CS109 Midterm Solutions

Winter 2022

1a

$$F \rightarrow \text{R.V. for first person choice (1 through 5)}$$
$$S \rightarrow \text{R.V. for second person choice (1 through 5)}$$

$$P(match) = P(S=1, F=1) + P(S=2, F=2) + P(S=3, F=3) + P(S=4, F=4) + P(S=5, F=5)$$

$$P(match) = P(S=1)P(F=1) + P(S=2)P(F=2) + P(S=3)P(F=3)$$

$$+P(S=4)P(F=4) + P(S=5)P(F=5)$$

$$\left(\frac{1}{5}\right)^2 * 5$$

$$\frac{1}{5}$$

1b

In this problem, the unit of time we are interested in is 5 minutes. Thus, we have a rate of flights per "unit" of time, and we can use a Poisson distribution.

Y $\sim Poi(\lambda = 5)$

$$P(Y=0) + P(Y=2)$$

Plugging in the PMF of a poisson distribution we find:

$$\frac{5^0 e^{-5}}{0!} + \frac{5^2 e^{-5}}{2!}$$

2a

$P(K) = \frac{3}{4} \rightarrow$ Probability learner knows concepts

$P(C|K^{\complement}) = \frac{1}{4} \rightarrow$ Probability CORRECT given student does NOT know concepts

$P(C^{\complement}|K) = \frac{1}{10} \rightarrow$ Probability INCORRECT given student DOES know concepts

$$P(C) = P(C|K)P(K) + P(C|K^{\complement})P(K^{\complement})$$
$$P(C) = \frac{9}{10} * \frac{3}{4} + \frac{1}{4} * \frac{1}{4}$$
$$P(C) = \frac{27}{40} + \frac{1}{16}$$

2b

$$\binom{5}{3} * 0.6^3 * 0.4^2$$

3

There are $\binom{26}{3}$ unique sets of 3 letters and thus $\binom{\binom{26}{3}}{2}$ ways to choose a pair.

We also accepted solutions that accounted for order in the sets of 3: $\binom{\binom{26}{3}*3!}{2}$.

4a

Besides C, there are 25 remaining letters, each of which is equally likely.

$$\frac{1-0.95}{25}$$

4b

We want to find k such that $P(L_1 = C, L_2 = A, L_3 = T) = k \cdot P(L_1 = M, L_2 = A, L_3 = N)$. We can do so as follows:

$$\begin{aligned}
k &= \frac{P(L_1 = C, L_2 = A, L_3 = T)}{P(L_1 = M, L_2 = A, L_3 = N)} \\
&= \frac{P(L_1 = C) \cdot P(L_2 = A) \cdot P(L_3 = T)}{P(L_1 = M) \cdot P(L_2 = A) \cdot P(L_3 = N)} && \text{(Since each letter is independent)} \\
&= \frac{0.95 \cdot 0.4 \cdot 0.3}{0.002 \cdot 0.4 \cdot 0.6} \\
&= 237.5
\end{aligned}$$

We see that the letters "CAT" are 237.5 times more likely than "MAN".

5a

$X \sim Bin(5, 0.6)$

$$P(X = 3)$$

$$\binom{5}{3} 0.6^3 0.4^2$$

5b

$S \rightarrow$ first player has stone

$T \rightarrow$ first player says "Three"

$$P(S|T) = \frac{P(T|S)P(S)}{P(T|S)P(S) + P(T|S^{\complement})P(S^{\complement})}$$

6a.

From the problem we know that money won by the correct algorithm (Y) is distributed as: $Y \sim \mathcal{N}(16500, 800^2)$. We are looking for $P(Y \geq 15000)$.

$$
\begin{aligned}
&P(Y \geq 15000) \\
&= P\left(\frac{Y - 16500}{800} \geq \frac{15000 - 16500}{800}\right) \\
&= P\left(Z \geq \frac{15000 - 16500}{800}\right) \\
&= P\left(Z \geq \frac{-1500}{800}\right) \\
&= P\left(Z \geq \frac{-15}{8}\right) \\
&= 1 - P\left(Z < \frac{-15}{8}\right) \\
&= 1 - \Phi\left(\frac{-15}{8}\right)
\end{aligned}
$$

Note: given that money is discrete, a very small continuity correction could be applied and we awarded points for both answers with and without a continuity correction.

6b

$$X \sim \mathcal{N}(16000, 750^2) \qquad Y \sim \mathcal{N}(16500, 800^2)$$

$$\frac{f_X(15900) * 0.2}{f_X(15900) * 0.2 + f_Y(15900) * 0.8}$$

7

```
def probability_sum(p_list, k):
    n = len(p_list)
    assert n == 5 and k == 3
    res = 0
    for perm in unique_permutations([1,1,1,0,0]):
        pr = 1
        for i in range(len(perm)):
            pr *= p_list[i] if perm[i] == 1 else 1 - p_list[i]
        res += pr
    return res
```

Alternative:

```
def probability_sum(p_list, k):
    n = len(p_list)
    assert n == 5 and k == 3
    res = 0
    for p_perm in unique_permutations(p_list):
        res += p_perm[0] * p_perm[1] * p_perm[2] * (1 - p_perm[3]) * (1 - p_perm[4])
    return res / (3!*2!)
```