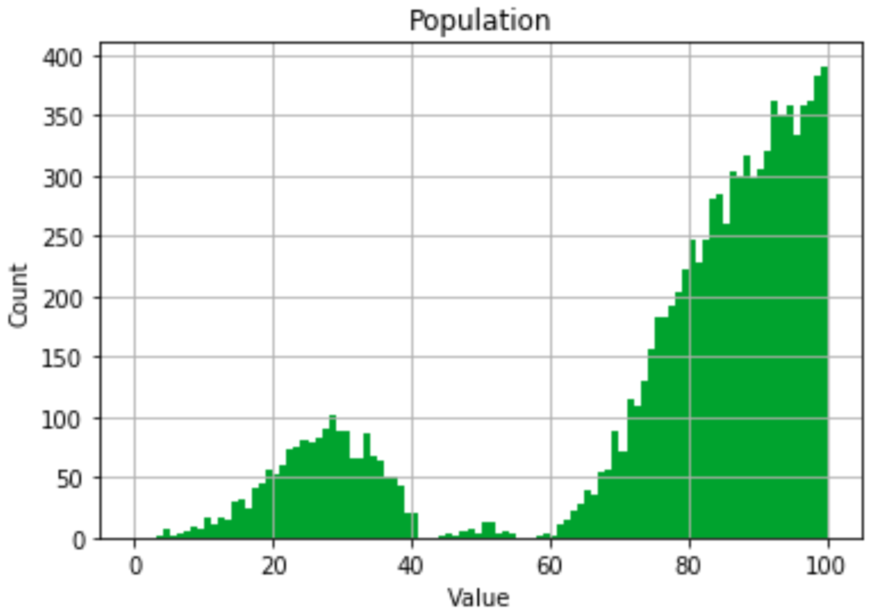
Distribution of $\sum_{i=1}^{15} X_i$

CLT explains a lot

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$$

As $n \to \infty$

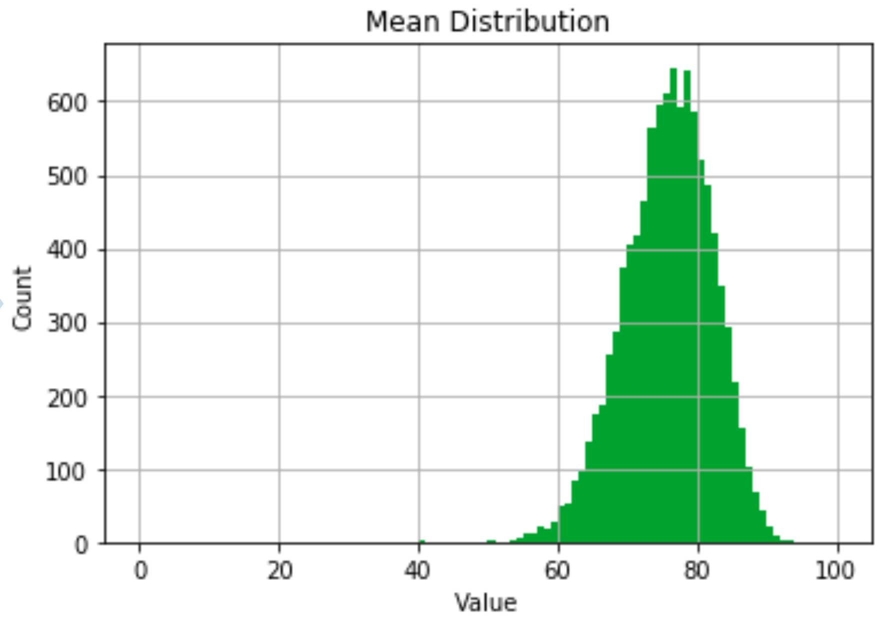
The sum of n **i.i.d.** random variables is normally distributed with mean $n\mu$ and variance $n\sigma^2$.



Distribution of X_i

Sample of size 15, **average** values

(sample mean)



Distribution of $\frac{1}{15}\sum_{i=1}^{15} X_i$

CLT example

Sum of n independent Uniform RVs

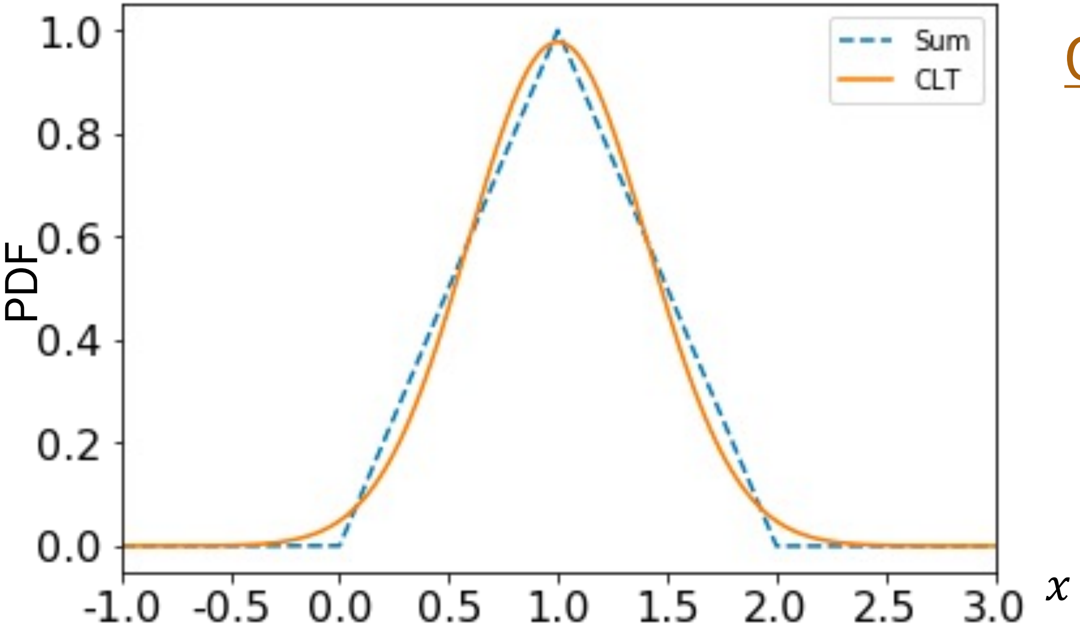
Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$.

$\mu = E[X_i] = 1/2$

$\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \leq n/3)$?

$n = 2$:



Exact $\quad P(X \leq 2/3) \approx 0.2222$

CLT approximation

$$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2) \quad \Rightarrow Y \sim \mathcal{N}(1, 1/6)$$

$$P(X \leq 2/3) \approx P(Y \leq 2/3)$$

$$= \Phi\left(\frac{2/3 - 1}{\sqrt{1/6}}\right) \quad \approx 0.2071$$

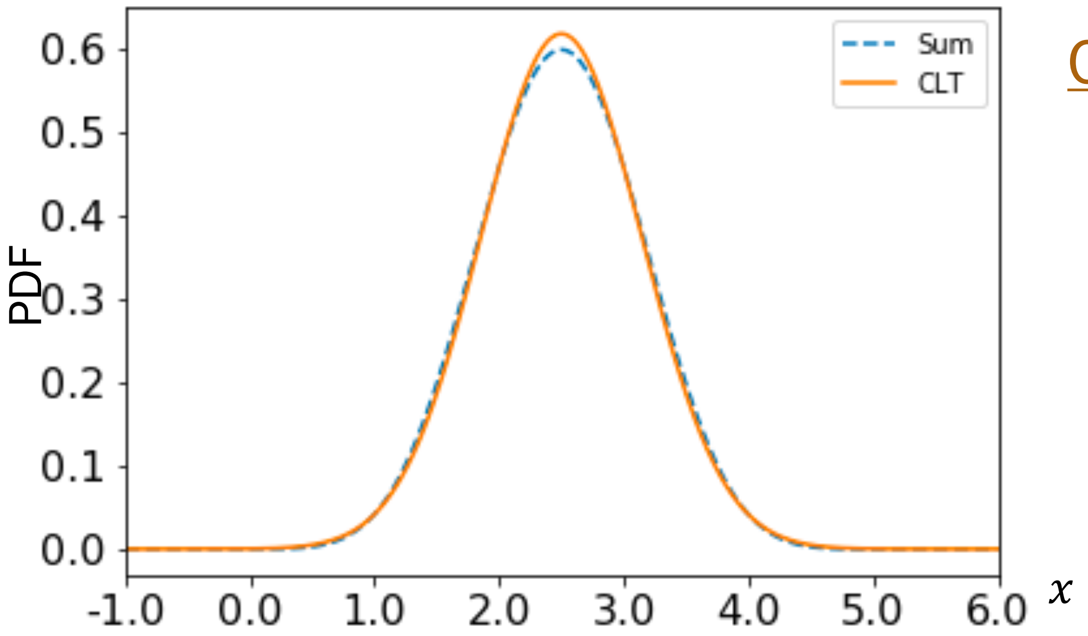
Sum of n independent Uniform RVs

Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$.

$\mu = E[X_i] = 1/2$
$\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \le n/3)$?

$n = 5$:



Exact $\quad P(X \le 5/3) \approx 0.1017$

CLT approximation

$$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2) \quad \Rightarrow Y \sim \mathcal{N}(5/2, 5/12)$$

$$P(X \le 5/3) \approx P(Y \le 5/3)$$

$$= \Phi\left(\frac{5/3 - 5/2}{\sqrt{5/12}}\right) \approx 0.0984$$

Sum of n independent Uniform RVs

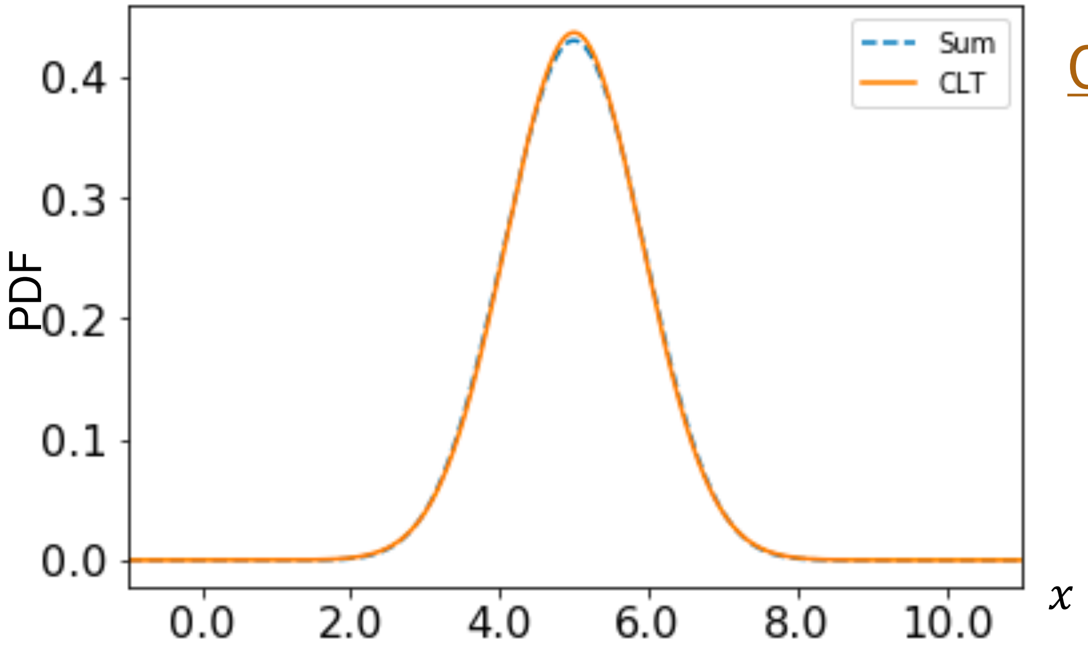
Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$.

$\mu = E[X_i] = 1/2$

$\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \le n/3)$?

$n = 10$:



Exact

$$P(X \le 10/3) \approx 0.0337$$

CLT approximation

$$X \approx Y \sim \mathcal{N}(n\mu, n\sigma^2) \quad \Longrightarrow Y \sim \mathcal{N}(5, 5/6)$$

$$P(X \le 10/3) \approx P(Y \le 10/3)$$

$$= \Phi\left(\frac{10/3 - 5}{\sqrt{5/6}}\right) \approx 0.0339$$

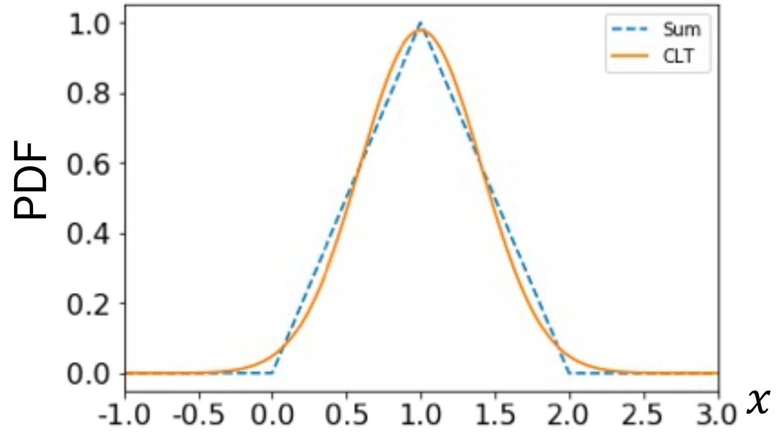
Sum of n independent Uniform RVs

Let $X = \sum_{i=1}^{n} X_i$ be sum of i.i.d. RVs, where $X_i \sim \text{Uni}(0,1)$.

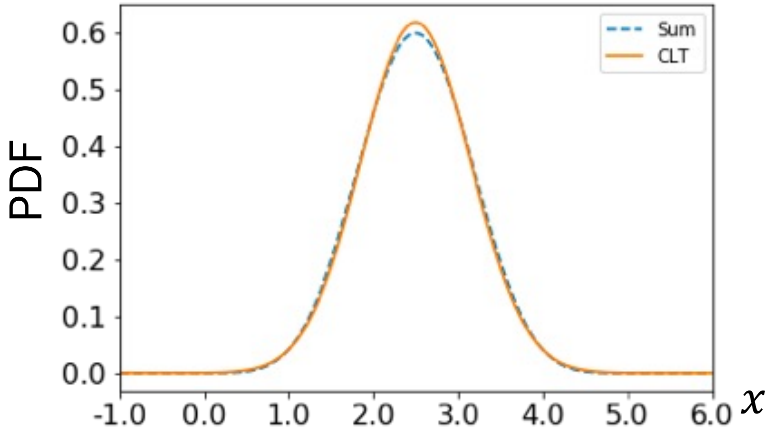
$\mu = E[X_i] = 1/2$
$\sigma^2 = \text{Var}(X_i) = 1/12$

For different n, how close is the CLT approximation of $P(X \leq n/3)$?

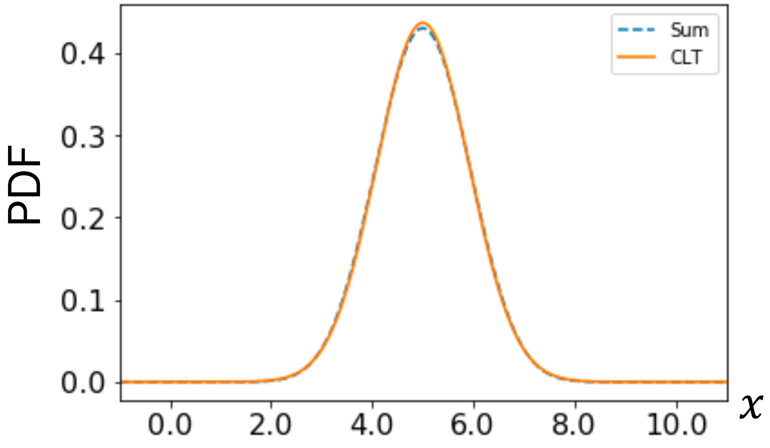
$n = 2$:



$n = 5$:



$n = 10$:



Most books will tell you that CLT holds if $n \geq 30$, but it can hold for smaller n depending on the distribution of your i.i.d. X_i's.

The sum of independent, identically distributed variables:

$$Y = \sum_{i=0}^{n} X_i$$

Is normally distributed:

$$Y \sim N(n\mu, n\sigma^2)$$

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where $\mu = E[X_i]$

$$\sigma^2 = \mathrm{Var}(X_i)$$

It's play time!

Sum of Dice

- You will roll 10 6-sided dice ($X_1, X_2, \ldots, X_{10}$)
  - X = total value of all 10 dice = $X_1 + X_2 + \ldots + X_{10}$
  - Win if: $X \leq 25$ or $X \geq 45$
  - Roll!

- And now the truth (according to the CLT)…

Sum of Dice

- You will roll 10 6-sided dice ($X_1, X_2, \ldots, X_{10}$)
  - X = total value of all 10 dice = $X_1 + X_2 + \ldots + X_{10}$
  - Win if: $X \le 25$ or $X \ge 45$

- Recall CLT: $X = \sum_i^n X_i \to N(n\mu, n\sigma^2)$ As $n \to \infty$
  - Determine P($X \le 25$ or $X \ge 45$) using CLT:

$\mu = E[X_i] = 3.5$ $\sigma^2 = \text{Var}(X_i) = \frac{35}{12}$ $X \approx N(35, 29.2)$

$$1 - P(25.5 < X < 44.5) = 1 - P\left(\frac{25.5 - 35}{\sqrt{29.2}} < Z < \frac{44.5 - 35}{\sqrt{29.2}}\right)$$

$$\approx 1 - (2\Phi(1.76) - 1) \approx 2(1 - 0.9608) = 0.0784$$

Wonderful Form of Cosmic Order

I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the ”[Central limit theorem]". The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.

- Sir Francis Galton