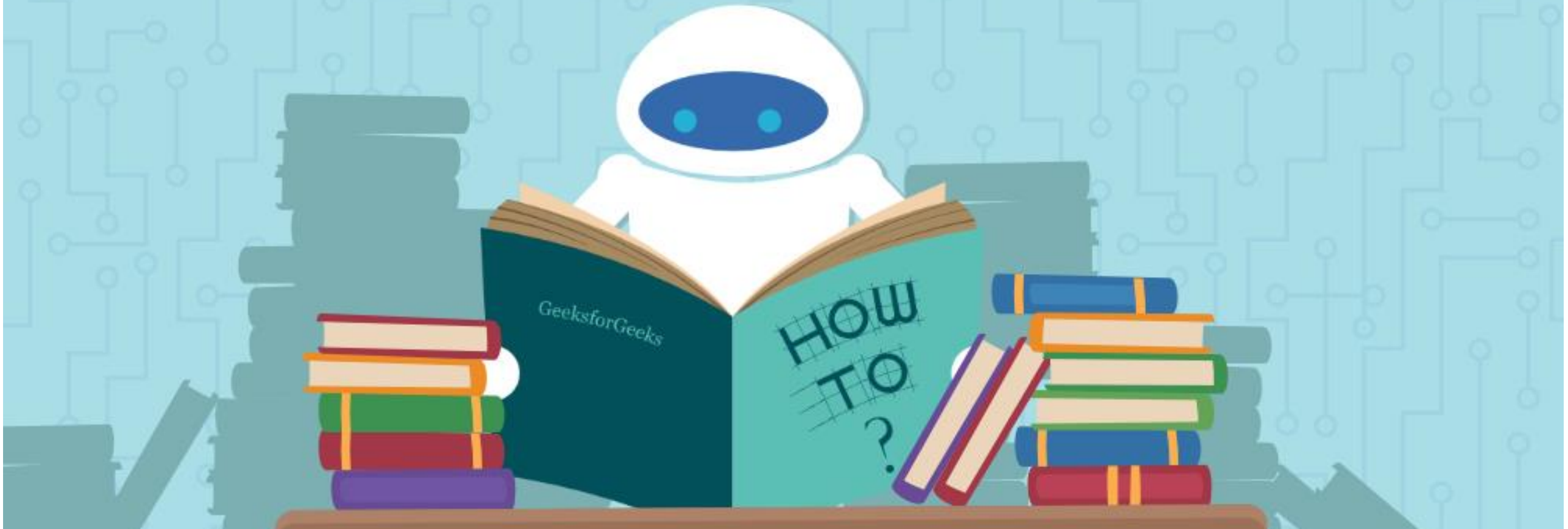
Parameter Estimation

Juliette Woodrow
CS109, Stanford University

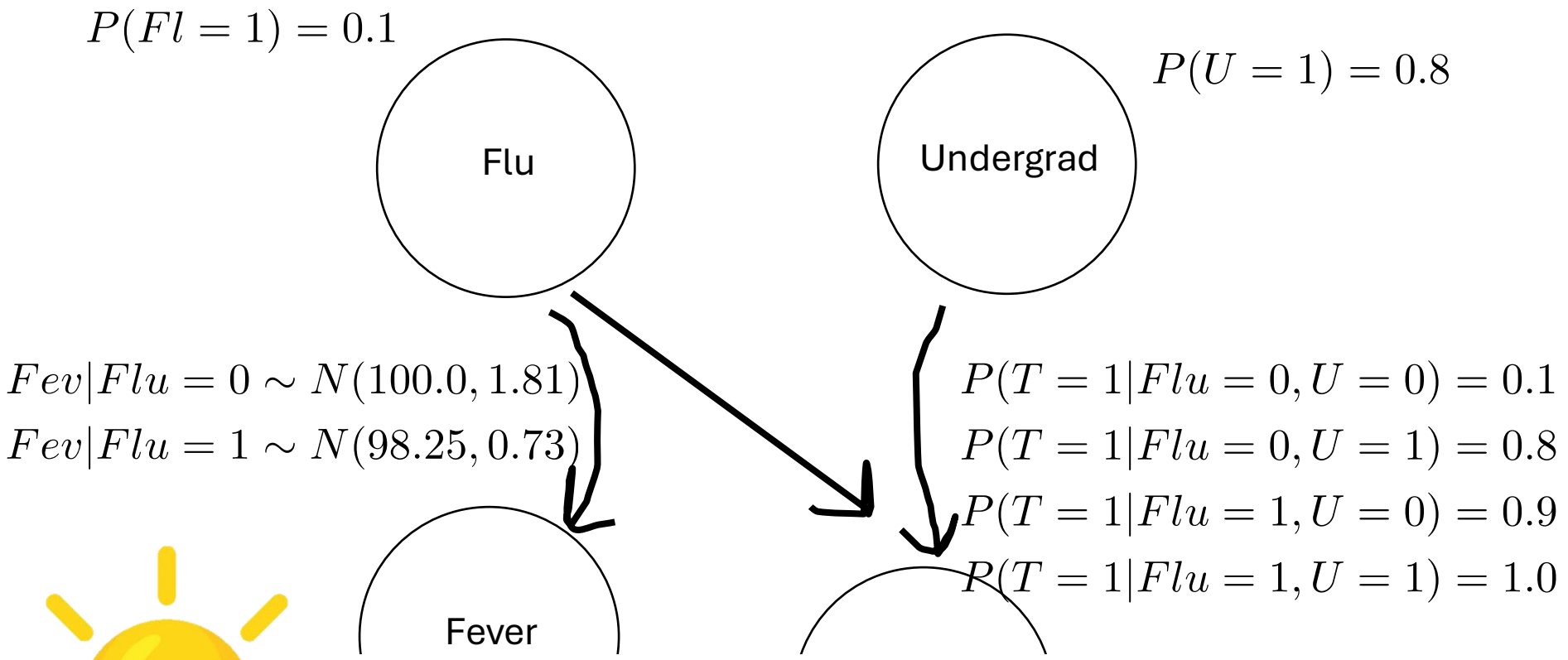
Where are we in CS109?

MACHINE LEARNING



General “Inference”

Probabilistic Model



If you know the probability of each random variables given the ones that directly cause it, you can joint sample!

But where do those numbers come from?

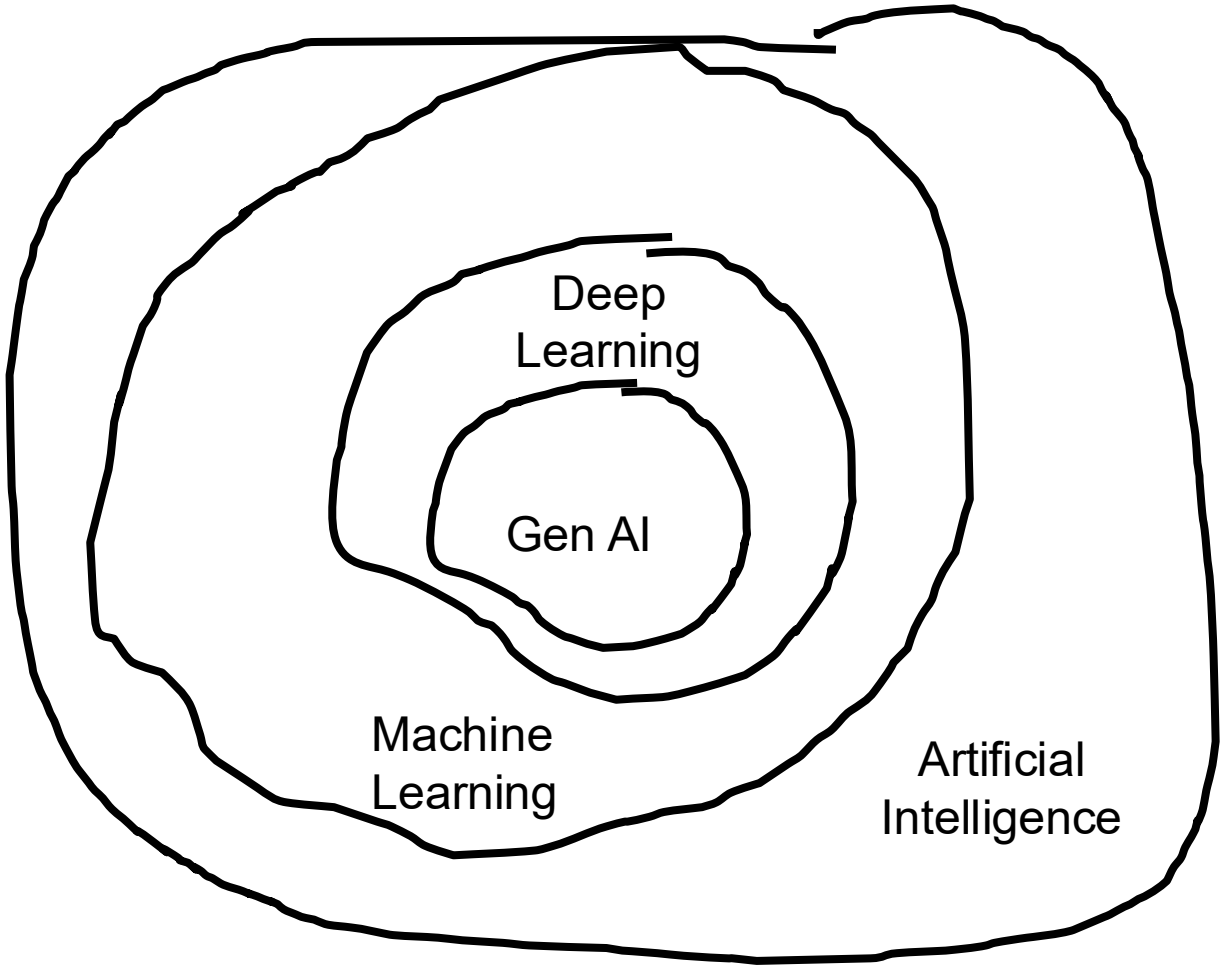
Suspense

At this point, if you are given a *model*, with all the involved probabilities, you can make predictions

But what if you want to *learn* the probabilities in the model?

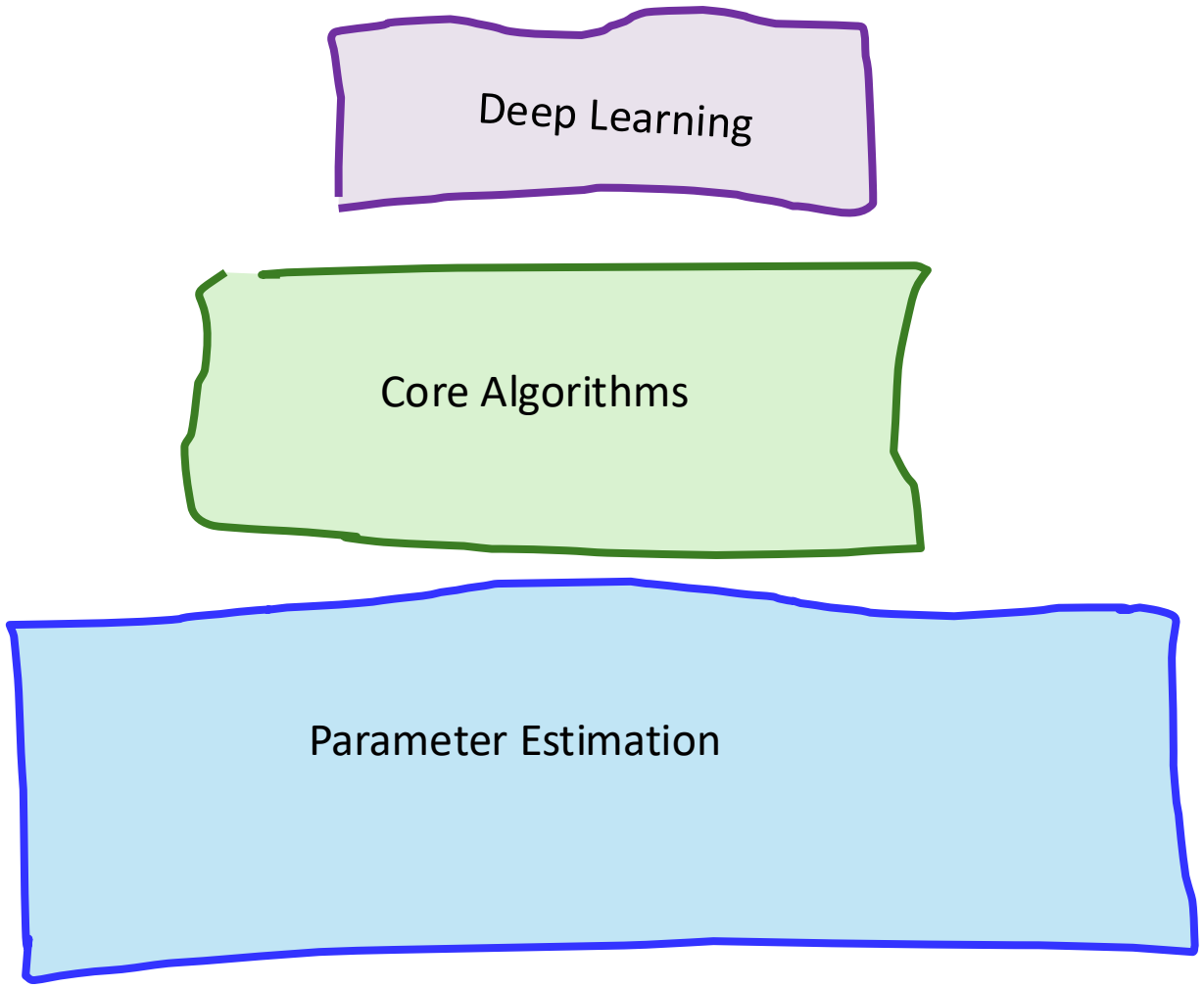
Machine Learning

AI and Machine Learning

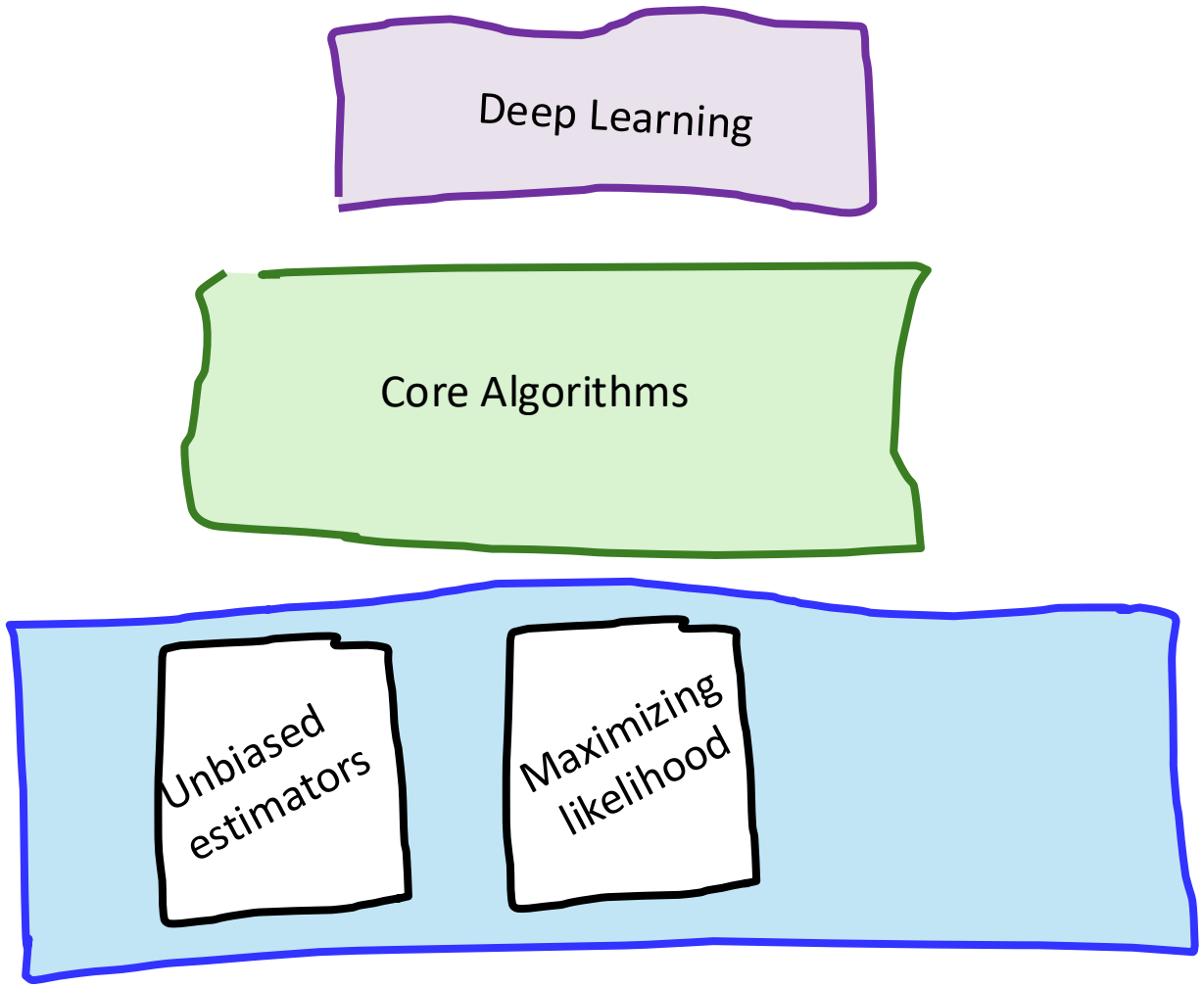


ML: Rooted in probability theory

Our Path



Our Path


Deep Learning
Core Algorithms
Unbiased estimators
Maximizing likelihood


WALT DISNEY
PICTURES

Review

Shorthand for Equality Events

Our shorthand notation

$$f(x|\theta)$$

x Is shorthand for the event $X = x$

θ Is shorthand for the event $\Theta = \theta$

Full Notation

$$f(X = x|\Theta = \theta)$$

End Review

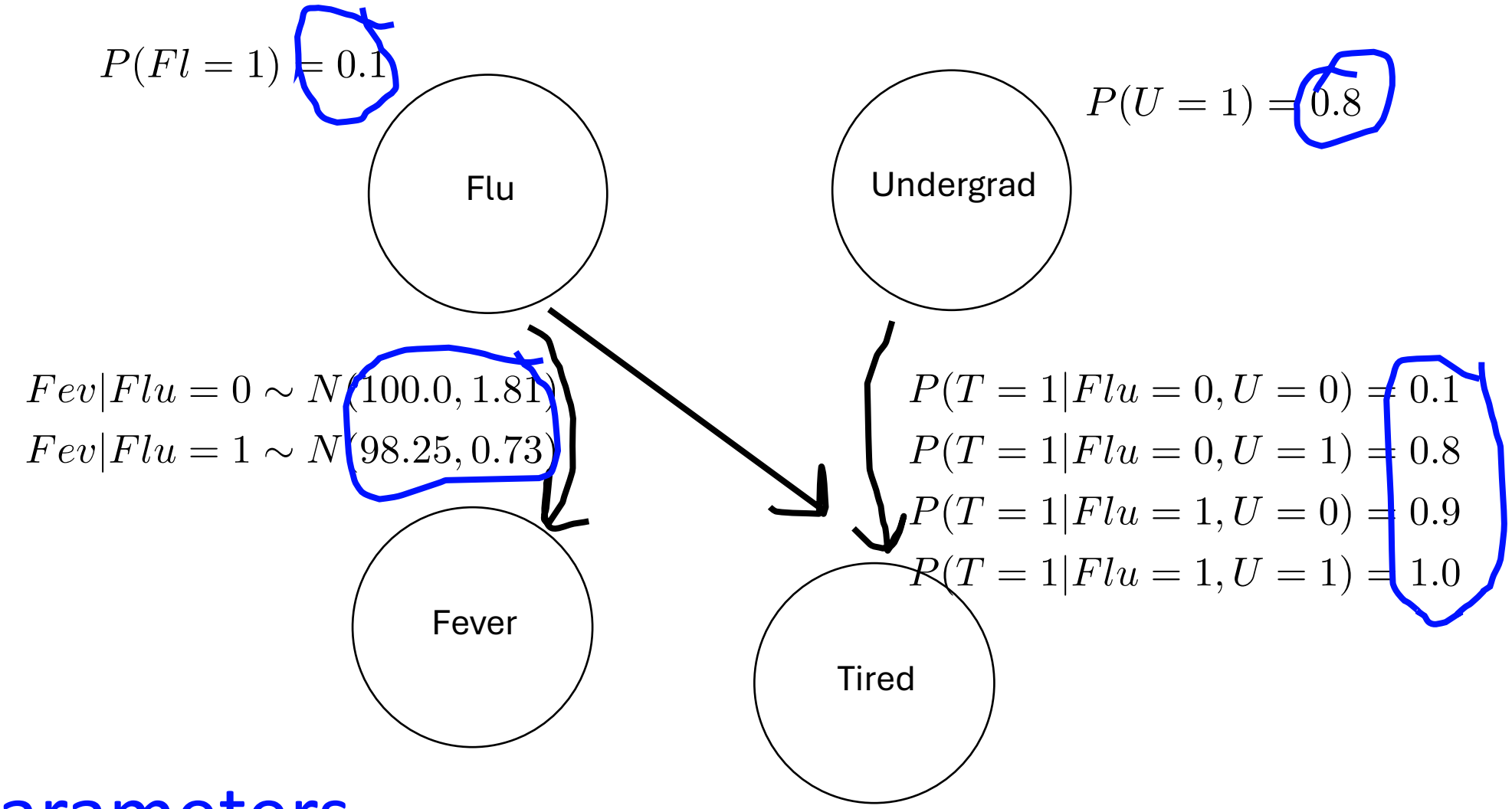
Once upon a time…

…there was parameter estimation

What are Parameters?

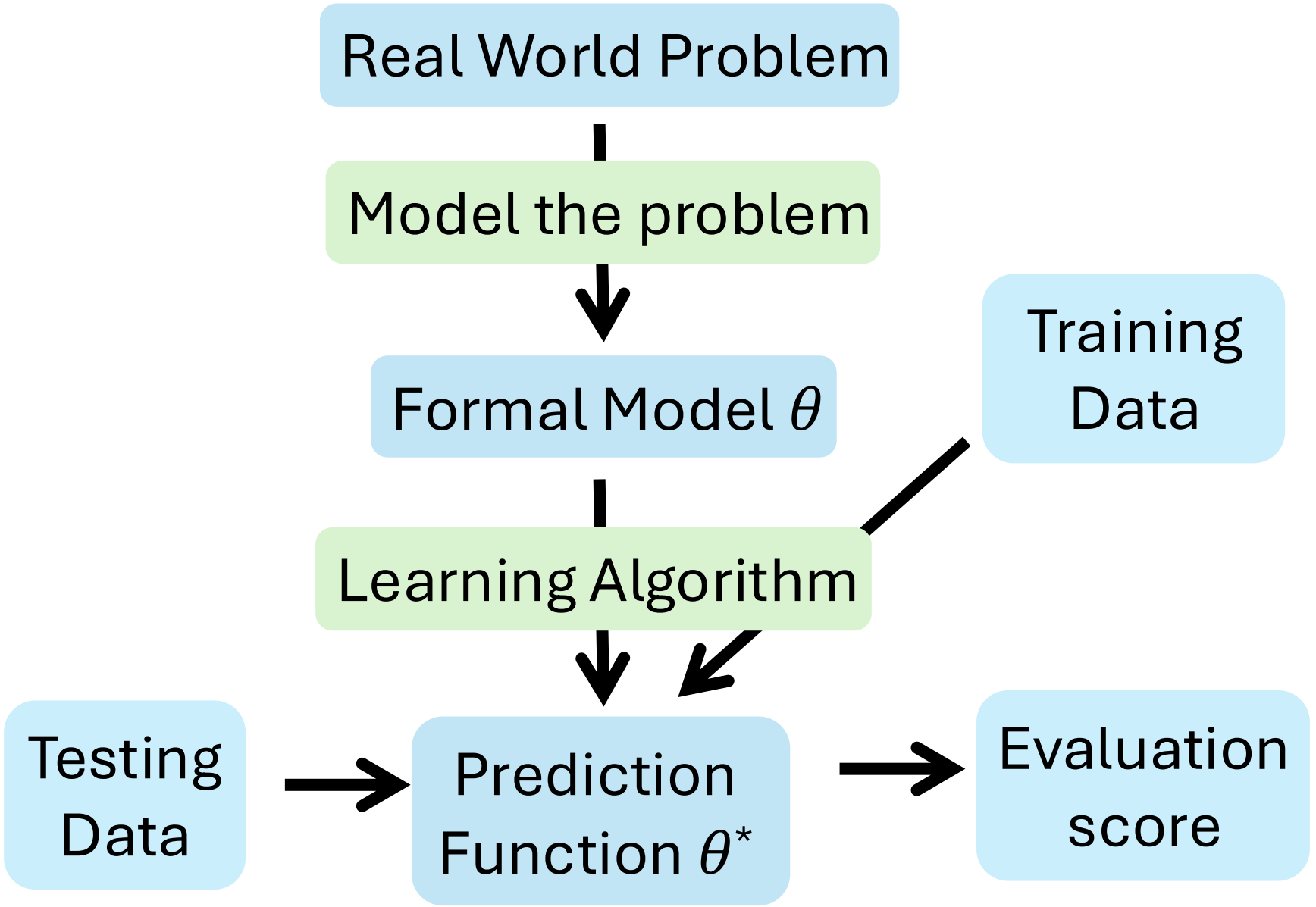
- Consider some probability distributions:
  - Ber(p) — $\theta = p$
  - Poi(λ) — $\theta = \lambda$
  - Uni(α, β) — $\theta = (\alpha, \beta)$
  - Normal(μ, σ^2) — $\theta = (\mu, \sigma^2)$
  - $Y = mX + b$ — $\theta = (m, b)$
  - etc…
- Call these “parametric models”
- Given model, **parameters** yield actual distribution
  - Usually refer to parameters of distribution as θ
  - Note that θ that can be a vector of parameters

What are Parameters?

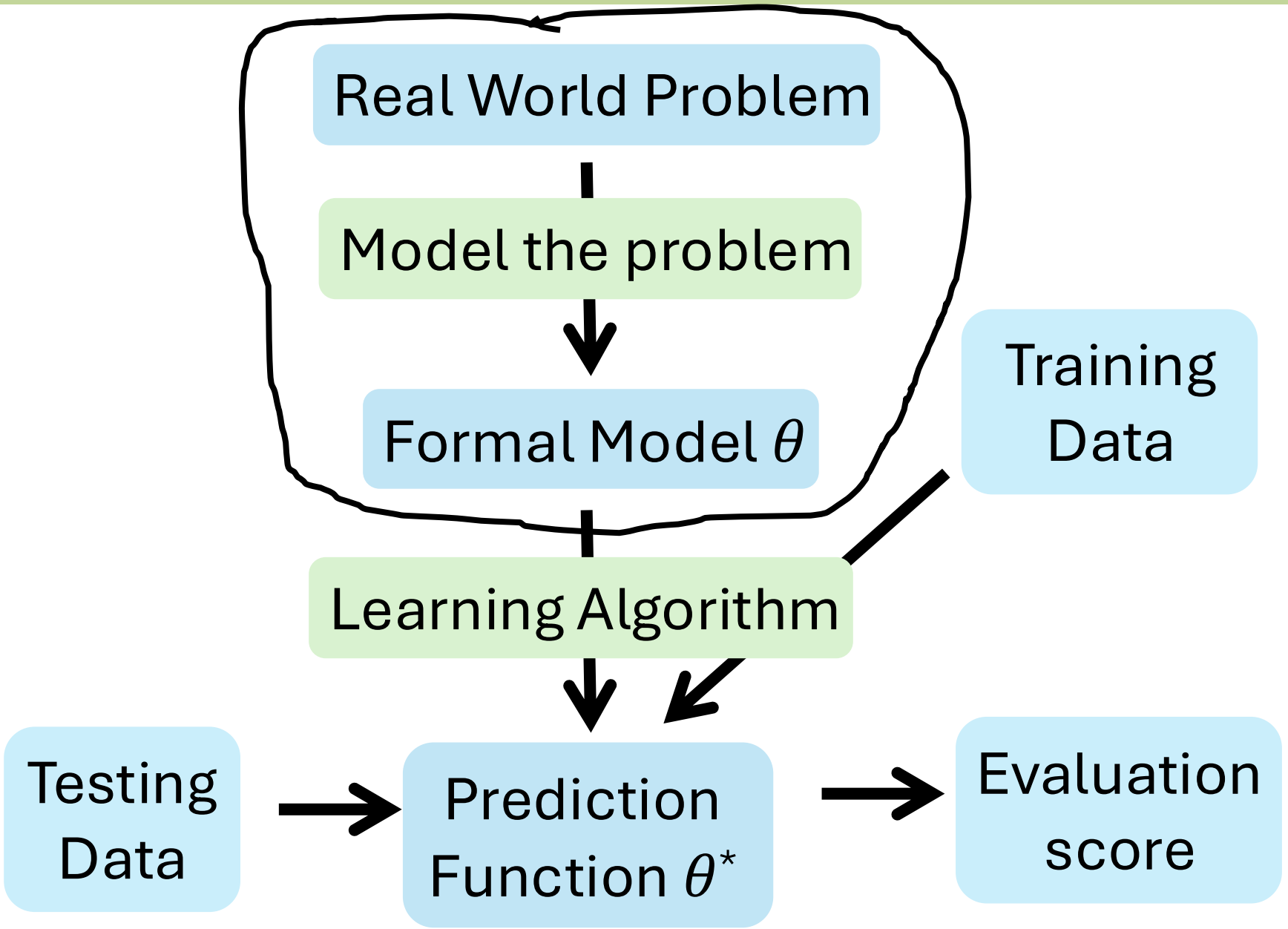


Parameters

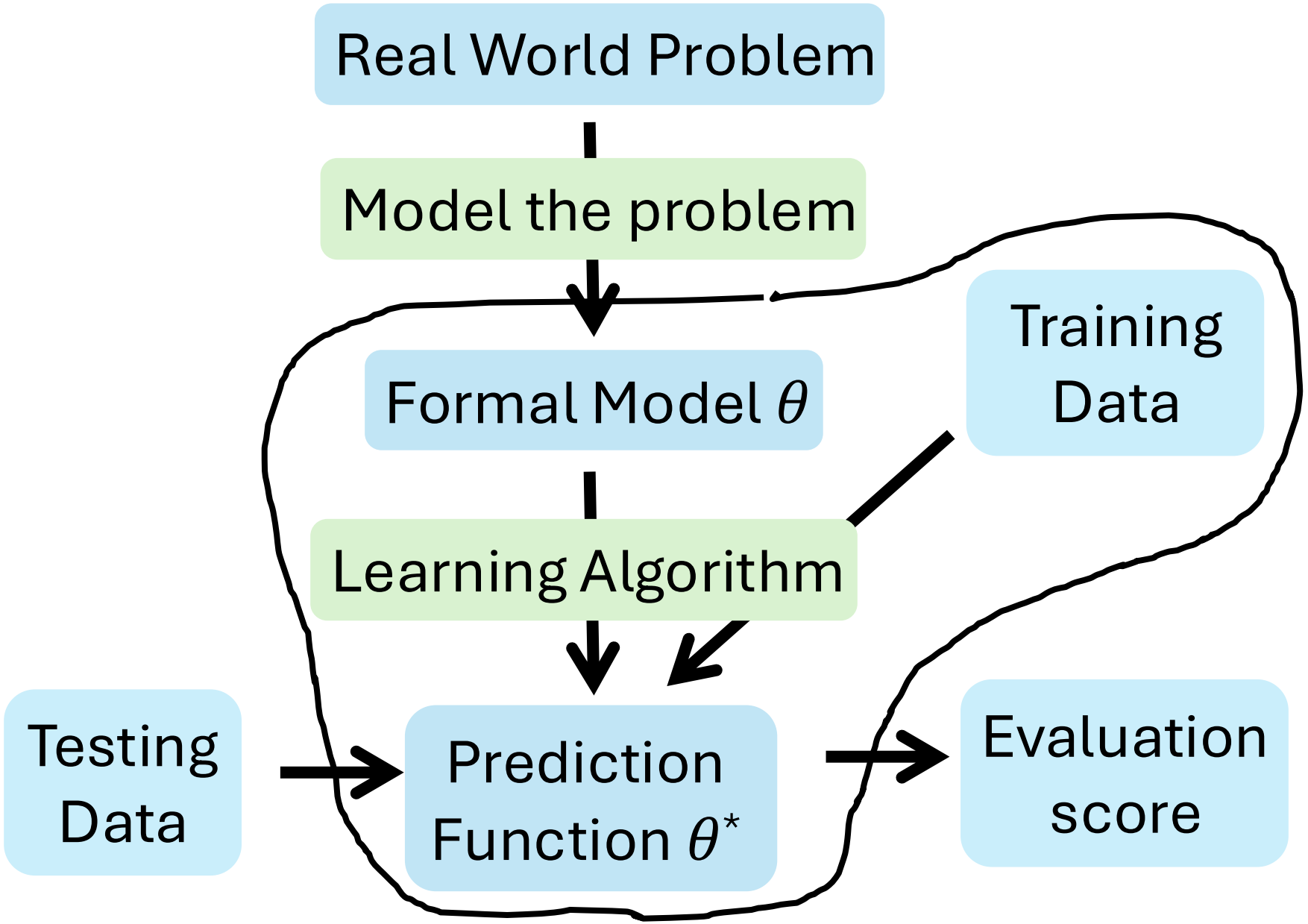
Why Do We Care?



Modelling


Real World Problem
Model the problem
Formal Model θ
Training Data
Learning Algorithm
Testing Data
Prediction Function θ^*
Evaluation score

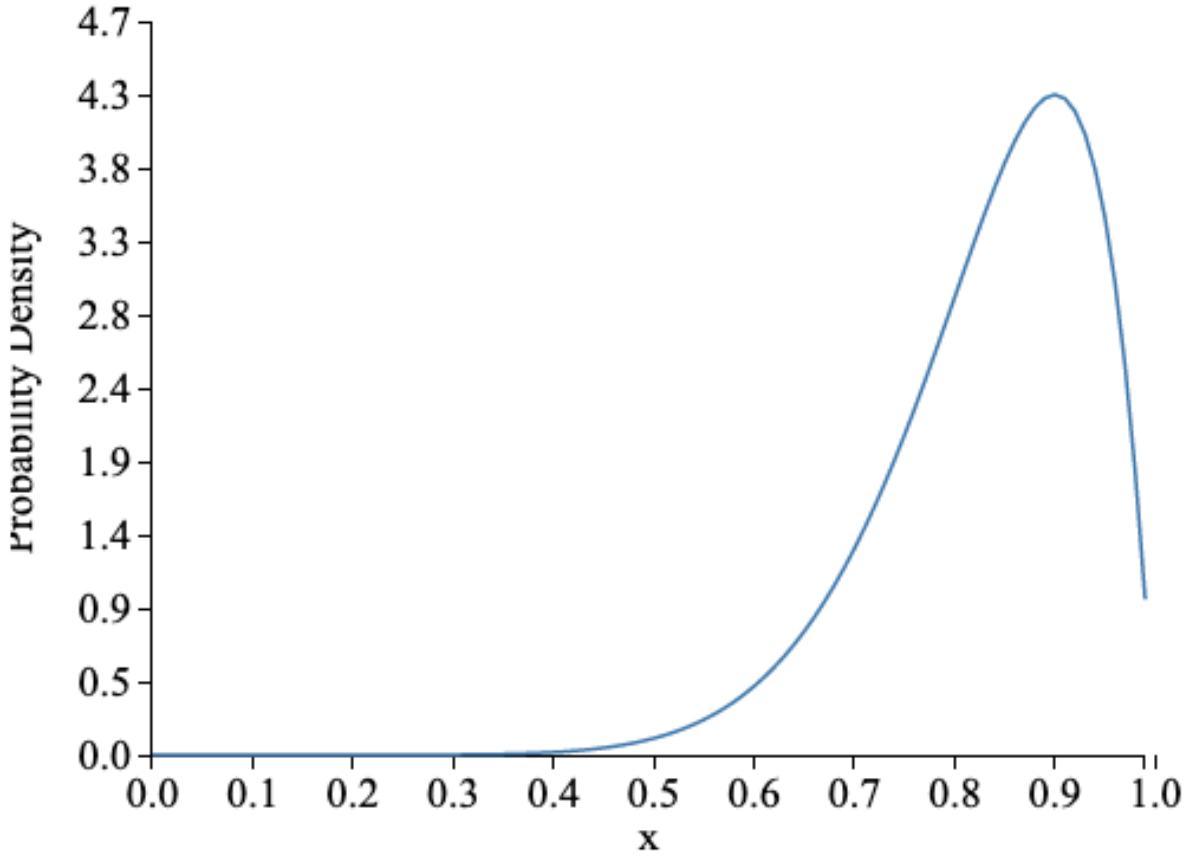
Parameter Estimation (aka Training)



We have already seen some parameter estimators!

9 Heads out of 10 Flips. What is your Belief in p?

$$f(X = x | H = 9, T = 1)$$



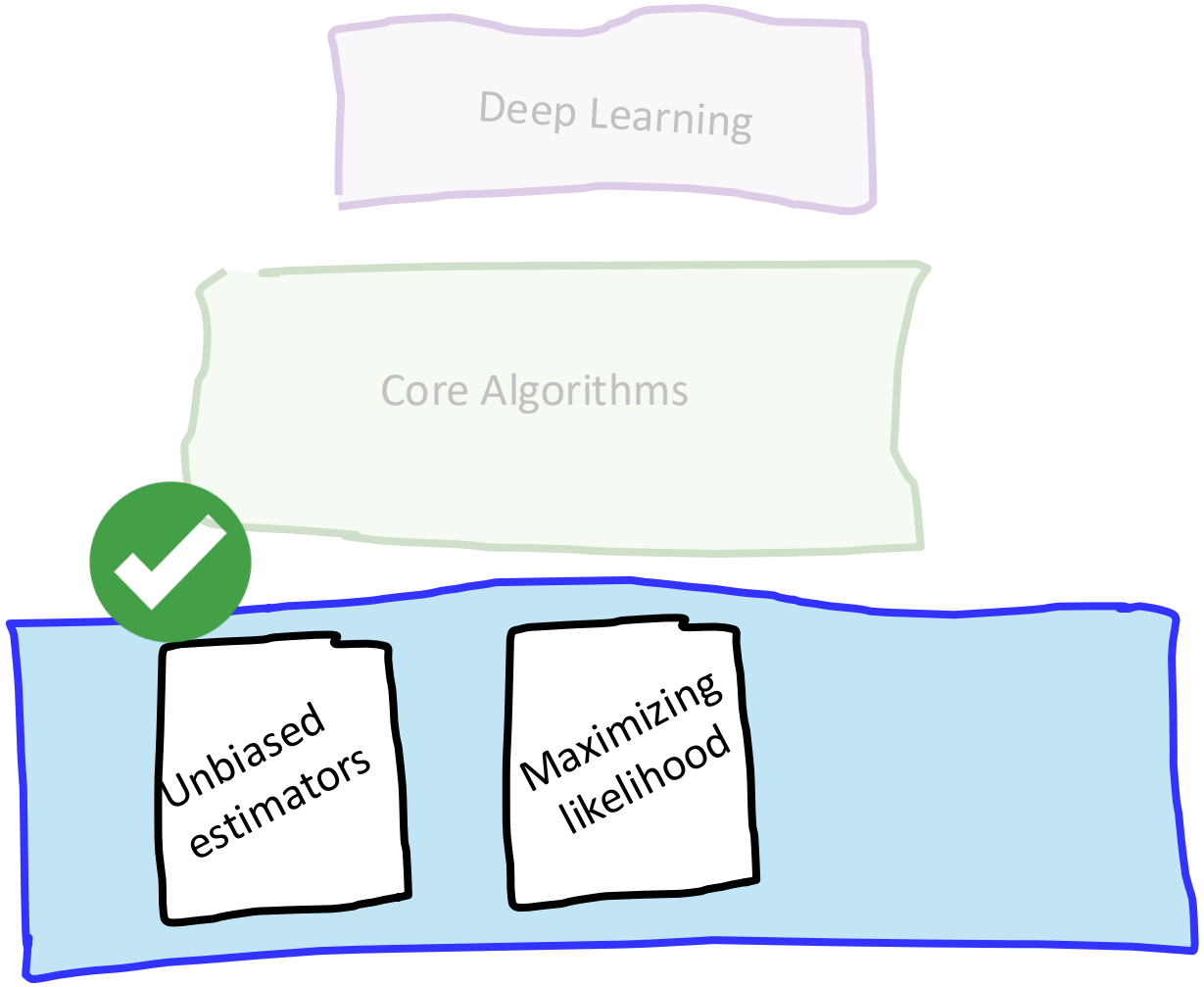
Unbiased Estimators of Mean and Variance

- $X_1, X_2, \ldots, X_n$ are n i.i.d. random variables, where X_i drawn from distribution F with $E[X_i] = \mu, \text{Var}(X_i) = \sigma^2$.

- Sample mean: $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$ unbiased **estimate** of μ

Sample variance: $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$ unbiased **estimate** of σ^2

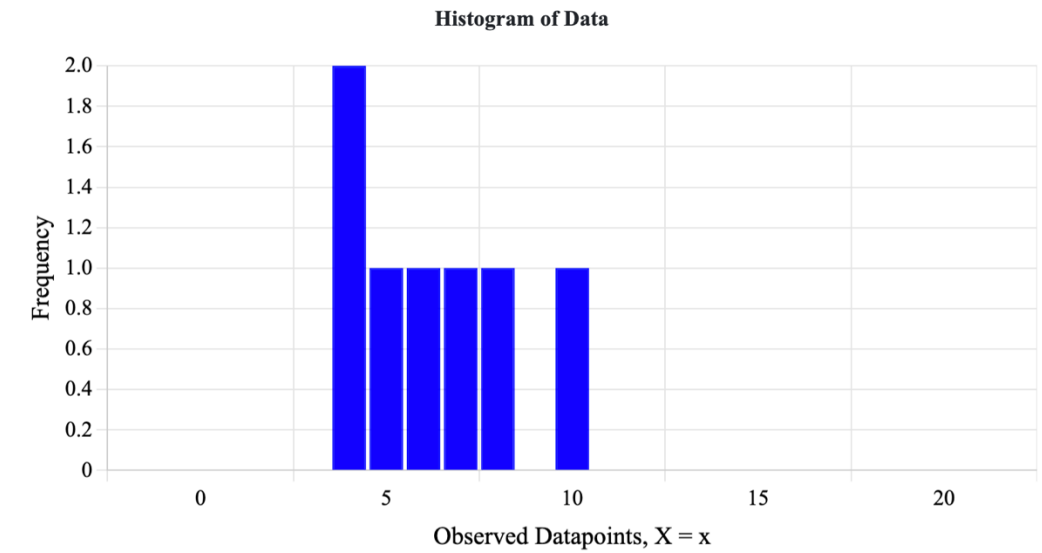
Our Path

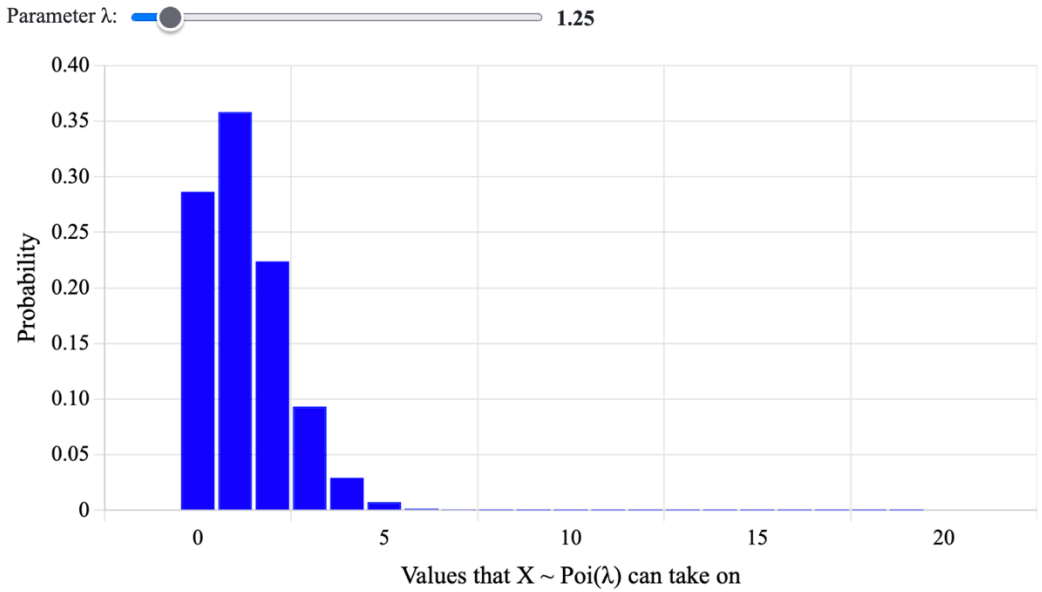

Deep Learning
Core Algorithms
Unbiased estimators
Maximizing likelihood

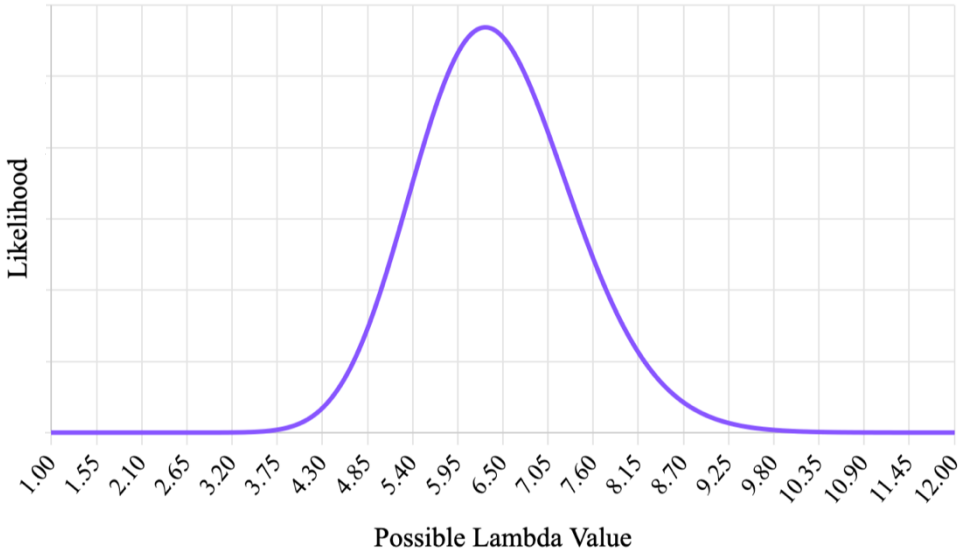
Unbiased Estimation is a limited tool:
how could we use that for fitting WebMD?

Great idea in Machine Learning

To the Course Reader!







How to Choose the “Best” Parameters: MLE

We want to choose the parameter value that maximizes the probability of the data.

How to Choose the "Best" Parameters: MLE

We want to choose the parameter value that maximizes the probability of the data.

Maximum **Likelihood** **Estimation!**

“I feel seen”

How to Choose the "Best" Parameters: MLE

We want to choose the parameter value that maximizes the probability of the data.

Maximum **Likelihood** **Estimation!**

How do we quantify "probability of the data"?

Likelihood Definition

Wikipedia:

Likelihood function

Article Talk

From Wikipedia, the free encyclopedia

The **likelihood function** (often simply called the **likelihood**) is the joint probability (or probability density) of observed data viewed as a function of the parameters of a statistical model.[1] [2] [3]

A generalized term for “PDF / PMF / Joint”
of data as a function of parameters

The Likelihood Function

Definition: The probability of our observed data if our parameters were θ.

$$L(\theta) = P(\text{data}|\theta)$$

θ is shorthand for parameter(s)
(if we have a Poisson, $\theta = \lambda$)

The Likelihood Function

Definition: The probability of our observed data if our parameters were θ.

If we had a single observation, $X = x$:

$$L(\theta) = P(X = x|\theta)$$

$$= P(x|\theta)$$

(in our example, this would just be the Poisson PMF with the specific parameter)

The Likelihood Function

Definition: The ***joint*** probability of our observed data if our parameters were θ.

For a list of observations, $[x_1, x_2, \ldots, x_n]$:

$$L(\theta) = P(x_1, x_2, \ldots, x_n | \theta)$$

The Likelihood Function

Definition: The ***joint*** probability of our observed data if our parameters were θ.

For a list of observations, $[x_1, x_2, \ldots, x_n]$:

$$L(\theta) = P(x_1, x_2, \ldots, x_n | \theta)$$

We assume that data points are I.I.D.

The Likelihood Function

Definition: The ***joint*** probability of our observed data if our parameters were θ.

For a list of observations, $[x_1, x_2, \ldots, x_n]$:

$$L(\theta) = P(x_1, x_2, \ldots, x_n | \theta)$$

We assume that data points are I.I.D.

$$= \prod_{i=1}^{n} P(x_i | \theta)$$

The Likelihood Function

Definition: The ***joint*** probability of our observed data if our parameters were θ.

For a list of observations, $[x_1, x_2, \ldots, x_n]$:

$$L(\theta) = P(x_1, x_2, \ldots, x_n | \theta)$$

We assume that data points are I.I.D.

$$= \prod_{i=1}^{n} f(x_i | \theta)$$

We always use f for likelihood in MLE (even for discrete) 🙃

The Likelihood Function

n I.I.D. data points $x_1, x_2, \ldots, x_n$

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

This is just a product since X_i are I.I.D.

We explicitly specify parameter θ of distribution

Likelihood (of data given parameters):

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

Either the
PDF (continuous) or
PMF (discrete), or
joint if multiple variables per datapoint

How to Choose the "Best" Parameters: MLE

We want to choose the parameter value that maximizes the probability of the data:

Likelihood $$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

How to Choose the "Best" Parameters: MLE

We want to choose the parameter value that maximizes the probability of the data:

Likelihood

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

To put words into math:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\ L(\theta)$$

Our best estimate

How to Choose the "Best" Parameters: MLE

We want to choose the parameter value that maximizes the probability of the data:

Likelihood $$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

Log Likelihood $$LL(\theta) = \sum_{i=1}^{n} \log f(x_i|\theta)$$

To put words into math: $$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\ LL(\theta)$$

Our best estimate

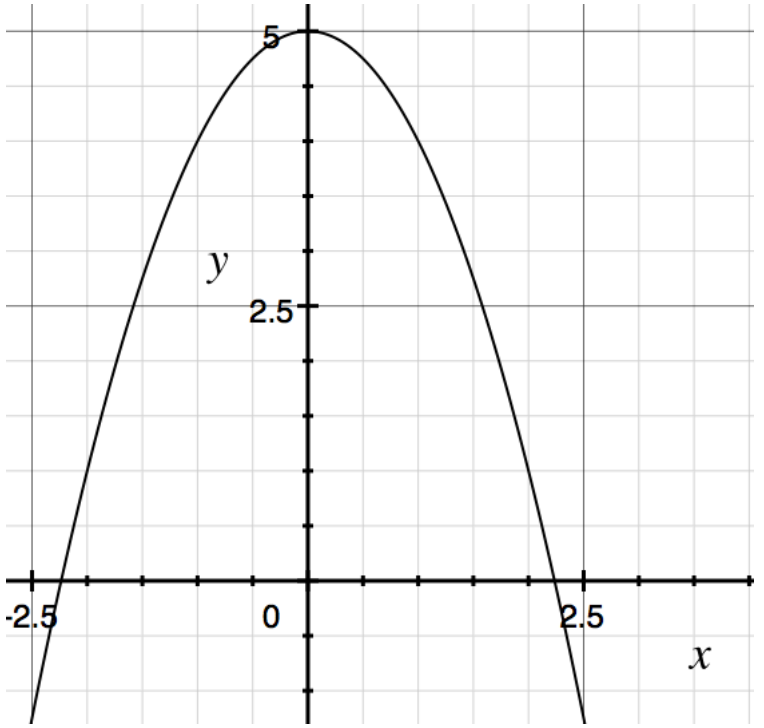
Sidequest: Review argmax

Argmax

$$f(x) = -x^2 + 5$$

$$\max_x \; -x^2 + 5 = 5$$

$$\operatorname*{argmax}_x \; -x^2 + 5 = 0$$




arg max

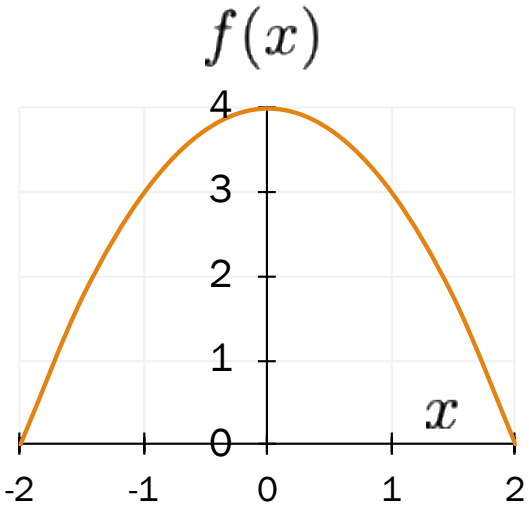
But how do we compute argmax?

Option #1: Straight optimization

Finding the Argmax with Calculus

$$f(x) = -x^2 + 4$$

$$\hat{x} = \arg\max_x f(x)$$

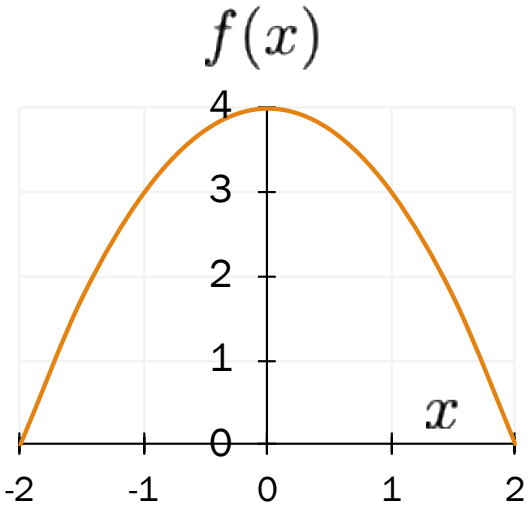


Finding the Argmax with Calculus

$$f(x) = -x^2 + 4 \qquad \hat{x} = \arg\max_x f(x)$$

Differentiate w.r.t. argmax's argument

$$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}\left[-x^2 + 4\right] = -2x$$



Finding the Argmax with Calculus

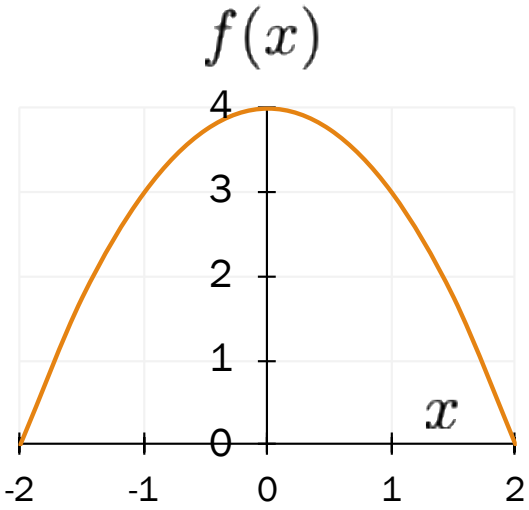
$$f(x) = -x^2 + 4 \qquad \hat{x} = \arg\max_x f(x)$$

Differentiate w.r.t. argmax's argument

$$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}\left[-x^2 + 4\right] = -2x$$

Set to 0 and solve

$$-2\hat{x} = 0 \quad \Rightarrow \quad \hat{x} = 0$$



Finding the Argmax with Calculus

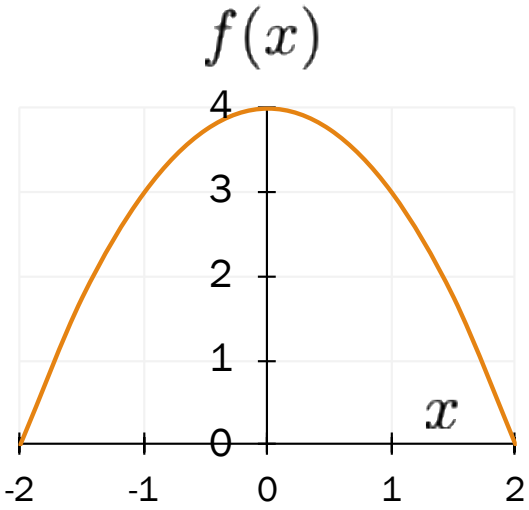
$$f(x) = -x^2 + 4 \qquad \hat{x} = \arg\max_x f(x)$$

Differentiate w.r.t. argmax's argument

$$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x}\left[-x^2 + 4\right] = -2x$$

Set to 0 and solve

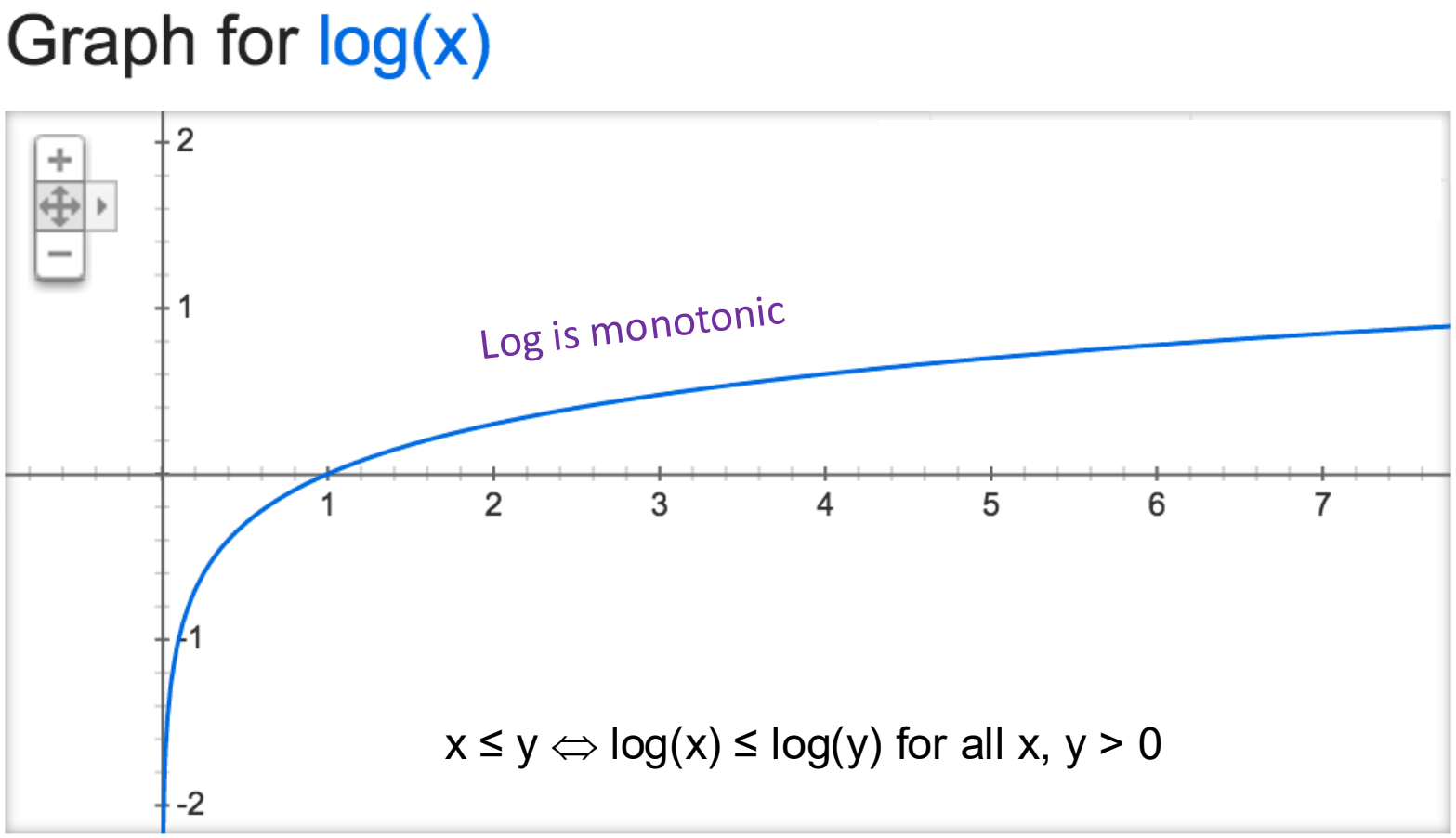
$$-2\hat{x} = 0 \quad \Rightarrow \quad \hat{x} = 0$$



Make sure $\hat{x}$ is a maximum

- Check the second derivative
- Generally ignored in expository derivations
- arg min is defined similarly, relevant for gradient descent

Argmax of Log



Claim: $\underset{x}{\operatorname{argmax}}\ f(x) = \underset{x}{\operatorname{argmax}}\ \log f(x)$

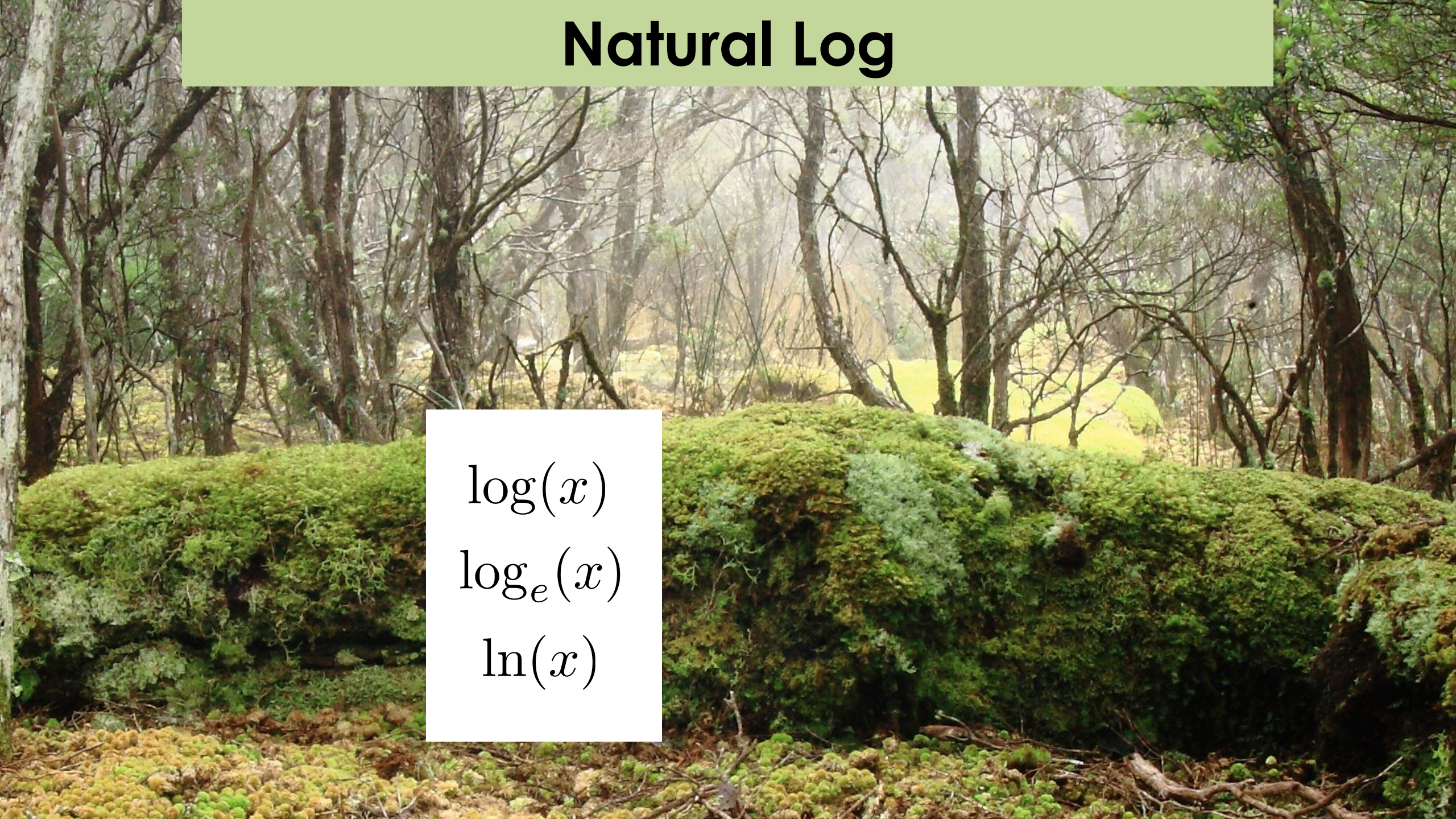
Argmax of Log

$$\underset{x}{\operatorname{argmax}}\ f(x) = \underset{x}{\operatorname{argmax}}\ \log f(x)$$

Log I Love You

$$\log(ab) = \log(a) + \log(b)$$

Natural Log



$$\log(x)$$

$$\log_e(x)$$

$$\ln(x)$$

End Sidequest

MLE For Poisson

$$X \sim \text{Poi}(\lambda)$$

MLE For Poisson

$$X \sim \text{Poi}(\lambda)$$

We observed the following samples:
[6, 1, 2, 1, 2, 3, 3, 2, 1, 3, 1, 3]

What is lambda, ? λ

x_i

MLE For Poisson

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$

  $X_i \sim \text{Poi}(\lambda)$ **Use Maximum Likelihood to estimate λ**

1. What is the likelihood of one X_i

2. What is the likelihood of all the *data*

3. What is the log-likelihood all the *data*

4. Find the value of λ which maximizes log likelihood

MLE For Poisson

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$

  $X_i \sim \text{Poi}(\lambda)$ **Use Maximum Likelihood to estimate λ**

  - Probability mass function can be written as: $f(x_i|\lambda) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

2. What is the likelihood of all the *data*

3. What is the log-likelihood all the *data*

4. Find the value of λ which maximizes log likelihood

MLE For Poisson

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$

  $X_i \sim \text{Poi}(\lambda)$ **Use Maximum Likelihood to estimate λ**

  - Probability mass function can be written as: $f(x_i|\lambda) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
  - Likelihood: $L(\lambda) = f(x_1 \dots x_n|\lambda) = \prod_{i=1}^{n} f(x_i|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
  - Log-likelihood:

3. What is the log-likelihood all the *data*

4. Find the value of λ which maximizes log likelihood

MLE For Poisson

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$

  $X_i \sim \text{Poi}(\lambda)$ **Use Maximum Likelihood to estimate λ**

  - Probability mass function can be written as: $f(x_i|\lambda) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
  - Likelihood: $L(\lambda) = f(x_1 \dots x_n|\lambda) = \prod_{i=1}^{n} f(x_i|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
  - Log-likelihood:

$$LL(\lambda) = \log \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \sum_{i=1}^{n} \log \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \sum_{i=1}^{n} -\lambda + x_i \log \lambda - \log x_i!$$

  - Differentiate w.r.t. λ and set to 0:

4. Find the value of λ which maximizes log likelihood

MLE For Poisson

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$

  $X_i \sim \text{Poi}(\lambda)$ **Use Maximum Likelihood to estimate λ**

  - Probability mass function can be written as: $f(x_i|\lambda) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
  - Likelihood: $L(\lambda) = f(x_1 \dots x_n|\lambda) = \prod_{i=1}^{n} f(x_i|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$
  - Log-likelihood:

$$LL(\lambda) = \log \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \sum_{i=1}^{n} \log \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \sum_{i=1}^{n} -\lambda + x_i \log \lambda - \log x_i!$$

  - Differentiate w.r.t. λ, and set to 0:

$$\frac{\partial LL(\lambda)}{\partial \lambda} = \sum_{i=1}^{n} -1 + \frac{x_i}{\lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i \qquad 0 = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i$$

$$\lambda = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Isn’t that the same as
the sample mean?

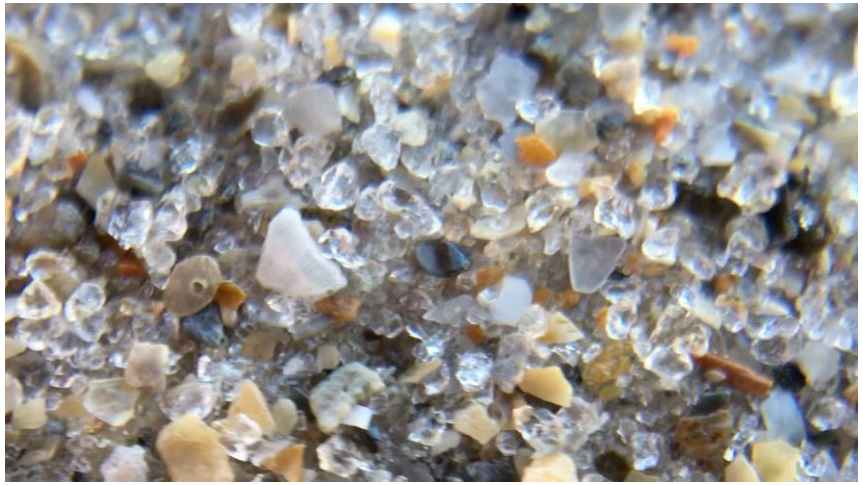
Yes. For Poisson.

MLE For Poisson

MLE For Pareto

```
observations = [1.677, 3.812, 1.463, 2.641, 1.256, 1.678, 1.157,
1.146, 1.323, 1.029, 1.238, 1.018, 1.171, 1.123, 1.074, 1.652,
1.873, 1.314, 1.309, 3.325, 1.045, 2.271, 1.305, 1.277, 1.114,
1.391, 3.728, 1.405, 1.054, 2.789, 1.019, 1.218, 1.033, 1.362,
1.058, 2.037, 1.171, 1.457, 1.518, 1.117, 1.153, 2.257, 1.022,
1.839, 1.706, 1.139, 1.501, 1.238, 2.53 , 1.414, 1.064, 1.097,
1.261, 1.784, 1.196, 1.169, 2.101, 1.132, 1.193, 1.239, 1.518,
2.764, 1.053, 1.267, 1.015, 1.789, 1.099, 1.25 , 1.253, 1.418,
1.494, 1.015, 1.459, 2.175, 2.044, 1.551, 4.095, 1.396, 1.262,
1.351, 1.121, 1.196, 1.391, 1.305, 1.141, 1.157, 1.155, 1.103,
1.048, 1.918, 1.889, 1.068, 1.811, 1.198, 1.361, 1.261, 4.093,
2.925, 1.133, 1.573]
```

```
def estimate_alpha(observations):
    print('your code here')
```



We know sand is distributed as a pareto with PDF

$$f(x) = \frac{\alpha}{x^{\alpha+1}}$$

https://probabilitycoders.stanford.edu/win26/mle_pareto

MLE for a Pareto

Consider I.I.D. random variables $X_1, X_2, ..., X_n$

- $X_i \sim$ Pareto(α). **Use Maximum Likelihood to estimate α.**

1. What is the likelihood of all the *data*

2. What is the log-likelihood all the *data*

3. Find the value of α which maximizes log likelihood

MLE for a Pareto

Consider I.I.D. random variables $X_1, X_2, ..., X_n$

- $X_i \sim$ Pareto(α). **Use Maximum Likelihood to estimate α.**
- Likelihood:

$$L(\alpha) = \prod_{i=1}^{n} \frac{\alpha}{x_i^{\alpha+1}}$$

2. What is the log-likelihood all the *data*

3. Find the value of α which maximizes log likelihood

MLE for a Pareto

Consider I.I.D. random variables $X_1, X_2, ..., X_n$

- $X_i \sim$ Pareto(α). **Use Maximum Likelihood to estimate α.**
- Likelihood:

$$L(\alpha) = \prod_{i=1}^{n} \frac{\alpha}{x_i^{\alpha+1}}$$

- Log-likelihood:

$$LL(\alpha) = \sum_{i=1}^{n} \log \alpha - (\alpha+1) \log x_i = n \log \alpha - (\alpha+1) \sum_{i=1}^{n} \log x_i$$

3. Find the value of **α** which maximizes log likelihood

MLE for a Pareto

Consider I.I.D. random variables $X_1, X_2, ..., X_n$

- $X_i \sim$ Pareto(α). **Use Maximum Likelihood to estimate α.**
- Likelihood:

$$L(\alpha) = \prod_{i=1}^{n} \frac{\alpha}{x_i^{\alpha+1}}$$

- Log-likelihood:

$$LL(\alpha) = \sum_{i=1}^{n} \log \alpha - (\alpha+1) \log x_i = n \log \alpha - (\alpha+1) \sum_{i=1}^{n} \log x_i$$

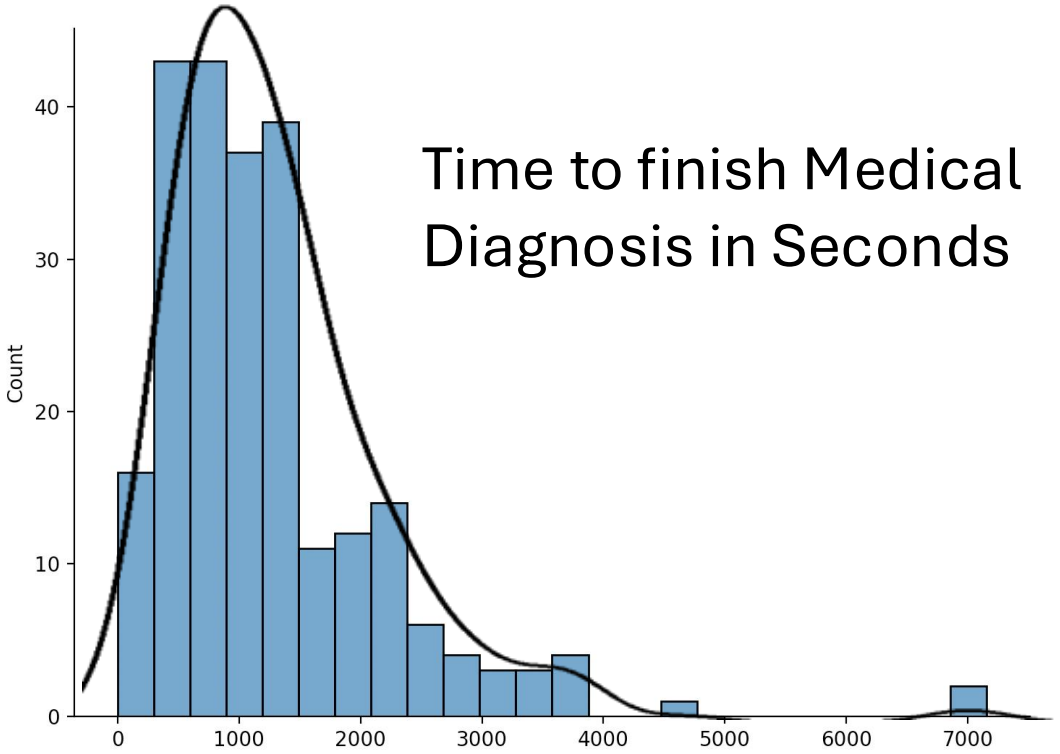
- Chose **α** to be the argmax of LL:

$$\frac{\partial LL(\alpha)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log x_i$$

Pause

MLE for Erlang

```
[3.002, 0.983, 2.186, 1.624, 3.997, 1.777,
2.809, 0.42, 0.515, 1.582, 0.948, 0.458, 1.
066, 0.8, 2.398, 0.794, 2.561, 2.61, 0.
595, 3.897, 1.852, 1.182, 3.043, 0.905, 1.
45, 0.405, 0.445, 2.103, 1.425, 3.12, 0.
973, 1.056, 3.715, 2.952, 1.817, 2.686, 4.
173, 0.358, 2.185, 2.581, 7.134, 0.206, 2.
049, 0.896, 2.095, 4.39, 2.199, 3.434, 5.
696, 0.819, 0.416, 1.571, 1.337, 2.79, 2.
701, 3.061, 4.677, 0.671, 1.594, 3.586, 2.
708, 1.417, 1.799, 1.137, 1.771, 2.12, 0.
93, 6.835, 3.213, 2.541, 2.505, 1.257, 1.
99, 1.5, 0.014, 3.856, 0.979, 2.413, 2.
596, 1.653, 0.881, 4.457, 0.717, 3.305, 2.
456, 3.462, 1.737, 0.968, 0.528, 0.18, 1.
626, 2.224, 1.466, 1.6, 1.572, 0.12, 2.86,
1.062, 2.139, 1.217]
```

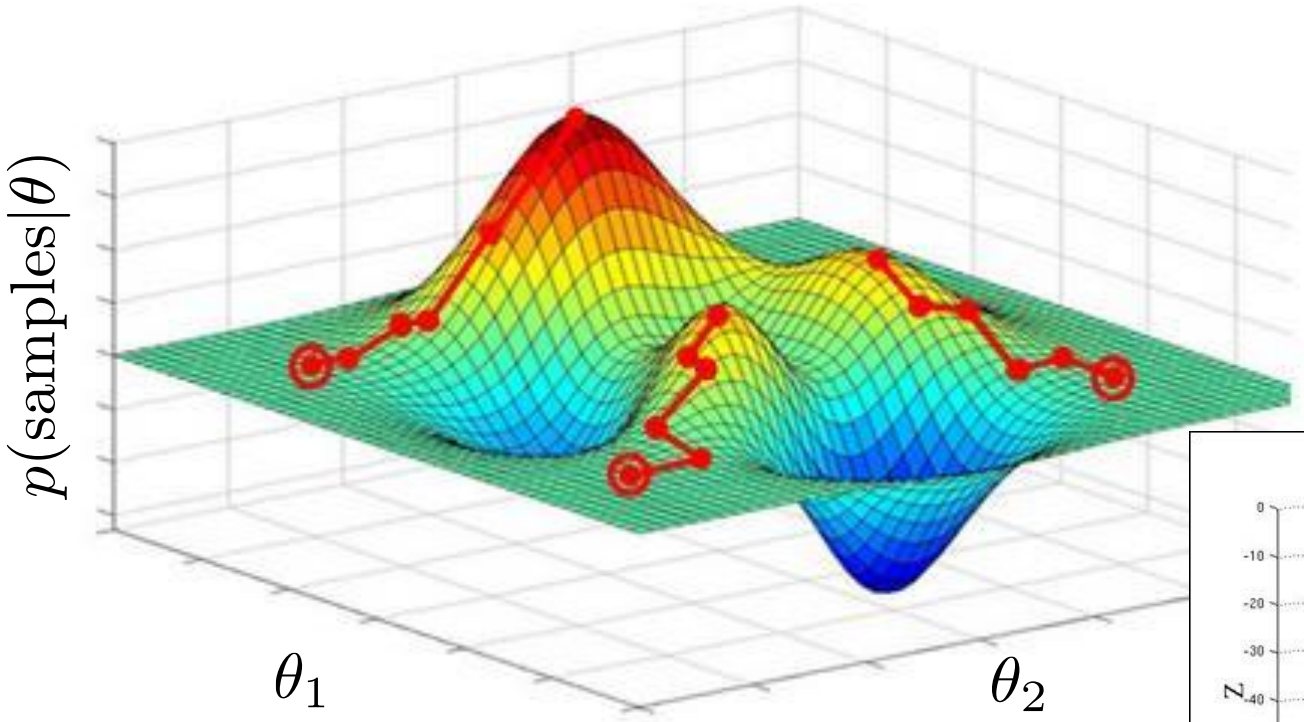


Time to finish Medical Diagnosis in Seconds

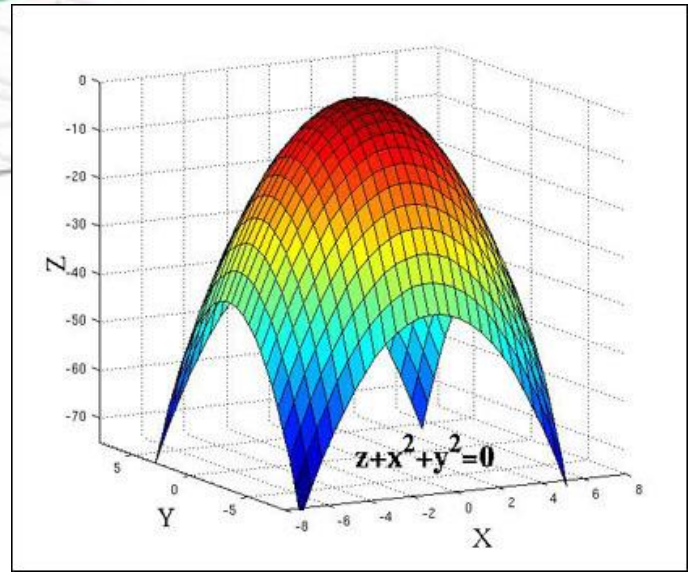
$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$


arg max

Gradient Ascent



Especially good if function is convex



Walk uphill and you will find a local maxima
(if your step size is small enough)

Gradient Ascent

Repeat many times

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \frac{\partial LL(\theta^{\text{old}})}{\partial \theta_j^{\text{old}}}$$

Step size constant

This is some **profound** life philosophy

Walk uphill and you will find a local maxima
(if your step size is small enough)

Gradient Ascent

Initialize: θ_j = random for all $0 \leq j \leq m$

Repeat many times:

Calculate all `gradient[j]`*'s based on data*

θ_j += η * gradient[j] for all $0 \leq j \leq m$

To the code!

Gradient Ascent for MLE of Erlang

```python
def fit_erlang(observations, n_iterations=10, initial_step_size=0.0001):
    step_size = initial_step_size
    k = 1.0  # Initial guess for shape parameter
    lambda_ = 2.0  # Initial guess for rate parameter
    n = len(observations)

    for i in range(n_iterations):
        # To debug: Calculate log-likelihood
        # ll = calc_log_likelihood_erlang(observations, k, lambda_)
        # print(f"Log-Likelihood at iteration {k, lambda_}: {ll}")

        # Calculate gradients
        # gradient_lambda = (n * k / lambda_) - sum(observations)
        gradient_lambda = 0
        for x_i in observations:
            gradient_lambda += k / lambda_ - x_i
        gradient_k = n * math.log(lambda_) + sum(math.log(x) for x in observations) - n * digamma(k)

        # Update parameters
        lambda_ += step_size * gradient_lambda
        k += step_size * gradient_k

    return k, lambda_
```

Derived these partial derivatives of log likelihood

Pause

MLE For Bernoulli

$$X \sim \text{Bern}(\text{p})$$

Don’t we already have the Beta?

Yes! But this example is critical for developing towards deep learning.

MLE For Bernoulli

- Consider I.I.D. random variables X_1, X_2, ..., X_n

  $X_i \sim \text{Bern}(p)$ **Use Maximum Likelihood to estimate *p***

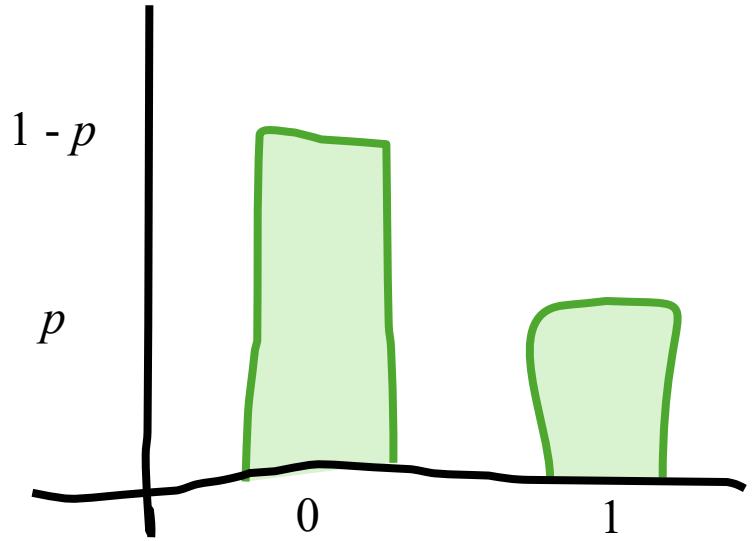
  - Probability mass function can be written as:

$$f(x_i|p) = \begin{cases} p & \text{if } x_i = 1 \\ 1-p & \text{if } x_i = 0 \end{cases}$$

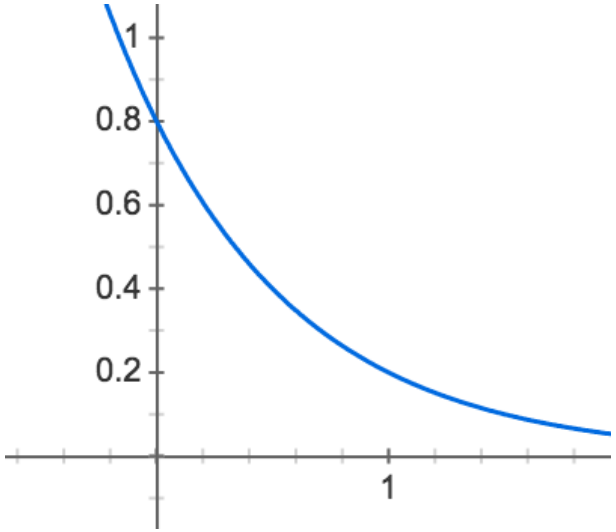
Differentiable PMF for Bernoulli

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Ber}(p)$
  - Probability mass function, $f(X_i = x_i | P = p)$

PMF of Bernoulli



PMF of Bernoulli (*p* = 0.2)



$$f(x_i|p) = p^{x_i}(1-p)^{1-x_i}$$

$$f(x_i|p = 0.2) = 0.2^{x_i}(1-0.2)^{1-x_i}$$

Bernoulli PMF

$$X \sim \text{Ber}(p)$$

$$f(X = x|p) = p^x(1-p)^{1-x}$$

Maximum Likelihood For Bernoulli

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$

$X_i \sim \text{Bern}(p)$ **Use Maximum Likelihood to estimate p**

1. What is the likelihood of one X_i

2. What is the likelihood of all the *data*

3. What is the log-likelihood all the *data*

4. Find the value of p which maximizes log likelihood

Maximum Likelihood For Bernoulli

- Consider I.I.D. random variables X_1, X_2, ..., X_n

  $X_i \sim \text{Bern}(p)$ **Use Maximum Likelihood to estimate p**

  - Probability mass function can be written as: $f(x_i|p) = p^{x_i}(1-p)^{1-x_i}$

2. What is the likelihood of all the *data*

3. What is the log-likelihood all the *data*

4. Find the value of p which maximizes log likelihood

Maximum Likelihood For Bernoulli

- Consider I.I.D. random variables X_1, X_2, ..., X_n

  $X_i \sim \text{Bern}(p)$ **Use Maximum Likelihood to estimate p**

  - Probability mass function can be written as: $f(x_i|p) = p^{x_i}(1-p)^{1-x_i}$
  - Likelihood: $L(p) = \prod_{i=1}^{n} f(x_i|p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$
  - Log-likelihood:

3. What is the log-likelihood all the *data*

4. Find the value of p which maximizes log likelihood

Maximum Likelihood For Bernoulli

- Consider I.I.D. random variables X_1, X_2, ..., X_n

  $X_i \sim \text{Bern}(p)$ **Use Maximum Likelihood to estimate p**

  - Probability mass function can be written as: $f(x_i|p) = p^{x_i}(1-p)^{1-x_i}$
  - Likelihood: $L(p) = \prod_{i=1}^{n} f(x_i|p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$
  - Log-likelihood:

$$LL(p) = \sum_{i=1}^{n} x_i \log p + (1-x_i)\log(1-p)$$

4. Find the value of p which maximizes log likelihood

Take Derivative: $LL(p) = \sum_{i=1}^{n} x_i \log p + (1 - x_i) \log(1 - p)$

$$\frac{\partial LL(p)}{\partial p} = \frac{\partial}{\partial p} \sum_{i=1}^{n} x_i \log p + (1 - x_i) \log(1 - p)$$

Take the derivative wrt p

$$= \sum_{i=1}^{n} \frac{\partial}{\partial p} \left[x_i \log p + (1 - x_i) \log(1 - p) \right]$$

Derivative of a sum!

$$= \sum_{i=1}^{n} \frac{\partial}{\partial p} x_i \log p + \frac{\partial}{\partial p} (1 - x_i) \log(1 - p)$$

Derivative of a sum!

$$= \sum_{i=1}^{n} \frac{x_i}{p} + \frac{\partial}{\partial p} (1 - x_i) \log(1 - p)$$

Derivative of log p

$$= \sum_{i=1}^{n} \frac{x_i}{p} - \frac{1 - x_i}{1 - p}$$

Derivative of log (1-p)

Set to Zero: $\frac{\partial LL(p)}{\partial p} = \sum_{i=1}^{n} \frac{x_i}{p} - \frac{1 - x_i}{1 - p}$

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} \frac{x_i}{\hat{p}} - \frac{1 - x_i}{1 - \hat{p}} \\
&= \sum_{i=1}^{n} \frac{x_i}{\hat{p}} - \sum_{i=1}^{n} \frac{1 - x_i}{1 - \hat{p}} \\
&= \frac{y}{\hat{p}} - \frac{n - y}{1 - \hat{p}} \\
\frac{n - y}{1 - \hat{p}} &= \frac{y}{\hat{p}} \\
\hat{p}(n - y) &= y(1 - \hat{p}) \\
\hat{p}n - \hat{p}y &= y - \hat{p}y \\
\hat{p}n &= y
\end{aligned}
$$

Let $\sum_{i=1}^{n} x_i = y$ To make life easier

And $\sum_{i=1}^{n} 1 - x_i = \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} x_i = n - y$

$$
\begin{aligned}
\hat{p} &= \frac{1}{n} y \\
&= \frac{1}{n} \sum_{i=1}^{n} x_i
\end{aligned}
$$

Isn’t that the same as unbiased estimator?

Yes. For Bernoulli.

MLE for Bernoulli is Sample Mean

MLE vs. Beta

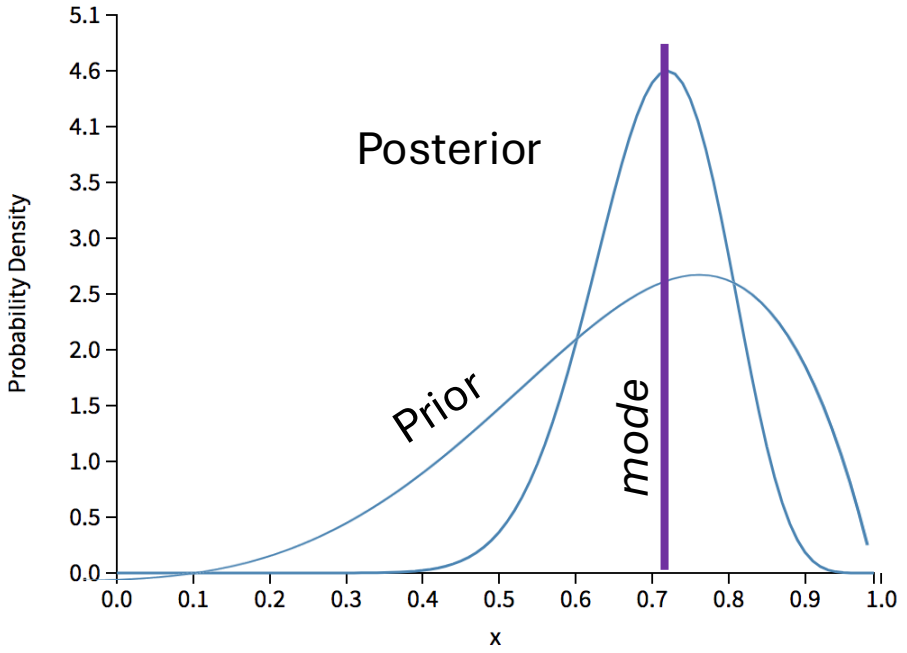
The medicine is tried on 20 patients. It "works" for 14 and "doesn't work" for 6. What is your new belief that the drug works?

In other words I have 20 IID samples from a Bernoulli. Estimate p. The data is [1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0]

MLE estimate:

$$p \approx \frac{14}{20} = 0.7$$

Beta estimate:



Think about the difference between a **point estimate** and a **distribution**

$p = 0.75$

$p =$