Continuous Random Variables

CS109

Announcements

PSet 1

- Solutions are out, grades out today; 1 week to request a regrade
- Your "Create A Problem" answers were awesome!!
  - Feel free to share with the class on Ed ☺

PSet2 - Due today

Pset 3 (Random Variable Practice)

- Released today, due next Sunday @ 2 pm
- Slightly longer than Psets 1 and 2: recommend starting early
- Extension deadline is shorter (until Monday) due to the midterm

ILL. No. 65. MEMORIAL ARCH, WITH CHURCH IN BACKGROUND, STANFORD UNIVERSITY, SHOWING TYPES OF CARVED WORK WITH THE SANDSTONE.

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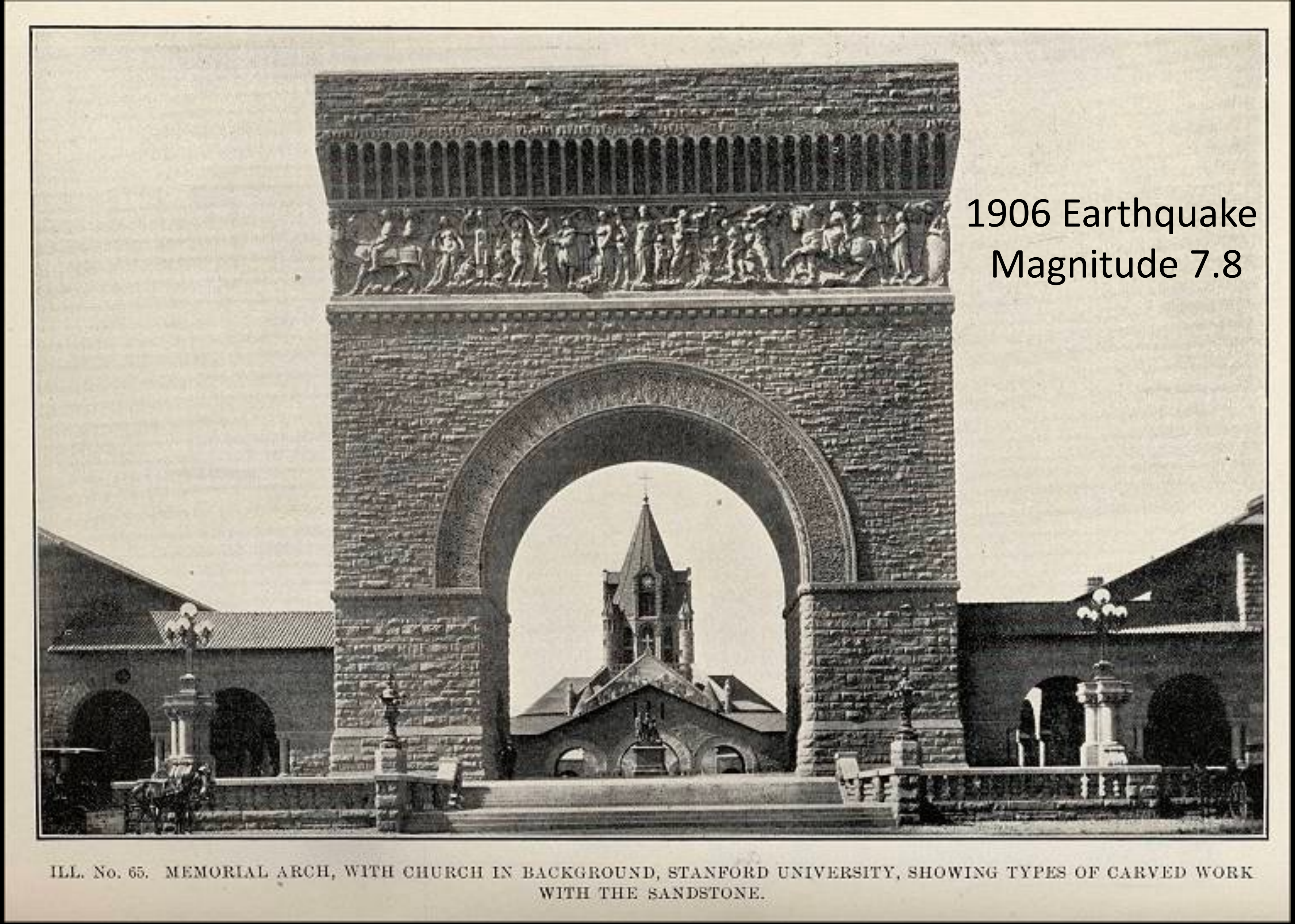
1906 Earthquake
Magnitude 7.8



How long until the next "big one"?

Review

Pokémon™

Gotta catch 'em all!™

(The Last Discrete Random Variable)

It’s Time
To Talk About Time

Probability of ***k* Requests** From This Area Each Minute

Idea: we can break a minute down into *infinitely small* buckets

In each bucket, you either get a request or don't.

Probability of ***k* Requests** From This Area Each Minute

Idea: we can break a minute down into *infinitely small* buckets

too small to draw ☹

1 ∞

In each bucket, you either get a request or don't.
Let *X* be the number of requests in a minute.

On average, λ = 5 requests per minute

$$X \sim \text{Bin}(n = \infty, p = \lambda/n)$$

$$p = \frac{\lambda}{n}$$

$$P(X = k) = \binom{n}{k}(\lambda/n)^k(1 - \lambda/n)^{n-k}$$

Is this impossible to work with? No?! Time for cool math!

The Poisson Random Variable

A **Poisson** random variable models the number of occurrences that happen in a *fixed* interval of time.

$$X \sim \text{Poi}(\lambda)$$

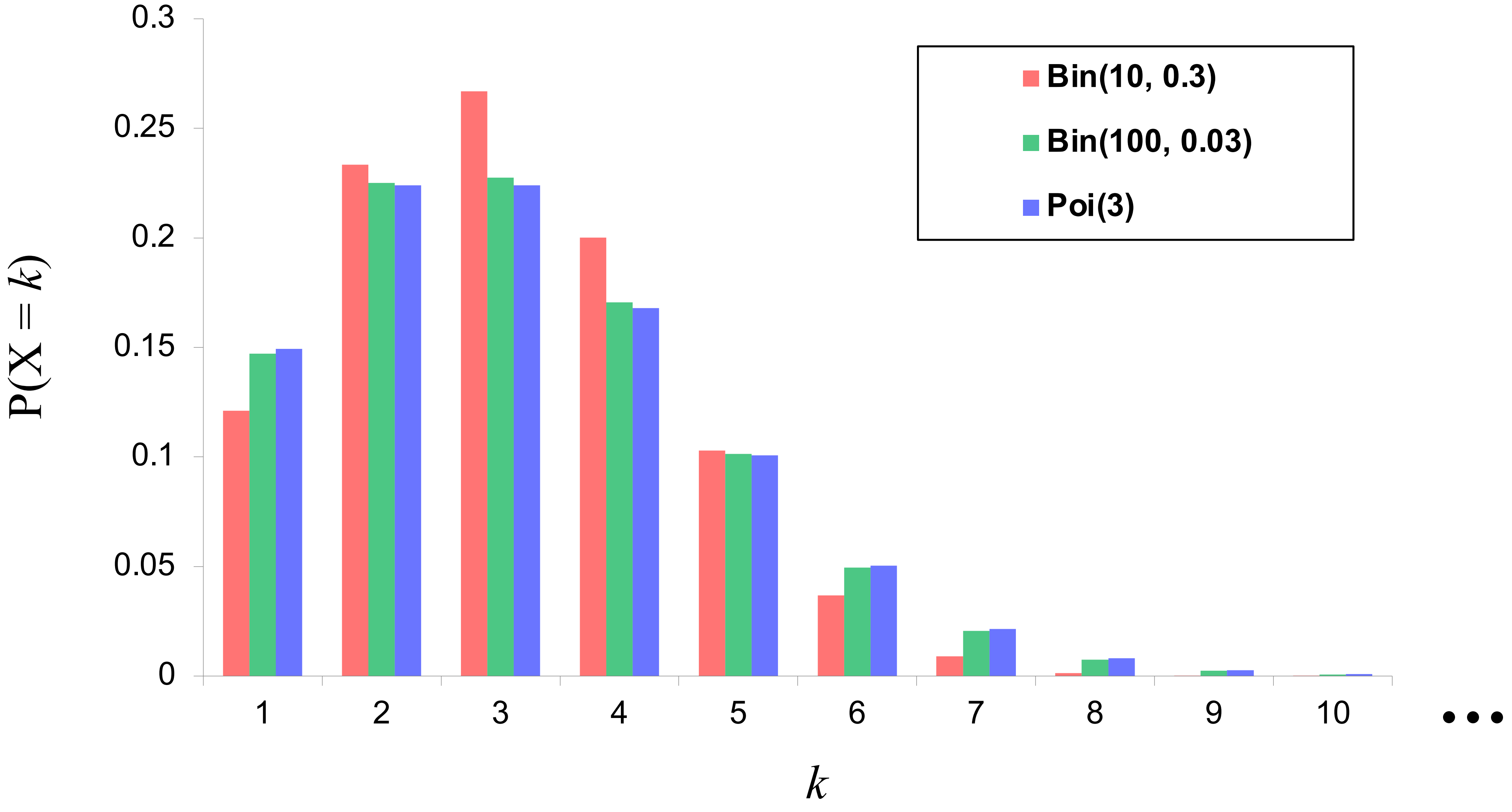
PMF:

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

The events you're modeling must follow a **Poisson Process**:

1. Events happen *independently* of one another
2. Events arrive at a *fixed* rate: λ events per interval of time

Poisson Approximates Binomial, With Extreme *n* and *p*



Variance

Variance is a formal definition of the **spread** of a random variable.

If *X* is a random variable with mean $\mu = E[X]$, then the **variance** of *X*, denoted Var(*X*), is:

$$\text{Var}(X) = E[(X - \mu)^2]$$

In practice, it is usually easier to calculate this equivalent:

$$\text{Var}(X) = E[X^2] - E[X]^2$$

How to calculate $E[X^2]$? Law of the Unconscious Statistician!

Variance of Classic Random Variables

$X \sim \text{Geo}(p)$

$$Var(X) = \frac{1-p}{p^2}$$

$X \sim \text{Bern}(p)$

$$Var(X) = p(1-p)$$

$Y \sim \text{NegBin}(r, p)$

$$Var(X) = \frac{r \cdot (1-p)}{p^2}$$

$Y \sim \text{Bin}(n, p)$

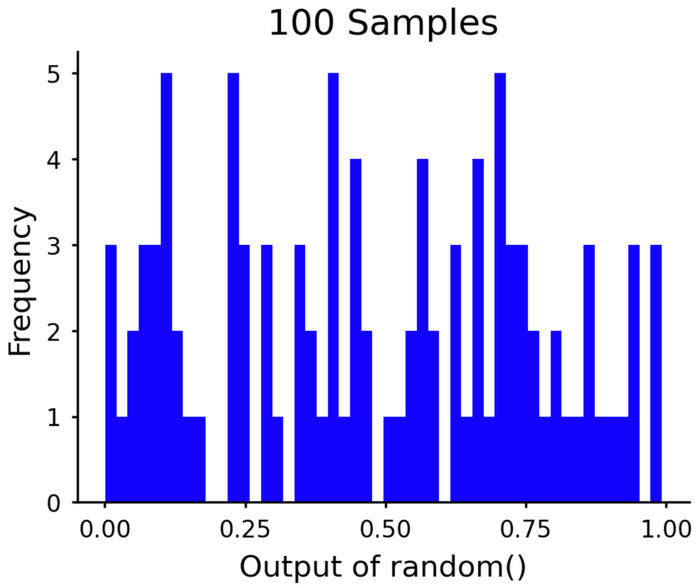
$$Var(Y) = n \cdot p(1-p)$$

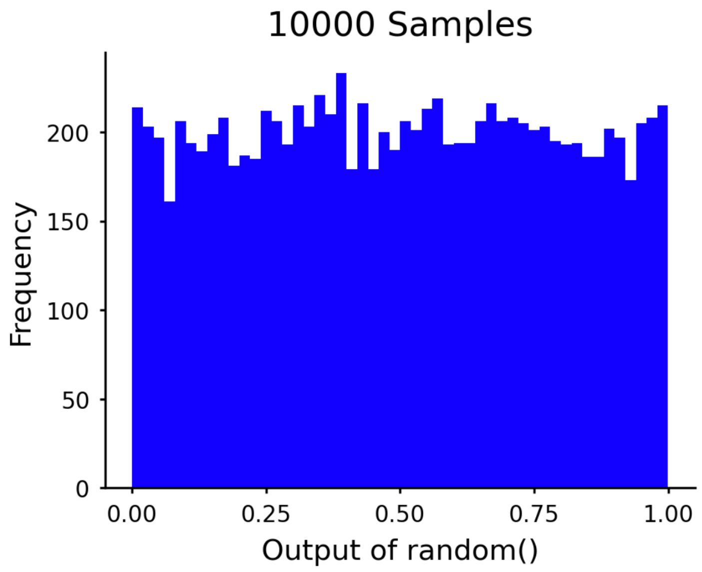
Time to patch the big hole in our knowledge random()?

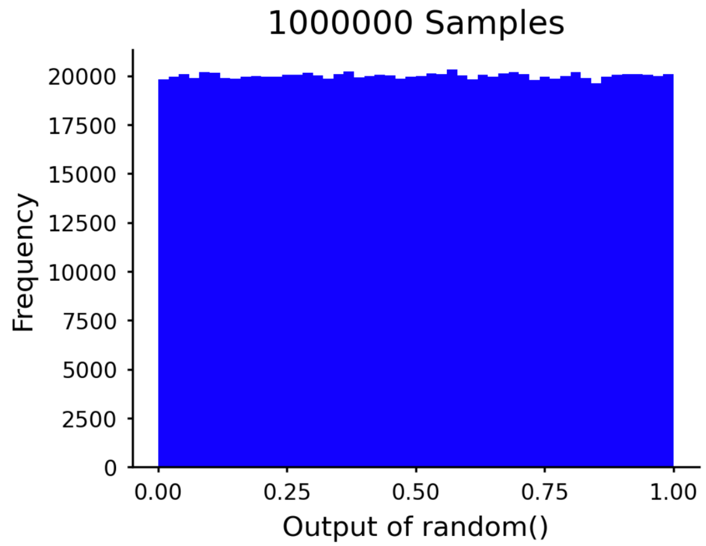
The `random()` Function

- Outputs values between 0 and 1
- All possible values are equally likely
- This is a continuous random variable!

```python
import random

samples_small = []
for i in range(100):
    samples_small.append(random.random())

samples_medium = []
for i in range(10000):
    samples_medium.append(random.random())

samples_large = []
for i in range(1000000):
    samples_large.append(random.random())
```







$X \sim \text{Uniform}(0,1)$: A Continuous Random Variable

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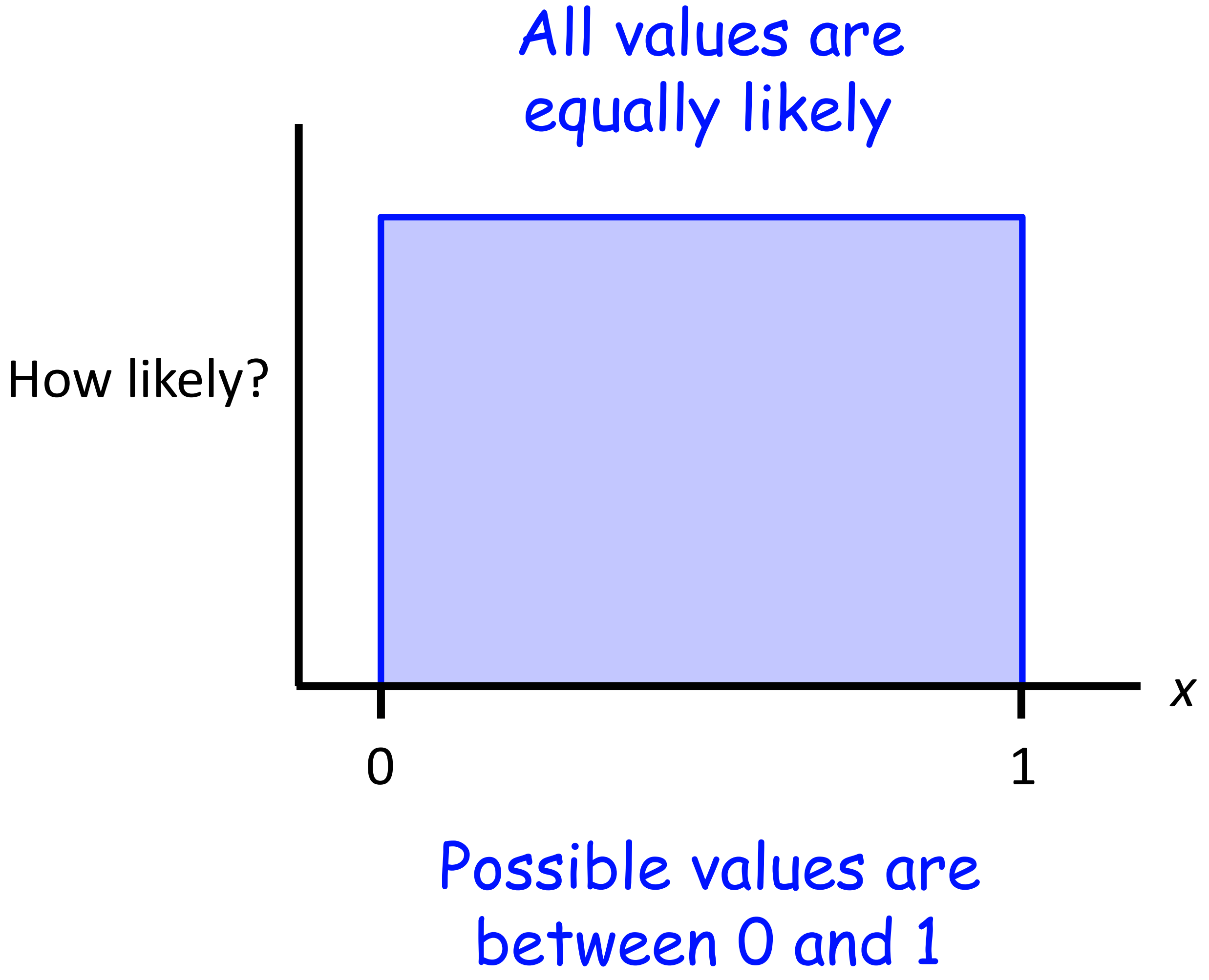

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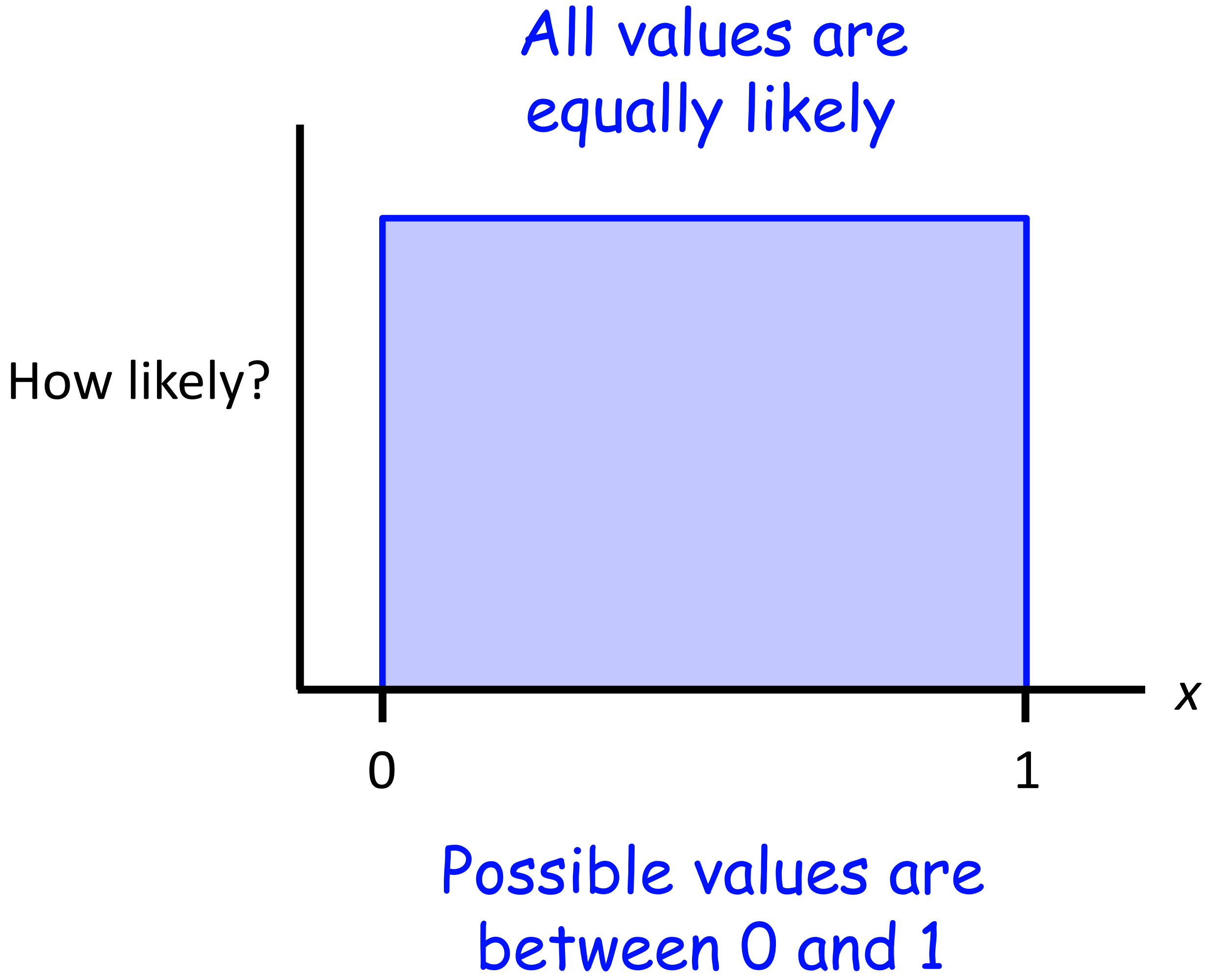
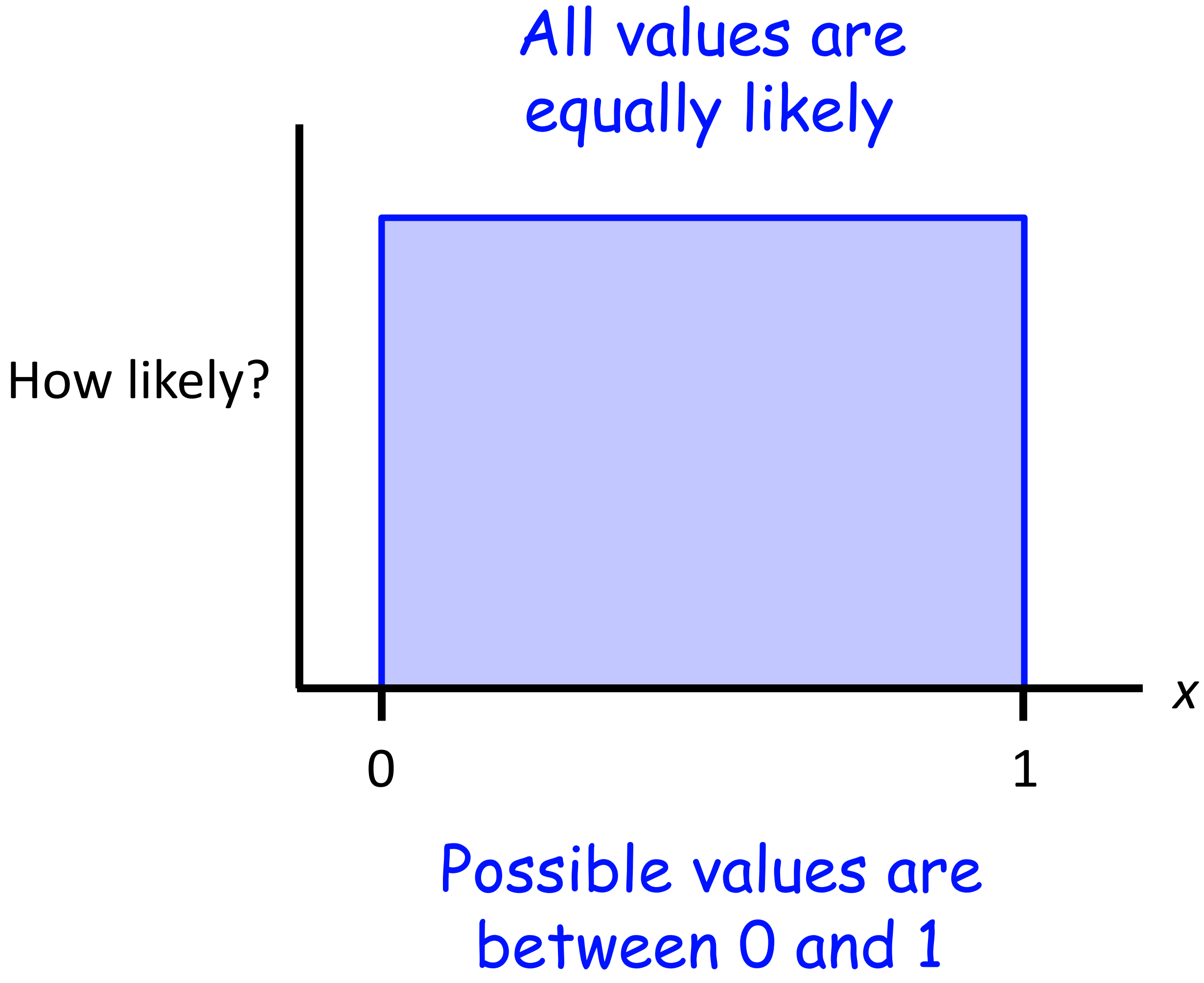


$$P(0 \leq X \leq 1) = \ ?$$

$X \sim \text{Uniform}(0,1)$: A Continuous Random Variable

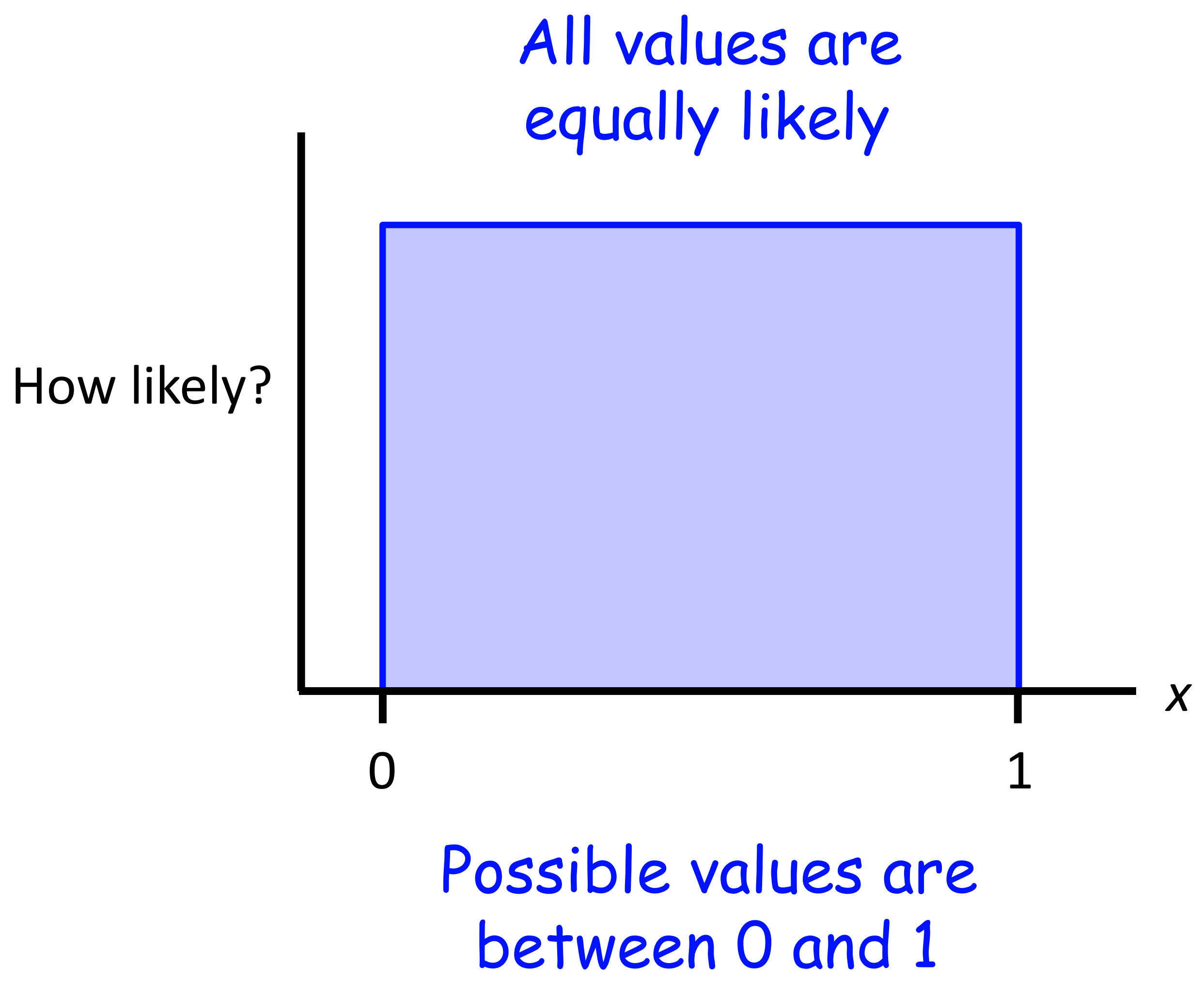


$$P(0 \leq X \leq 1) = 1$$

Probability of the whole sample space must equal 1

(Axiom 2)

$X \sim$ Uniform(0,1): A Continuous Random Variable



$$P(0 \leq X \leq 1) = 1$$

$$P(0.5 \leq X \leq 1) = \ ?$$

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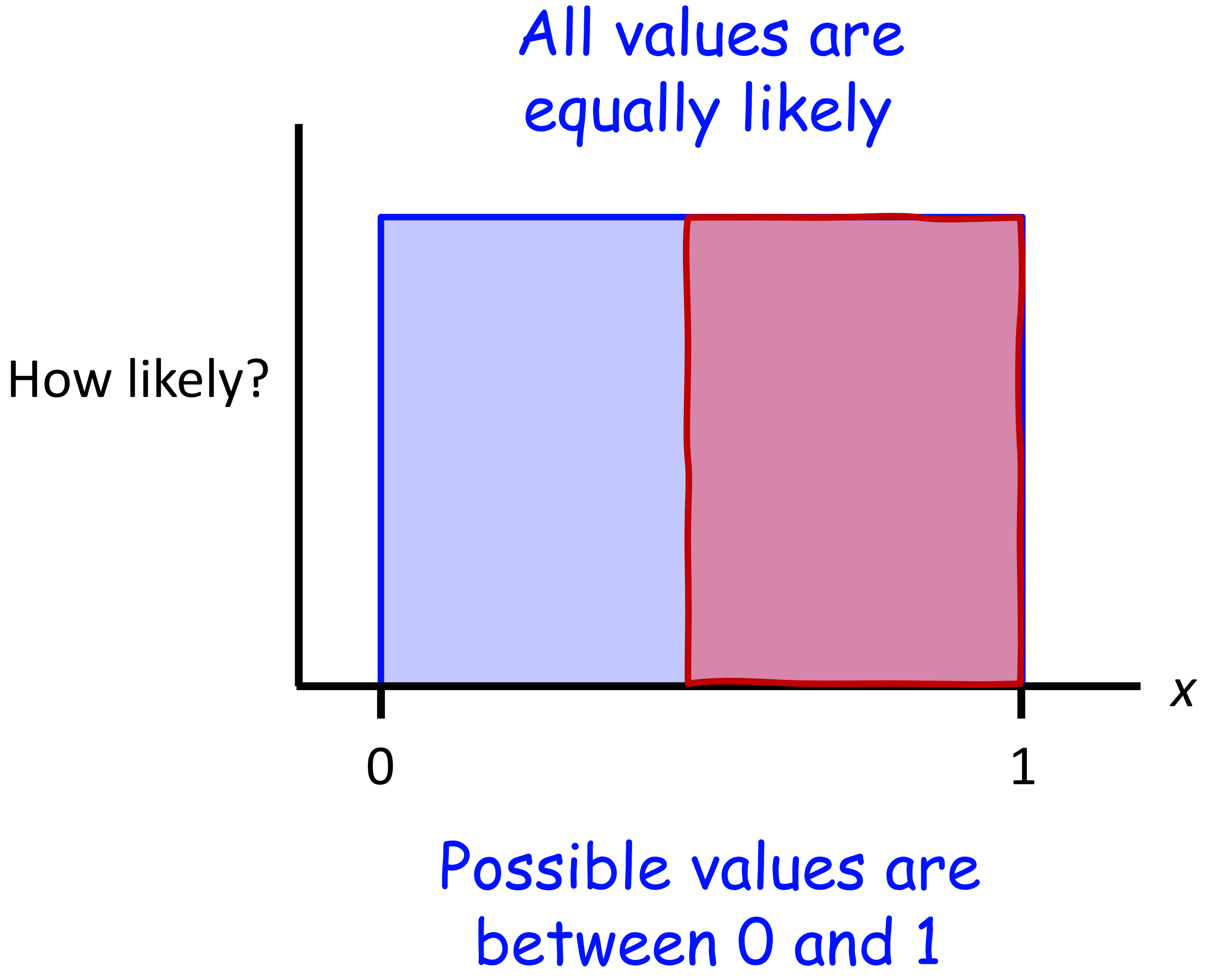
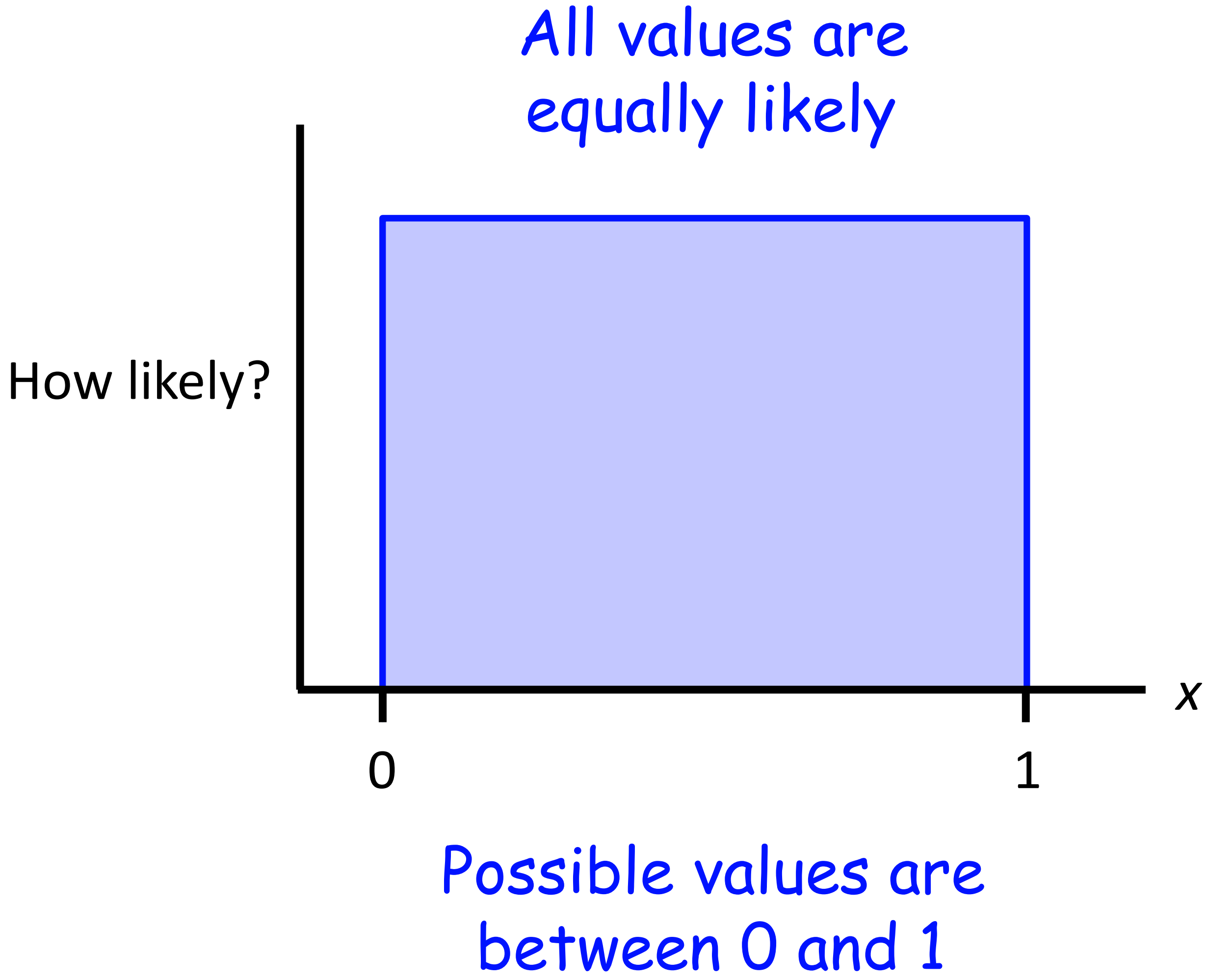


$$P(0 \leq X \leq 1) = 1$$

$$P(0.5 \leq X \leq 1) = 0.5$$

Half of all possible outcomes are between 0.5 and 1

$X \sim$ Uniform(0,1): A Continuous Random Variable

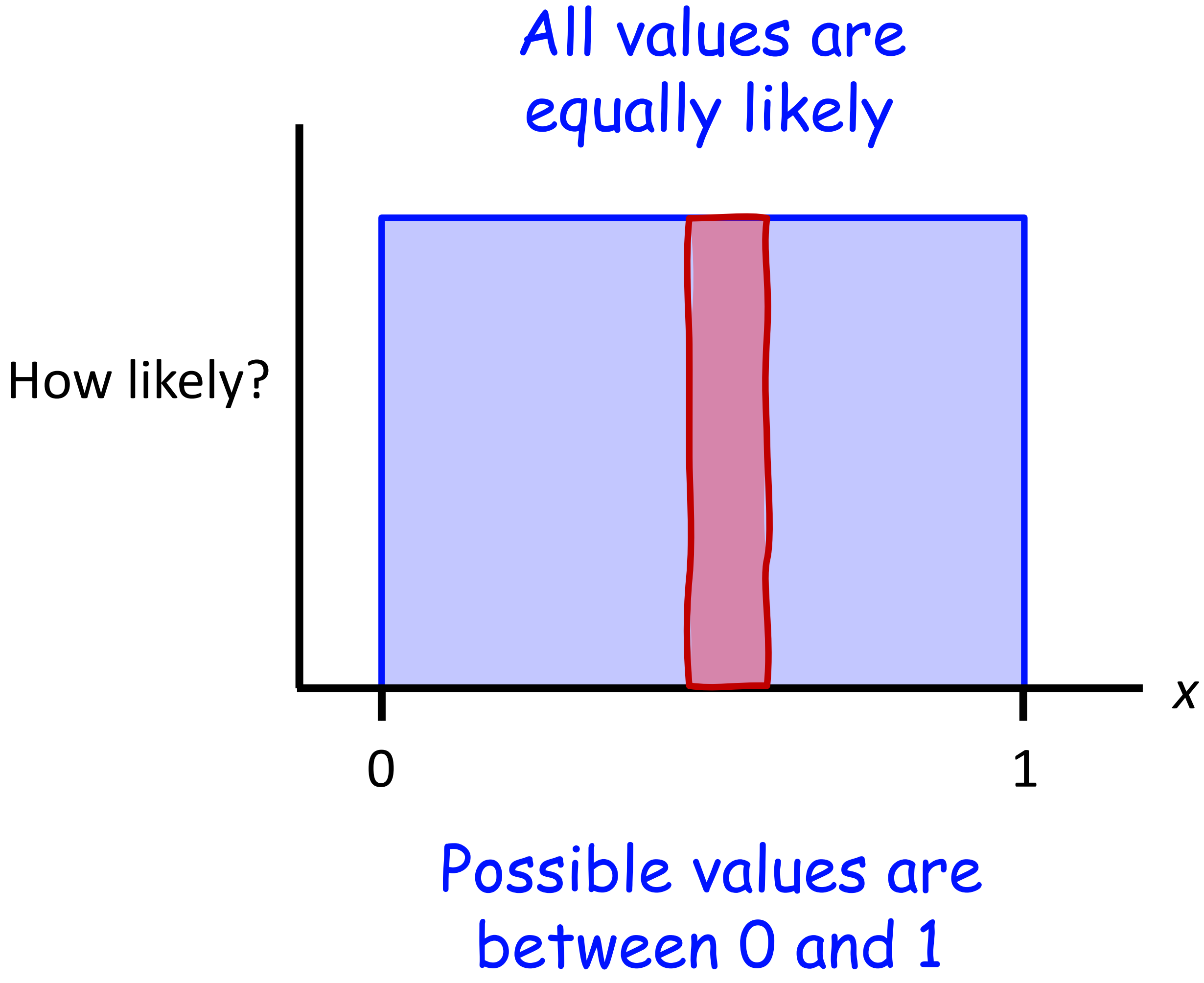


$$P(0 \leq X \leq 1) = 1$$

$$P(0.5 \leq X \leq 1) = 0.5$$

$$P(0.5 \leq X \leq 0.6) = ?$$

$X \sim$ Uniform(0,1): A Continuous Random Variable



$$P(0 \leq X \leq 1) = 1$$

$$P(0.5 \leq X \leq 1) = 0.5$$

$$P(0.5 \leq X \leq 0.6) = 0.1$$

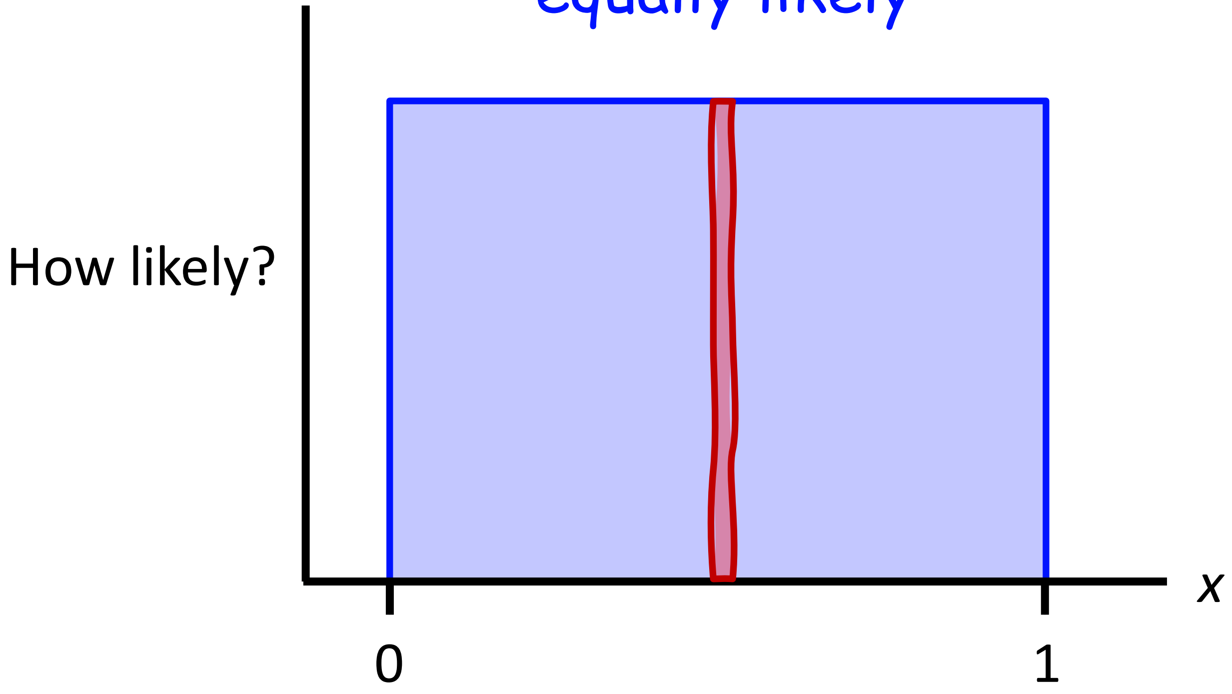
1/10 of all possible outcomes are between 0.5 and 0.6

So far, the pattern looks like:

P(start ≤ X ≤ end) = end - start

$X \sim$ Uniform(0,1): A Continuous Random Variable

All values are equally likely



Possible values are between 0 and 1

$$P(0 \leq X \leq 1) = 1$$

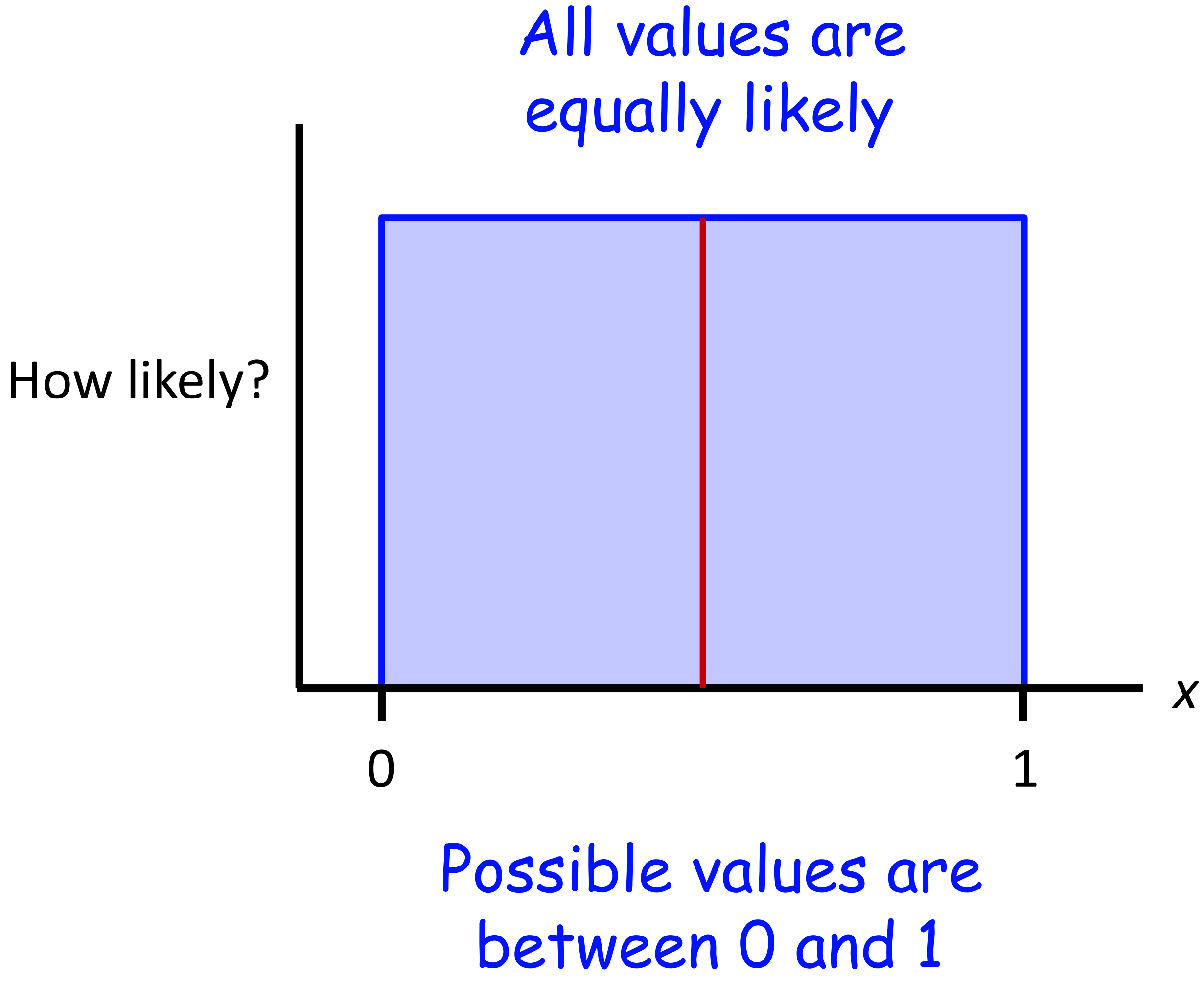
$$P(0.5 \leq X \leq 1) = 0.5$$

$$P(0.5 \leq X \leq 0.6) = 0.1$$

$$P(0.5 \leq X \leq 0.5001) = 0.0001$$

As we get more precise, probabilities keep shrinking…

$X \sim \text{Uniform}(0,1)$: A Continuous Random Variable



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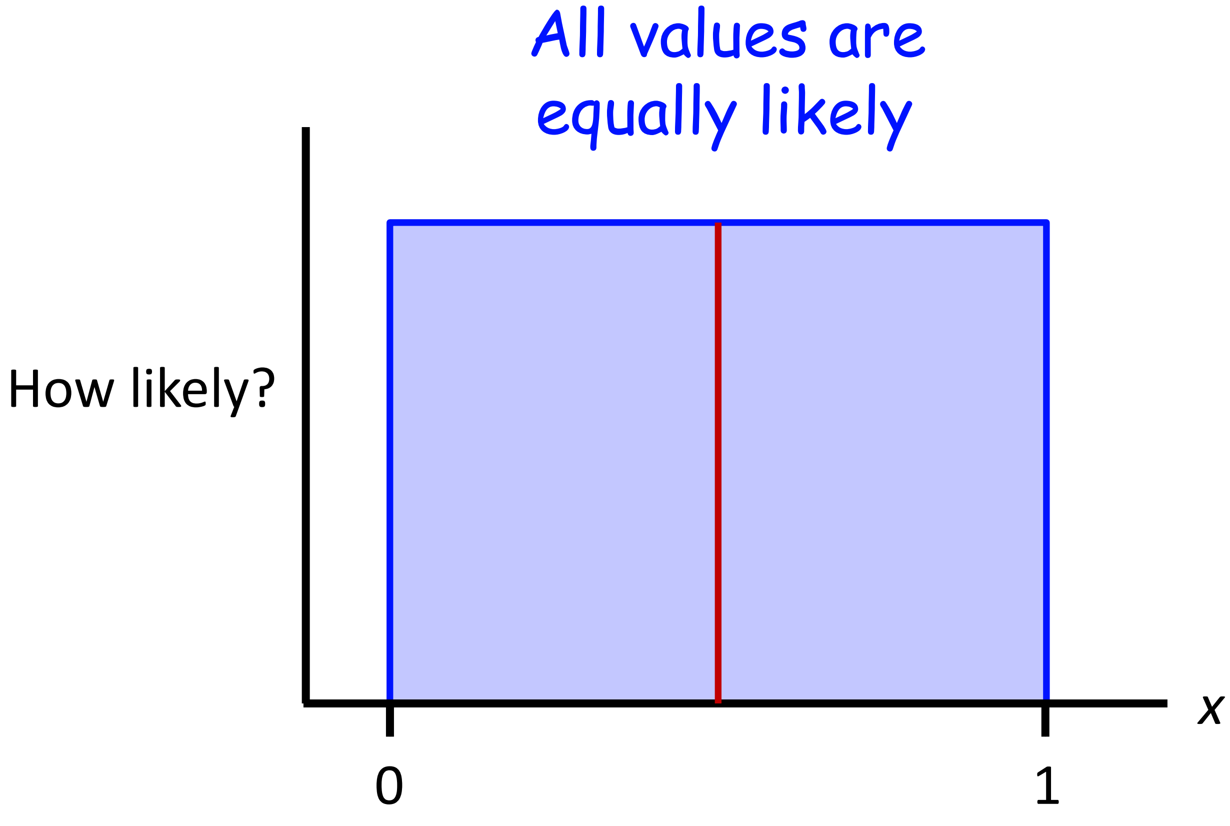
$$P(X = 0.5) = ?$$

$X \sim \text{Uniform}(0,1)$: A Continuous Random Variable



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$$P(X = 0.5) = 0$$

The probability of any exact outcome, with infinite precision…is zero

$X \sim$ Uniform(0,1): A Continuous Random Variable

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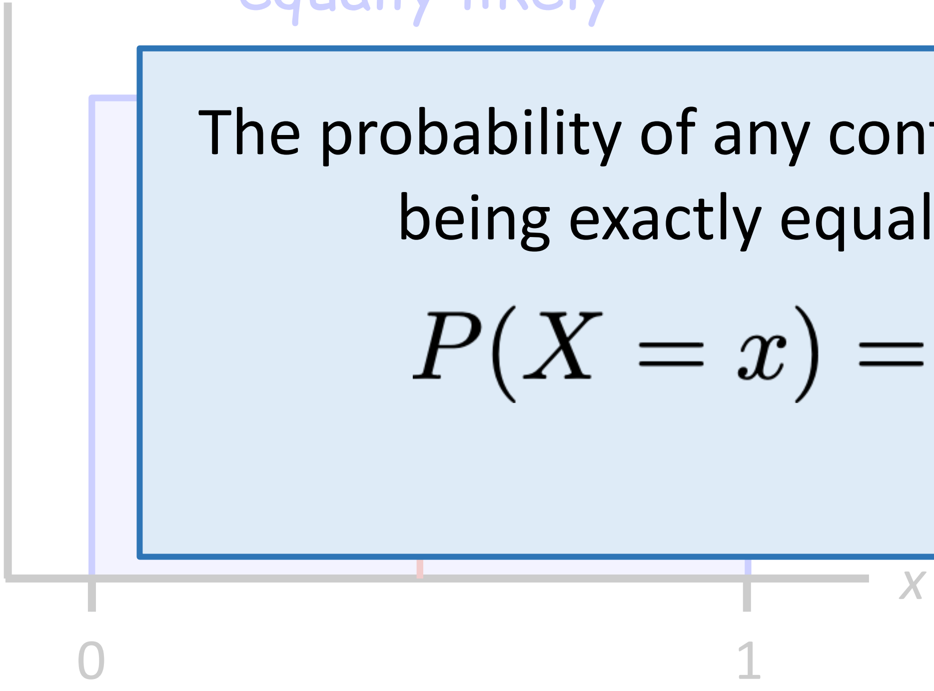


The probability of any continuous random variable being exactly equal to any value is 0.

$$P(X = x) = 0, \text{ for all } x$$

$$P(0 \leq X \leq 1) = 1$$

$= 0.5$

$= 0.1$

$= 0.0001$

$$P(X = 0.5) = 0$$

Possible values are between 0 and 1

The probability of any exact outcome, with infinite precision…is zero

$X \sim$ Uniform(0,1): A Continuous Random Variable

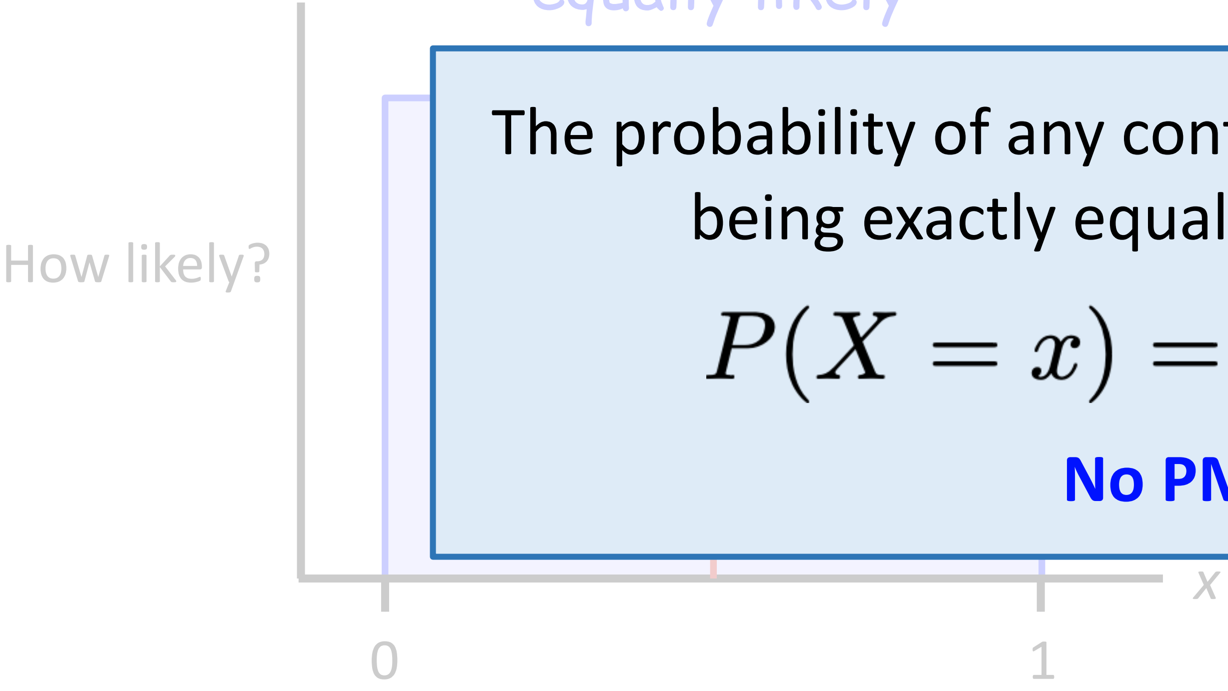
All values are equally likely

$$P(0 \leq X \leq 1) = 1$$

$$= 0.5$$

$$= 0.1$$

$$= 0.0001$$



The probability of any continuous random variable being exactly equal to any value is 0.

$$P(X = x) = 0, \text{ for all } x$$

No PMFs!

$$P(X = 0.5) = 0$$

Possible values are between 0 and 1

The probability of any exact outcome, with infinite precision…is zero

$X \sim$ Uniform(0,1): A Continuous Random Variable

$$P(0 \leq X \leq 1) = 1$$

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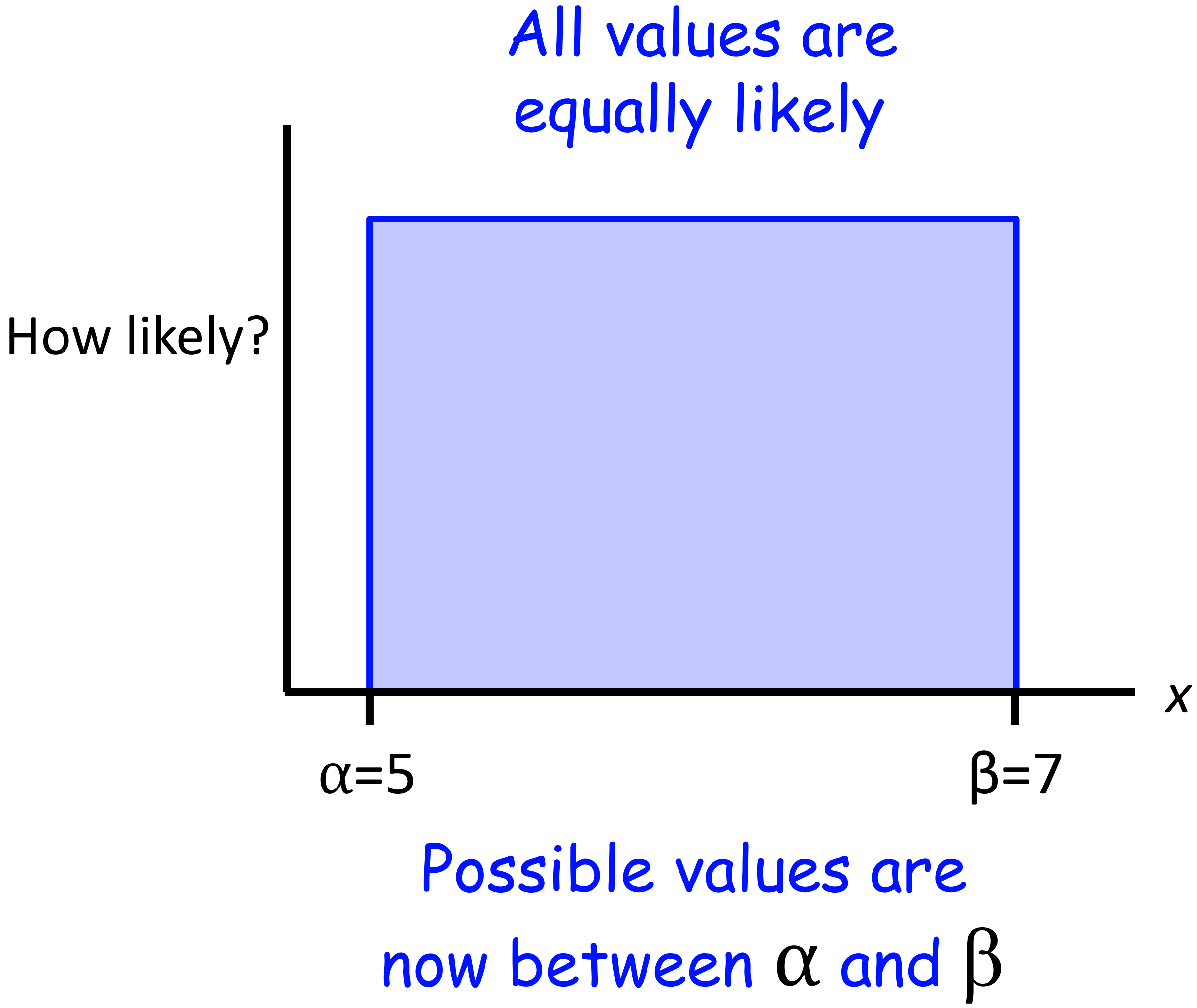
ability of any exact outcome,
infinite precision…is zero

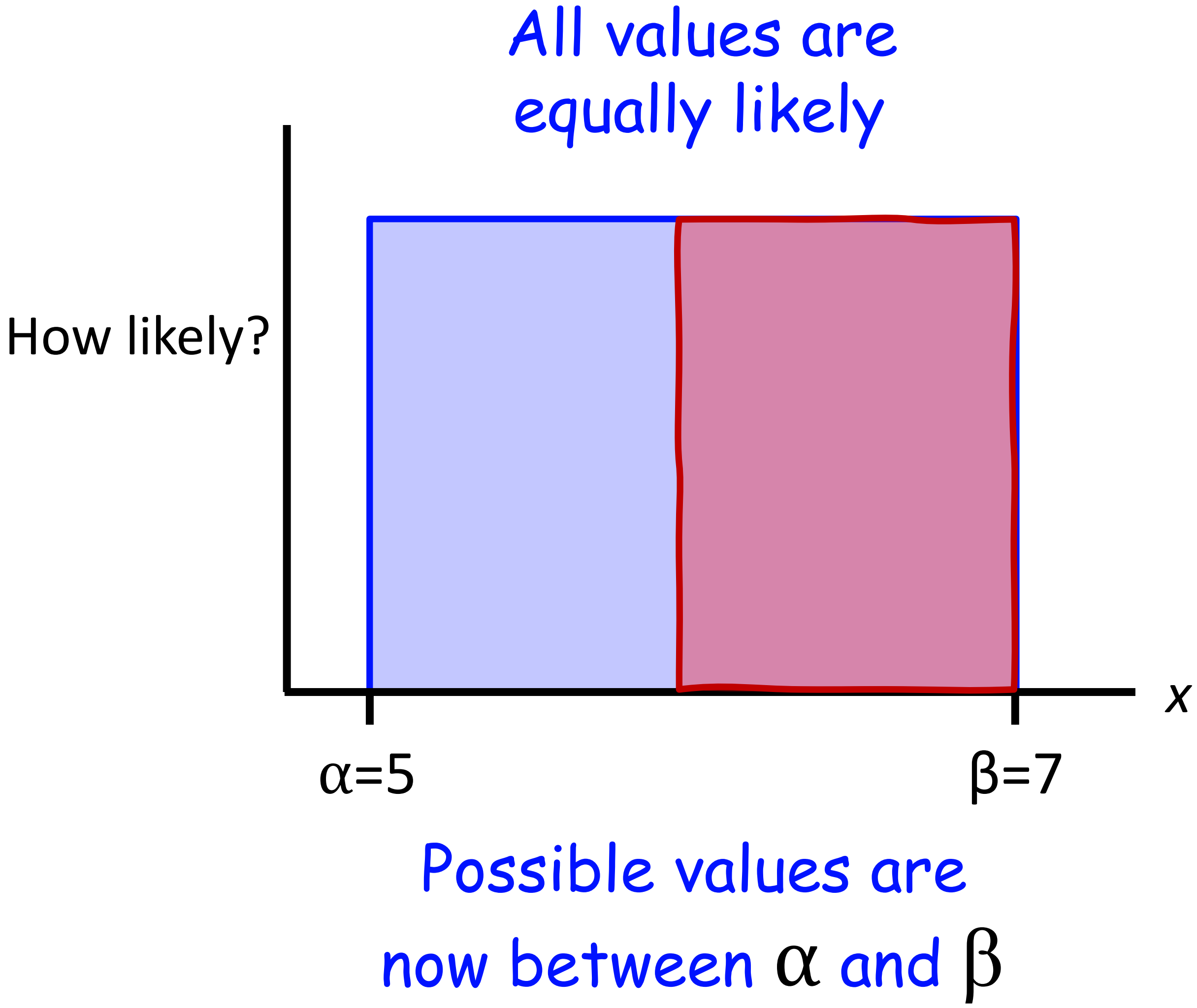
The only way to talk about probabilities of outcomes for *continuous* random variables is using ranges of possible values.

$X \sim$ Uniform(α, β): More General Case



$$P(5 \leq X \leq 7) = 1$$

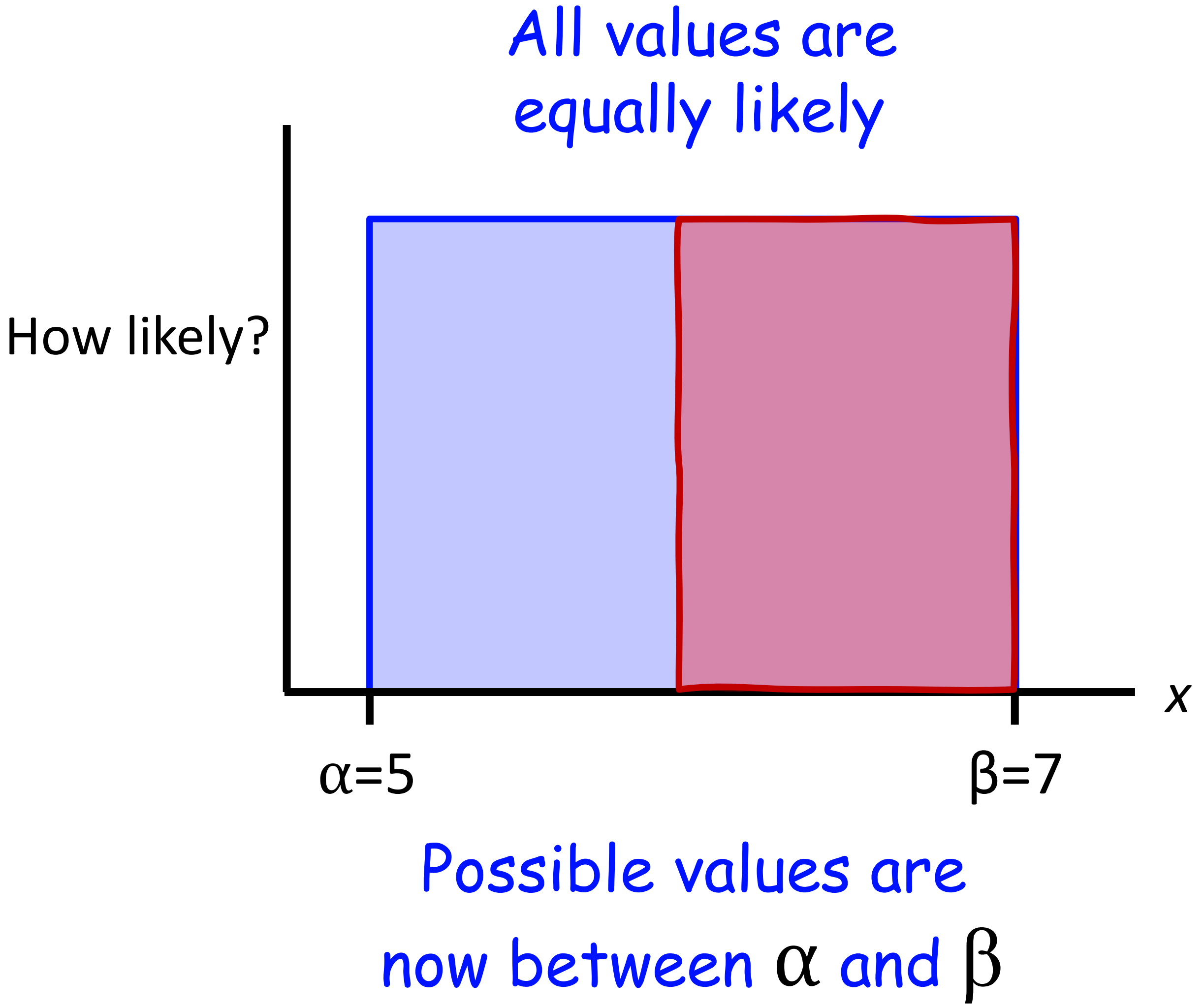
$X \sim$ Uniform(α, β): More General Case



$$P(5 \leq X \leq 7) = 1$$

$$P(6 \leq X \leq 7) = \ ?$$

$X \sim$ Uniform(α, β): More General Case



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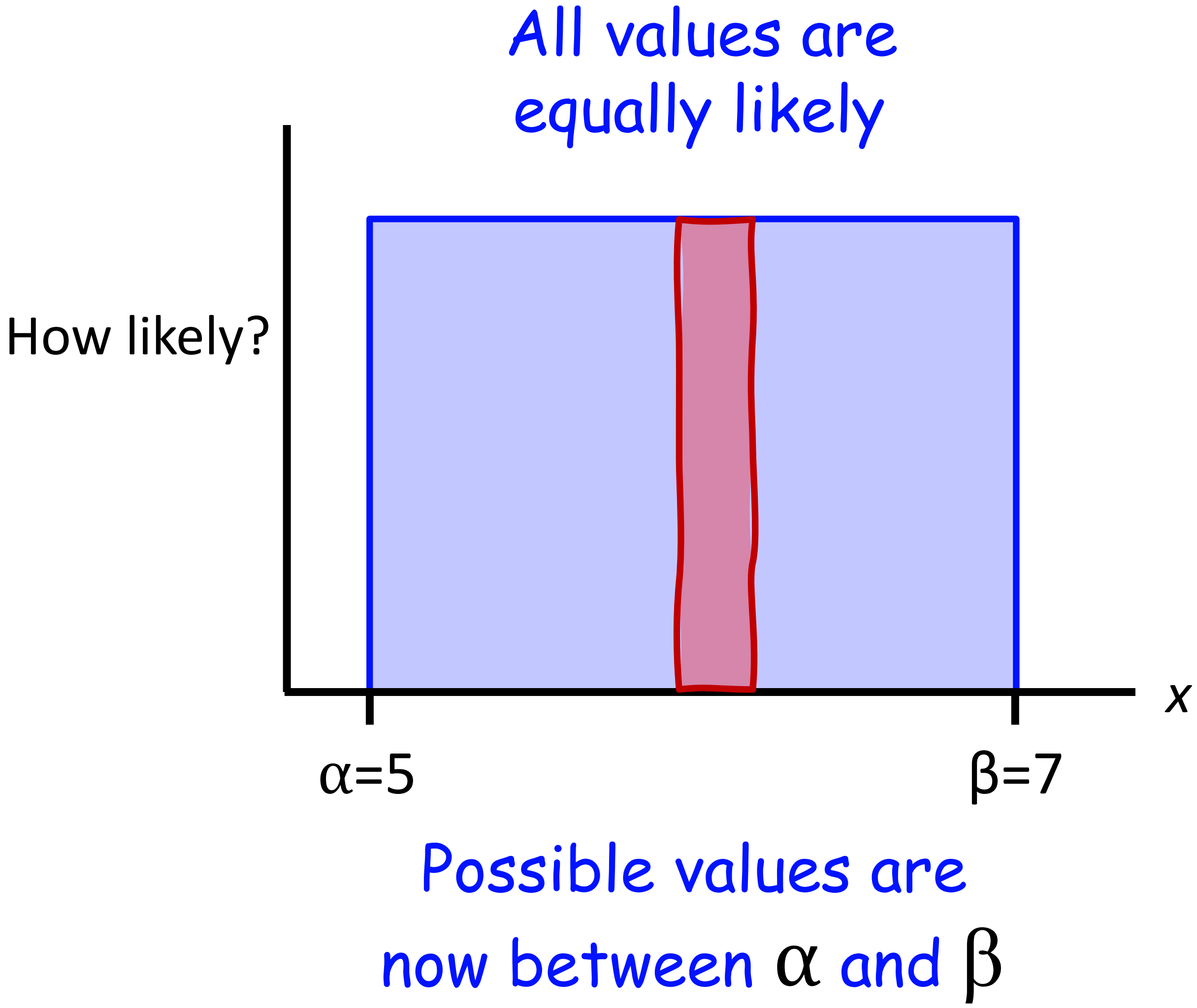

$$P(5 \leq X \leq 7) = 1$$

$$P(6 \leq X \leq 7) = 0.5$$

$$P(6 \leq X \leq 6.1) = \ ?$$

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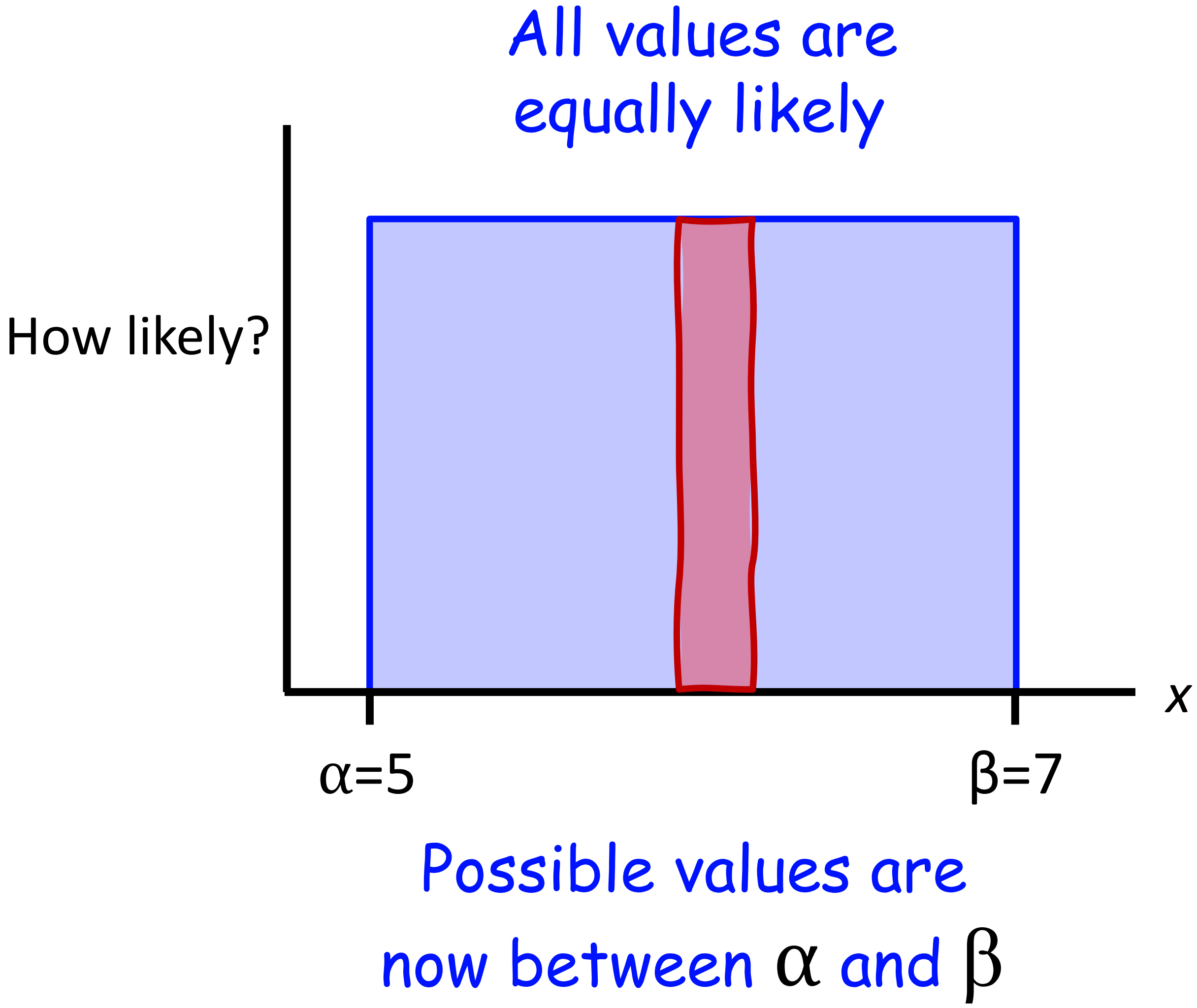
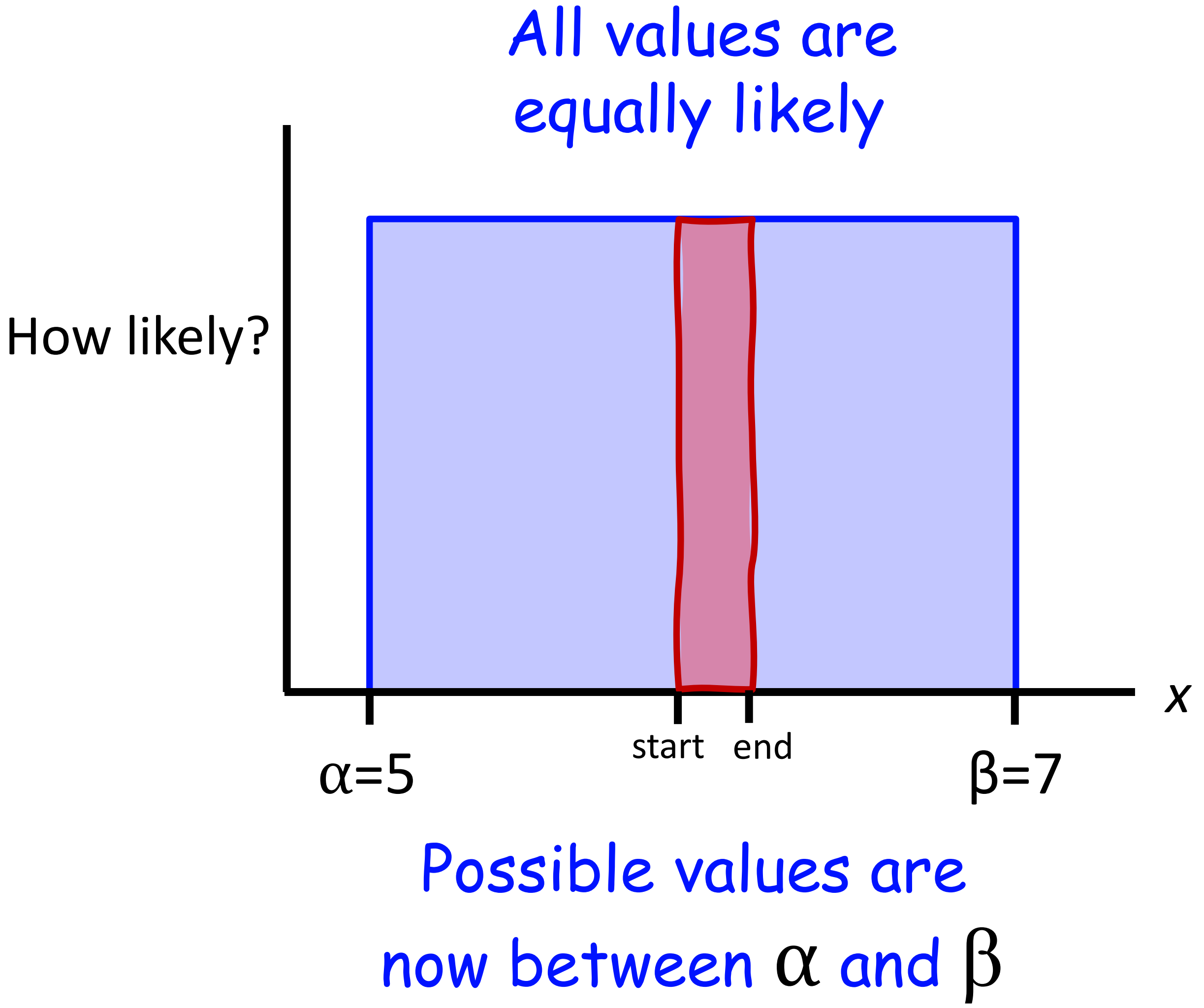


$$P(5 \leq X \leq 7) = 1$$

$$P(6 \leq X \leq 7) = 0.5$$

$$P(6 \leq X \leq 6.1) = 0.05$$

$X \sim$ Uniform(α, β): More General Case



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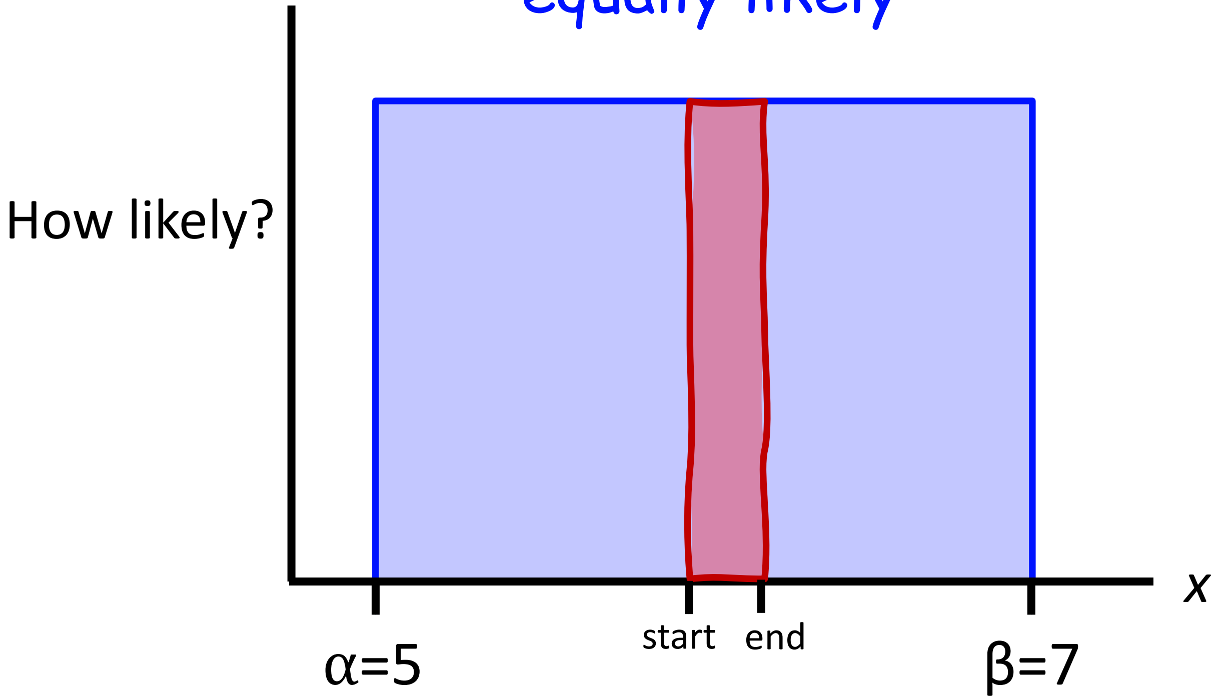
For Uniform(0,1):

P(start ≤ X ≤ end) = end - start

Does that still work?

$X \sim$ Uniform(α, β): More General Case

All values are equally likely



Possible values are now between α and β

$$P(5 \le X \le 7) = 1$$

$$P(6 \le X \le 7) = 0.5$$

$$P(6 \le X \le 6.1) = 0.05$$

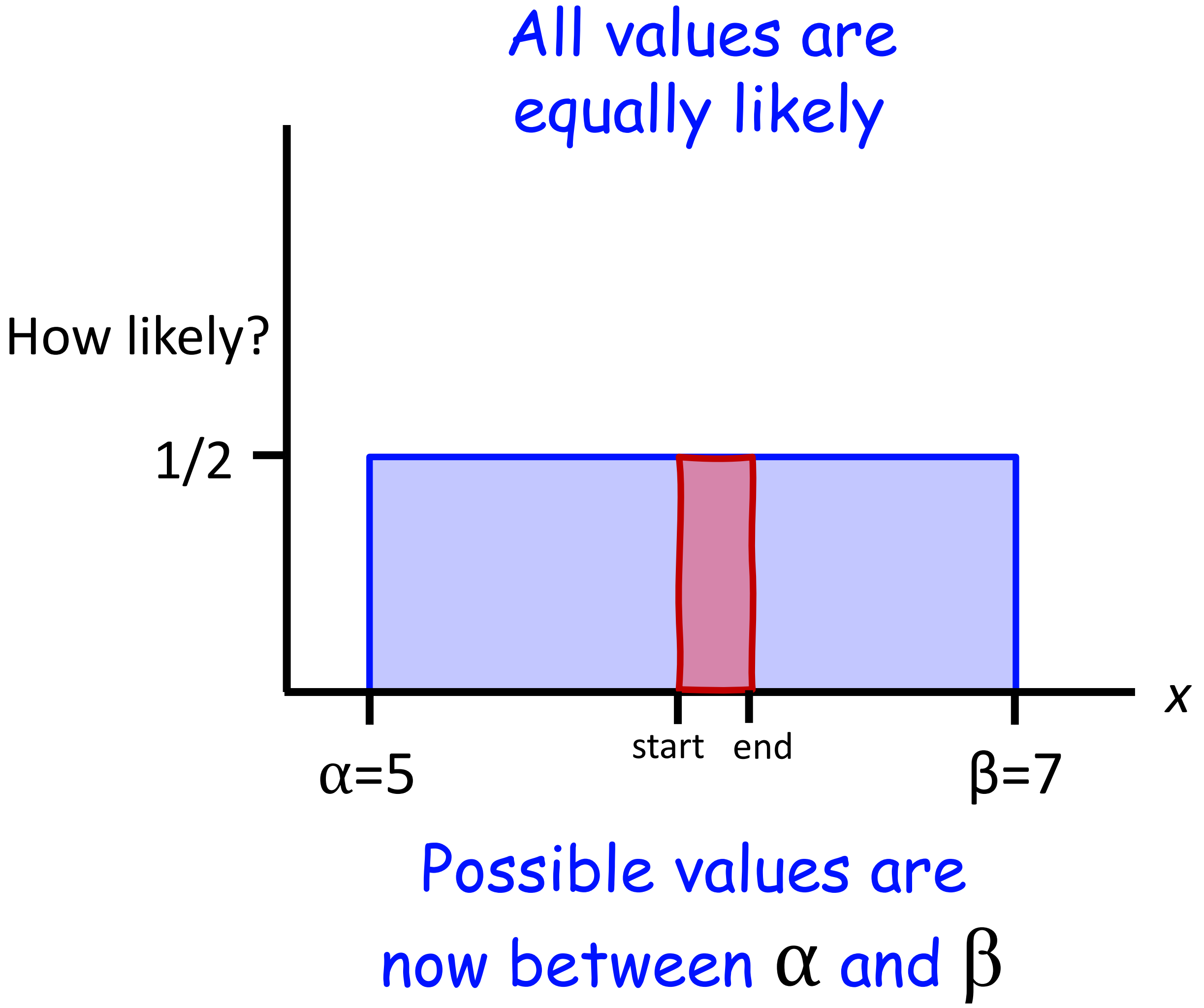
For Uniform(0,1):

P(start ≤ X ≤ end) = end - start

Does that still work? No!

Need to divide by 2?

$X \sim \text{Uniform}(\alpha, \beta)$: More General Case



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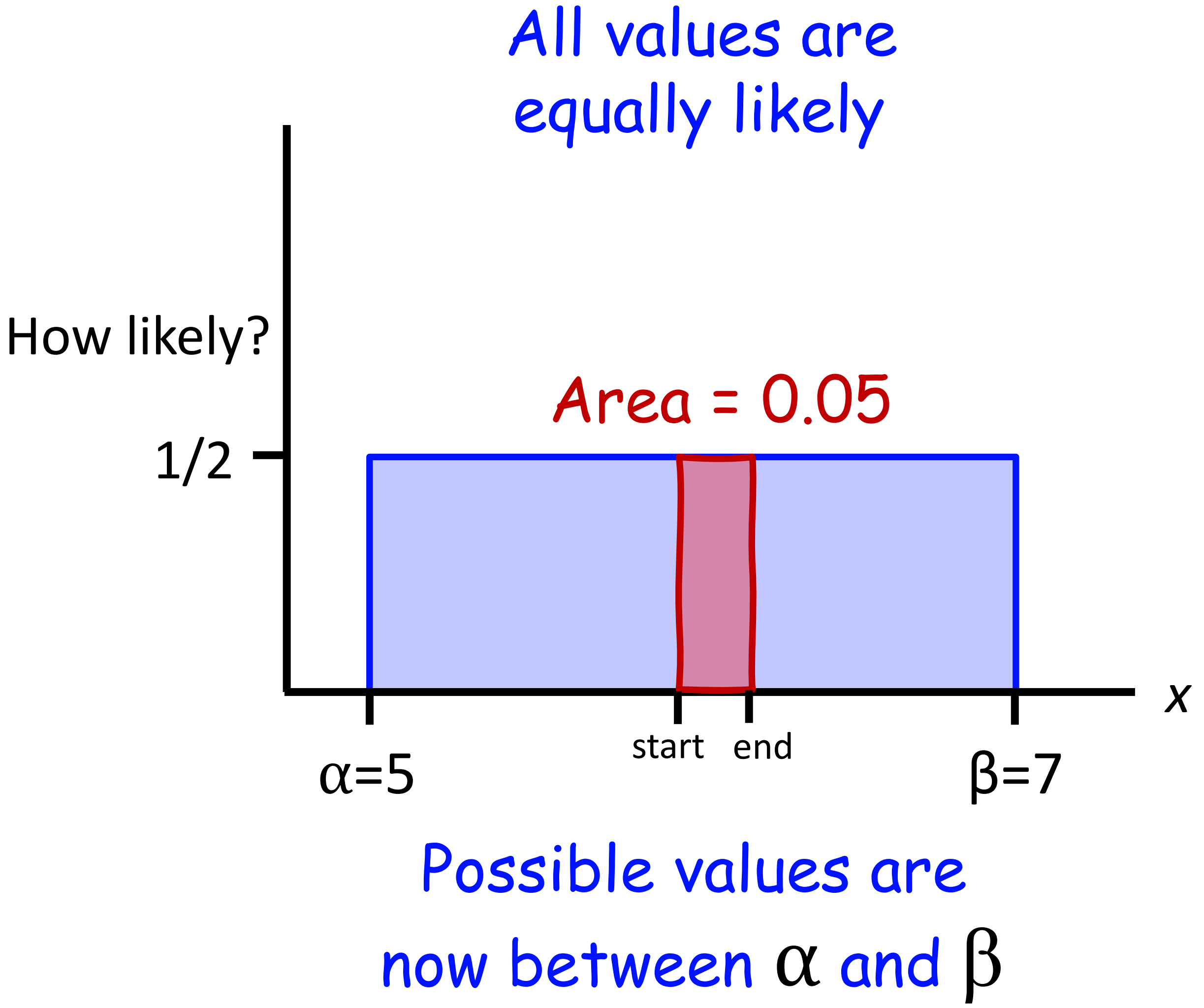
For Uniform(α, β):

$$P(\text{start} \le X \le \text{end}) = \frac{\text{end} - \text{start}}{\beta - \alpha}$$

$X \sim$ Uniform(α, β): More General Case



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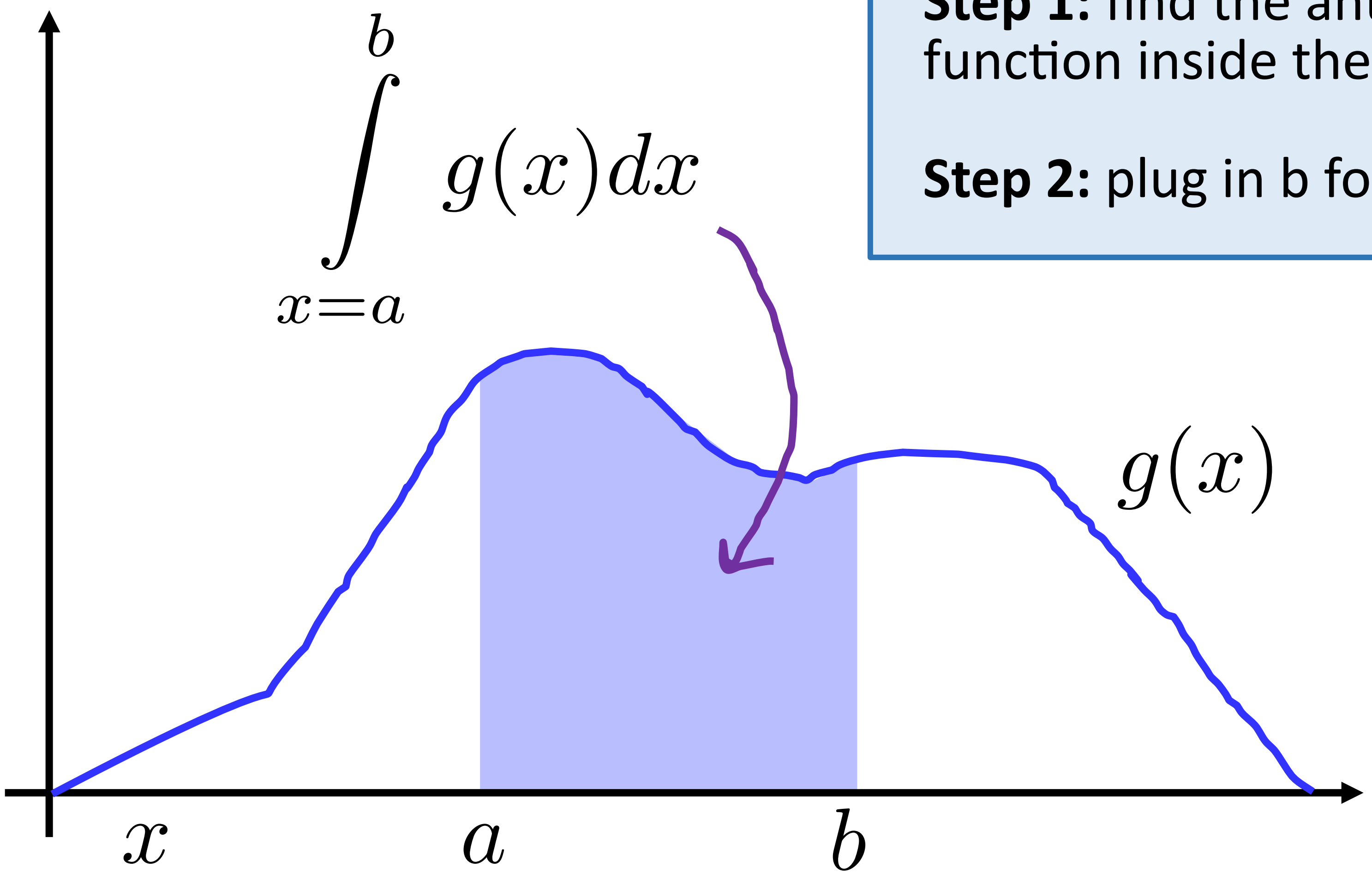
For Uniform(α, β):

$$P(\text{start} \le X \le \text{end}) = \frac{\text{end - start}}{\beta - \alpha}$$

If we set y = 1/2 between α and β, then probabilities are "slices of the whole box"

Time For Integrals!!!!

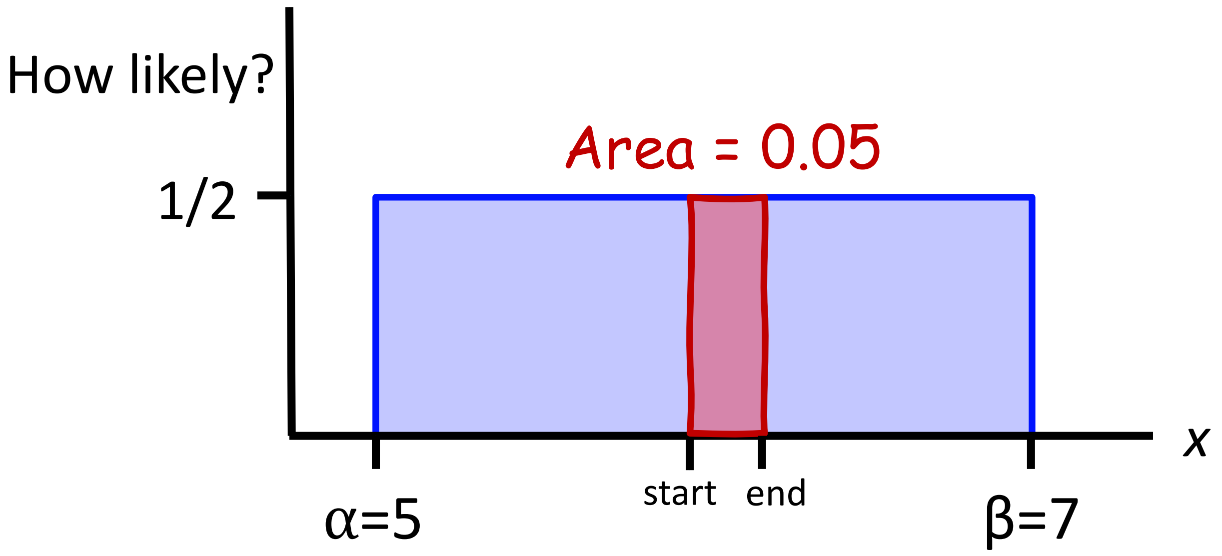
(Definite) Integrals For Any Curve



Step 1: find the anti-derivative for g(x), the function inside the integral.

Step 2: plug in b for x, plug in a for x, subtract.

Integrals: The Area Under A Curve



For Uniform(α, β):

$$P(\text{start} \leq X \leq \text{end}) = \frac{\text{end - start}}{\beta - \alpha}$$

$$P(\text{start} \leq X \leq \text{end}) = \int_{\text{start}}^{\text{end}} \frac{1}{\beta - \alpha} dx$$

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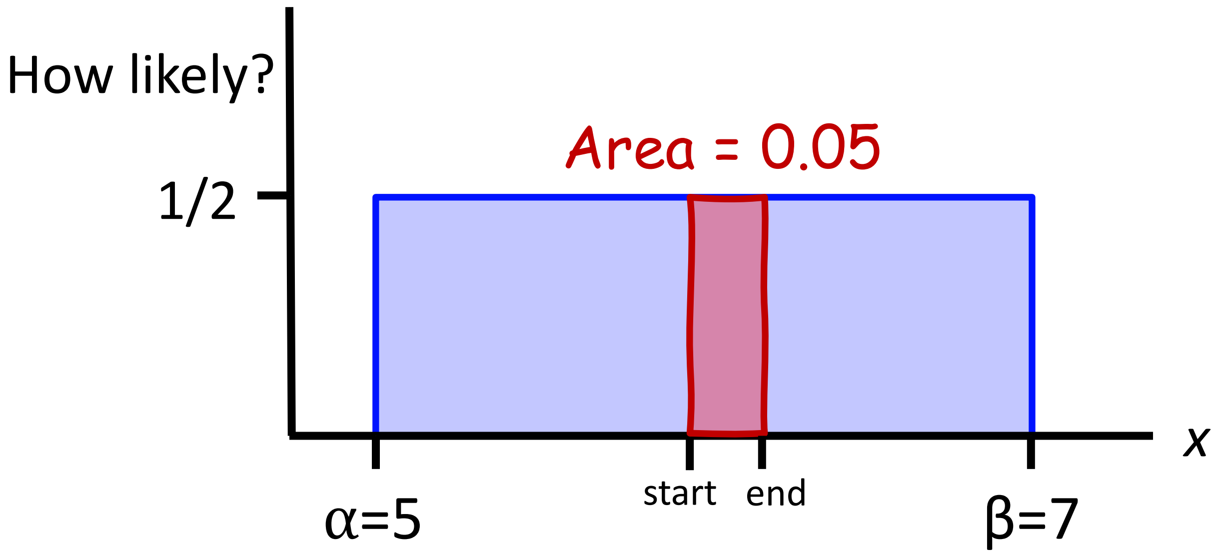

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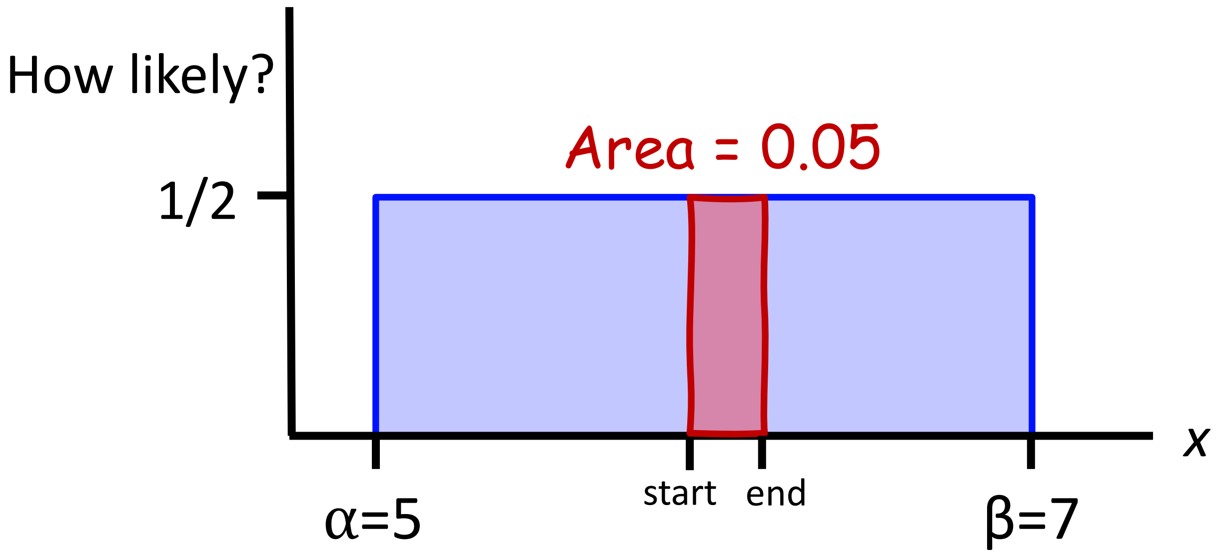
The range of values we want the probability of…

…are the bounds of the integral

Integrals: The Area Under A Curve



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This function is the blue box: it represents how likely different outcomes are

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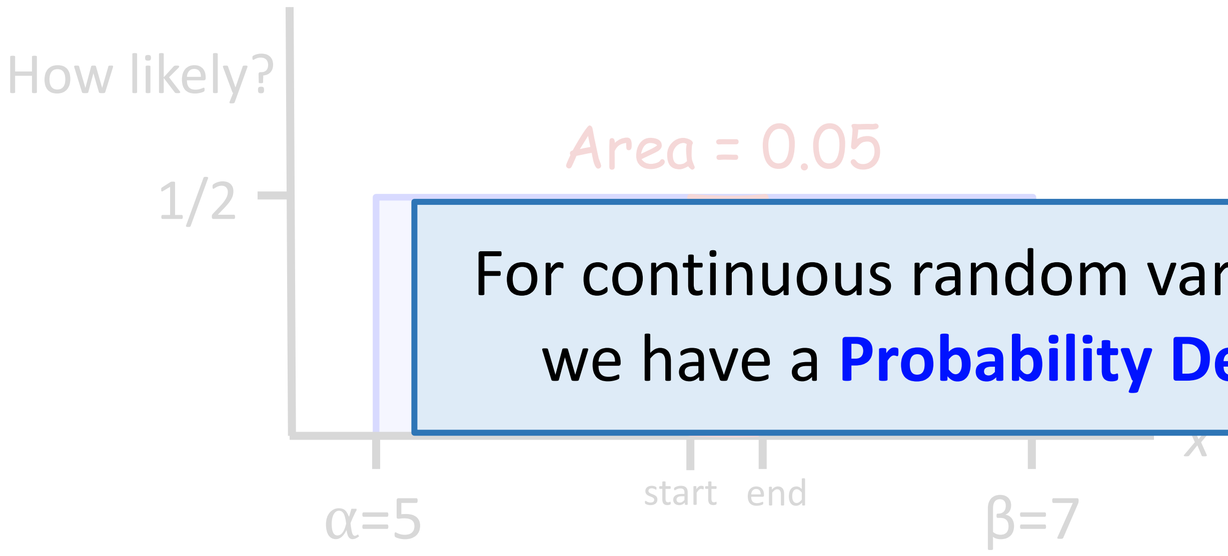

For Uniform(α, β):

$$\frac{\text{end - start}}{\beta - \alpha}$$

For continuous random variables, instead of a PMF, we have a **Probability Density Function (PDF).**

$$P(\text{start} \leq X \leq \text{end}) = \int_{\text{start}}^{\text{end}} \boxed{\frac{1}{\beta - \alpha}} dx$$

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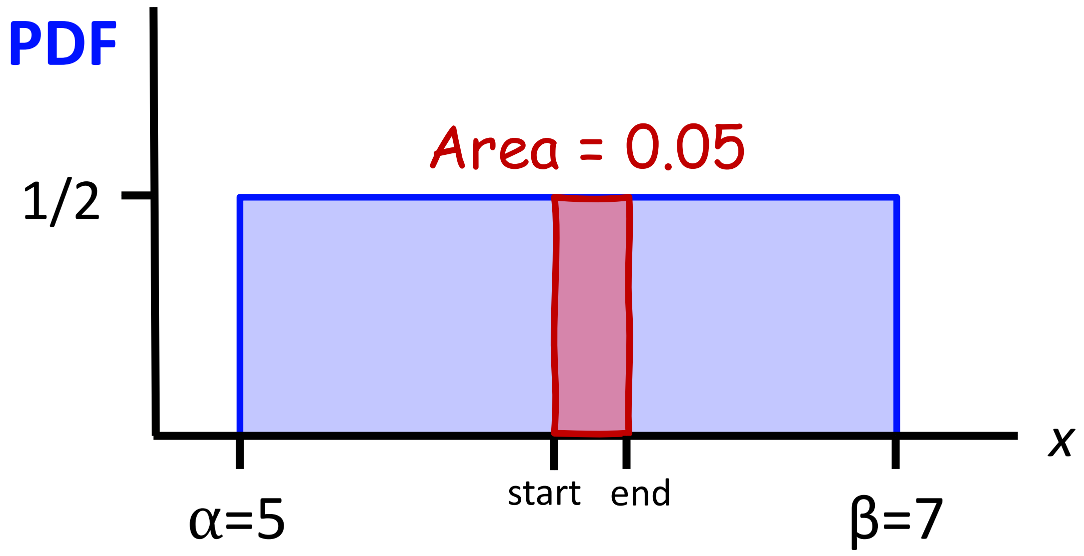
Probability Density Functions

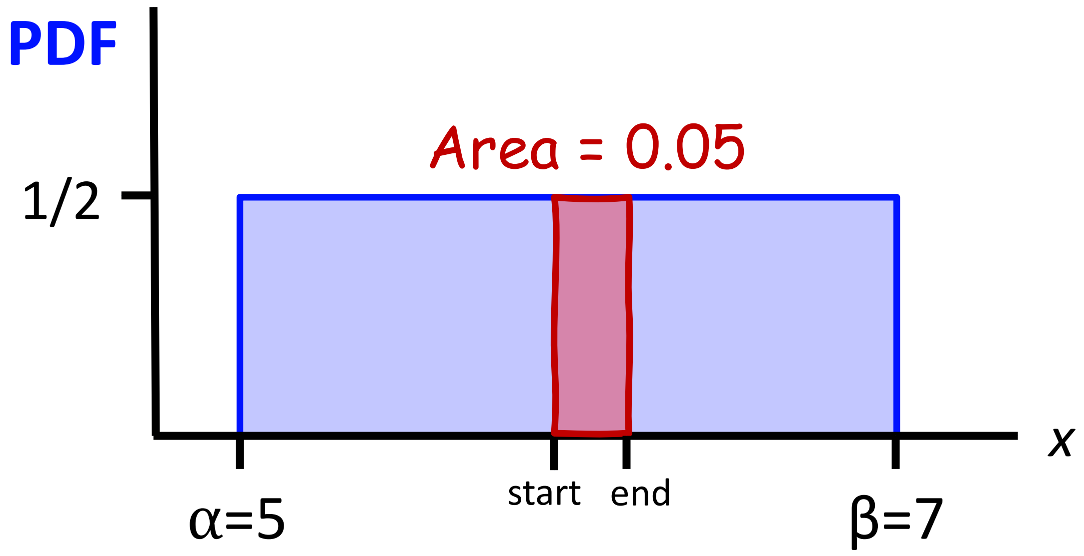


The **probability density function** (PDF) of a continuous random variable represents the relative likelihood of various values.

Units: probability *divided by units of X, or the derivative of the probability of x.*
Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^{b} f(X = x)\,dx$$

Probability Density Functions: Uniform



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Integrate it to get probabilities!

$$P(a < X < b) = \int_{x=a}^{b} f(X = x)\,dx$$

For Uniform(α, β):

$$f(X = x) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Riding the Marguerite

Marguerite Arrival Times

You're running to the bus stop. You don't know exactly when the bus arrives.

You believe all times between 2 and 2:30 are equally likely.

You show up at 2:15pm. What is P(wait < 5 min)?

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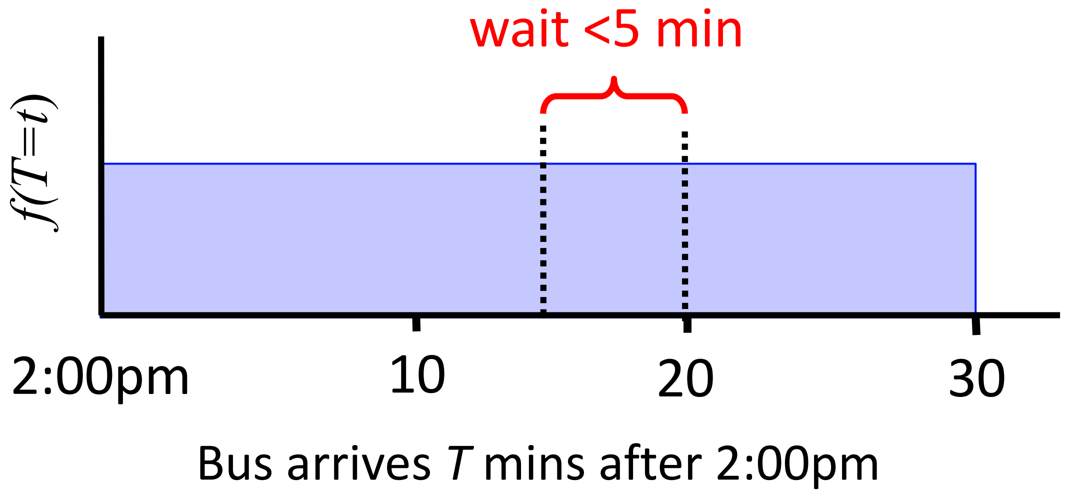
You show up at 2:15pm. What is P(wait < 5 min)?

$$T \sim \text{Uni}(\alpha = 0, \beta = 30)$$

Want: $P(15 \leq \boldsymbol{T} \leq 20)$



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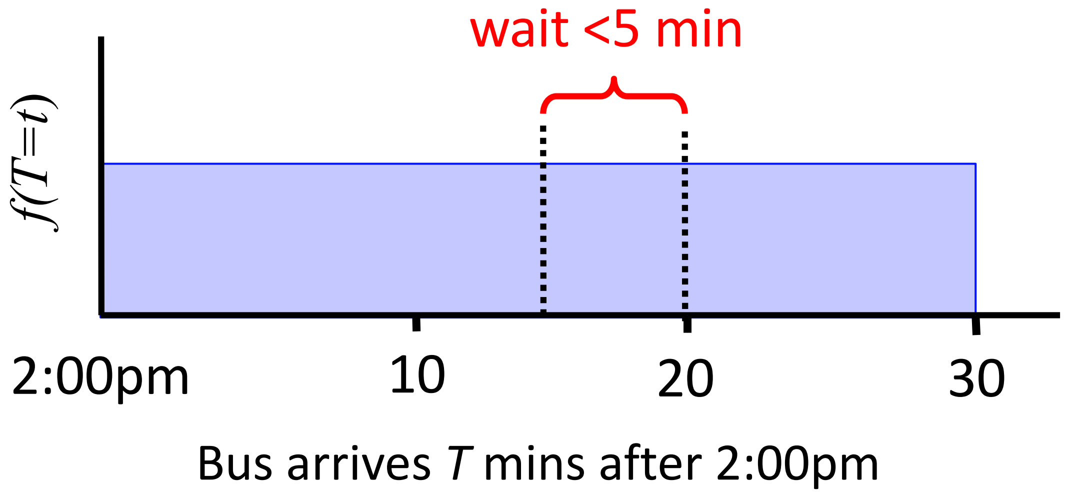


$$P(\text{Wait} \; < 5) = \int_{15}^{20} \frac{1}{\beta - \alpha} \, dx$$

$$= \frac{x}{\beta - \alpha}\Big|_{15}^{20} = \frac{x}{30 - 0}\Big|_{15}^{20} = \frac{5}{30}$$

Uniform Random Variable

- We have a PDF instead of a PMF, like all continuous random variables
- We compute probabilities of a continuous RV being within a range (6 < X < 7), rather than an exact value (X = k), using integrals of the PDF
  - For the uniform, the integral is chill, because the PDF is flat
- Continuous RVs have expectation and variance just like discrete RVs

Uniform Random Variable

Notation: $X \sim \text{Uni}(\alpha, \beta)$

Description: A continuous random variable that takes on values, with equal likelihood, between α and β

Parameters: $\alpha \in \mathbb{R}$, the minimum value of the variable.
$\beta \in \mathbb{R}, \beta > \alpha$, the maximum value of the variable.

Support: $x \in [\alpha, \beta]$

PDF equation: $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{else} \end{cases}$

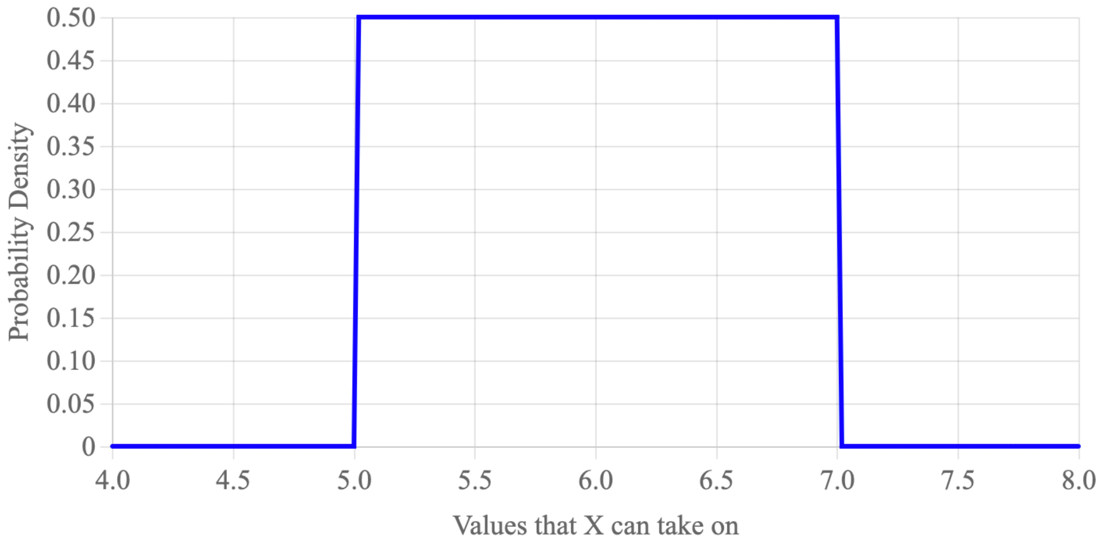
CDF equation: $F(x) = \begin{cases} \frac{x-\alpha}{\beta-\alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{for } x < \alpha \\ 1 & \text{for } x > \beta \end{cases}$

Expectation: $\text{E}[X] = \frac{1}{2}(\alpha + \beta)$

Variance: $\text{Var}(X) = \frac{1}{12}(\beta - \alpha)^2$

PDF graph:

Parameter α: 5 Parameter β: 7



Marguerite Arrival Times: Not Uniform

You're running to the bus stop. You don't know exactly when the bus arrives.

You have a probability distribution for bus arrival times -- some times are more likely than others.

You show up at 2:15pm. What is P(wait < 5 min)?

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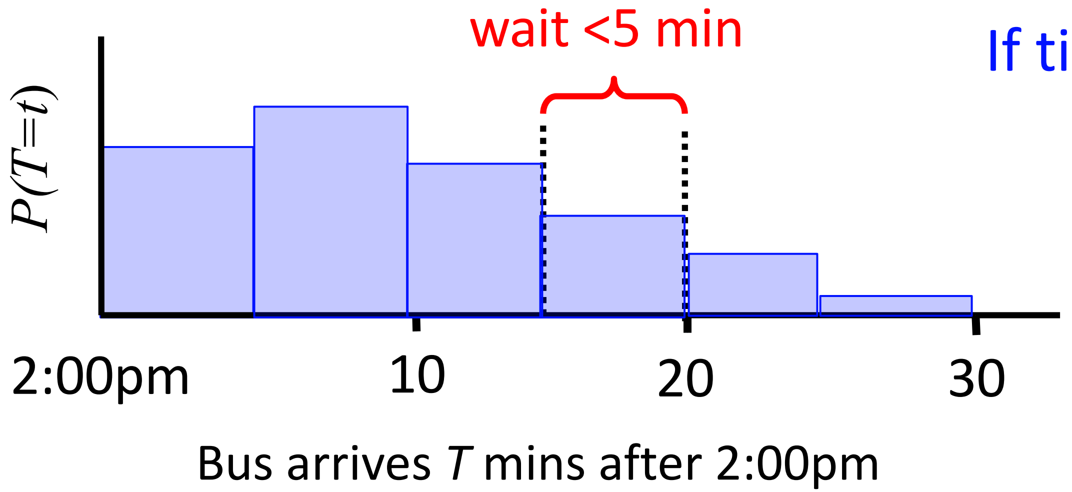

If time was discrete: a PMF could look like this.

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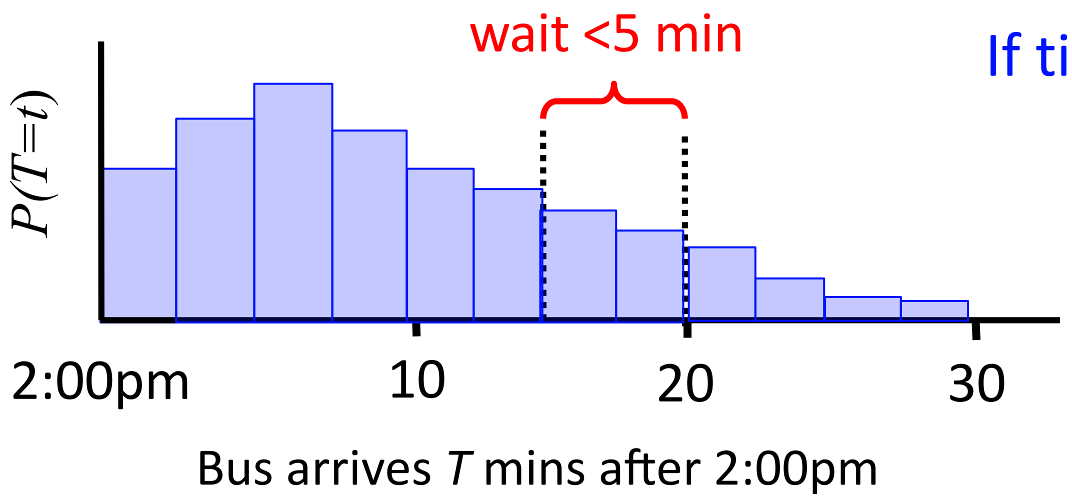

If time was discrete: a PMF could look like this.

But in order to talk about more precise time intervals, we would need to keep making bins smaller and smaller, adding up probabilities for all of bins in the interval we care about…

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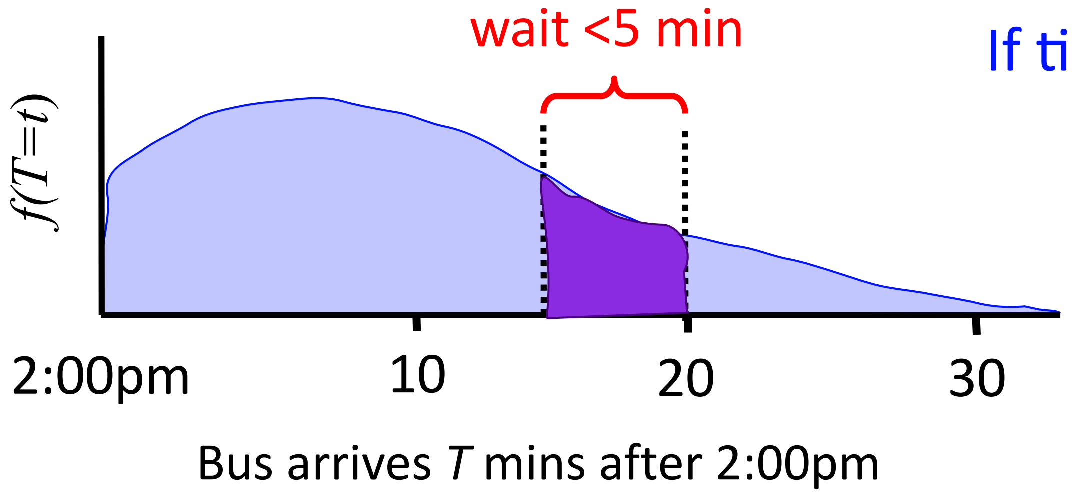

If time was discrete: a PMF could look like this.

The integral of the PDF is this idea, taken to the limit.

PDFs - $f(X = x)$ vs. PMFs - $P(X = x)$

$$P(X = x)$$

"The probability that a **discrete** random variable X takes on the value *x*."

$$f(X = x)$$

"The ***derivative*** of the probability that a **continuous** random variable X takes at the value *x*."

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What do you get if you integrate over a probability *density* function?

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What do you get if you integrate over a probability *density* function?

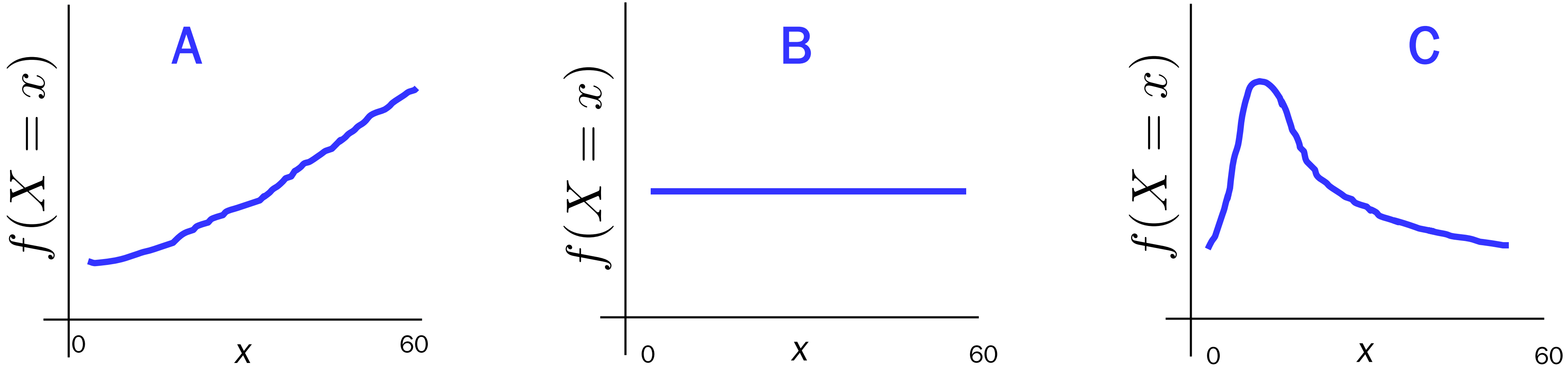
A probability!

They are *both* measures of how **likely** *X* is to take on the value *x*.

The Relative Values of PDFs Are Meaningful

Probability density functions are derivatives that articulate ***relative*** belief.

Let X be the # of minutes after 2pm that the bus arrives at a stop.



Which of these represent that the bus's arrival
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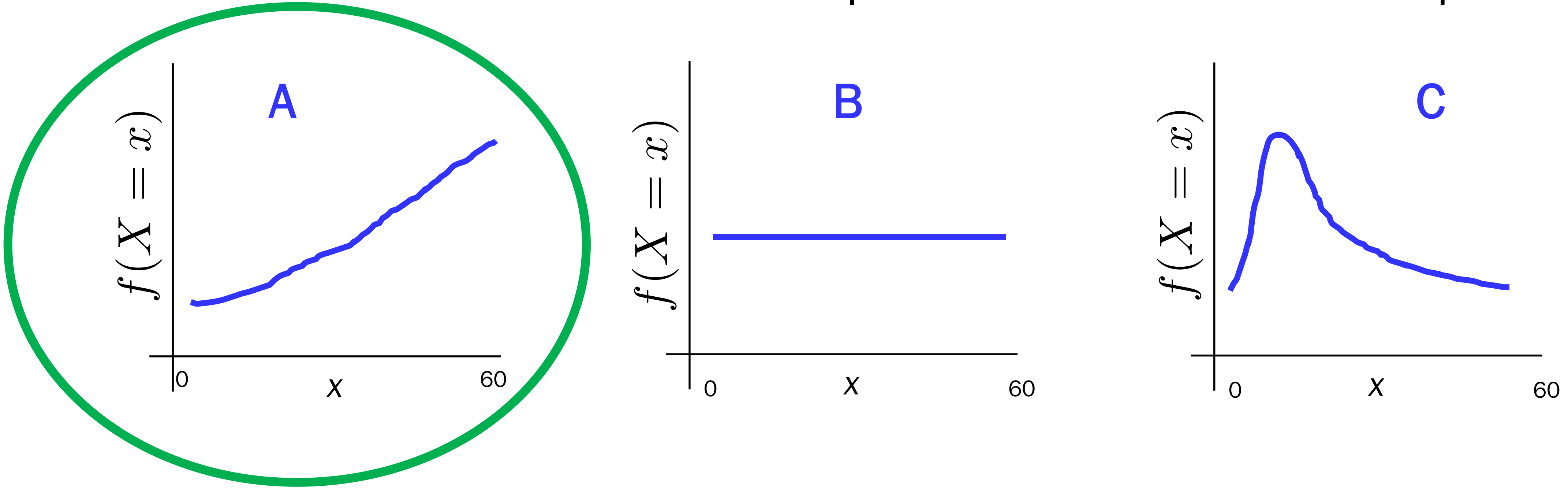

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Axioms of Probability For PDFs

Axiom 1:

$$0 \leq \int_{x=a}^{b} f(X = x)\, dx \leq 1$$

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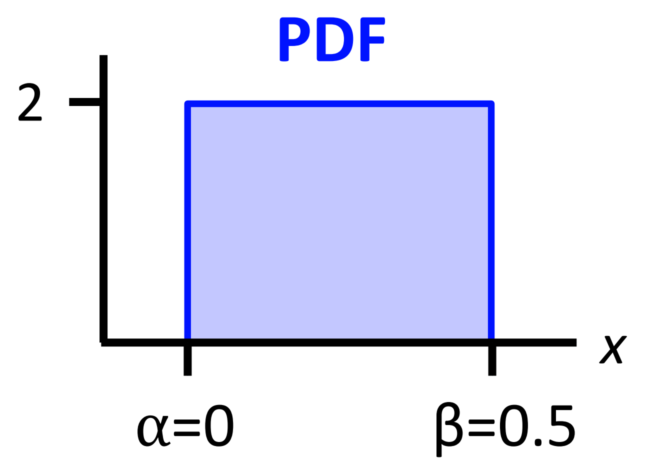
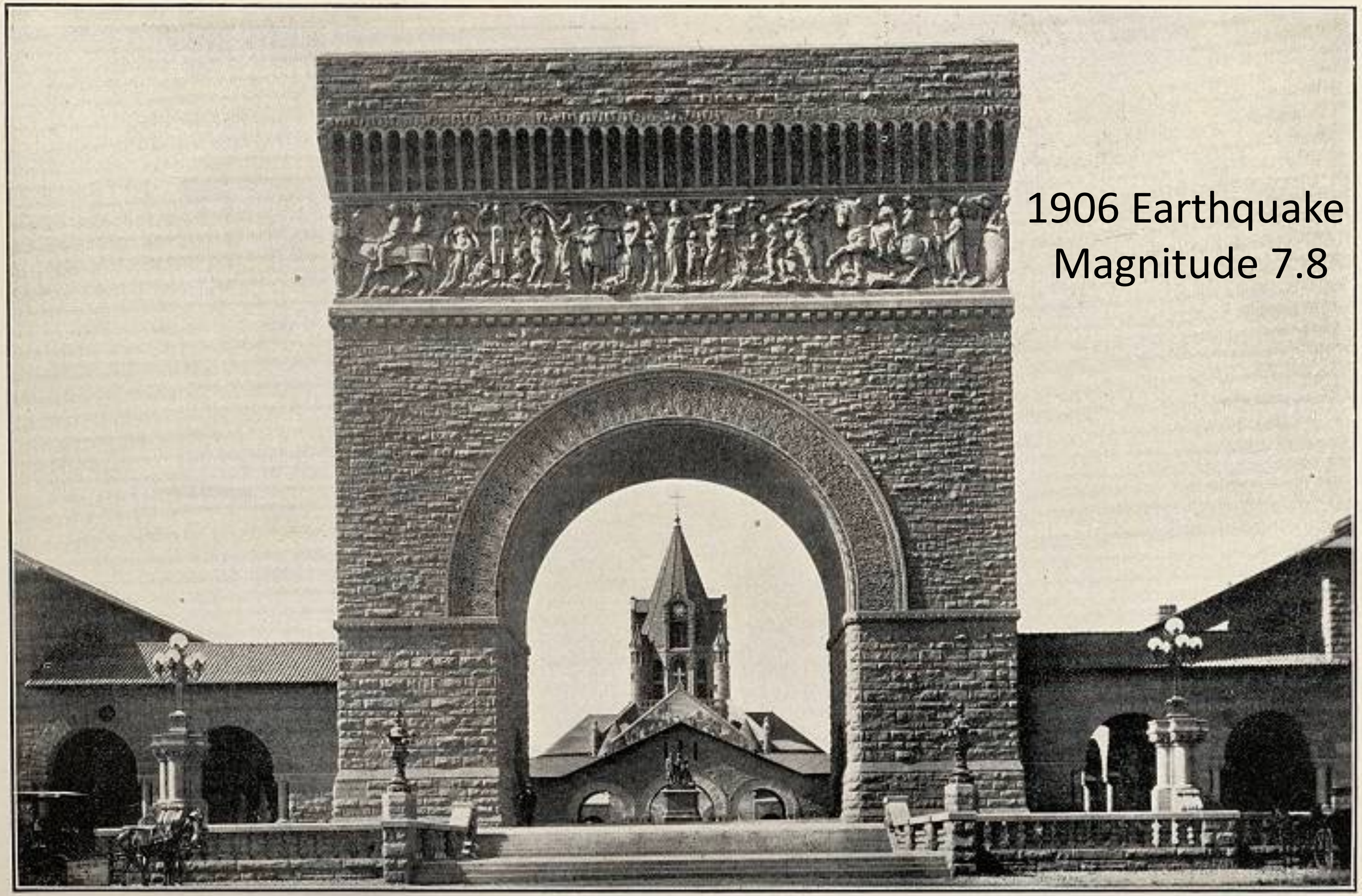


Can a PDF ever have a value > 1?

Yes!

It’s Time
To Talk About Time, Again



ILL. No. 65. MEMORIAL ARCH, WITH CHURCH IN BACKGROUND, STANFORD UNIVERSITY, SHOWING TYPES OF CARVED WORK WITH THE SANDSTONE.

How long until the next "big one"?

Exponential Random Variable

For any Poisson Process, the **Exponential** RV models *time until an event*:

$$X \sim \text{Exp}(\lambda)$$

PDF: $$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Examples:

- Time until next earthquake
- Time until a ping reaches a web server
- Time until next Uber request

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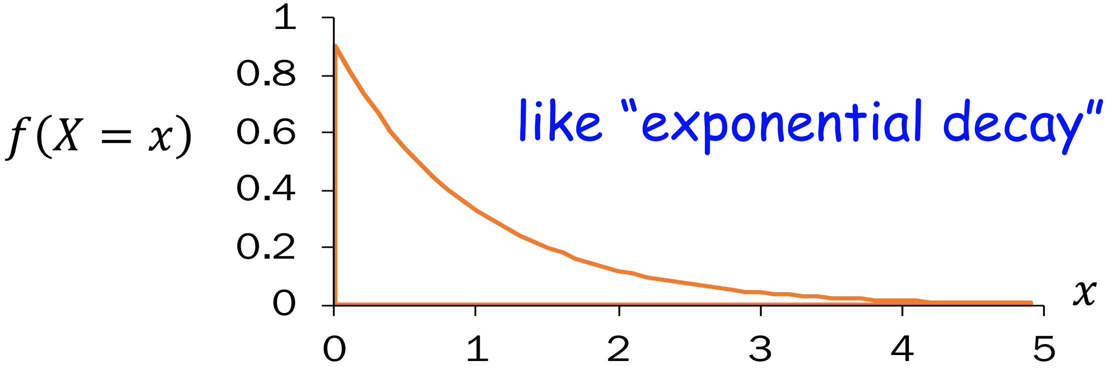
Examples:

- Time until next earthquake
- Time until a ping reaches a web server
- Time until a Uranium atom decays



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Based on historical data, major earthquakes (with magnitude 8.0+) happen at a **rate of 0.002** per year*.

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*In California, according to the USGS, 2015

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$$Y \sim \text{Exp}(\lambda = \frac{1}{500})$$

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$$Y \sim \text{Exp}(\lambda = \frac{1}{500})$$

$$P(Y < 30) = \int_0^{30} \frac{1}{500} e^{-\frac{y}{500}} \, dy$$

Exponential PDF:

$$f_Y(y) = \lambda e^{-\lambda y}$$

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Integral Fun Fact:

$$\int e^{cx}\,dx = \frac{1}{c}e^{cx}$$

$$Y \sim \text{Exp}(\lambda = \frac{1}{500})$$

$$P(Y < 30) = \int_0^{30} \frac{1}{500} e^{-\frac{y}{500}}\,dy$$

$$= \left[-e^{-\frac{y}{500}}\right]_0^{30} = -e^{\frac{30}{500}} + e^0 \approx 0.058$$

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How Long Until the Next Big Earthquake?

Based on historical data, major earthquakes (with magnitude 8.0+) happen at a **rate of 0.002** per year*.
What is the expected number of years before **the next major earthquake?**

Let *Y* be years until the next earthquake of magnitude 8.0+.

$$Y \sim \text{Exp}(\lambda = \frac{1}{500})$$

$$E[Y] = \frac{1}{\lambda} = \frac{1}{0.002} = 500$$

Expectation of an Exponential:

$$E[Y] = \frac{1}{\lambda}$$

Concept Check

*In California, according to the USGS, 2015

Is there a way to avoid integrals?

please?

Cumulative Density Functions

A *cumulative* density function (CDF) is a "closed-form" equation for the probability that a continuous random variable is less than a given value.

$$F(x) = P(X < x)$$

$$P(X < x) = \int_{y=-\infty}^{x} f(y)\, dy$$

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For random variables that have cumulative density functions, we can avoid integrals!

CDF For an Exponential

$$F_X(x) = 1 - e^{-\lambda x}$$

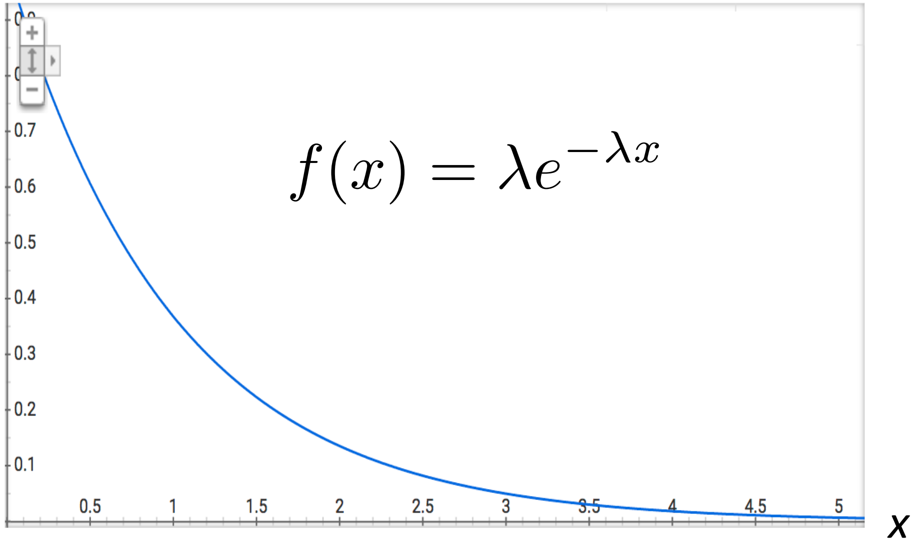
CDF For an Exponential

$$F_X(x) = 1 - e^{-\lambda x}$$

$$
\begin{aligned}
P(X < x) &= \int_{y=-\infty}^{x} f(y)\ dy \\
&= \int_{y=0}^{x} \lambda e^{-\lambda y}\ dy \\
&= \frac{\lambda}{\lambda}\Big[-e^{-\lambda y}\Big]_0^x \\
&= [-e^{-\lambda x}] - [-e^{\lambda 0}] \\
&= 1 - e^{-\lambda x}
\end{aligned}
$$

Example: $X \sim \text{Exp}(\lambda = 1)$

Probability
Density
Function



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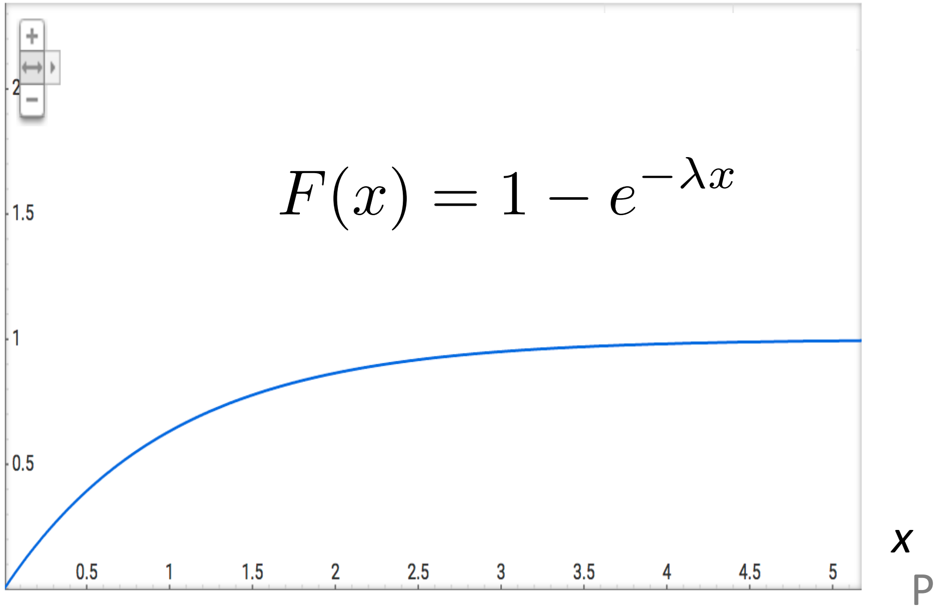

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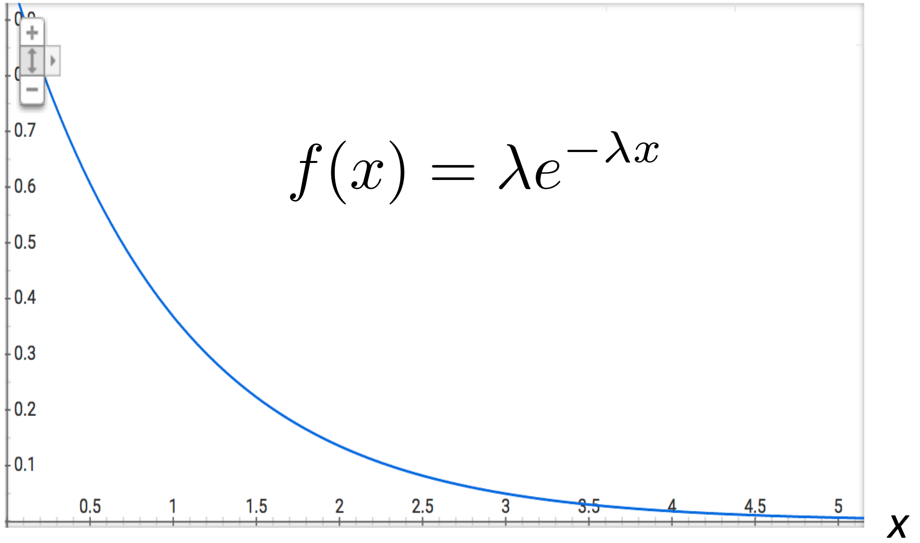
$P(X < 2)$

Probability
Density
Function



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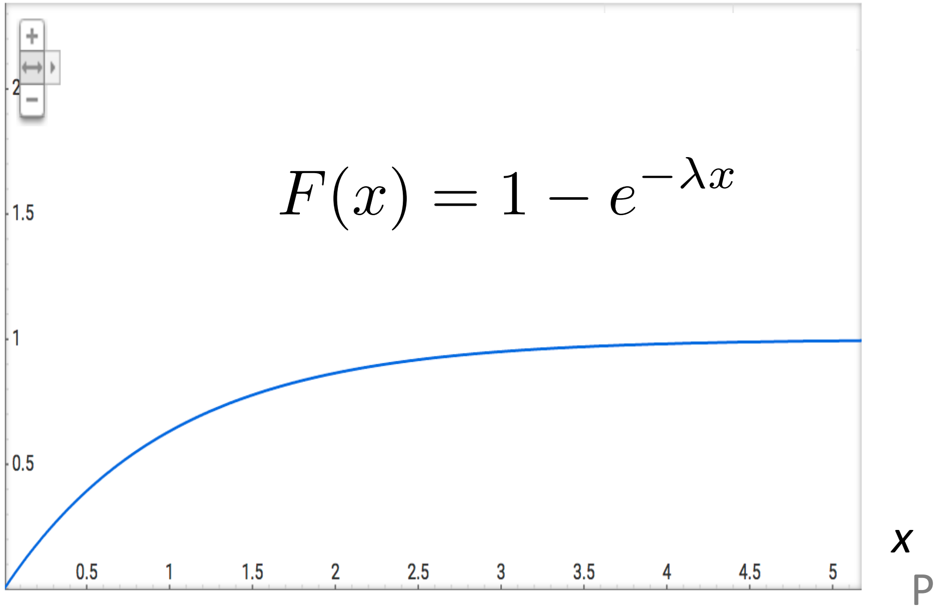

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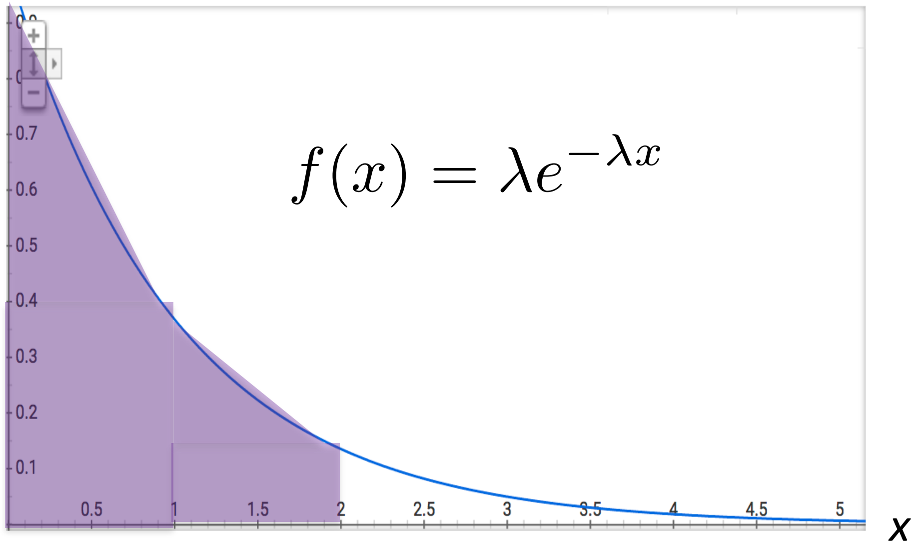

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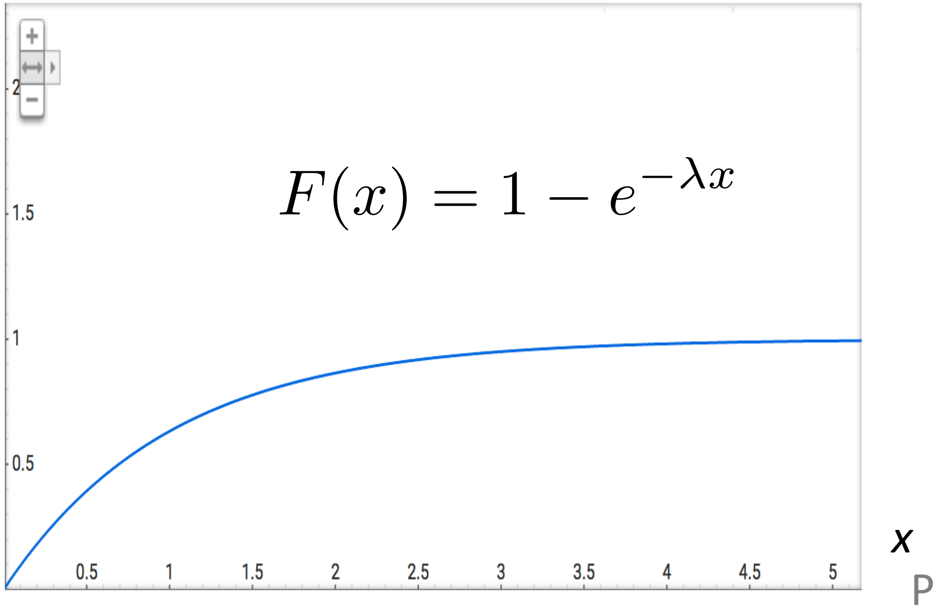

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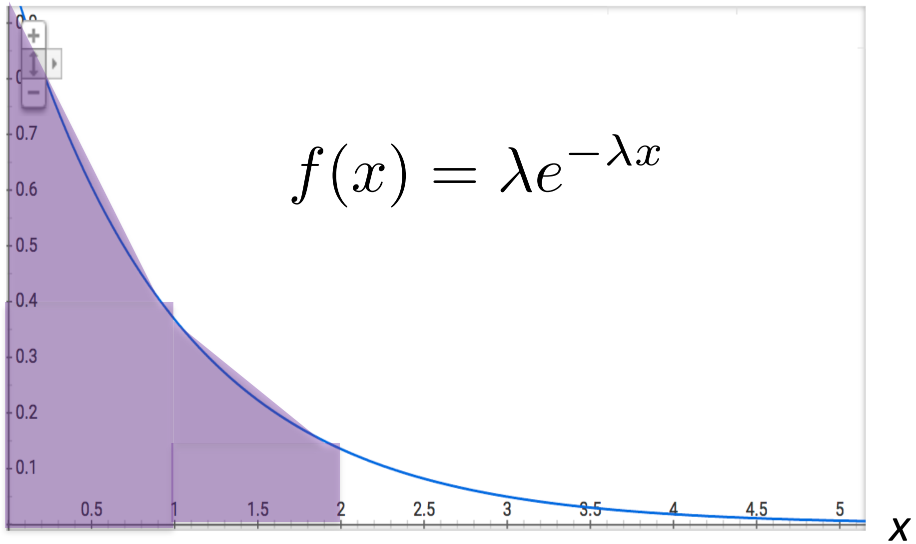

$P(X < 2)$

$$= \int_{x=-\infty}^{2} f(x)\, dx$$

or

Cumulative
Density
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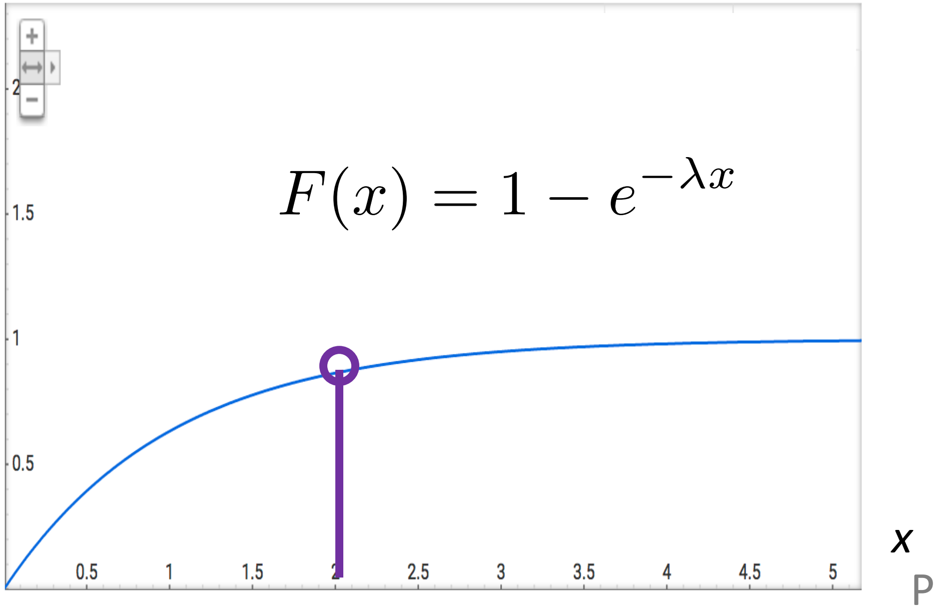
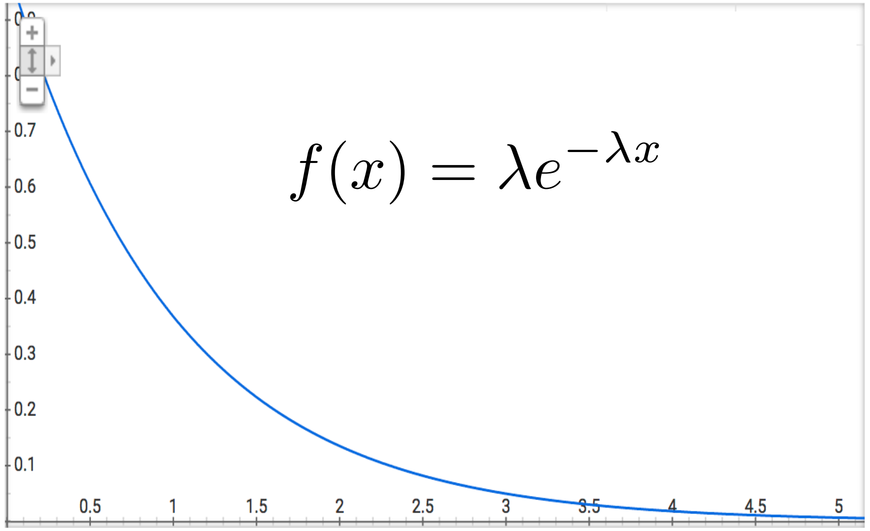


$$F(2) = 1 - e^{-2}$$

$$\approx 0.84$$

Example: $X \sim \text{Exp}(\lambda = 1)$

Probability
Density
Function



$$f(x) = \lambda e^{-\lambda x}$$

x

Cumulative
Density
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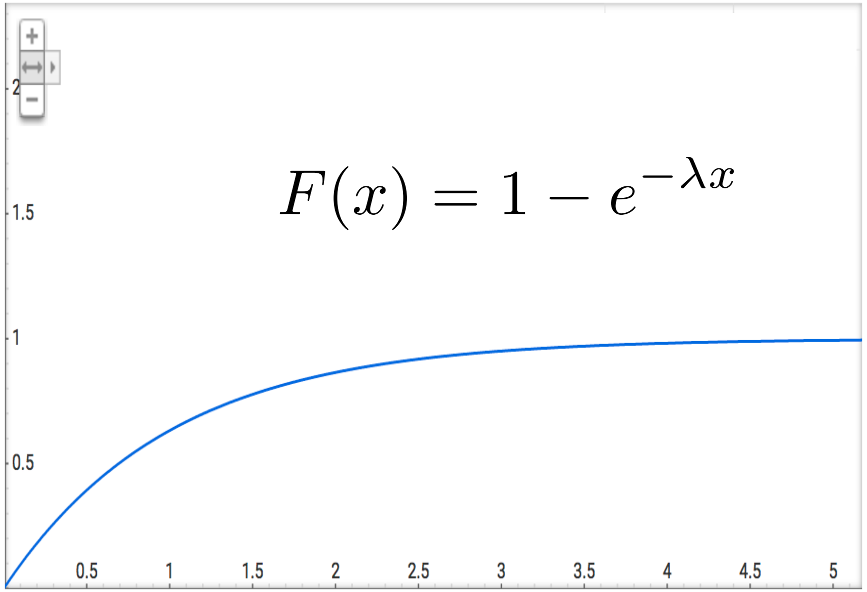

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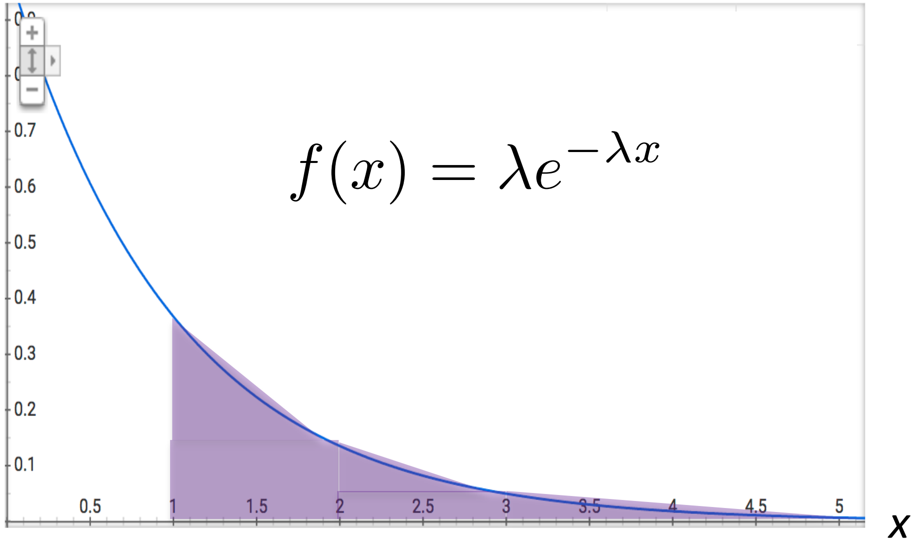
$P(X > 1)$

Example: $X \sim \text{Exp}(\lambda = 1)$

Probability
Density
Function



$P(X > 1)$

$$= \int_{x=1}^{\infty} f(x)\, dx$$

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Density
Function

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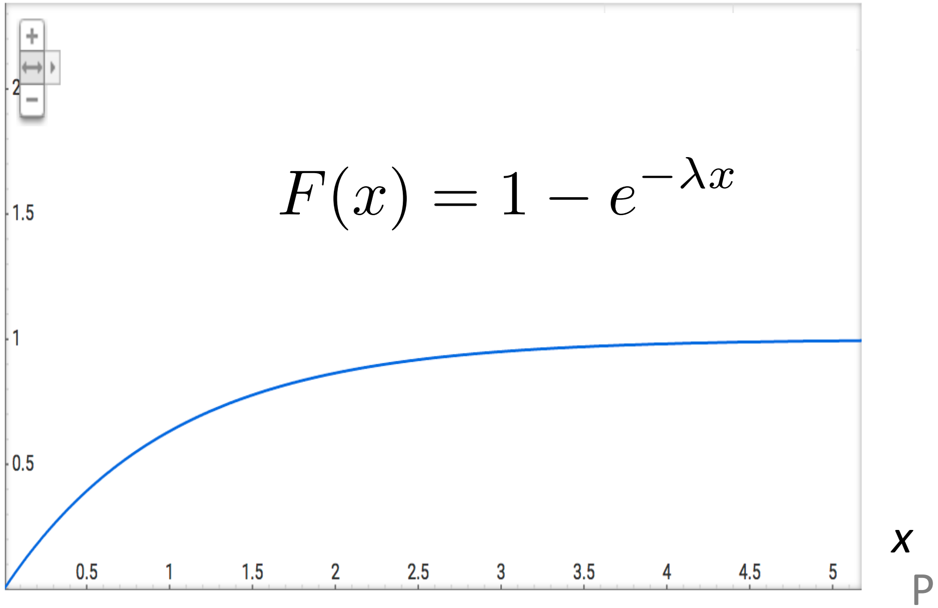

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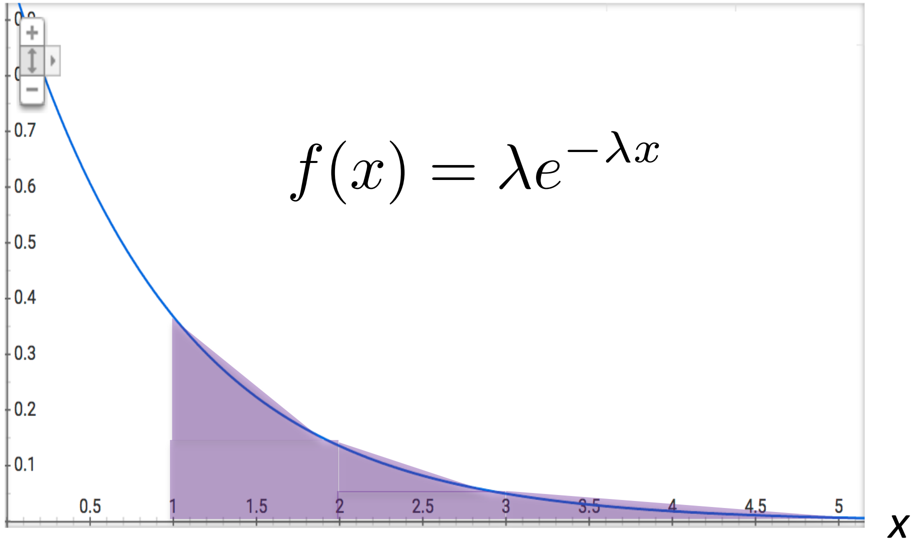

$$f(x) = \lambda e^{-\lambda x}$$

Cumulative
Density
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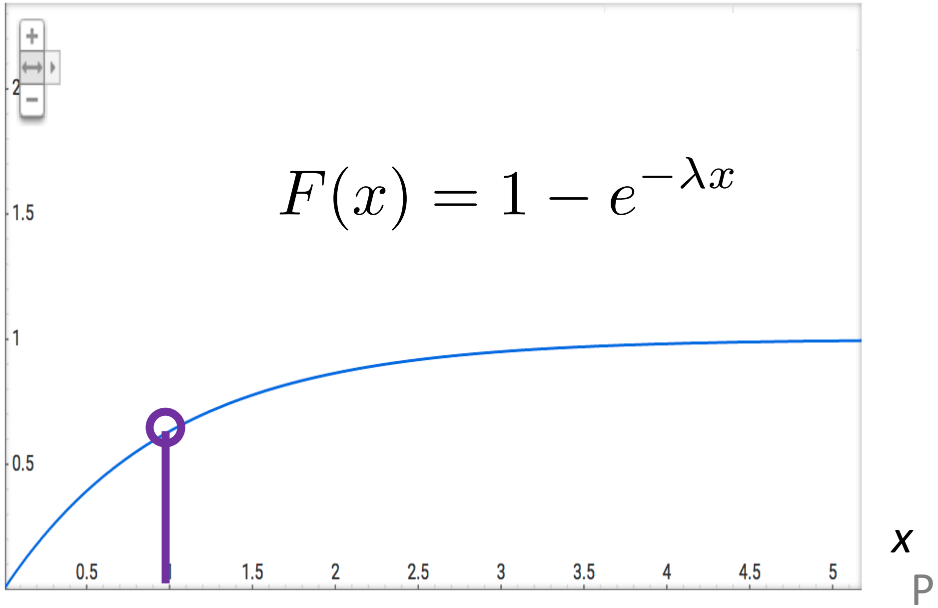


$$F(x) = 1 - e^{-\lambda x}$$

P(*X* > 1)

$$= \int_{x=1}^{\infty} f(x)\, dx$$

or

$$1 - F(1) = e^{-1}$$

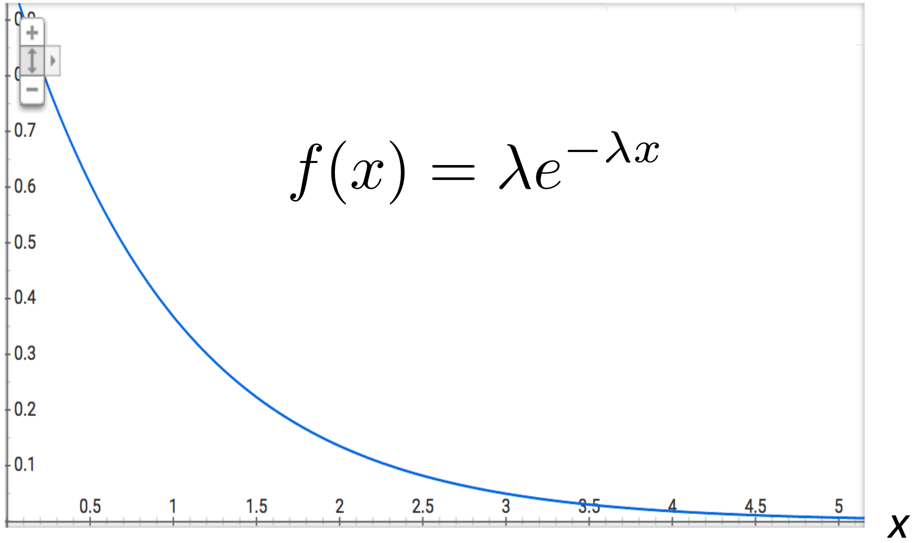
$$\approx 0.37$$

Example: $X \sim \text{Exp}(\lambda = 1)$

$P(1 < X < 2)$

____

Probability
Density
Function



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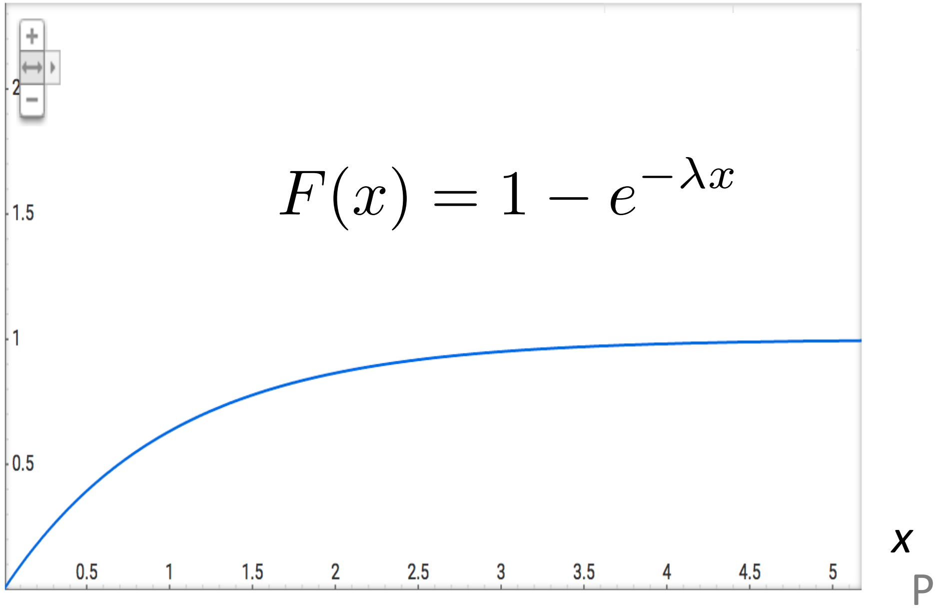

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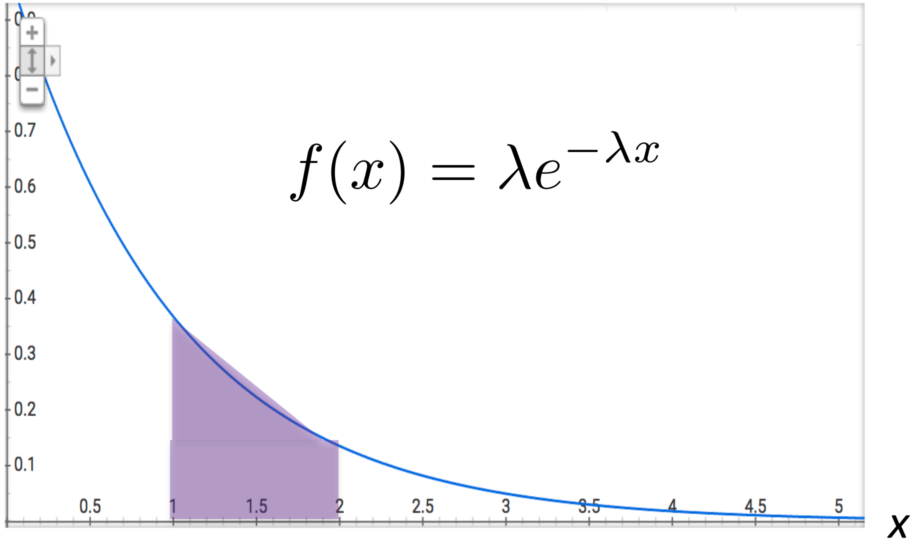

$P(1 < X < 2)$

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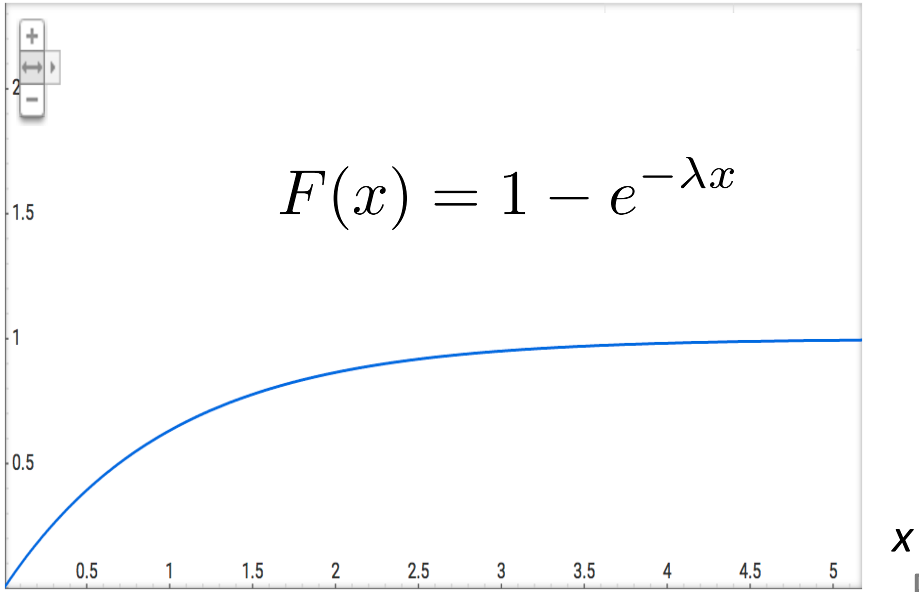

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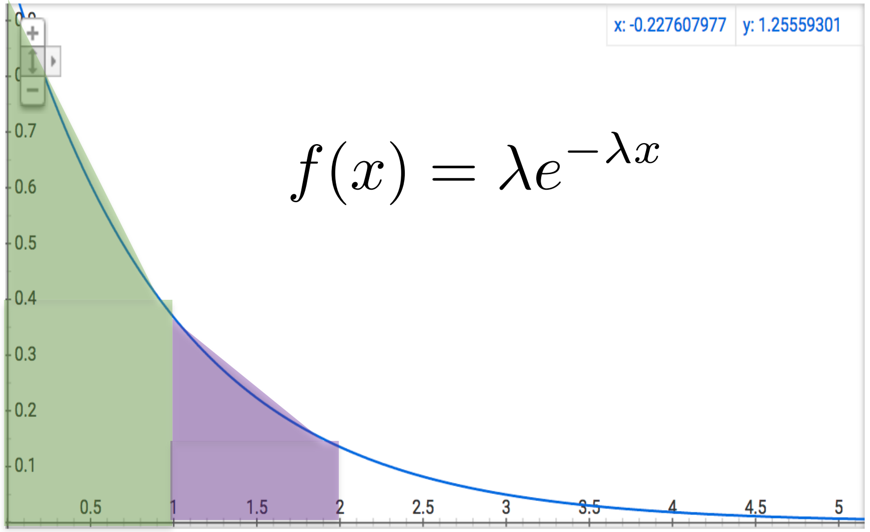

x

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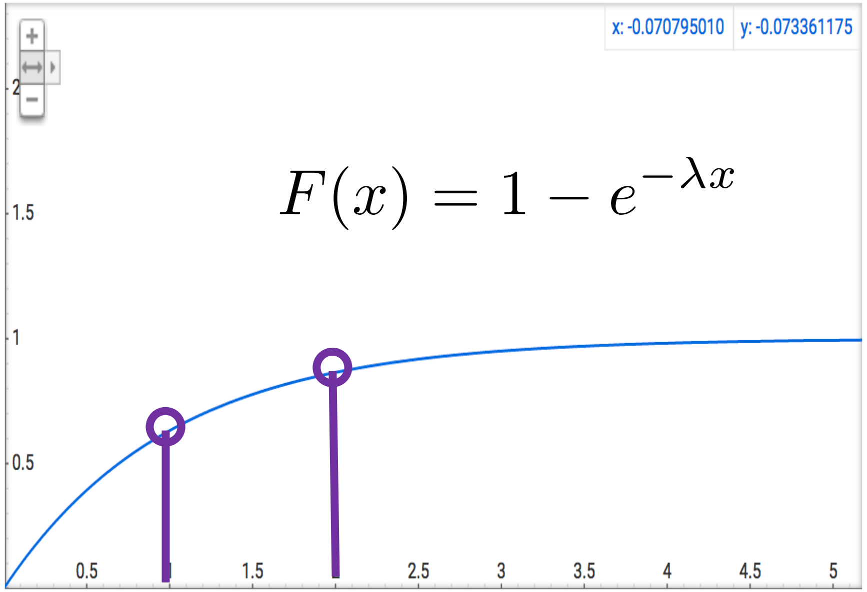

x

$$F(2) - F(1) = (1 - e^{-2})$$
$$- (1 - e^{-1})$$
$$\approx 0.23$$

How Long Until the Next Big Earthquake?

Based on historical data, major earthquakes (with magnitude 8.0+) happen at a **rate of 0.002** per year*.
What is the probability of **a major earthquake in the next 30 years?**

Let Y be years until the next earthquake of magnitude 8.0+.

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Exponential CDF:

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CDF: $F(x) = 1 - e^{-\lambda x}$

Funnest Fact: Exponential is Memoryless!

$$X \sim \text{Exp}(\lambda)$$

$$\text{P}(X > s + t | X > s) = \; ?$$

What if s time has passed?

"How long until the next big earthquake,
if it's been 50 years since the last one?"

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$$\text{P}(X > s + t | X > s) = \text{P}(X > t)$$

What if s time has passed?

"How long until the next big earthquake, if it's been 50 years since the last one?"

Answer: It doesn't matter how long it's been. The Exponential will look the same!

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$$\text{P}(X > s + t | X > s) = \text{P}(X > t)$$

What if *s* time has passed?

Which is something we can prove:

$$\begin{aligned}
\text{P}(X > s+t | X > s) &= \frac{\text{P}(X > s+t \text{ and } X > s)}{\text{P}(X > s)} && \text{Def of conditional prob.} \\
&= \frac{\text{P}(X > s+t)}{\text{P}(X > s)} && \text{Because } X > s+t \text{ implies } X > s \\
&= \frac{1 - F_X(s+t)}{1 - F_X(s)} && \text{Def of CDF} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} && \text{By CDF of Exp} \\
&= e^{-\lambda t} && \text{Simplify} \\
&= 1 - F_X(t) && \text{By CDF of Exp} \\
&= \text{P}(X > t) && \text{Def of CDF}
\end{aligned}$$

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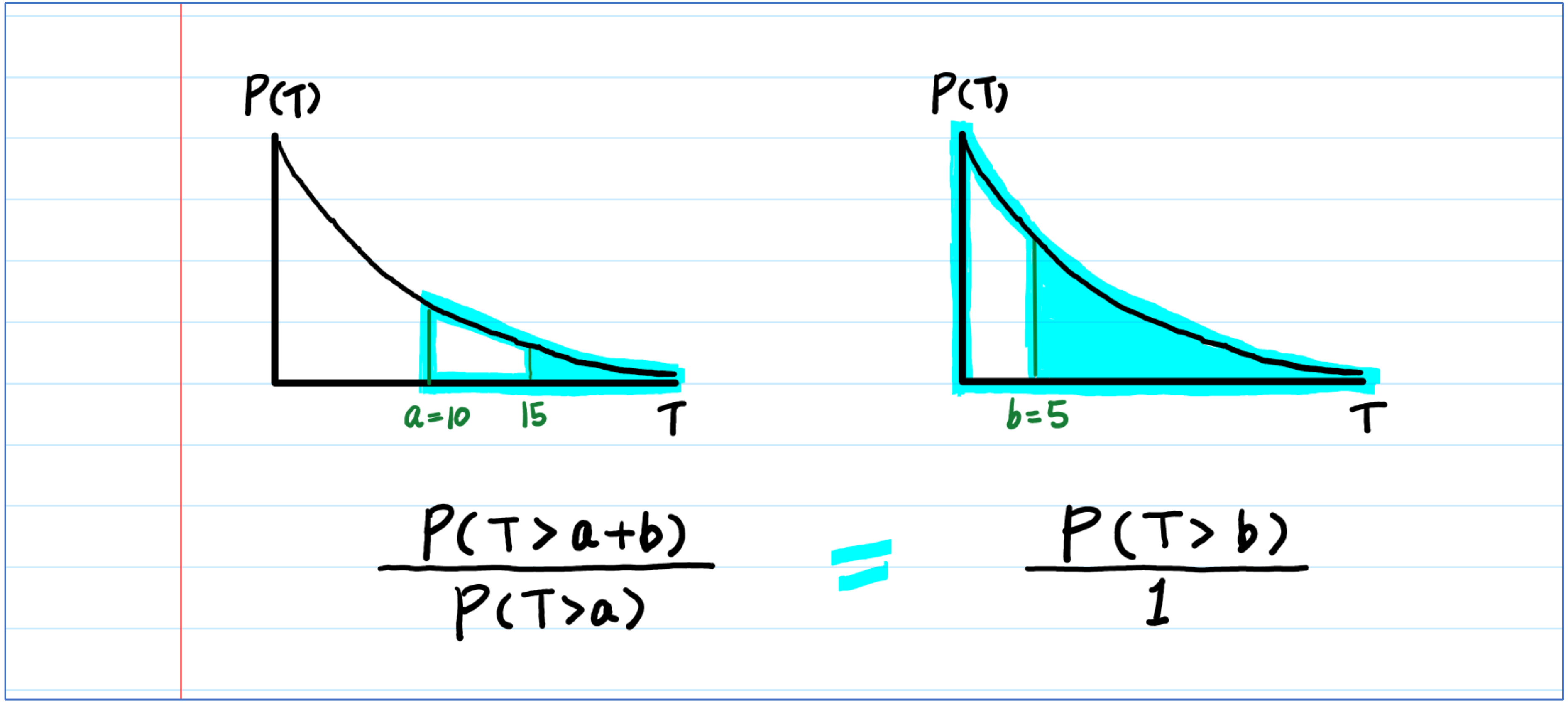

https://towardsdatascience.com/what-is-exponential-distribution-7bdd08590e2a

Have a Lovely Weekend!