Algorithmic Analysis

CS109, Stanford University

Learning Goals

1. Be able to compute conditional Expectation
2. Solve problems using the Law of Total Expectation
3. Use Conditional Expectation for Algorithmic Analysis

PS5

Algorithmic Analysis: Dice Rolls

Consider the following function, which simulates repea
(where each integer value from 1 to 6 is equally likely t
is "rolled".

```
def roll():
  total = 0
  while True:
    # equally likely to return 1,...,6
    roll = random_integer(1, 6)
    total += roll
    # exit condition
    if roll >= 3: break
  return total
```

Let X be the value returned by the function roll(). Wha

Previous Question

18

PS5

Llama Flu

Our ability to fight contagious diseases depends on our ability t
person is exposed to llama-flu (a made up disease). The metho
number of individuals who will get infected.

```
def num_infected():
  """
  Returns the number of people infected by one ind
  """

  # most people are immune to llama-flu
  immune = bernoulli(p = 0.99)
  if immune: return 0

  # people who are not immune spread the disease f
  spread = 0

  # they make contact with k people (up to 100)
  k = binomial(n = 100, p = 0.25)
  for i in range(k)
    spread += num_infected():

  # total infections should include this individua
  return spread + 1
```

What is the expected return value of `num_infected`?

Previous Question

19

PS5

Approximate Counting

What if you needed a counter that could count up to the number of atoms in the universe, but you wanted to store the counter in 8 bits? You could use the algorithm below:

```
def stochastic_counter(true_count):
  n = -1
  for i in range(true_count):
    n += count(n)
  return 2 ** n  # 2^n, aka 2 to the power of n

def count(n):
  # To return 1 you need n heads. Always returns 1 if n is <= 0
  for i in range(n):
    if not coin_flip():
      return 0
  return 1

def coin_flip():
  # returns true 50% of the time
  return random.random () < 0.5
```

Show that the expected return value of `stochastic_counter(4)`, where `count` is called 4 times, is in fact equal to 4.

Previous Question

Next Question

19

Def: Conditional Expectation

$$E[X|Y = y] = \sum_x xP(X = x|Y = y)$$

Def: Law of Total Expectation

$$E[X] = \sum_y E[X|Y = y]P(Y = y)$$

<review>

Expectation

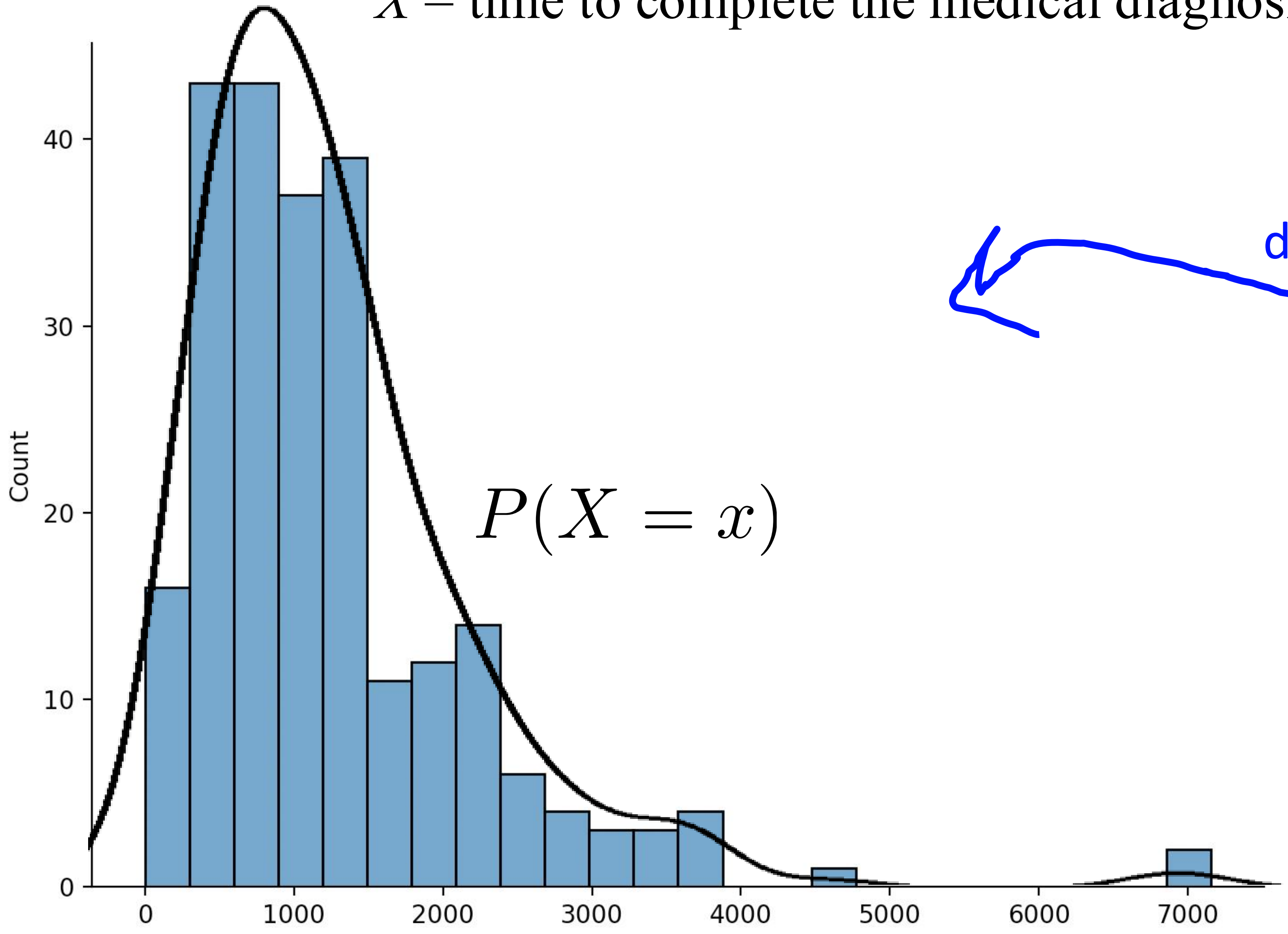
The probability that X takes on that value

$$\mathrm{E}[X] = \sum_x x \cdot P(X = x)$$

All the values that X can take on

Limitation of Expectation

X = time to complete the medical diagnosis problem (in seconds)



$P(X = x)$

Summarize the distribution in one number

$\mathrm{E}[X] = 1260$

Expectation of a Sum

$$E[X + Y] = E[X] + E[Y]$$

Generalized: $$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Holds regardless of dependency between X_i's

Expectation of a Function

Law of unconscious statistician

$$\mathrm{E}[g(X)] = \sum_x g(x) \cdot P(X = x)$$

So for example…

$$\mathrm{E}[X^2] = \sum_x x^2 \cdot P(X = x)$$

End Review

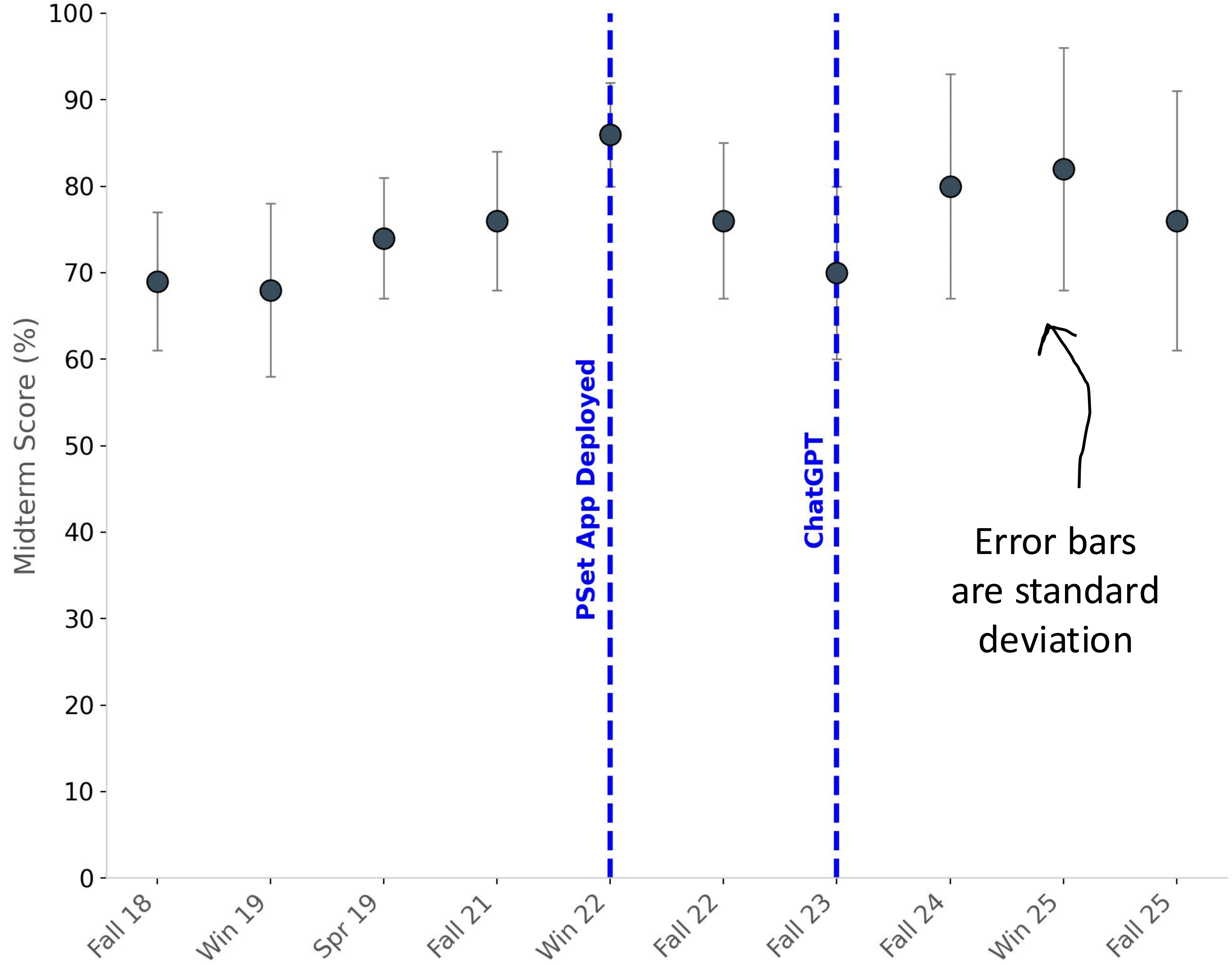
How was the midterm? Who knows until Wednesday.

We are here to make you hit heights you didn't think you could -- not to judge you.

Be easy on yourself too. But don't sell yourself short.

Midterm is a diagnostic instead of just summative assessment.

History of Midterms



Course grades are independent of midterm difficulty

Partial credit is a thing!

MAR
11TH

Four practice examples to warm us up

Bool was Cool

Let E_1, E_2, ... E_n be events with indicator RVs X_i

- If event E_i occurs, then X_i = 1, else X_i = 0
- Recall $E[X_i] = P(E_i)$

- Why?

$$E[X_i] = 0 \cdot (1 - P(E_i)) + 1 \cdot P(E_i)$$

Bernoulli aka Indicator Random Variables were studied extensively by George Boole

Boole died of being too cool

Differential Privacy

Aims to provide means to **maximize the accuracy** of probabilistic queries while minimizing the **probability** of identifying its records.

Cynthia Dwork's celebrity lookalike is Cynthia Dwork.

Differential Privacy

100 independent values X_1 … X_{100} where $X_i \sim \text{Bern}(p)$

Y_i

What is returned

```
# Maximize accuracy, while preserving privacy.
def calculateYi(Xi):
  obfuscate = random()
  if obfuscate:
    return indicator(random())
  else:
    return Xi
```

random() returns True or False with equal likelihood

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What is E[Y_i]?

$$E[Y_i] = P(Y_i = 1) = \frac{p}{2} + \frac{1}{4}$$

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```

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Let $$Z = \sum_{i=1}^{100} Y_i$$

What is E[Z]?

$$E[Z] = E\left[\sum_{i=1}^{100} Y_i\right] = \sum_{i=1}^{100} E[Y_i] = \sum_{i=1}^{100} \left(\frac{p}{2} + \frac{1}{4}\right) = 50p + 25$$

Differential Privacy

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```

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Let $Z = \sum_{i=1}^{100} Y_i \quad E[Z] = 50p + 25$ How do you estimate p?

$$\hat{p} \approx \frac{Z - 25}{50}$$

Challenge: What is the probability that our estimate is good?

Differential Privacy

Story which continues to unfold…

Generalization in Adaptive Data Analysis and Holdout Reuse*

Cynthia Dwork
Microsoft Research

Vitaly Feldman
IBM Almaden Research Center†

Moritz Hardt
Google Research

Toniann Pitassi
University of Toronto

Omer Reingold
Samsung Research America

Aaron Roth
University of Pennsylvania

Professor at Stanford

What is Amazon?

Cape Town, 2006

What is Amazon?

amazon
web services™

* ~~52%~~ 74% of Amazons Profits

**More profitable than Amazon’s North America commerce operations

https://www.fool.com/investing/2022/07/07/aws-chief-says-amazons-most-profitable-segment-is/

Computer Cluster Utilization

Computer cluster with *k* servers

- Requests independently go to server *i* with probability p_i
- Let event A_i = server *i* receives no requests
- X = # of events $A_1, A_2, \ldots A_k$ that occur
- E[X] after first *n* requests?

Since X is an expectation, can you express it as a sum?

- Let Bernoulli B_i be an **indicator** for A_i
- Since requests independent:

$$X = \sum_{i=1}^{k} B_i$$

$$P(A_i) = (1 - p_i)^n$$

$$E[X] = E\left[\sum_{I=1}^{k} B_i\right] = \sum_{i=1}^{k} E[B_i] = \sum_{i=1}^{k} P(A_i) = \sum_{i=1}^{k} (1 - p_i)^n$$

When stuck, brainstorm
about random variables

Toy Collection

You are trying to collect *n* distinct toys

- Each purchase, each toy is equally likely
- Let X = # purchases until you have ≥ 1 of each toy. What is E[X]?

Toy Collection

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- Each purchase, each toy is equally likely
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Let X_i = # of **trials to get success *after i-th success*** where "success" is getting an unseen toy.

$$X = X_0 + X_1 + \ldots + X_{n-1} \Rightarrow \mathrm{E}[X] = \mathrm{E}[X_0] + \mathrm{E}[X_1] + \ldots + \mathrm{E}[X_{n-1}]$$

After i successes, the probability of the next success is $p = (n - i) / n$

$$X_i \sim \mathrm{Geo}(\, p = (n - i) / n)$$

$$\mathrm{E}[X_i] = 1 / p = n / (n - i)$$

$$E[X] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} = n\left[\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right] = O(n \log n)$$

Conditional Expectation

Conditional Expectation

X and Y are discrete random variables:

$$E[X|Y=y] = \sum_x xP(X=x|Y=y)$$

Analogously, continuous random variables:

$$E[X|Y=y] = \int_x xP(X=x|Y=y)$$

Conditional Expectation

$$\mathrm{E}[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$$

Roll two 6-sided dice D_1 and D_2

- X = value of $D_1 + D_2$ Y = value of D_2
- What is E[X | Y = 6]?

$$E[X \mid Y=6] = \sum_x x P(X=x \mid Y=6)$$

$$= \left(\frac{1}{6}\right)(7+8+9+10+11+12) = \frac{57}{6} = 9.5$$

- Intuitively makes sense: 6 + E[value of D_1] = 6 + 3.5

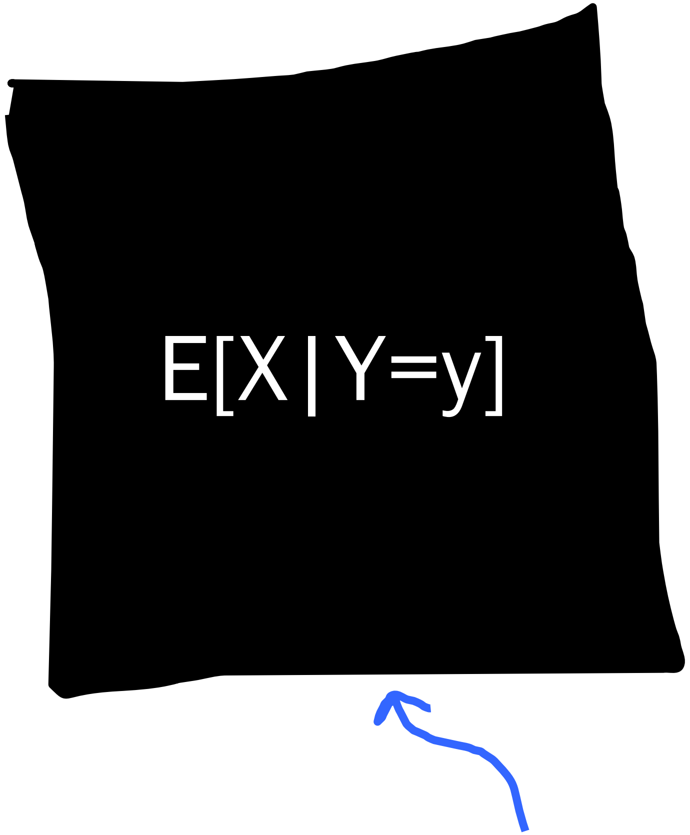
Conditional Expectation as a Function

Conditional Expectation

$$E[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$$

Define $g(Y) = E[X \mid Y]$

This is a function of Y



This is a function with Y as input

Conditional Expectation

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Define $g(Y) = E[X \mid Y]$

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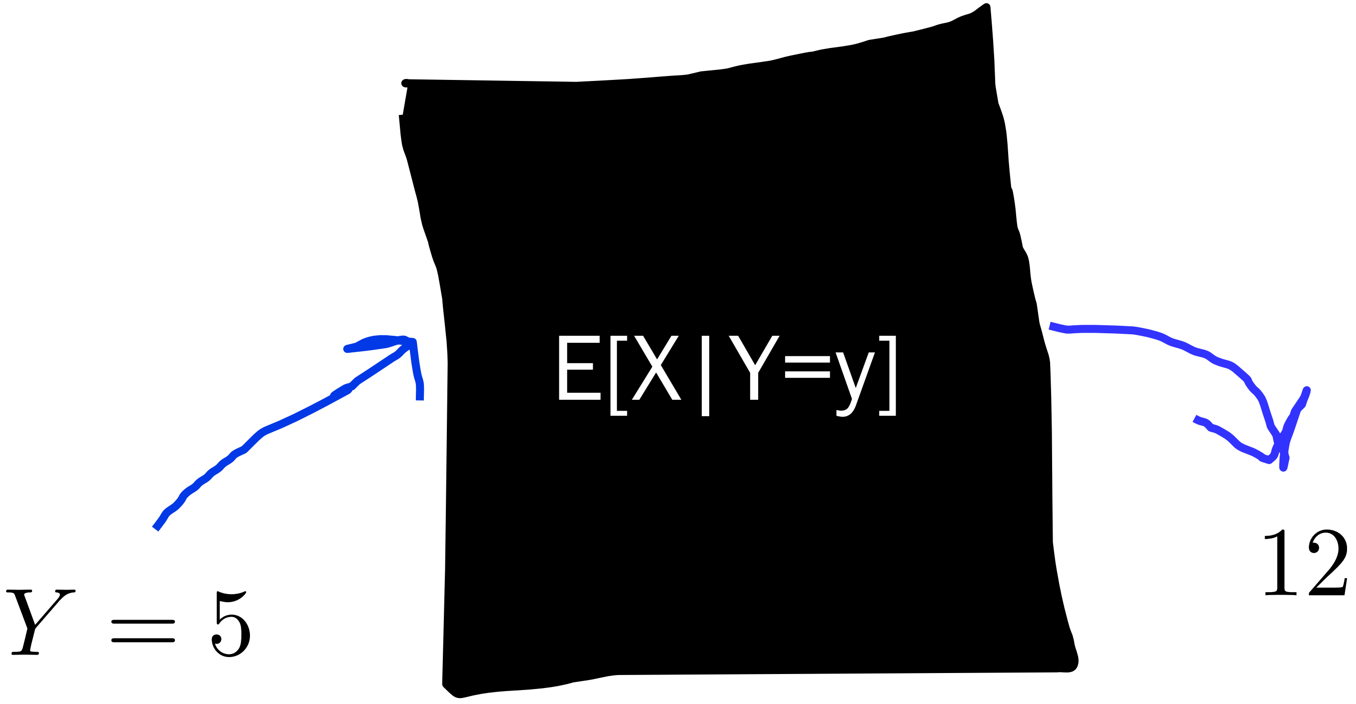

Conditional Expectation

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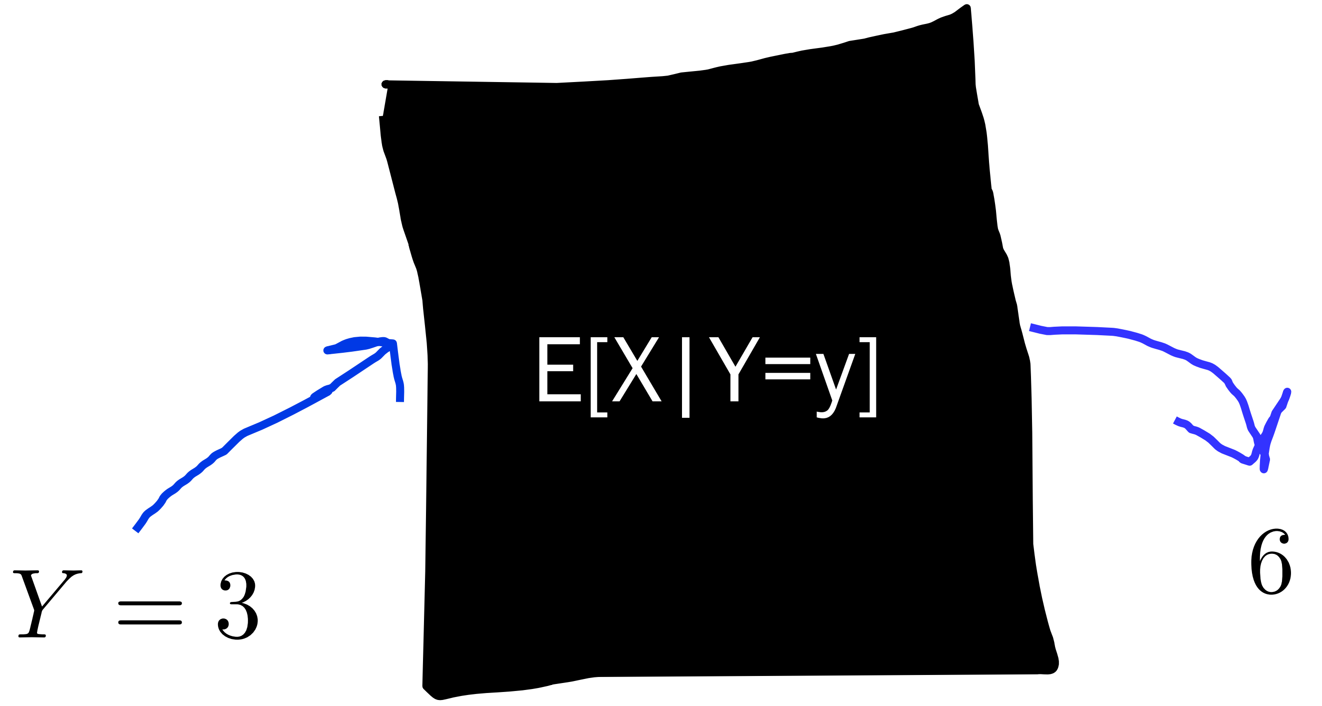


Conditional Expectation

$$\mathrm{E}[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$$

This is a number:

$$E[X]$$

This is a function of y:

$$E[X|Y=y]$$

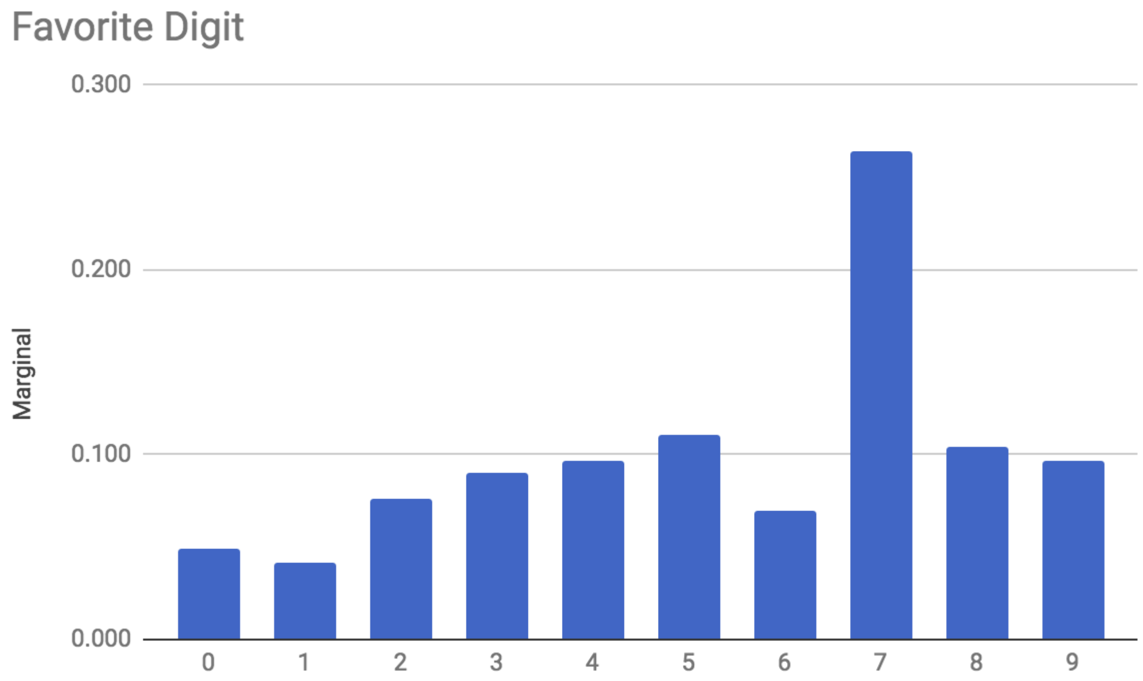
$$E[X=5]$$

Doesn't make sense. Take expectation of random variables, not events

Expectation

$$E[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$$

X = favorite number
Y = year in school



E[X] = 0 * 0.05 + … + 9 * 0.10 = 5.38

Conditional Expectation

$$E[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$$

X = favorite number
Y = year in school

$$E[\ X \mid Y\ ]\ ?$$

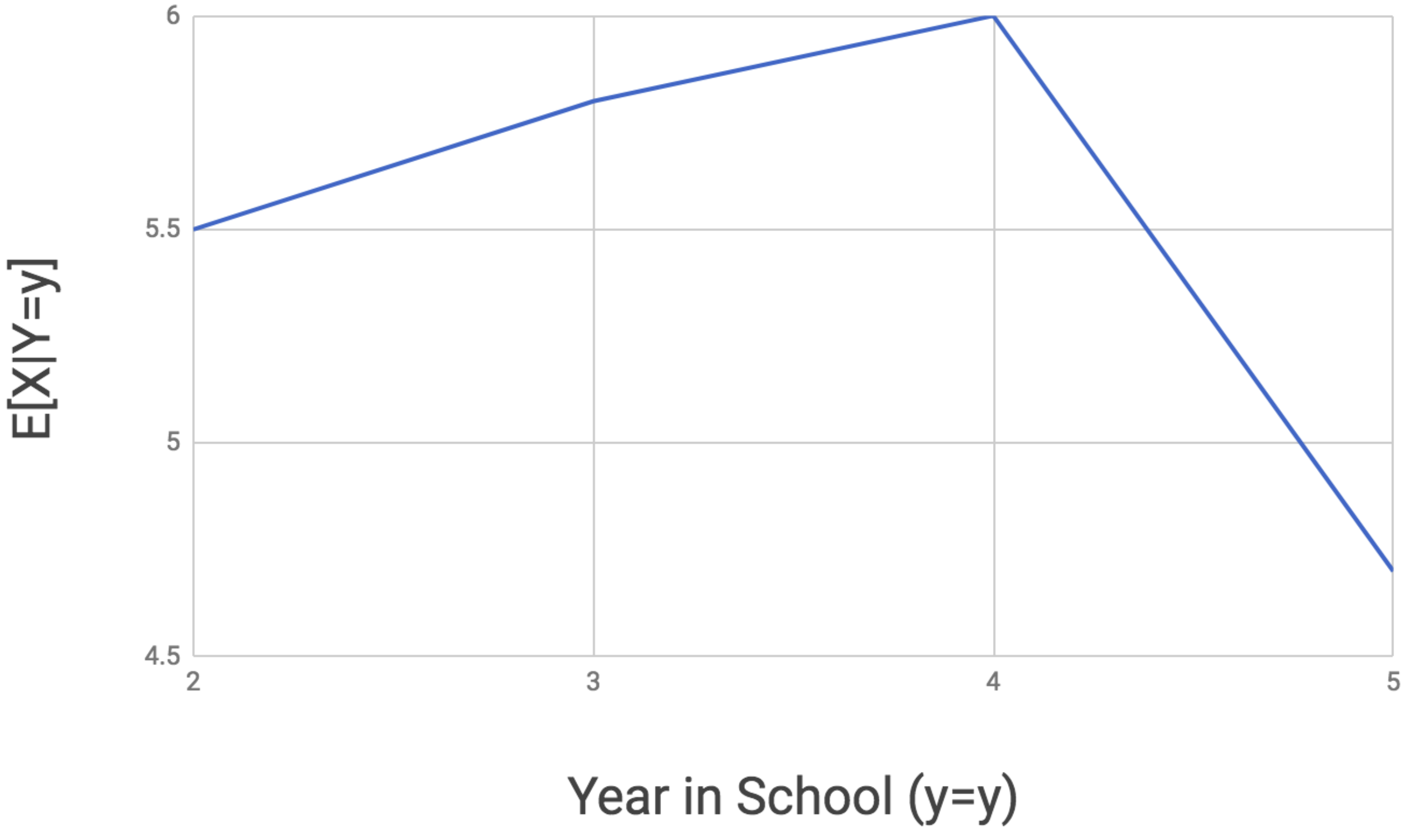
Year in school, Y = y	E[X | Y = y]
2	5.5
3	5.8
4	6.0
5+	4.7

Conditional Expectation

$$E[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$$

X = favorite number
Y = year in school

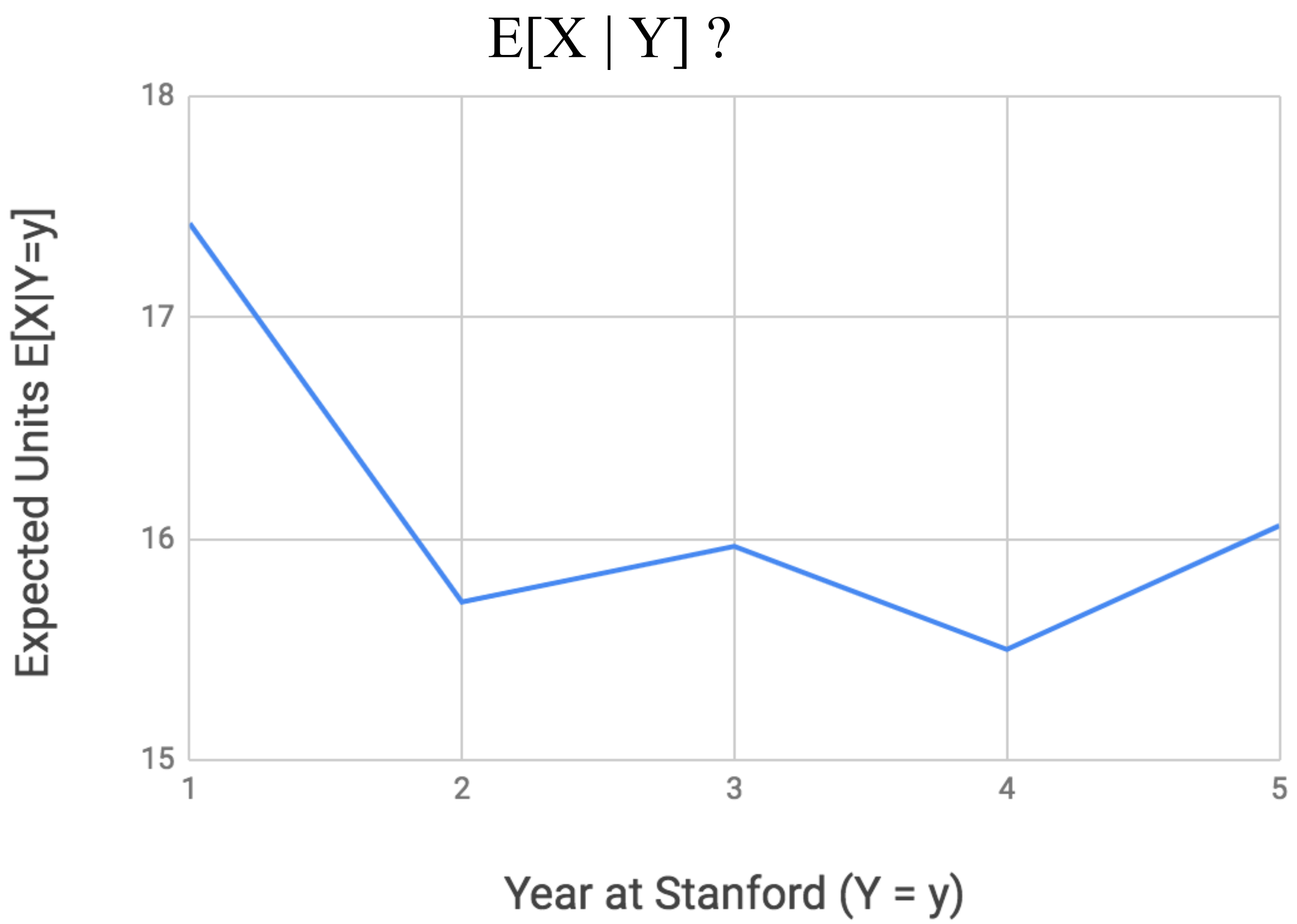
E[X | Y] ?



Conditional Expectation

$$E[X|Y=y] = \sum_x x \cdot P(X=x|Y=y)$$

X = units in fall quarter
Y = year in school



Want to see something cool?

What the heck does this give you?

$$\sum_y E[X|Y = y]P(Y = y)$$

Law of Total Expectation

$$E[X] = \sum_y E[X|Y=y]P(Y=y)?$$

$$\sum_y E[X|Y=y]P(Y=y) = \sum_y \sum_x xP(X=x|Y=y)P(Y=y)$$

Def of E[X|Y]

Law of Total Expectation

$$E[X] = \sum_y E[X|Y=y]P(Y=y)?$$

$$\sum_y E[X|Y=y]P(Y=y) = \sum_y \sum_x xP(X=x|Y=y)P(Y=y)$$

Def of E[X|Y]

$$= \sum_y \sum_x xP(X=x, Y=y)$$

Chain rule!

Law of Total Expectation

$$E[X] = \sum_y E[X|Y = y]P(Y = y)?$$

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I switch the order of the sums

Law of Total Expectation

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Chain rule!

$$= \sum_x \sum_y xP(X=x, Y=y)$$

I switch the order of the sums

$$= \sum_x x \sum_y P(X=x, Y=y)$$

Move that x outside the y sum

Law of Total Expectation

$$E[X] = \sum_y E[X|Y=y]P(Y=y)?$$

$$\sum_y E[X|Y=y]P(Y=y) = \sum_y \sum_x xP(X=x|Y=y)P(Y=y)$$ Def of E[X|Y]

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$$= \sum_x x \sum_y P(X=x, Y=y)$$ Move that x outside the y sum

$$= \sum_x xP(X=x)$$ Marginalization

Law of Total Expectation

$$E[X] = \sum_y E[X|Y=y]P(Y=y)?$$

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$$= \sum_x xP(X=x)$$ Marginalization

$$= E[X]$$ Def of E[X]

Law of Total Expectation

Law of Total Expectation

For any discrete random variable X
and any discrete random variable Y

$$E[X] = \sum_y E[X|Y = y]P(Y = y)$$

Recall the Law of Total Probability

$$P(X = x) = \sum_y P(X = x|Y = y)P(Y = y)$$

NETFLIX

(The Streaming Part)

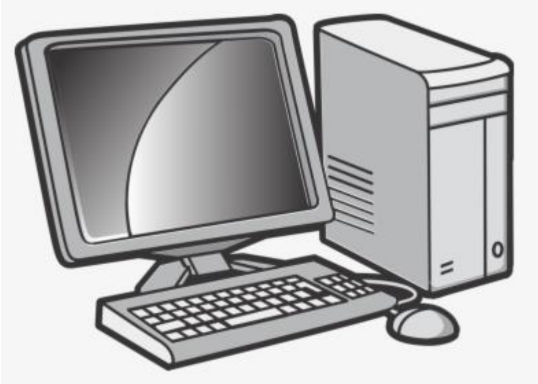
How long does this code take to run?

Netflix streams millions of hours of videos per day. They REALLY care about the speed of the following code:

```
database.get_movie(movie_name)
```

How long does this line of code take? Say 512 MB movie.

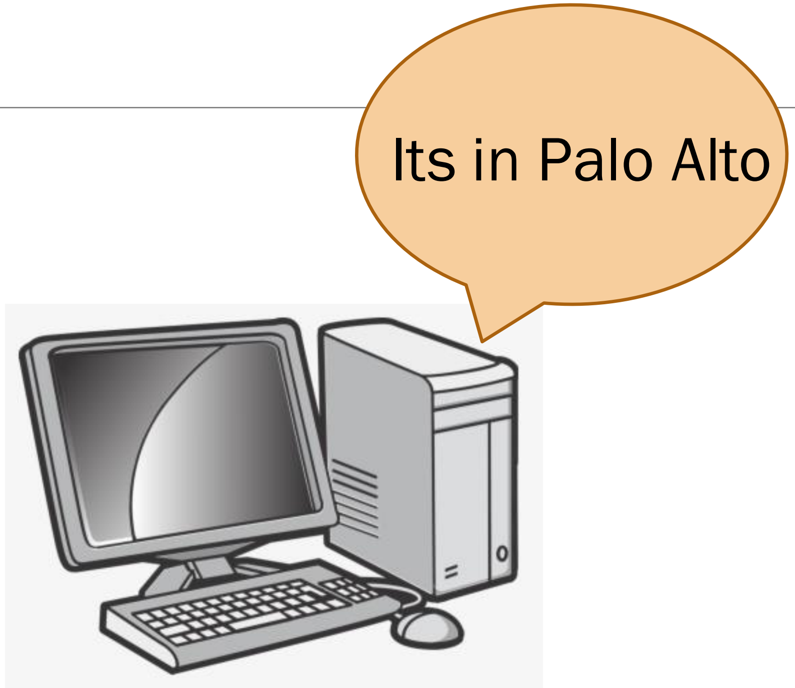
How long does this code take to run?



database.get_movie(movie_name)

1. 0.3s
2. 1.6 mins
3. 5 mins
4. 2 hours

Millisecond Latency



database.get_movie(movie_name)

1. 0.3s
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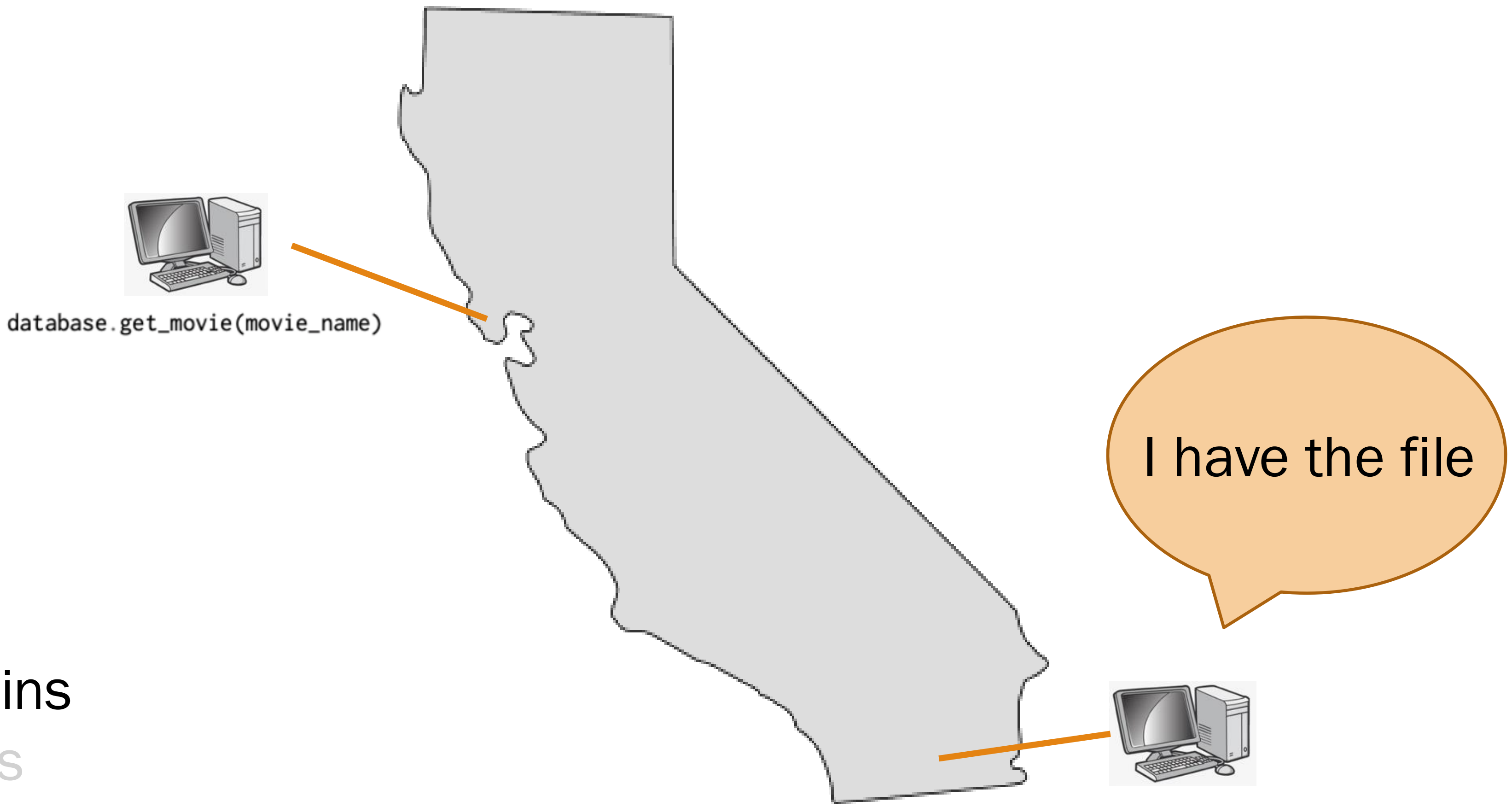
Minute Latency



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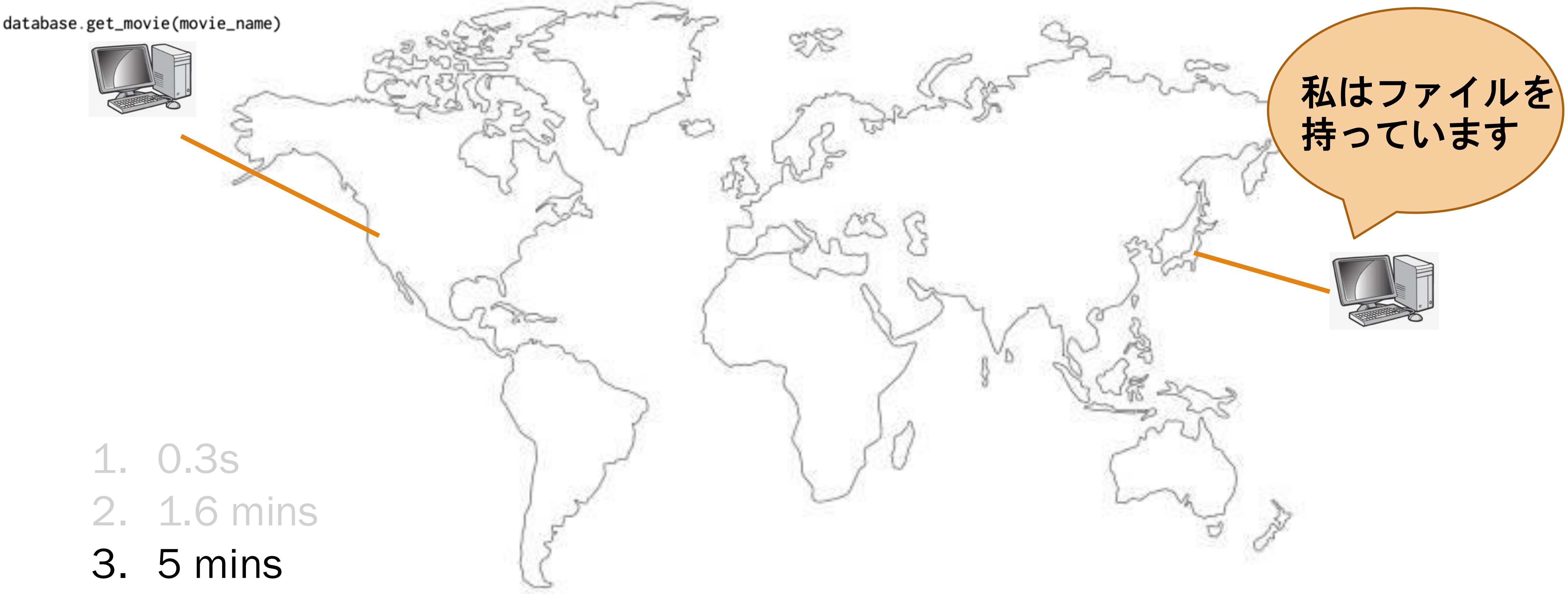
4. 2 hours

Many Minutes Latency



1. 0.3s
2. 1.6 mins
3. 5 mins
4. 2 hours

Are we done?

```
database.get_movie(movie_name)
```

5mins across the world!!!!!!

2 hours

```
database.get_movie(movie_name)
```

Expected Run Time

Expected runtime

Location, *l*	E[Runtime | Location = *l*]	P(Location = *l*)
Palo Alto	0.3s	0.10
SoCal	1.6s	0.50
Japan	300.0s	0.37
Space	7200.0s	0.03

Let Location (L for short) be the location where the movie file is stored

What is the expected runtime of `database.get_movie(movie_name)`

Using the Law of Total Expectation

$$
\begin{aligned}
E[\text{Runtime}] = & E[\text{Runtime}|L = \text{Palo Alto}] \cdot P(L = \text{Palo Alto}) \\
& + E[\text{Runtime}|L = \text{SoCal}] \cdot P(L = \text{SoCal}) \\
& + E[\text{Runtime}|L = \text{Japan}] \cdot P(L = \text{Japan}) \\
& + E[\text{Runtime}|L = \text{Space}] \cdot P(L = \text{Space}) = 327s
\end{aligned}
$$

Analyze Recursive Code

Analyzing Recursive Code

```
int Recurse() {
    int x = randomInt(1, 3);  // Equally likely values

    if (x == 1) return 3;
    else if (x == 2) return (5 + Recurse());
    else return (7 + Recurse());
}
```

Let Y = value returned by **`Recurse()`**. **What is E[Y]?**

Analyzing Recursive Code

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$$E[Y] = E[Y \mid X = 1]P(X = 1) + E[Y \mid X = 2]P(X = 2) + E[Y \mid X = 3]P(X = 3)$$

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$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

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$E[Y \mid X = 1] = 3$, $P(X = 1) = \frac{1}{3}$, $P(X = 2) = \frac{1}{3}$, $P(X = 3) = \frac{1}{3}$

Analyzing Recursive Code

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$E[Y \mid X = 1] = 3$, $P(X = 1) = \frac{1}{3}$, $E[Y \mid X = 2] = (5 + E[Y])$, $P(X = 2) = \frac{1}{3}$, $P(X = 3) = \frac{1}{3}$

Analyzing Recursive Code

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$$E[Y] = E[Y \mid X=1]P(X=1) + E[Y \mid X=2]P(X=2) + E[Y \mid X=3]P(X=3)$$

3 $\quad \frac{1}{3}$ $\quad (5 + E[Y])$ $\quad \frac{1}{3}$ $\quad (7 + E[Y])$ $\quad \frac{1}{3}$

Analyzing Recursive Code

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$$= 3 \cdot \frac{1}{3} + (5 + E[Y]) \cdot \frac{1}{3} + (7 + E[Y]) \cdot \frac{1}{3}$$

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$$= 3 \cdot \frac{1}{3} + (5 + E[Y]) \cdot \frac{1}{3} + (7 + E[Y]) \cdot \frac{1}{3}$$

$$= \frac{15}{3} + \frac{2}{3}E[Y]$$

$$E[Y] = 15$$

Algorithmic Analysis: Dice Rolls

Consider the following function, which simulates repeatedly rolling a 6-sided die (where each integer value from 1 to 6 is equally likely to be "rolled") until a value ≥ 3 is "rolled".

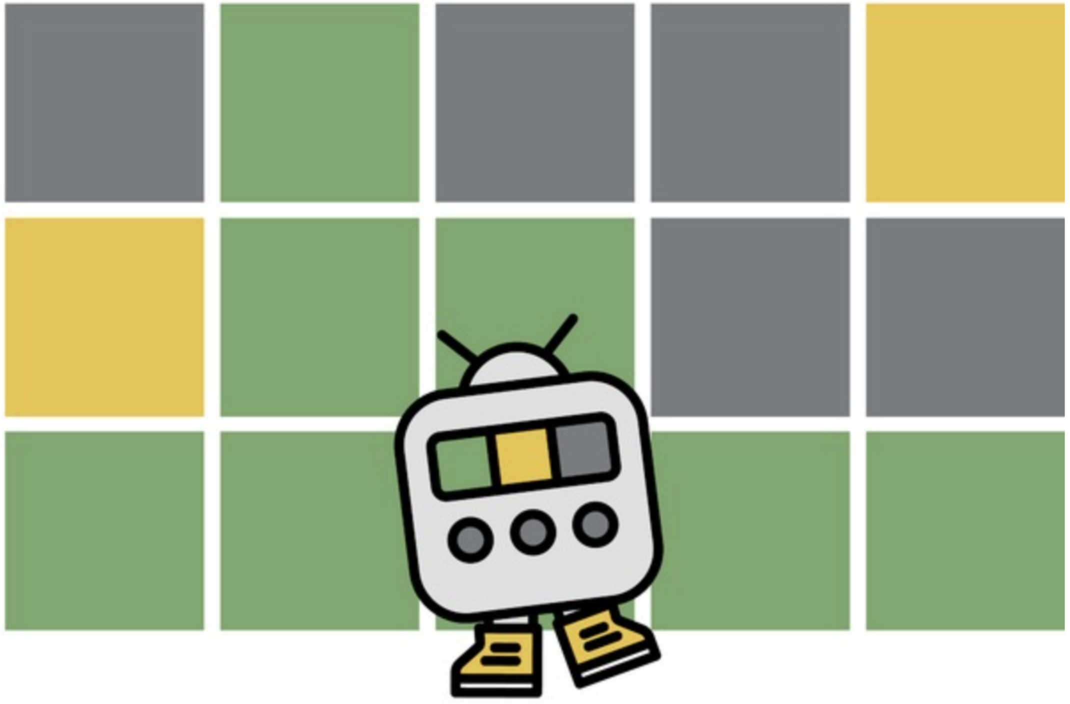
```
def roll():
  total = 0
  while True:
    # equally likely to return 1,...,6
    roll = random_integer(1, 6)
    total += roll
    # exit condition
    if roll >= 3: break
  return total
```

Let X be the value returned by the function roll(). What is $\mathrm{E}[X]$?

Information Theory

Wordle

The goal of the game Wordle is to guess a five-letter secret word in as few tries as possible. Let's say we are playing a game of Wordle and we have narrowed down the possible secret words to these seven: bring, girls, storm, tears, rates, grind, reign. Which word should we guess next?



We are almost ready to learn information theory where we can make decisions under uncertainty!

This will be a really fun time if you have a strong understanding of expectation and conditional expectation.

Uncertainty Theory

Beta Distributions	Thompson Sampling	Adding Random Vars	Central Limit Theorem
Sampling	Bootstrapping	Algorithmic Analysis	Information Theory

As requested by CS faculty

No lecture on Monday due to holiday

Have a good weekend ☺