Chris Piech
CS109

Section 1: Analytic Probability

Chris Piech and the CS109 teaching teams

1 The Birthday Problem

When solving a counting problem, it can often be useful to come up with a generative process, a series of steps that "generates" examples. A correct generative process to count the elements of set A will (1) *generate every element of A* and (2) *not generate any element of A more than once*. If our process has the added property that (3) *any given step always has the same number of possible outcomes*, then we can use the product rule of counting.

Assume that birthdays happen on any of the 365 days of the year with equal likelihood (we'll ignore leap years).

a. **Warmup:** How many ways can you choose birthdays for two (distinct) people?

There are 365 choices for one person's birthday. If we break down the generative process of choosing two birthdays into the steps 1) choose the first person's birthday and 2) choose the second person's birthday, then we can use the step/product rule of counting to say that the total number of ways to assign two birthdays is $365 \cdot 365 = 365^2$.

Note: because the two people are distinct, we don't need to do any correcting for overcounting – "person 1's birthday is 1/1, person 2's birthday is 2/1" and "person 1's birthday is 2/1, person 2's birthday is 1/1" are two different outcomes.

b. What is the probability that of the n people in class, at least two people share the same birthday?

For a more complete explanation of this problem, check out this chapter in the course reader: https://probabilityforcs.firebaseapp.com/book/birthday_paradox

Computing $P(\text{at least 2 people share a birthday})$ is difficult. We realize that this can be thought of as

$$P(\text{exactly 2 people share birthday} \cup \text{exactly 3} \cup \text{exactly 4} \cup \ldots \cup \text{exactly } n \text{ people share birthday})$$

Using the additivity axiom of probability, we realize that this can be split up because the events are mutually exclusive.

$$P(\text{exactly 2 people share birthday}) + P(\text{exactly 3}) + P(\text{exactly 4}) + \ldots + P(\text{exactly } n \text{ people share birthday})$$

However, this is very tedious!

It is much easier to calculate $1 - P(\text{no one shares a birthday})$. Let our sample space, S be the set of all possible assignments of birthdays to the students in section. By the assumptions of this problem, each of those assignments is equally likely, so this is a good choice of sample space. We can use the product rule of counting to calculate $|S|$:

$$|S| = (365)^n$$

Our event E will be the set of assignments in which there are no matches (i.e. everyone has a different birthday). We can think of this as a generative process where there are 365 choices of birthdays for the first student, 364 for the second (since it can't be the same birthday as the first student), and so on. Verify for yourself that this process satisfies the three conditions listed above. We can then use the product rule of counting:

$$|E| = (365) \cdot (364) \cdot \cdots \cdot (365 - n + 1)$$

$$\begin{aligned} P(\text{birthday match}) &= 1 - P(\text{no matches}) \\ &= 1 - \frac{|E|}{|S|} \\ &= 1 - \frac{(365) \cdot (364) \dots (365 - n + 1)}{(365)^n} \end{aligned}$$

A common misconception is that the size of the event E can be computed as $|E| = \binom{365}{n}$ by choosing n distinct birthdays from 365 options. However, outcomes in this event (n unordered distinct dates) cannot recreate any outcomes in the sample space $|S| = 365^n$ (n distinct dates, one for each distinct person). However if we compute the size of event E as $|E| = \binom{365}{n} n!$ (equivalent to the number above), then we can assign the n birthdays to each person in a way consistent with the sample space. The expression $\binom{365}{n} n!$ is equivalent to $\frac{365!}{(365-n)!}$ which is known as a "falling factorial" and also as "365 permute n" outside of this class.

Interesting values: $(n = 13 : p \approx 0.19)$, $(n = 23 : p \approx 0.5)$, $(n = 70 : p \geq 0.99)$.

c. What is the probability that this class contains exactly one pair of people who share a birthday?

We can use the same sample space, but our event is a little bit trickier. Now E is the set of birthday assignments in which exactly two students share a birthday and the rest have different birthdays. One generative process that works for this is (1) choose the two students who share a birthday, (2) choose $n - 1$ birthdays in the same manner as in part a (i.e. one for the pair of students and one for each of the remaining students). We then have:

$$P(\text{exactly one match}) = \frac{|E|}{|S|} = \frac{\binom{n}{2}(365) \cdot (364) \cdot \ldots \cdot (365 - n + 2)}{(365)^n}$$

Many other generative processes work for this problem. Try to think of some other ones and make sure you get the same answer!

2 Conditional Probability Warmup

What is the difference between these two terms $P(B|A)$ and $P(A \cap B)$? Imagine that B is the event that a student "correctly answer a multiple choice question" and A is the event that the same student "guesses randomly". Provide an explanation as well as a mathematical relationship between the two.

They are very different concepts! Students often confuse the two. $P(B|A)$ is the chance that the student gets the problem correct **given** that we **already know** that they are guessing. $P(A \cap B)$ is very different. means that we are curious if both events will occur and that we don't know if the student has guessed. The relationship between the two is given by the chain rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

The key piece of information which would allow you to go from one value to the other is $P(A)$: the probability that a student guesses randomly.

3 Self-Driving Car

A self-driving car has a 60% belief that there is a motorcycle to its left based on all the information it has received up until this point in time. Then, it receives a new, independent report from its left camera. The camera reports that there is **no** motorcycle. What is the updated belief that there is a motorcycle to the left of the car? The camera is an imperfect instrument. When there is truly no motorcycle, the camera will report "no motorcycle" 90% of the time. When there actually is a motorcycle, the camera will report "no motorcycle" 5% of the time.

We need Bayes' rule to solve this problem. Let's define M to be the event that there is a motorcyle to our left. Let N be the event that the camera reports no motorcycle.

$$\begin{aligned}
P(M|N) &= \frac{P(N|M) \cdot P(M)}{P(N)} && \text{Bayes' Theorem} \\
&= \frac{P(N|M) \cdot P(M)}{P(N|M) \cdot P(M) + P(N|M^C) \cdot P(M^C)} && \text{Law of Total Probability} \\
&= \frac{0.05 \cdot 0.6}{0.05 \cdot 0.6 + 0.9 \cdot 0.4} \approx 0.077 && \text{Law of Plugging in}
\end{aligned}$$

4 Extra Practice: Axioms of Probability

Decide whether each of the three statements below is true or false:

a. $P(A) + P(A^C) = 1$. Recall that A^C means A "complement" or "not" A

b. $P(A \cap B) + P(A \cap B^C) = 1$. Recall that $\cap$ means "and"

c. If $P(A) = 0.4$ and $P(B) = 0.6$ then it must be the case that $A = B^C$

T F F

The first one is simply saying that an event either falls inside of an event space A or outside of it. The second is false, as the left hand side, in general, is $P(A)$, which isn't guaranteed to be 1. And the fact that $P(A)$ and $P(B^C)$ are each 0.4 doesn't require the event spaces to be the same.