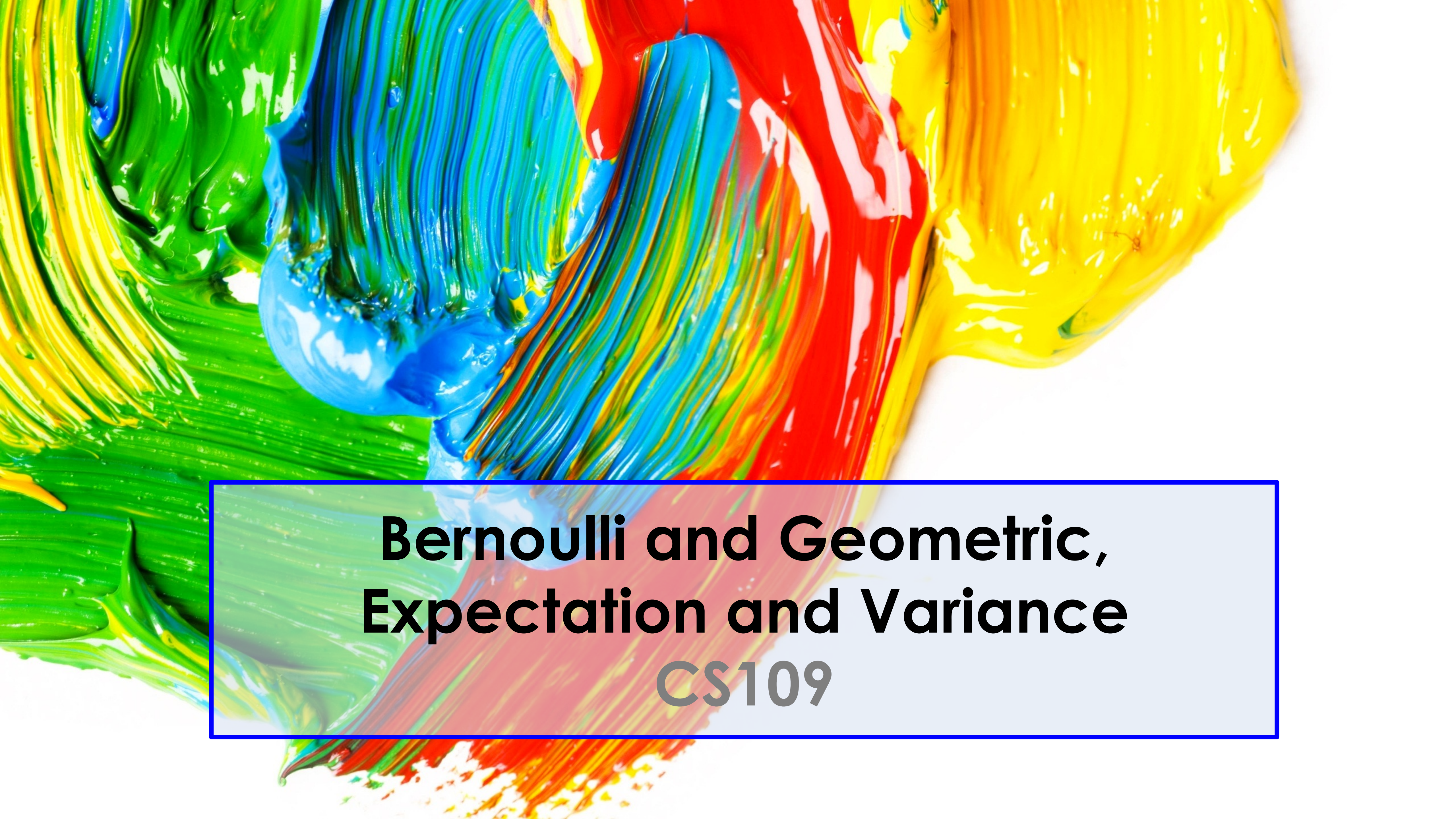


Bernoulli and Geometric, Expectation and Variance

CS109

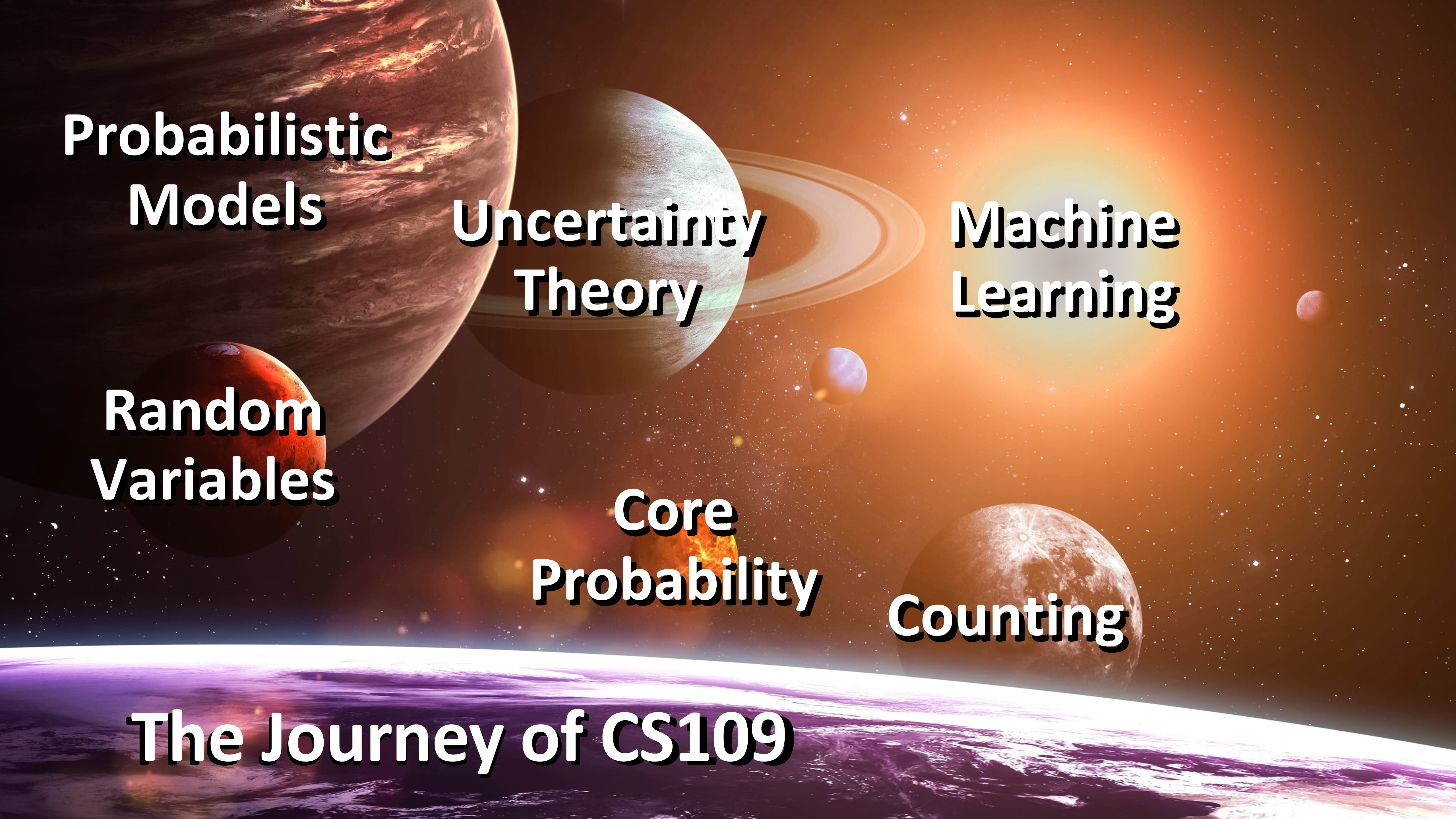
St. Petersburg Paradox

The Game:

- We have a fair coin (lands on heads with p = 0.5)
- Let n = number of coin flips (tails) to get the first heads
- You will win: $\$2^n$

How much would you pay to play?

Review


Probabilistic Models
Uncertainty Theory
Machine Learning
Random Variables
Core Probability
Counting
The Journey of CS109

Random Variables

A **random variable** is a variable whose value is uncertain.

Random Variables

It is an event when
X takes on a value

$$X \qquad\qquad X = 2$$

Let X be a
random variable

A **random variable** is a variable whose value is uncertain.

Random Variables

It is an event when
X takes on a value

$$X \qquad X = 2 \qquad P(X = 2)$$

Let X be a
random variable

So we can still work with
probabilities of events

A **random variable** is a variable whose value is uncertain.

Probability Mass Functions (PMFs)

Random variables are fully described by their **probability mass function**.

"Let Z be the sum of rolling two dice."

- $P(Z = 2) = 1/36$
- $P(Z = 3) = 2/36$
- $P(Z = 4) = 3/36$
- $P(Z = 5) = 4/36$
- $P(Z = 6) = 5/36$
- $P(Z = 7) = 6/36$
- $P(Z = 8) = 5/36$
- $P(Z = 9) = 4/36$
- $P(Z = 10) = 3/36$
- $P(Z = 11) = 2/36$
- $P(Z = 12) = 1/36$

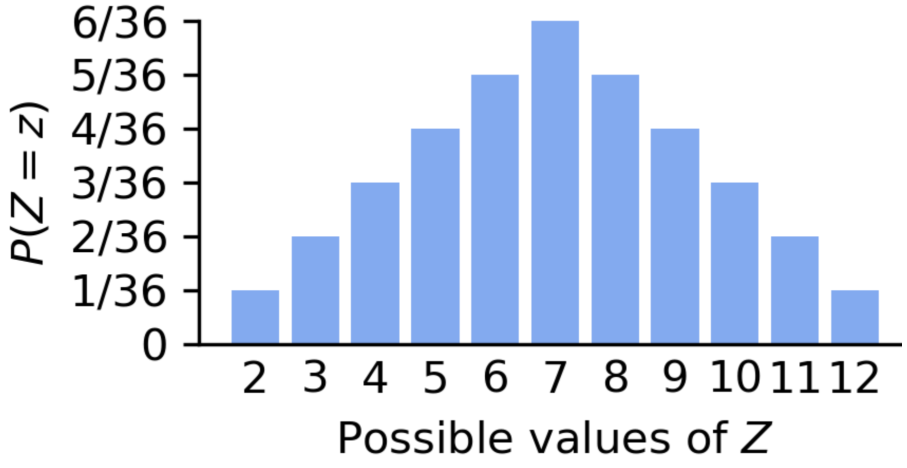
Probability mass function = all possible outcomes + their probabilities

Probability Mass Functions (PMFs)

Random variables are fully described by their **probability mass function**.

"Let Z be the sum of rolling two dice."

- $P(Z = 2) = 1/36$
- $P(Z = 3) = 2/36$
- $P(Z = 4) = 3/36$
- $P(Z = 5) = 4/36$
- $P(Z = 6) = 5/36$
- $P(Z = 7) = 6/36$
- $P(Z = 8) = 5/36$
- $P(Z = 9) = 4/36$
- $P(Z = 10) = 3/36$
- $P(Z = 11) = 2/36$
- $P(Z = 12) = 1/36$



$$P(Z = z) = \begin{cases} \frac{z-1}{36} & z \in \mathbb{Z}, 2 \leq z \leq 6 \\ \frac{13-z}{36} & z \in \mathbb{Z}, 7 \leq z \leq 12 \\ 0 & \text{else} \end{cases}$$

Random Variables Are Awesome

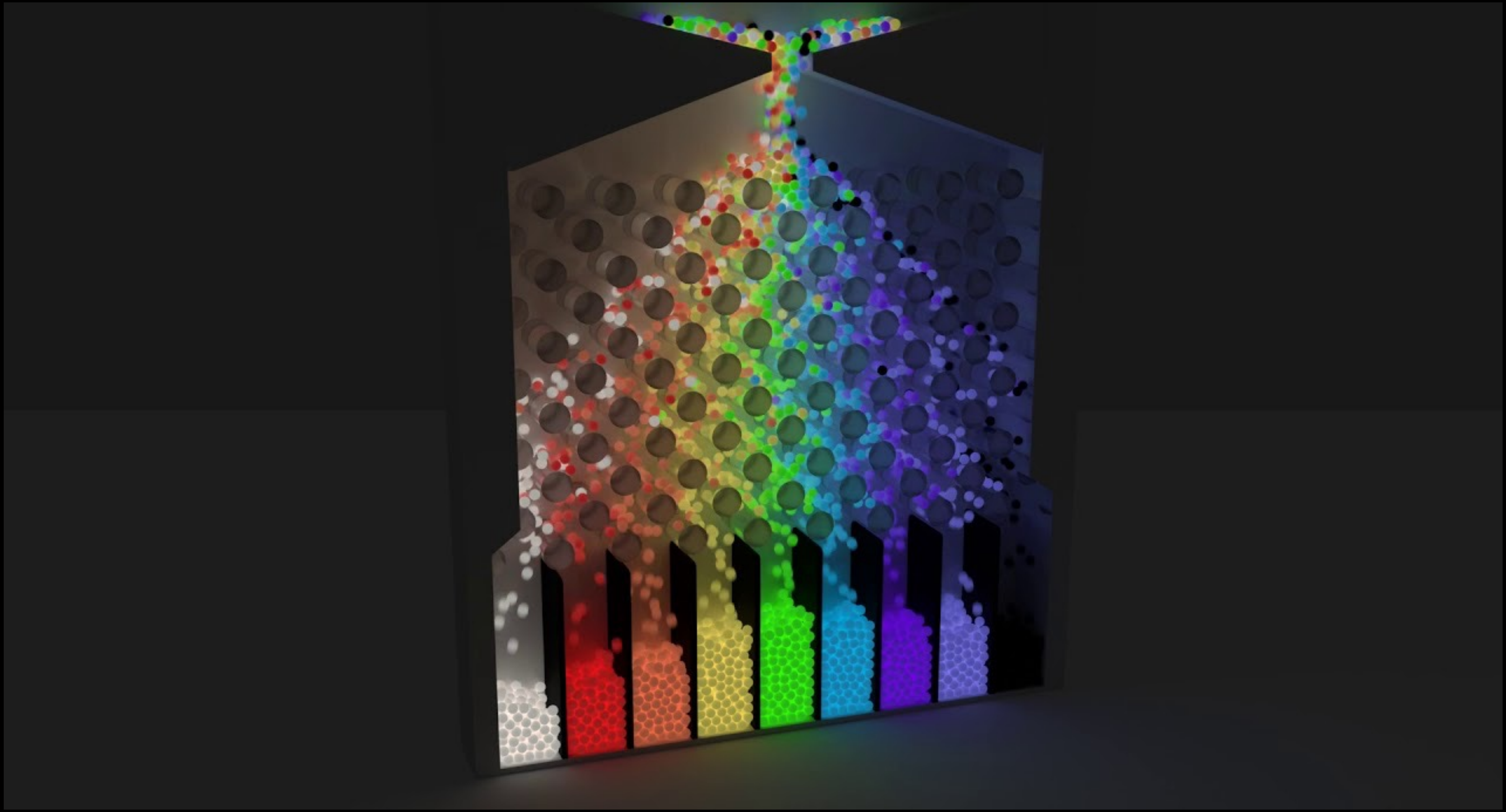
They compactly represent an entire experiment!

You can still do core probability with them!

Random Variables Are Awesome

You can use PMFs other people invent!

You can re-use random variables over and over!



Some Random Variables Are “Classics”

The Binomial: Probability of k Heads in n Coin Flips

Imagine flipping a coin n times and counting the number of heads.

1. We will flip a coin n times: ***n* independent trials** of the same experiment
2. Each coin flip has a **probability *p*** of being heads
3. What we want to model: what is the probability of **exactly *k* heads**?

The Binomial: Probability of *k* Heads in *n* Coin Flips

Imagine flipping a coin *n* times and counting the number of heads.

1. We will flip a coin ***n*** times: ***n* independent trials** of the same experiment

2. Each coin flip has a **probability *p*** of being heads

3. What we want to model: what is the probability of **exactly *k* heads**?

Tip: knowing the "generative story" behind each random variable helps you recognize when to apply it.

The Binomial: Probability of *k* Heads in *n* Coin Flips

If any single outcome with *k* heads has probability $p^k(1-p)^{n-k}$,

(H, H, H, H, T, T, T, T, T, T)

The Binomial: Probability of *k* Heads in *n* Coin Flips

If any single outcome with *k* heads has probability $p^k(1-p)^{n-k}$,

And there are $\binom{n}{k}$ possible outcomes with *k* heads,

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
```

The Binomial: Probability of *k* Heads in *n* Coin Flips

If any single outcome with *k* heads has probability $p^k(1-p)^{n-k}$,

And there are $\binom{n}{k}$ possible outcomes with *k* heads,

Then the probability of *k* heads (in any order) is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

PMF of Binomial

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
```

Declaring a Random Variable to be Binomial

Our random variable

Num trials

Probability of success on each trial

$$X \sim \text{Bin}(n, p)$$

Is distributed as a

Binomial

With these parameters

Declaring a Random Variable to be Binomial

Our random variable

Num trials

Probability of success on each trial

$$X \sim \text{Bin}(n, p)$$

Is distributed as a

Binomial

With these parameters

Then we automatically get the PMF for free!

End Review

Boss Battle Binomial Problem


PANTHERS
PANTHERS

Probability of Winning a 7-Game Series?

The Florida Panthers recently beat the Edmonton Oilers in the "best of 7" Stanley Cup final (ice hockey).

- A team wins the Stanley Cup if they win at least 4 games.
- Each game is **independent**, and the Panthers had a 0.55 probability of winning each game.

What was the probability of the Panthers winning the series?

Probability of Winning a 7-Game Series?

The Florida Panthers recently beat the Edmonton Oilers in the "best of 7" Stanley Cup final (ice hockey).

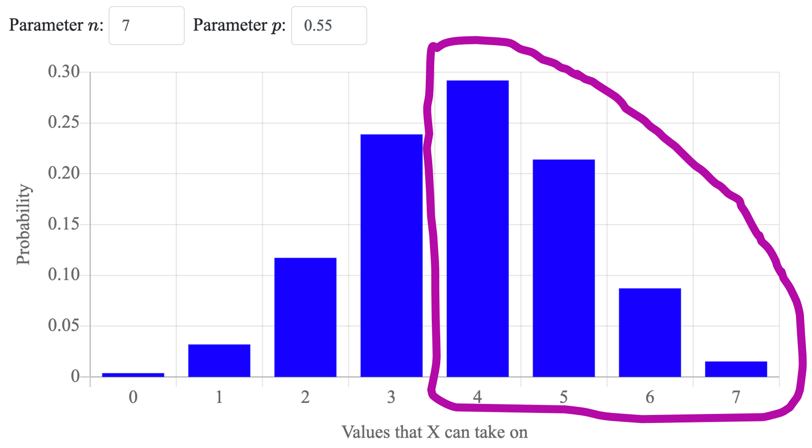
- A team wins the Stanley Cup if they win at least 4 games.
- Each game is **independent**, and the Panthers had a 0.55 probability of winning each game.

What was the probability of the Panthers winning the series?

Let X be the number of games the Panthers win. $X \sim \text{Bin}(n = 7, p = 0.55)$

$$P(X \geq 4) = \sum_{i=4}^{7} P(X = i) = \sum_{i=4}^{7} \binom{7}{i} p^i (1-p)^{7-i}$$

$$= \sum_{i=4}^{7} \binom{7}{i} 0.55^i (0.45)^{7-i}$$



Probability of Winning a 7-Game Series?

The Florida Panthers recently beat the Edmonton Oilers in the "best of 7" Stanley Cup final (ice hockey).

- A team wins the Stanley Cup if they win at least 4 games.
- Each game is **independent**, and the Panthers had a 0.55 probability of winning each game.

What was the probability of the Panthers winning the series?

Let X be the number of games the Panthers win. $X \sim \text{Bin}(n = 7, p = 0.55)$

$$P(X \geq 4) = \sum_{i=4}^{7} P(X = i) = \sum_{i=4}^{7} \binom{7}{i} p^i (1-p)^{7-i}$$

$$= \sum_{i=4}^{7} \binom{7}{i} 0.55^i (0.45)^{7-i}$$

```
from scipy import stats

prob_sum = 0
for i in range(4, 8):
    prob_sum += stats.binom.pmf(i, 7, 0.55)

print(prob_sum)
```

Pokémon™

Gotta catch 'em all!™

(More Classic Random Variables)

The Geometric Random Variable

Imagine flipping a coin *until you see your first heads*.

Each coin flip is an independent trial, with probability *p* of getting heads.

Want to model: how many coin flips until the first heads?

$$X \sim \text{Geo}(p)$$

Like throwing pokeballs
until you catch a pokemon!

The Geometric Random Variable

Imagine flipping a coin *until you see your first heads*.

Each coin flip is an independent trial, with probability p of getting heads.

Want to model: how many coin flips until the first heads?

$$X \sim \mathrm{Geo}(p)$$

Deriving the PMF:

P(heads on first flip) = p

P(tails, then heads) = $(1 – p) * p$

P(tails, tails, heads) = $(1 – p)^2 * p$

…

The Geometric Random Variable

Imagine flipping a coin *until you see your first heads*.

Each coin flip is an independent trial, with probability *p* of getting heads.

Want to model: how many coin flips until the first heads?

$$X \sim \text{Geo}(p)$$

Deriving the PMF:

P(heads on first flip) = p

P(tails, then heads) = $(1 – p) * p$

P(tails, tails, heads) = $(1 – p)^2 * p$

…

$$P(X = n) = (1 - p)^{n-1}p$$

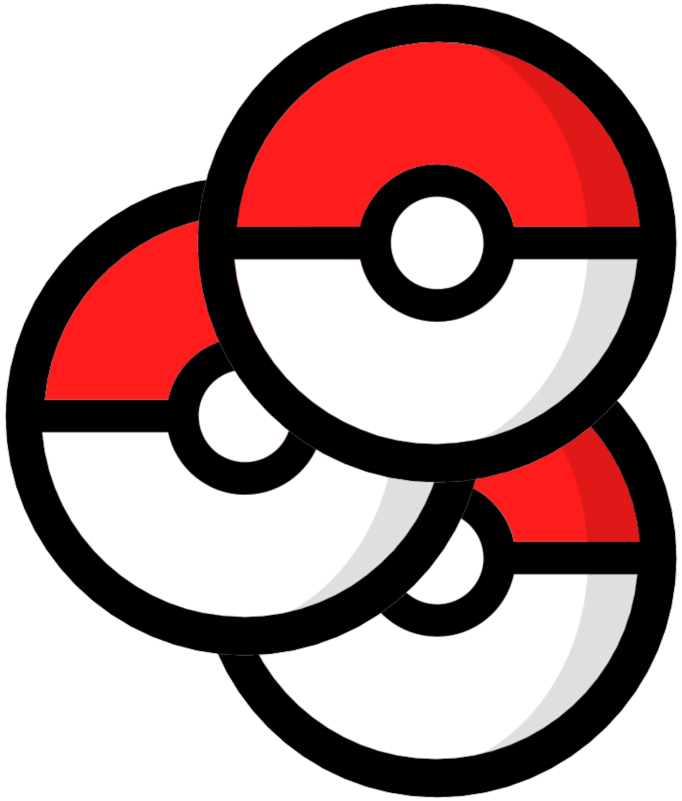
The Negative Binomial Random Variable

Imagine flipping a coin *until you see* ***r*** *heads*.

Each coin flip is an independent trial, with probability *p* of getting heads.

Want to model: how many coin flips until ***r*** heads?

Like catching *r* pokemon



The Negative Binomial Random Variable

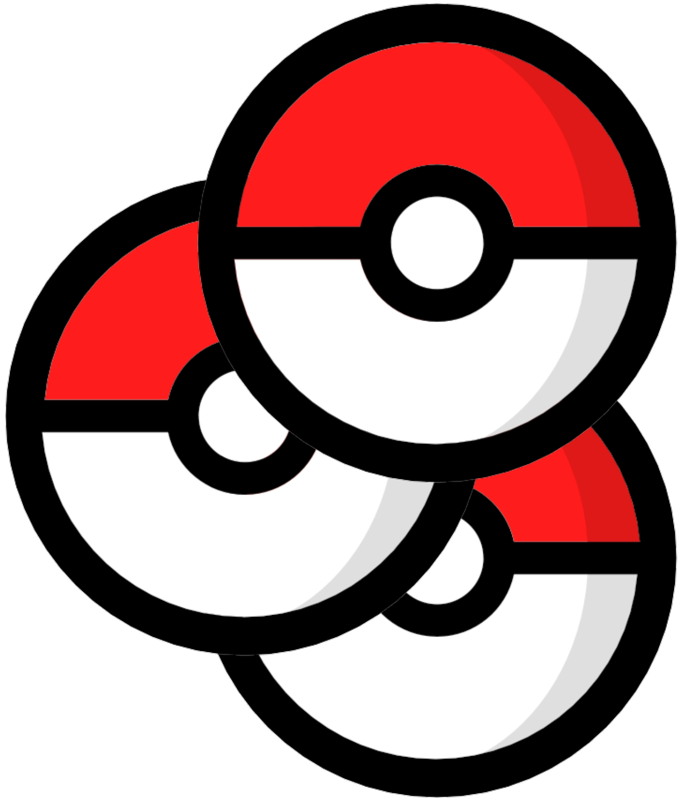
Imagine flipping a coin *until you see **r** heads*.

Each coin flip is an independent trial, with probability *p* of getting heads.

Want to model: how many coin flips until ***r*** heads?

$$X \sim \text{NegBin}(r, p)$$

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$



Pokemon: Actually Catching Them All

To catch a Pokemon, you throw a pokeball repeatedly until it's caught.

Each pokeball has a 1/3 chance of catching the Pokemon.

What is the probability that you catch a pokemon using fewer than 3 pokeballs?

Pokemon: Actually Catching Them All

To catch a Pokemon, you throw a pokeball repeatedly until it's caught.

Each pokeball has a 1/3 chance of catching the Pokemon.

What is the probability that you catch a pokemon using fewer than 3 pokeballs?

Let *X* be the number of pokeballs we use.

$X \sim \text{Geo}(p = 1/3)$

$$P(X = 1 \text{ or } X = 2) = p + p(1-p) = \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{5}{9}$$

Pokemon: Actually Catching Them All

To catch a Pokemon, you throw a pokeball repeatedly until it's caught.

Each pokeball has a 1/3 chance of catching the Pokemon.

What is the probability that you catch a pokemon using fewer than 3 pokeballs?

Let X be the number of pokeballs we use.

$X \sim \text{Geo}(p = 1/3)$

$$P(X = 1 \text{ or } X = 2) = p + p(1-p) = \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{5}{9}$$

There are 151 Pokemon to catch in the game Pokemon Diamond.

What is the probability that you need 300 pokeballs to catch *every* Pokemon?

Pokemon: Actually Catching Them All

To catch a Pokemon, you throw a pokeball repeatedly until it's caught.

Each pokeball has a 1/3 chance of catching the Pokemon.

What is the probability that you catch a pokemon using fewer than 3 pokeballs?

Let X be the number of pokeballs we use.

$X \sim \text{Geo}(p = 1/3)$

$$P(X = 1 \text{ or } X = 2) = p + p(1-p) = \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{5}{9}$$

There are 151 Pokemon to catch in the game Pokemon Diamond.

What is the probability that you need 300 pokeballs to catch *every* Pokemon?

Let Y be the number of pokeballs we use in total.

$Y \sim \text{NegBin}(r = 151, p = 1/3)$

$$P(Y = 300) = \binom{300-1}{151-1} \left(\frac{1}{3}\right)^{151} \left(1 - \frac{1}{3}\right)^{300-151}$$

Can Jacob Bernoulli Have a Variable Named After Him?

Here yee. I want to have a random variable named after myself. Huzzah.

Can Jacob Bernoulli Have a Variable Named After Him?

Here yee. I want to have a random variable named after myself. Huzzah.

Yes - the Bernoulli random variable: $X \sim \text{Bern}(p)$

Can Jacob Bernoulli Have a Variable Named After Him?

Here yee. I want to have a random variable named after myself. Huzzah.

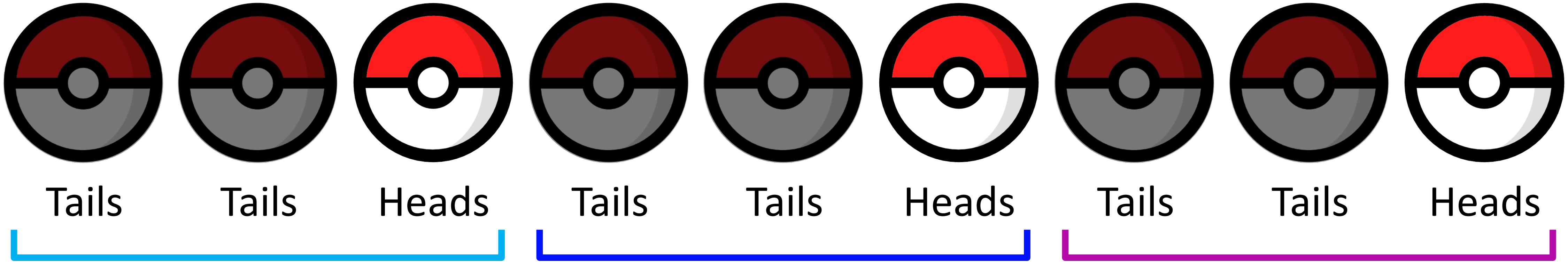
Yes - the Bernoulli random variable: $X \sim \text{Bern}(p)$

- The Bernoulli is an **indicator** random variable (value is either 0 or 1).
- $P(X = 1) = p$
- $P(X = 0) = 1 - p$
- Examples: a single coin flip, one ad click, any binary event

(this is the whole PMF)

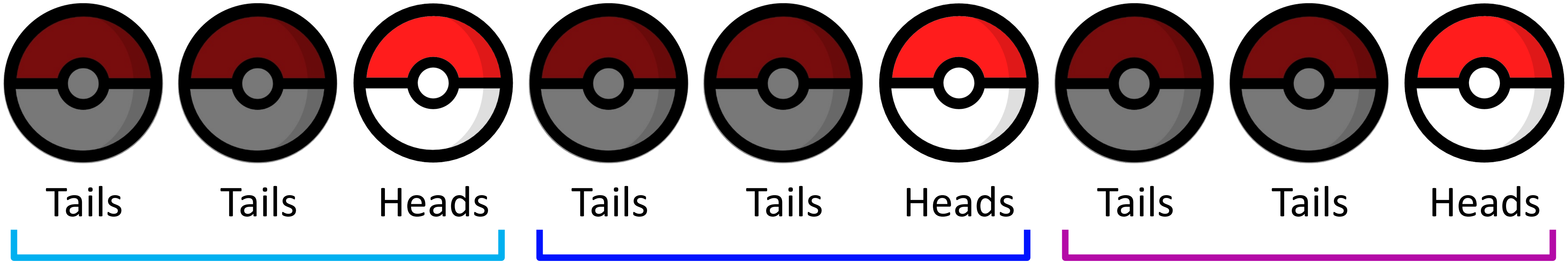
Random Variable Sums

The Negative Binomial



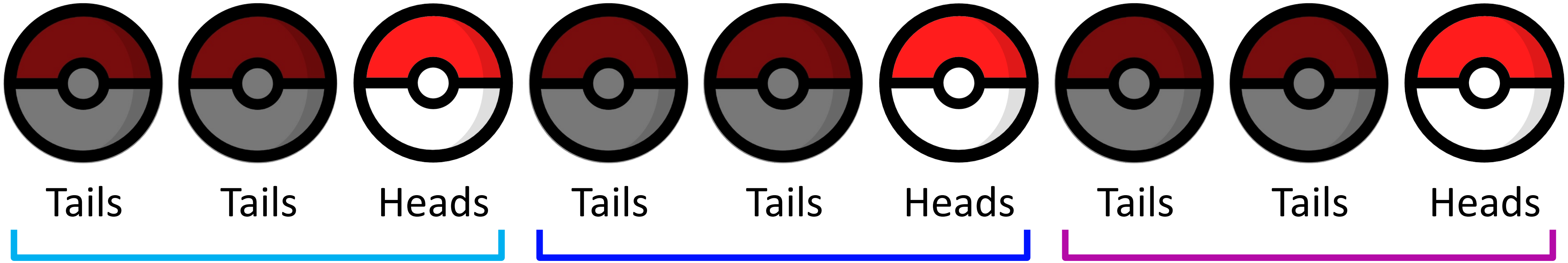
Random Variable Sums

The Negative Binomial ...is a sum of Geometric random variables



Random Variable Sums

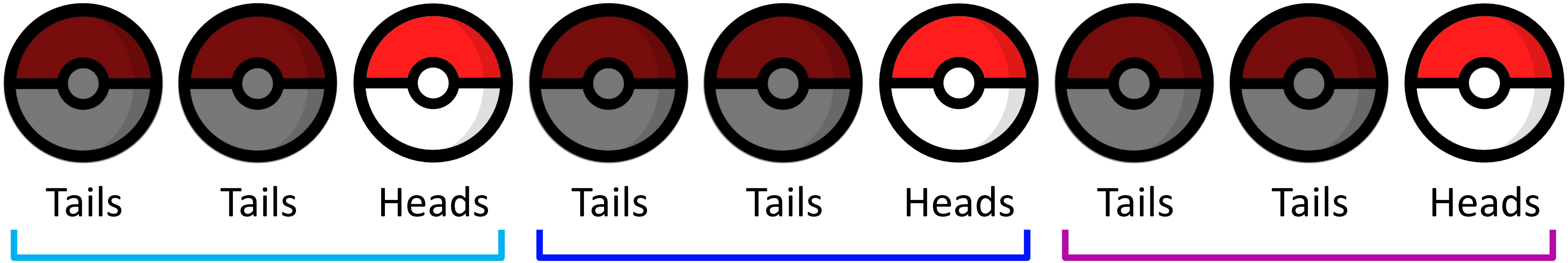
The Negative Binomial …is a sum of Geometric random variables



Let $X_1 \sim \text{Geo}(p = 1/3)$, $X_2 \sim \text{Geo}(p = 1/3)$, and $X_3 \sim \text{Geo}(p = 1/3)$.

Random Variable Sums

The Negative Binomial ...is a sum of Geometric random variables

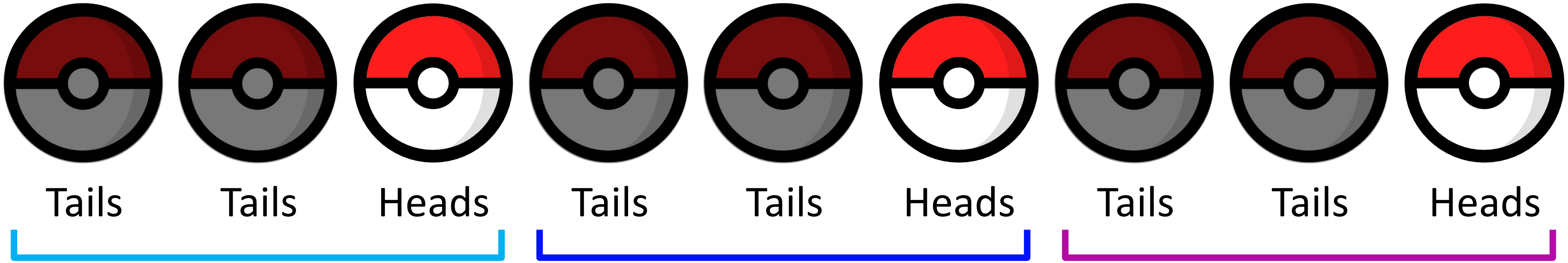


Let $X_1 \sim \text{Geo}(p = 1/3)$, $X_2 \sim \text{Geo}(p = 1/3)$, and $X_3 \sim \text{Geo}(p = 1/3)$.

$$Y \sim \text{NegBin}(r = 3, p = 1/3)$$

Random Variable Sums

The Negative Binomial ...is a sum of Geometric random variables



Let $X_1 \sim \text{Geo}(p = 1/3)$, $X_2 \sim \text{Geo}(p = 1/3)$, and $X_3 \sim \text{Geo}(p = 1/3)$.

$$Y \sim \text{NegBin}(r = 3, p = 1/3)$$

$$Y = X_1 + X_2 + X_3$$

Random Variable Sums

The Binomial

Heads Tails Heads Tails Heads Heads Tails Tails Heads

Random Variable Sums

The Binomial

…is a sum of Bernoulli random variables

Random Variable Sums

The Binomial ...is a sum of Bernoulli random variables

Let $X_1 \sim \text{Bern}(p = 1/2)$ and $X_2 \sim \text{Bern}(p = 1/2)$.

$$Y \sim \text{Bin}(n = 2, p = 1/2)$$

$$Y = X_1 + X_2$$

Expectation

Expected Value or Expectation

Expected value answers the question:

What is the average value we could expect some random variable to be?

Expected Value or Expectation

Expected value answers the question:

What is the average value we could expect some random variable to be?

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Expected Value or Expectation

Expected value answers the question:

What is the average value we could expect some random variable to be?

A possible value of X

The probability of that value

$$E[X] = \sum_{x} x \cdot \mathrm{P}(X = x)$$

Loop over all values x that X can take on

Example: Expected Value of Dice Roll

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

Example: Expected Value of Dice Roll

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$E[X] = \sum_{x=1}^{6} x \cdot P(X = x)$$

Example: Expected Value of Dice Roll

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$E[X] = \sum_{x=1}^{6} x \cdot P(X = x)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 3.5$$

Example: Expected Value of Dice Roll

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$\begin{aligned} E[X] &= \sum_{x=1}^{6} x \cdot P(X = x) \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \\ &= 3.5 \end{aligned}$$

E[X] is not always an actual possible outcome for X

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **class** with equal probability.

Let X be the chosen class's size. What is $\mathrm{E}[X]$?

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **class** with equal probability.

Let X be the chosen class's size. What is $\mathrm{E}[X]$?

$$P(X = 5) = 1/3$$
$$P(X = 10) = 1/3$$
$$P(X = 150) = 1/3$$

$$E[X] = \sum_{x \in \{5,10,150\}} x \cdot P(X = x)$$
$$= 5 \cdot \frac{1}{3} + 10 \cdot \frac{1}{3} + 150 \cdot \frac{1}{3}$$
$$= 55$$

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **student** with equal probability.

Let X be the chosen student's class size. What is $\mathrm{E}[X]$?

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **student** with equal probability.

Let X be the chosen student's class size. What is $\mathrm{E}[X]$?

$$P(X = 5) = 5/165$$
$$P(X = 10) = 10/165$$
$$P(X = 150) = 150/165$$

$$E[X] = \sum_{x \in \{5,10,150\}} x \cdot P(X = x)$$
$$= 5 \cdot \frac{5}{165} + 10 \cdot \frac{10}{165} + 150 \cdot \frac{150}{165}$$
$$= 137$$

Helpful Properties of Expectation

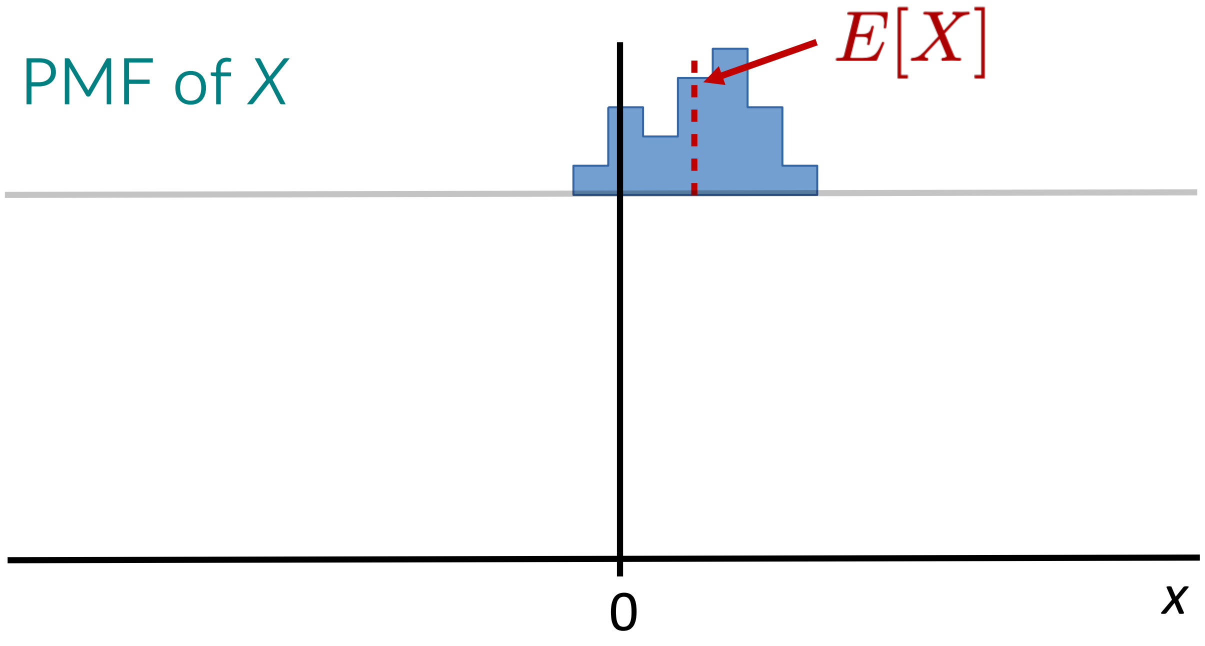
1. Linearity:

$$E[aX + b] = aE[X] + b$$

Linearity of Expectation

1. Linearity:

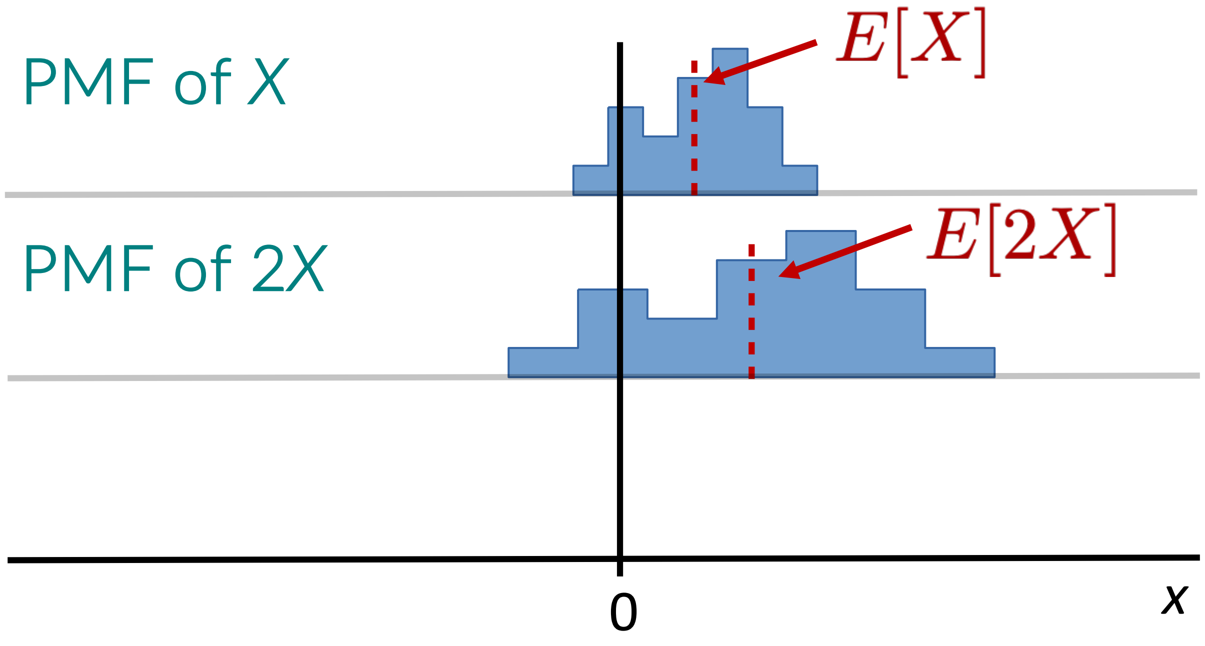
$$E[aX + b] = aE[X] + b$$



Linearity of Expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$



Multiplying by a constant stretches the whole distribution

The expectation is stretched out by the same amount

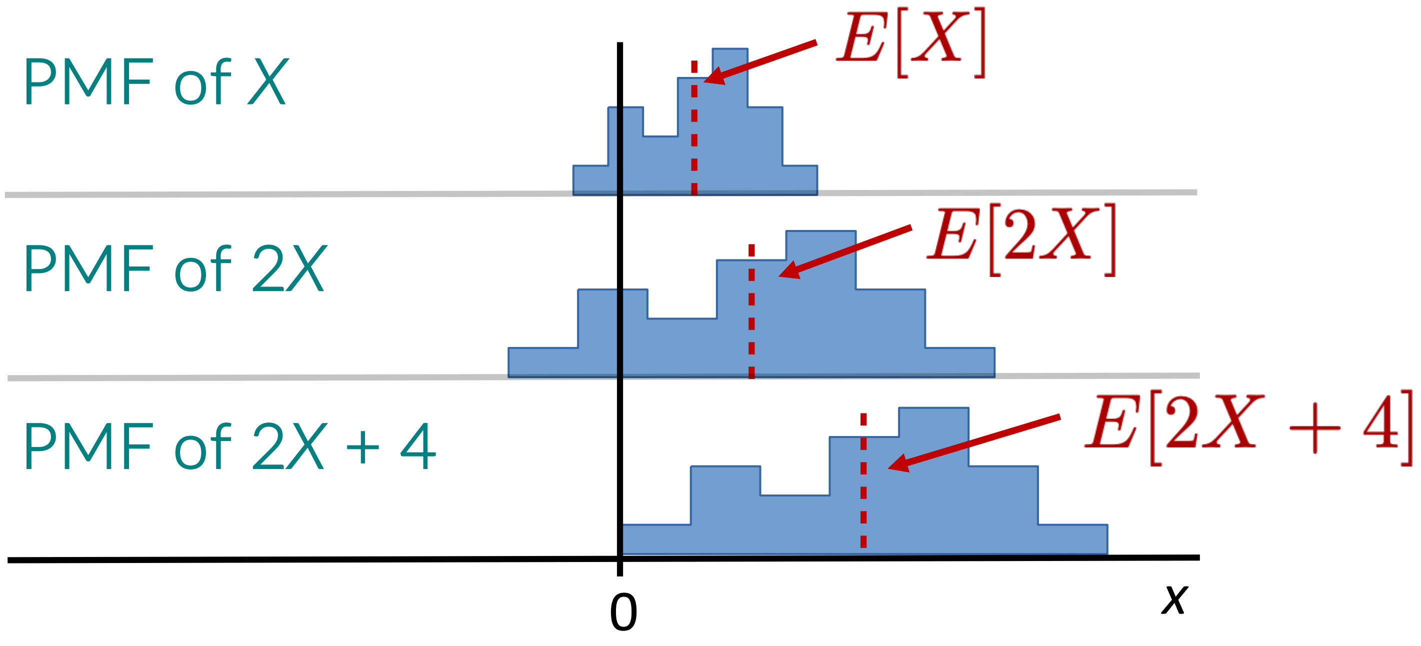
Linearity of Expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$

Adding a constant shifts the whole distribution

The expectation is shifted by the same amount



Helpful Properties of Expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$

2. Expectation of a sum is the sum of expectations:

$$E[X + Y] = E[X] + E[Y]$$

These are all true, no matter what random variables X and Y are

Helpful Properties of Expectation

1. Linearity:

$$E[aX + b] = aE[X] + b$$

2. Expectation of a sum is the sum of expectations:

$$E[X + Y] = E[X] + E[Y]$$

These are all true, no matter what random variables X and Y are

3. Law of the Unconscious Statistician:

$$E[g(x)] = \sum_{x \in X} g(x) P(X = x)$$

Law of the Unconscious Statistician (LOTUS)

$$E[g(X)] = \sum_x g(x) P(X = x)$$

This lets you get the expectation of **any** function of a random variable.

Examples:

$$E[X^2] = \sum_x x^2 \cdot P(X = x)$$

$$E[\sin(X)] = \sum_x \sin(x) \cdot P(X = x)$$

$$E[\sqrt{}(X)] = \sum_x \sqrt{x} \cdot P(X = x)$$

Expectation of Classic Random Variables

Expected Value of Free Throws

In basketball, players sometimes get a chance to shoot a free throw. If they make it, the team gets 1 point; otherwise they get no points.

Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

Let X be the points gained from Shaq attempting a free throw. What is $\mathrm{E}[X]$?

Expected Value of Free Throws

In basketball, players sometimes get a chance to shoot a free throw. If they make it, the team gets 1 point; otherwise they get no points.

Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

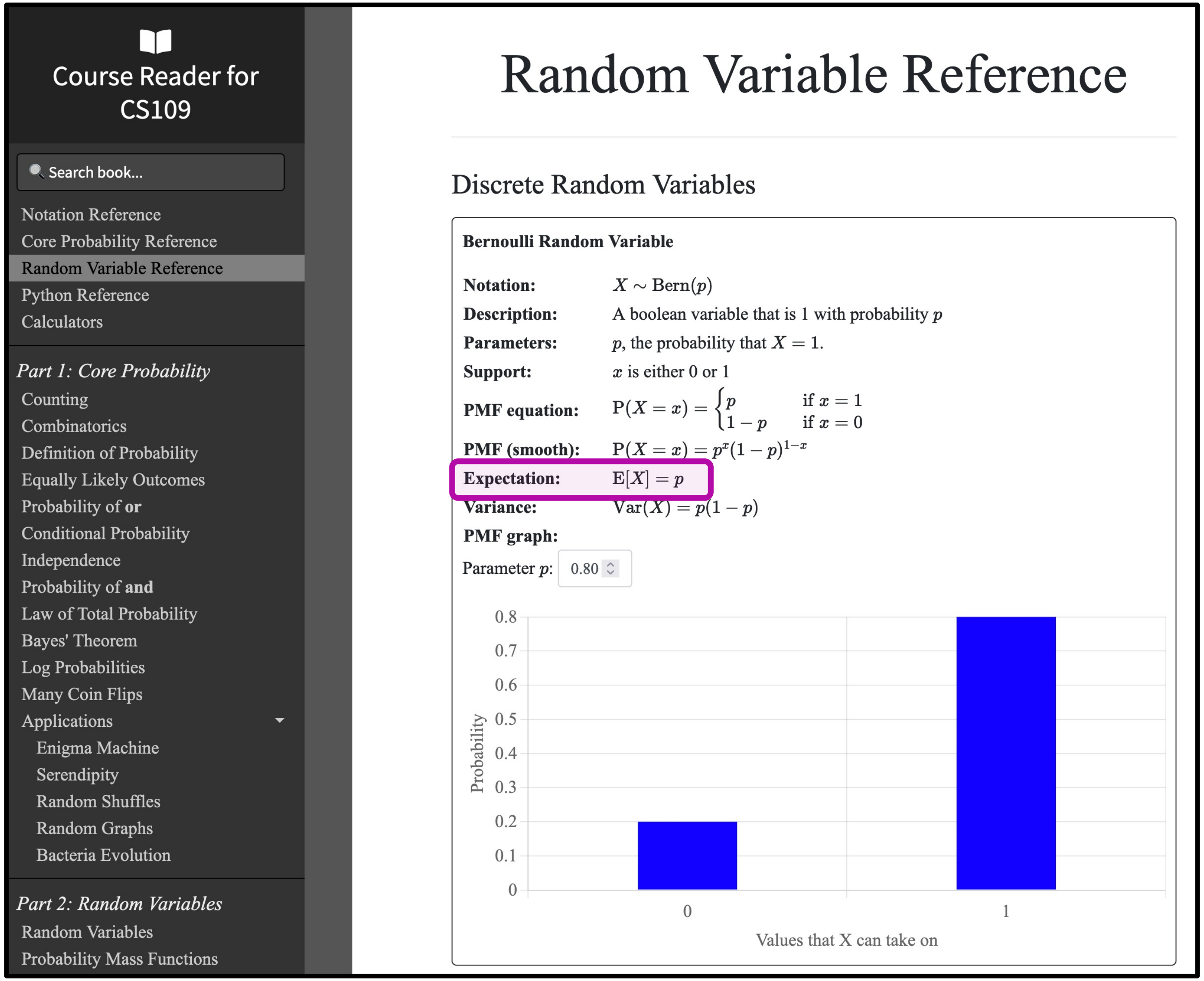
Let X be the points gained from Shaq attempting a free throw. What is $\mathrm{E}[X]$?

$$X \sim \mathrm{Bern}(p = 0.53)$$

$$\begin{aligned} E[X] &= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) \\ &= 0 \cdot 0.47 + 1 \cdot 0.53 = 0.53 \end{aligned}$$

For Bernoulli random variables, $\mathrm{E}[X] = p$ (always)

With Classic RVs, You Get Expectations For Free Too!

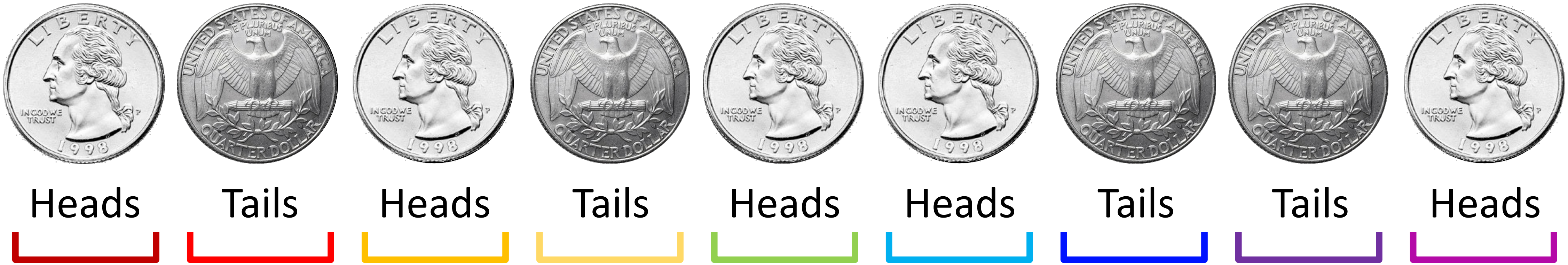


We Can Now Calculate Expectation of Binomial

$$X \sim \text{Bin}(n, p)$$

Let Y_i be 1 if trial i was a success, otherwise 0, with *i* from 1 to *n*. $Y_i \sim \text{Bern}(p)$.

The Binomial …is a sum of Bernoulli random variables



We Can Now Calculate Expectation of Binomial

$$X \sim \text{Bin}(n, p)$$

Let Y_i be 1 if trial i was a success, otherwise 0, with *i* from 1 to *n*. $Y_i \sim \text{Bern}(p)$.

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} Y_i\right] \qquad \text{Since } X = \sum_{i=1}^{n} Y_i$$

We Can Now Calculate Expectation of Binomial

$$X \sim \mathrm{Bin}(n, p)$$

Let Y_i be 1 if trial i was a success, otherwise 0, with i from 1 to n. $Y_i \sim \mathrm{Bern}(p)$.

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} Y_i\right] \qquad \text{Since } X = \sum_{i=1}^{n} Y_i$$

$$= \sum_{i=1}^{n} \mathrm{E}[Y_i] \qquad \text{Expectation of sum}$$

Expectation of a sum is the sum of expectations: $E[X + Y] = E[X] + E[Y]$

We Can Now Calculate Expectation of Binomial

$$X \sim \text{Bin}(n, p)$$

Let Y_i be 1 if trial i was a success, otherwise 0, with i from 1 to n. $Y_i \sim \text{Bern}(p)$.

$$\begin{aligned}
\text{E}[X] &= \text{E}\left[\sum_{i=1}^{n} Y_i\right] && \text{Since } X = \sum_{i=1}^{n} Y_i \\
&= \sum_{i=1}^{n} \text{E}[Y_i] && \text{Expectation of sum} \\
&= \sum_{i=1}^{n} p && \text{Expectation of Bernoulli} \\
&= n \cdot p && \text{Sum } n \text{ times}
\end{aligned}$$

We Can Now Calculate Expectation of Binomial

$$X \sim \text{Bin}(n, p)$$

Let Y_i be 1 if trial i was a success, otherwise 0, with i from 1 to n. $Y_i \sim \text{Bern}(p)$.

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n} Y_i\right] && \text{Since } X = \sum_{i=1}^{n} Y_i \\
&= \sum_{i=1}^{n} \mathrm{E}[Y_i] && \text{Expectation of sum} \\
&= \sum_{i=1}^{n} p && \text{Expectation of Bernoulli} \\
&= n \cdot p && \text{Sum } n \text{ times}
\end{aligned}$$

True for every binomial ever

Expected Value of Free Throws

In basketball, players sometimes get a chance to shoot a free throw. If they make it, the team gets 1 point; otherwise they get no points.

Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

Let Y be the points gained from Shaq attempting **500** free throws. What is $\mathrm{E}[Y]$?

Expected Value of Free Throws

In basketball, players sometimes get a chance to shoot a free throw. If they make it, the team gets 1 point; otherwise they get no points.

Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

Let Y be the points gained from Shaq attempting **500** free throws. What is $\mathrm{E}[Y]$?

$$Y \sim Bin(n = 500, p = 0.53)$$

Expected Value of Free Throws

In basketball, players sometimes get a chance to shoot a free throw. If they make it, the team gets 1 point; otherwise they get no points.

Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

Let Y be the points gained from Shaq attempting **500** free throws. What is $\mathrm{E}[Y]$?

$$Y \sim Bin(n = 500, p = 0.53)$$

$$E[Y] = n \cdot p = 500 \cdot 0.53 = 265$$

Expected Value of Free Throws

In basketball, players sometimes get a chance to shoot a free throw. If they make it, the team gets 1 point; otherwise they get no points.

Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

Let Y be the points gained from Shaq attempting **500** free throws. What is $\mathrm{E}[Y]$?

$$Y \sim Bin(n = 500, p = 0.53)$$

$$E[Y] = n \cdot p = 500 \cdot 0.53 = 265$$

Challenge: If Shaq was 10% better at shooting free throws, how many *more* free throws would you expect him to make, out of 500?

Expected Value of The Geometric

If $X \sim \text{Geo}(p)$, then $$E[X] = \frac{1}{p}$$

This definition has intuition built in:

- If Shaq makes about half his free throws, then on average, it will take him two shots to make one free throw. $E[X] = (1/2)^{-1} = 2$.

Expected Value of The Geometric

If $X \sim \text{Geo}(p)$, then $E[X] = \frac{1}{p}$

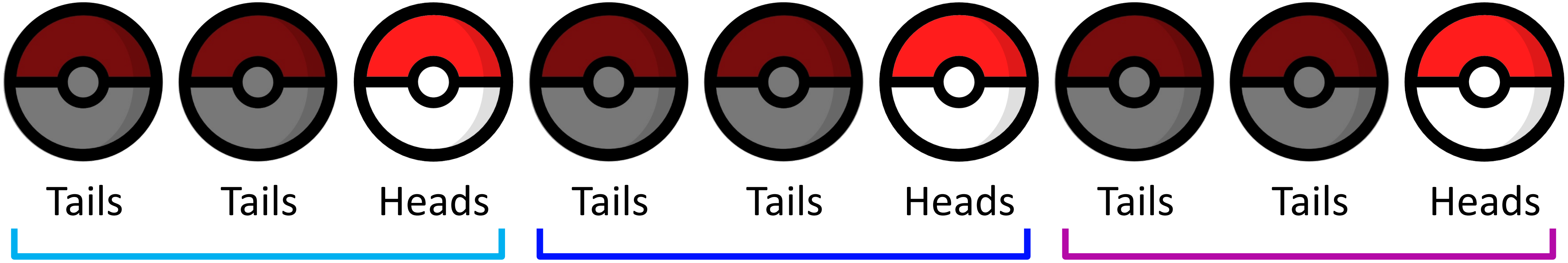
This definition has intuition built in:

- If Shaq makes about half his free throws, then on average, it will take him two shots to make one free throw. E[X] = $(1/2)^{-1}$ = 2.

Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

The Negative Binomial ...is a sum of Geometric random variables



Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

Let $X_i \sim \text{Geo}(p)$, for each i from 1 to r.

$$E[X_i] = \frac{1}{p}$$

Let $Y \sim \text{NegBin}(r, p)$.

Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

Let $X_i \sim \text{Geo}(p)$, for each i from 1 to r.

$$E[X_i] = \frac{1}{p}$$

Let $Y \sim \text{NegBin}(r, p)$.

$$E[Y] = E\left[\sum_{i=1}^{r} X_i\right]$$

Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

Let $X_i \sim \text{Geo}(p)$, for each i from 1 to r.

$$E[X_i] = \frac{1}{p}$$

Let $Y \sim \text{NegBin}(r, p)$.

$$E[Y] = E\left[\sum_{i=1}^{r} X_i\right]$$

$$= \sum_{i=1}^{r} E[X_i]$$

Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

Let $X_i \sim \text{Geo}(p)$, for each i from 1 to r.

$$E[X_i] = \frac{1}{p}$$

Let $Y \sim \text{NegBin}(r, p)$.

$$E[Y] = E\left[\sum_{i=1}^{r} X_i\right]$$

$$= \sum_{i=1}^{r} E[X_i]$$

$$= \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$

Pokemon: Actually Catching Them All

To catch a Pokemon, you throw a pokeball repeatedly until it's caught.

Each pokeball has a 1/3 chance of catching the Pokemon.

What is the **expected number** of pokeballs needed to catch 1 Pokemon?

There are 151 Pokemon to catch in the game Pokemon Diamond.

What is the **expected number** of pokeballs needed to catch *every* Pokemon?

Pokemon: Actually Catching Them All

To catch a Pokemon, you throw a pokeball repeatedly until it's caught.

Each pokeball has a 1/3 chance of catching the Pokemon.

What is the **expected number** of pokeballs needed to catch 1 Pokemon?

Let *X* be the number of pokeballs we use.

$X \sim \text{Geo}(p = 1/3)$

$$E[X] = \frac{1}{p} = 3$$

There are 151 Pokemon to catch in the game Pokemon Diamond.

What is the **expected number** of pokeballs needed to catch *every* Pokemon?

Pokemon: Actually Catching Them All

To catch a Pokemon, you throw a pokeball repeatedly until it's caught.

Each pokeball has a 1/3 chance of catching the Pokemon.

What is the **expected number** of pokeballs needed to catch 1 Pokemon?

Let X be the number of pokeballs we use.

$X \sim \text{Geo}(p = 1/3)$

$$E[X] = \frac{1}{p} = 3$$

There are 151 Pokemon to catch in the game Pokemon Diamond.

What is the **expected number** of pokeballs needed to catch *every* Pokemon?

Let Y be the number of pokeballs we use in total.

$Y \sim \text{NegBin}(r = 151, p = 1/3)$

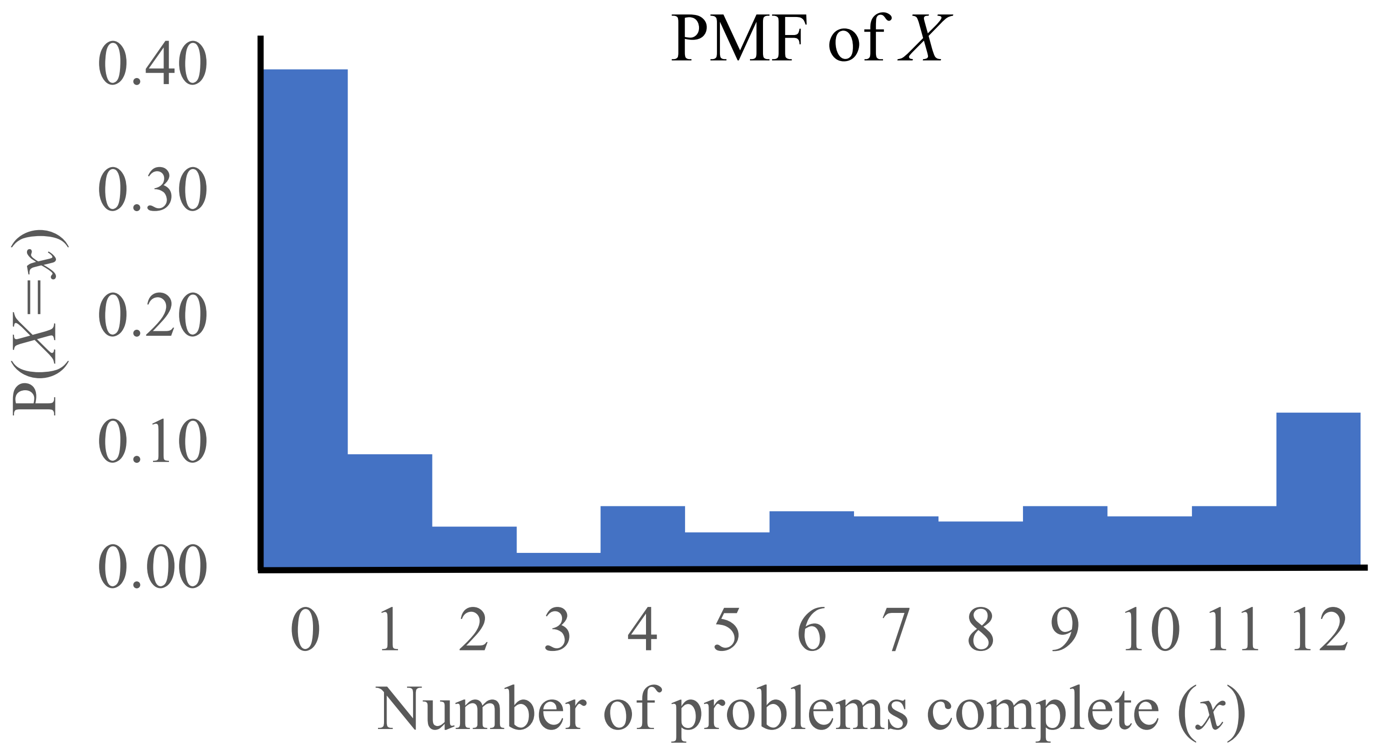
$$E[Y] = \frac{r}{p} = 151 \cdot 3 = 453$$

Expectation is only a single number summary…

Expectation Is Not All You Need

Let *X* be the number of problems on pset2 that a randomly selected student has completed, as of this morning.

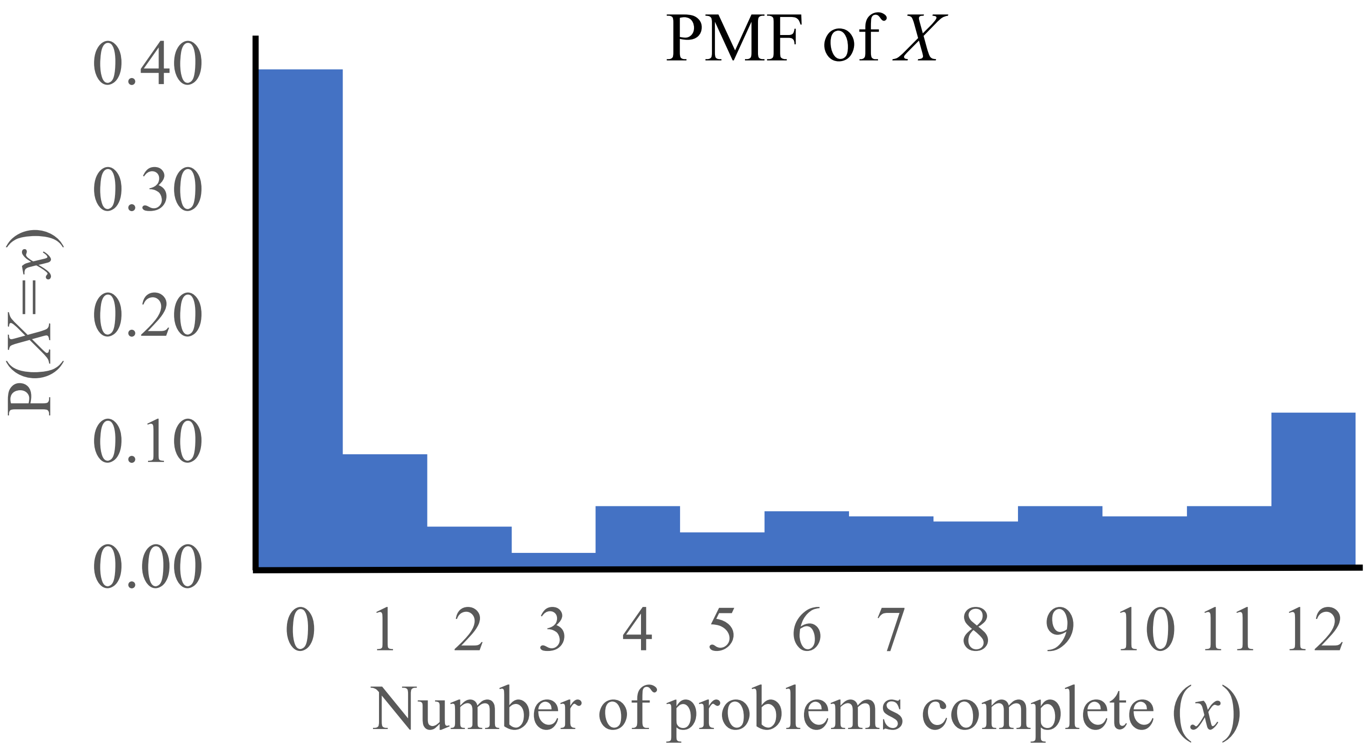
X takes on values with uncertainty, so *X* is a random variable.



Expectation Is Not All You Need

Let *X* be the number of problems on pset2 that a randomly selected student has completed, as of this morning.

X takes on values with uncertainty, so *X* is a random variable.

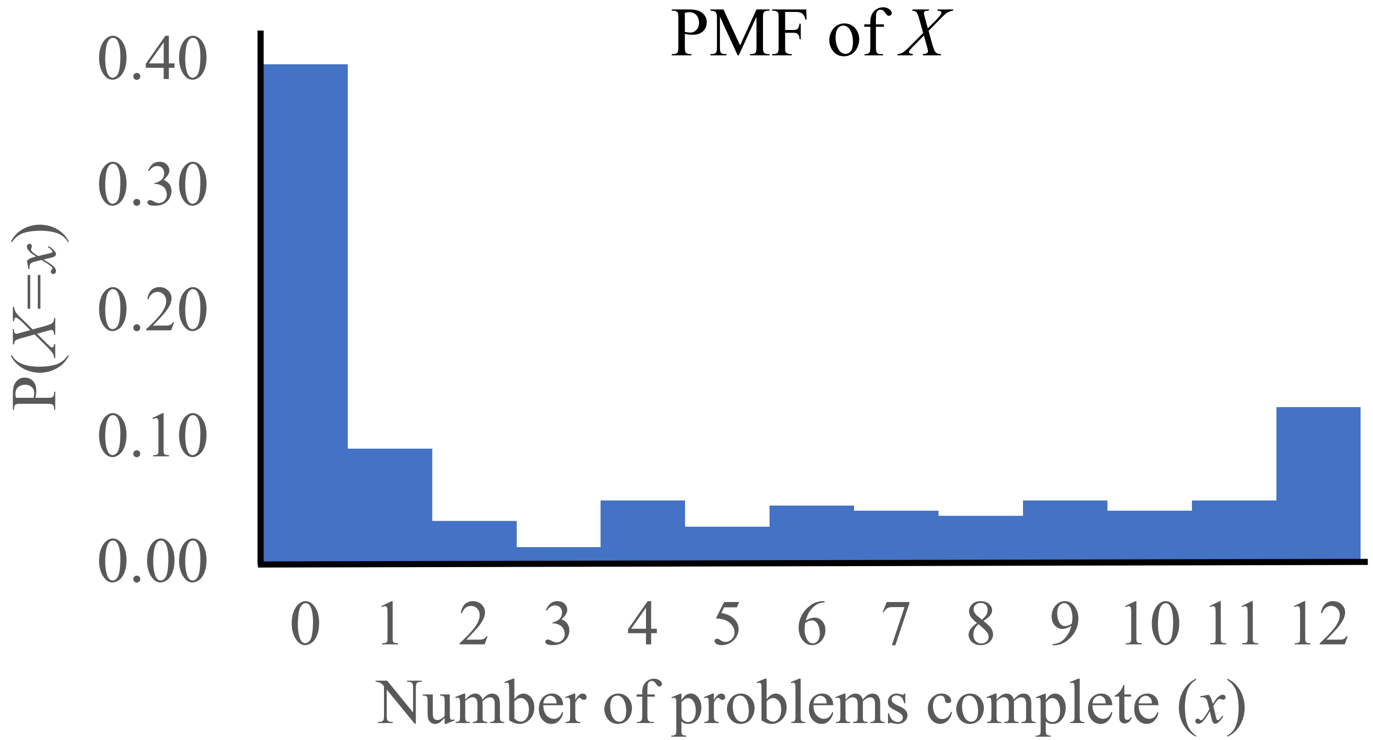


$$E[X] = 6$$

Expectation Is Not All You Need

Let X be the number of problems on pset2 that a randomly selected student has completed, as of this morning.

X takes on values with uncertainty, so X is a random variable.



$$E[X] = 6$$

Does this expected value capture all the information in the data?
No!

Back To Our Paradox…

St. Petersburg Paradox

The Game:

- We have a fair coin (lands on heads with p = 0.5)
- Let *n* = number of coin flips (tails) to get the first heads
- You will win: $\$2^n$

How much would you pay to play?

St. Petersburg Paradox

The Game:

- We have a fair coin (lands on heads with p = 0.5)
- Let n = number of coin flips (tails) to get the first heads
- You will win: \$$2^n$

How much would you pay to play?

Let X be your winnings.

$$E[X] = \left(\frac{1}{2}\right)^1 2^1 + \left(\frac{1}{2}\right)^2 2^2 + \left(\frac{1}{2}\right)^3 2^3 + \ldots = \sum_{i=0}^{\infty} 1 = \infty$$

St. Petersburg Paradox

The Game:

- We have a fair coin (lands on heads with $p = 0.5$)
- Let n = number of coin flips (tails) to get the first heads
- You will win: $\$2^n$

How much would you pay to play?

Let X be your winnings.

$$E[X] = \left(\frac{1}{2}\right)^1 2^1 + \left(\frac{1}{2}\right)^2 2^2 + \left(\frac{1}{2}\right)^3 2^3 + \ldots = \sum_{i=0}^{\infty} 1 = \infty$$

What if you could play this game for only $1000…but just once?

Next Time: The Final Discrete Random Variable!

Next Time: The Final Discrete Random Variable!