

Independence

Today, start with a cool program

G_1

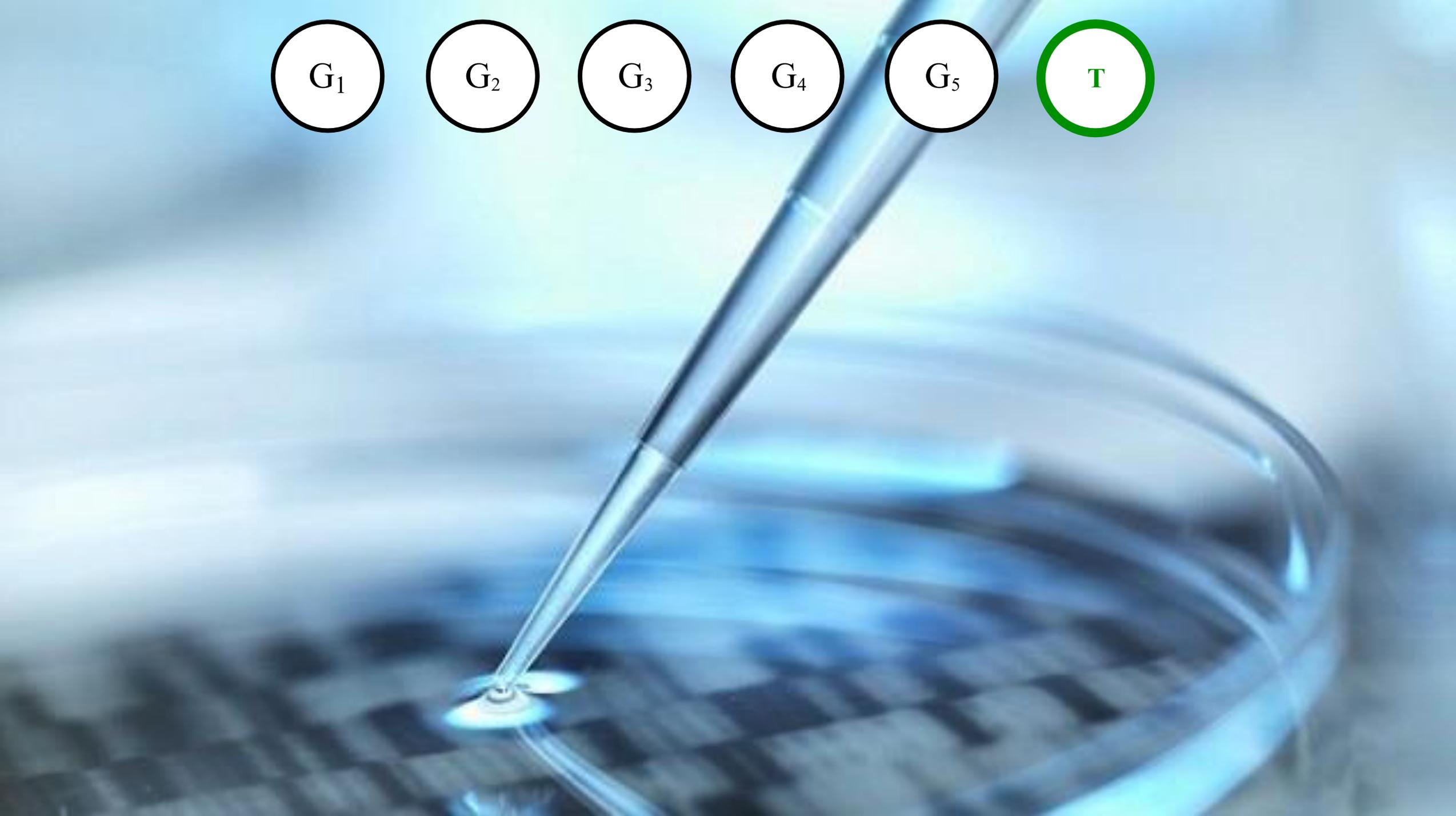
G_2

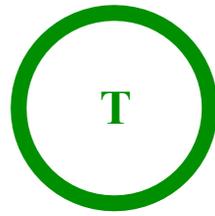
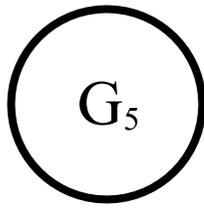
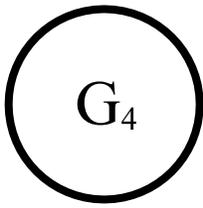
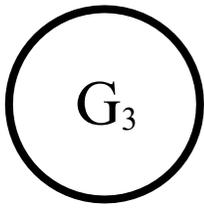
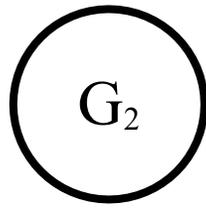
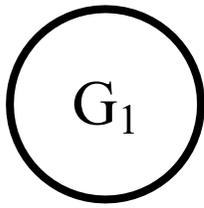
G_3

G_4

G_5

T





```
dna.txt — dna
dna.txt
1 False, True, False, False, True, False
2 True, True, False, True, True, False
3 True, True, False, True, True, True
4 False, True, False, True, True, False
5 False, True, False, False, True, False
6 True, True, False, True, True, True
7 False, False, True, False, False, False
8 False, False, True, False, True, False
9 True, False, False, True, False, False
10 False, True, False, True, True, False
11 True, False, False, True, False, False
12 True, False, True, True, False, False
13 False, True, False, False, True, False
14 False, False, True, True, False, False
15 True, True, False, False, True, True
16 True, False, True, True, False, False
17 True, True, True, True, True, True |
18 True, False, True, False, False, True
19 False, True, False, True, True, True
20 False, False, True, False, False, False
21 False, False, False, True, True, False
22 False, True, False, False, True, False
23 True, True, False, True, True, True
24 False, True, False, True, True, False
25 True, False, False, False, False, True
26 False, False, True, True, False, True
27 False, False, False, True, False, False
28 False, True, True, False, False, True
29 False, True, False, False, True, True
30 False, False, False, False, False, True
31 False, True, False, True, True, False
32 True, False, False, True, False, False
33 True, True, False, True, True, True
34 True, True, False, False, True, True
35 True, True, False, True, True, True
36 False, False, False, True, False, False
--
```

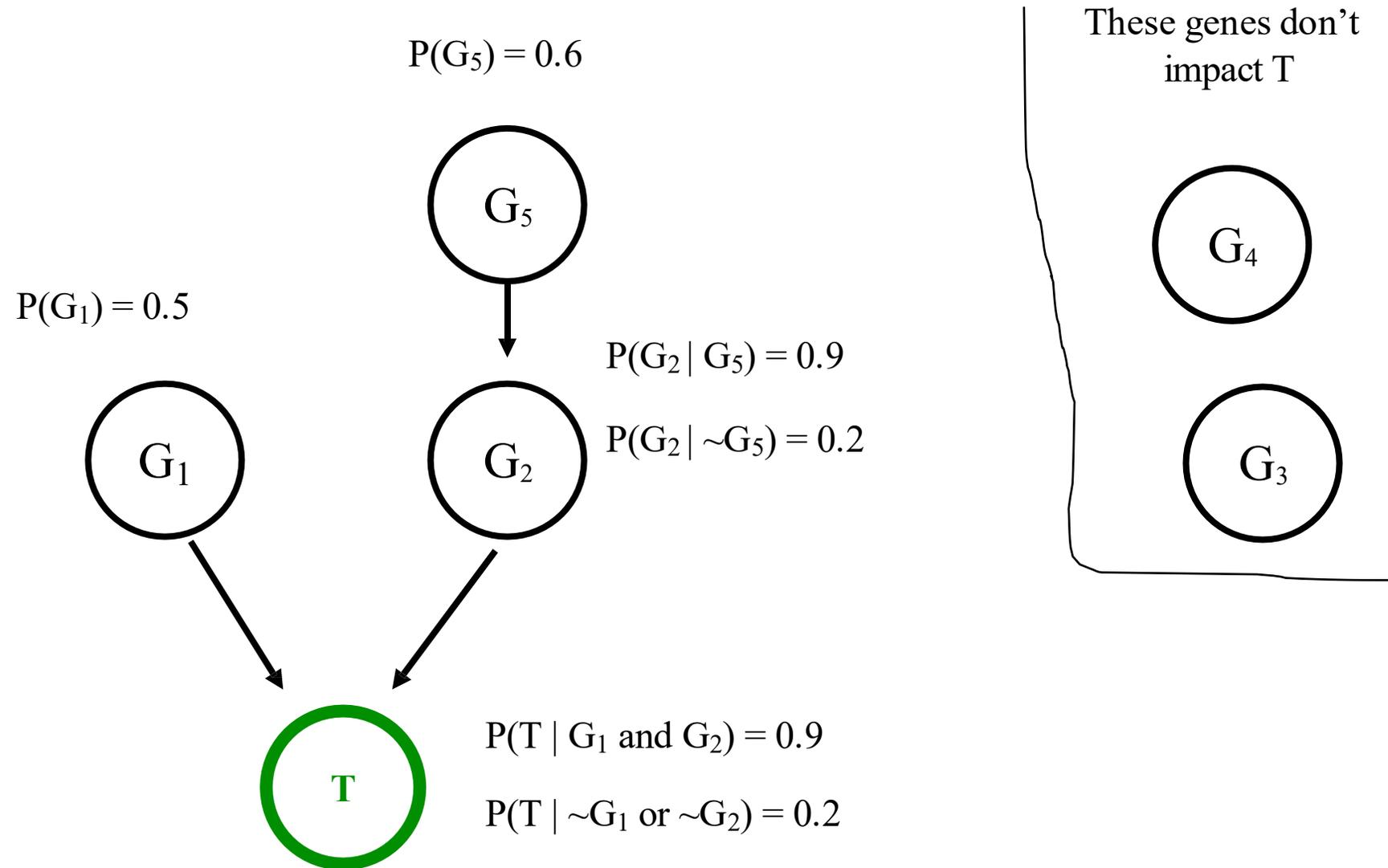


100,000 samples

6 observations per sample



Discovered Hypothesis



We've Gotten Ahead of Ourselves



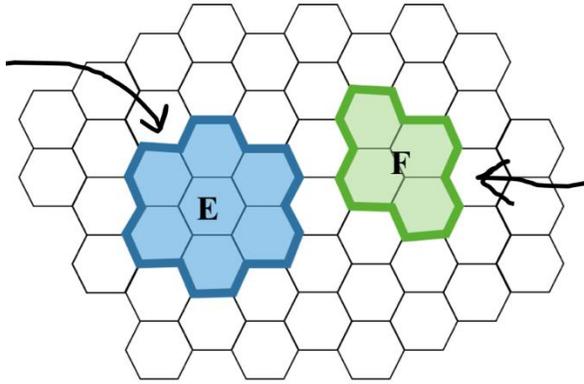
Source: The Hobbit

Start at the Beginning



Source: The Hobbit

Learning Goals of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Review

Notation

And

Or

Given

$$P(E \text{ and } F)$$

$$P(E \text{ or } F)$$

$$P(E|F)$$

$$P(E, F)$$

$$P(E \cup F)$$

$$P(EF)$$

$$P(E \cap F)$$

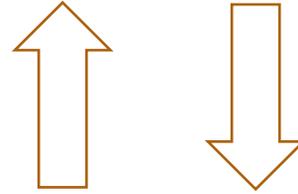


Relationship Between Probabilities



$$P(E \text{ and } F)$$

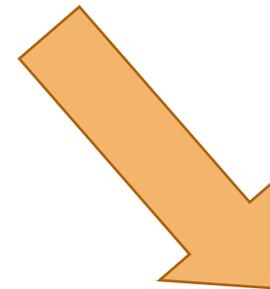
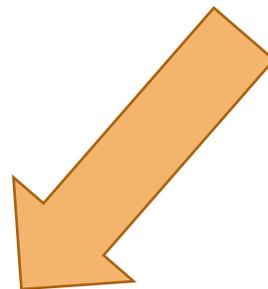
Chain rule
(Product rule)



Definition of
conditional probability

$$P(E|F)$$

Law of Total
Probability



Bayes'
Theorem

$$P(E)$$

$$P(F|E)$$



Review: Chain Rule

Definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule:

$$P(EF) = P(E|F)P(F)$$

Bayes' Theorem

$$P(E|F) \Rightarrow P(F|E)$$

Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

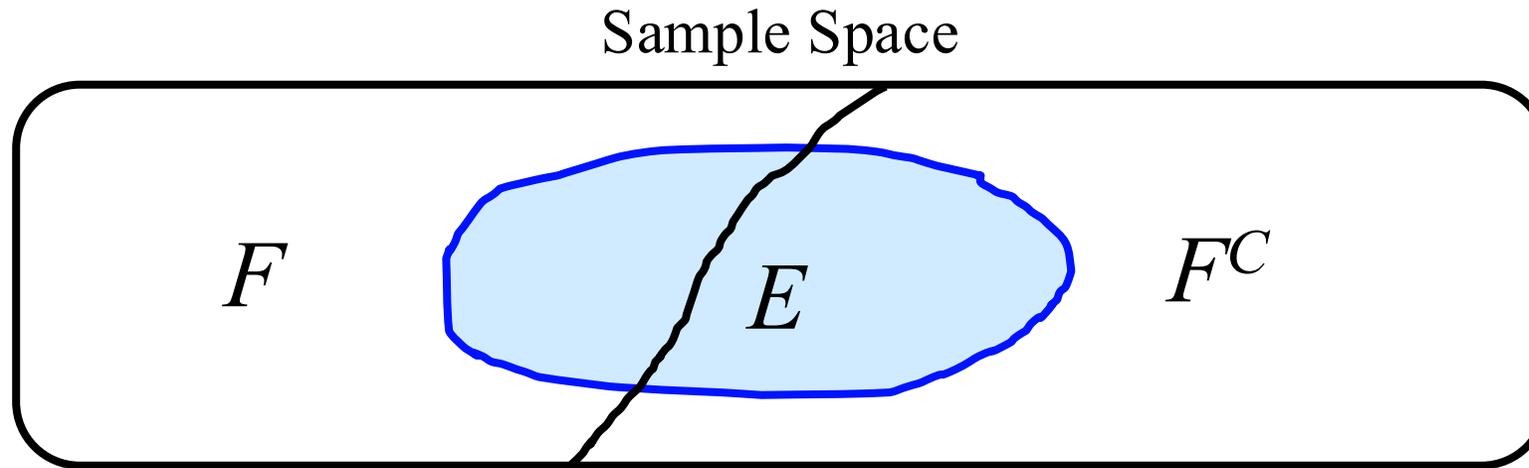


Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$



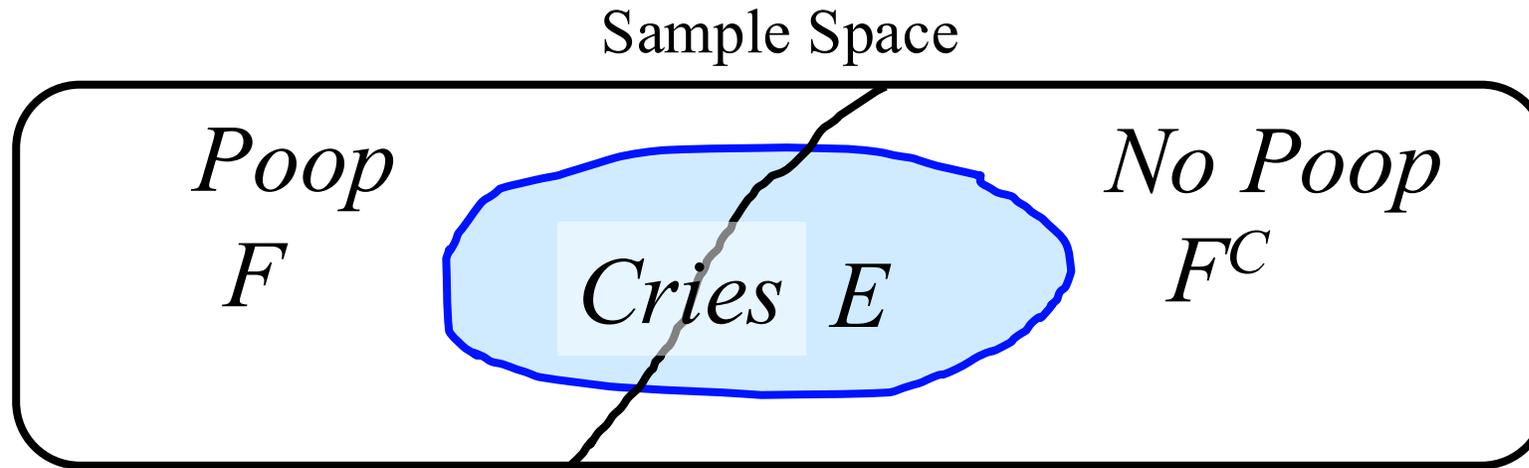
Law of Total Probability



$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E|F)P(F) + P(E|F^C)P(F^C) \end{aligned}$$



Law of Total Probability



$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E|F)P(F) + P(E|F^C)P(F^C) \end{aligned}$$



Detecting Spam Email

- 60% of all email in 2016 is spam.
- 20% of spam has the word “Dear”
- 1% of non-spam (aka ham) has the word “Dear”

You get an email with the word “Dear” in it.

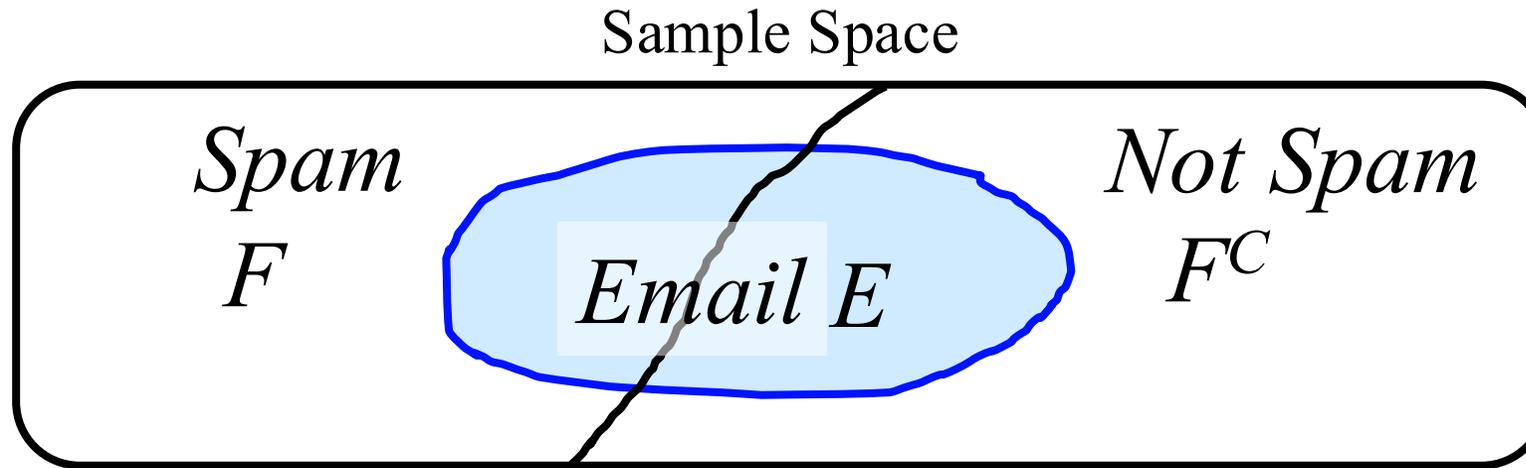
What is the probability that the email is spam?

Solution:

$$P(F | E) = \frac{P(E | F) P(F)}{P(E)}$$
$$P(F | E) = \frac{P(E | F) P(F)}{P(E | F) P(F) + P(E | F^c) P(F^c)}$$
$$P(F | E) = \frac{(0.20)(0.60)}{(0.20)(0.60) + (0.01)(0.4)} \approx 0.968$$



Law of Total Probability

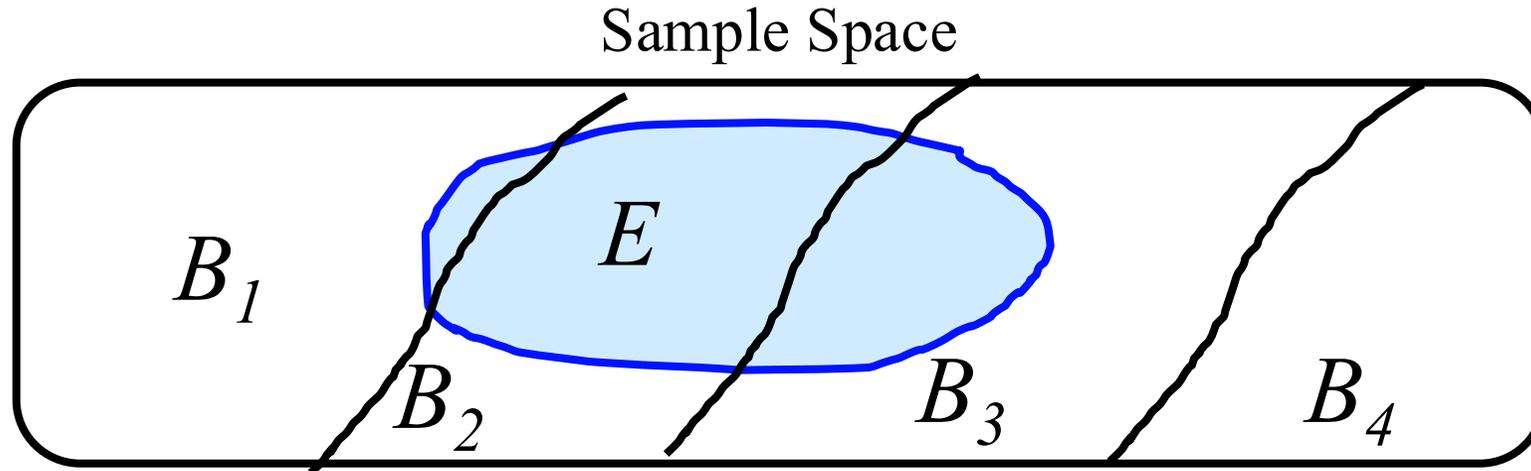


$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E|F)P(F) + P(E|F^C)P(F^C) \end{aligned}$$



End Review

Law of Total Probability



Thm For **partition events** B_1, B_2, \dots, B_n

$$\begin{aligned} P(E) &= \sum_i P(B_i \text{ and } E) \\ &= \sum_i P(E|B_i)P(B_i) \end{aligned}$$

partition events:

B_1, B_2, \dots, B_n
are mutually exclusive

B_1 or B_2 or ... or $B_n = S$



Results for San Francisco, CA



49 °F | °C

Precipitation: 90%

Humidity: 74%

Wind: 8 mph

Weather

Monday

Rain showers

Background event.
Where is the person in
San Francisco?

Results for San Francisco, CA



49 °F | °C

Precipitation: 90%

Humidity: 74%

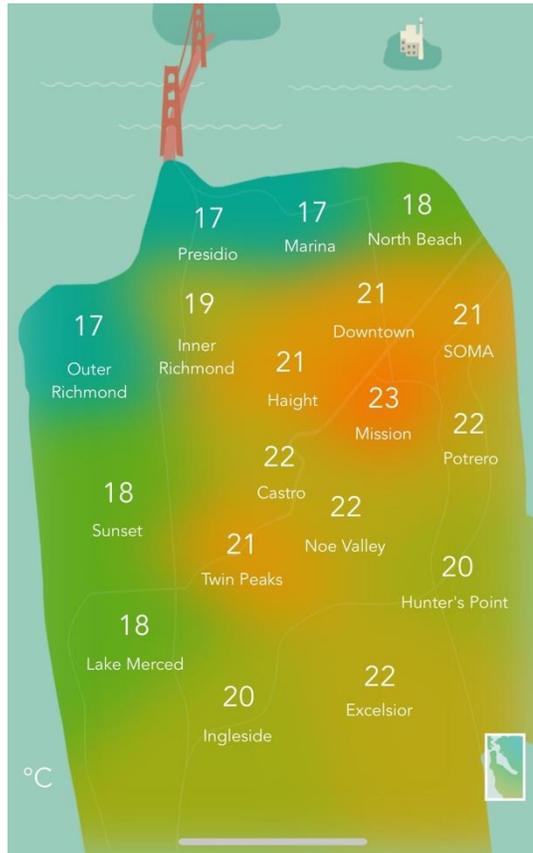
Wind: 8 mph

Weather

Monday

Rain showers

Background event.
Where is the person in
San Francisco?



From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

It rains tomorrow

$$P(R|D_i)$$

Person is in district i

$$P(D_i)$$

Want:

$$P(R)$$

Results for San Francisco, CA



49 °F | °C

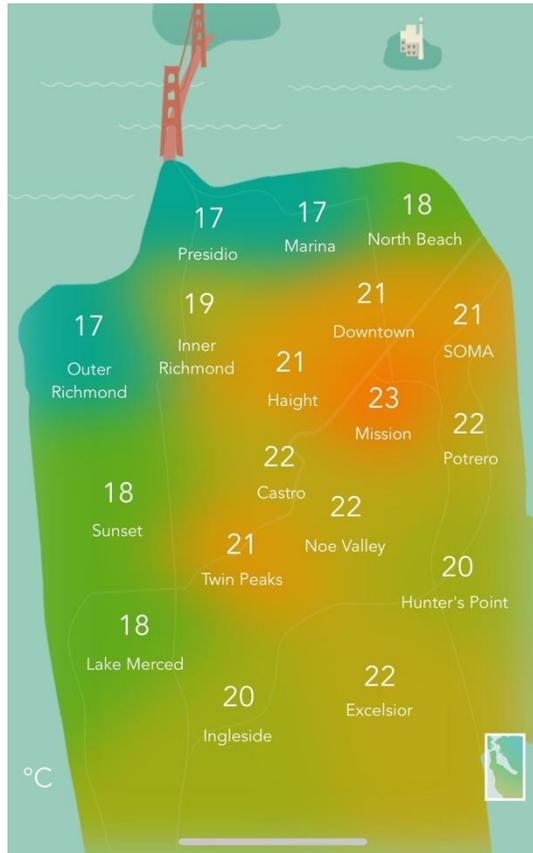
Precipitation: 90%
Humidity: 74%
Wind: 8 mph

Weather

Monday

Rain showers

Background event.
Where is the person in
San Francisco?



From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

	Mission District	Presidio	...	SOMA
$P(R D_i)$	0.23	0.84	...	0.52
$P(D_i)$	0.15	0.02		0.24

Want:

$$P(R)$$

Results for San Francisco, CA



49 °F | °C

Precipitation: 90%

Humidity: 74%

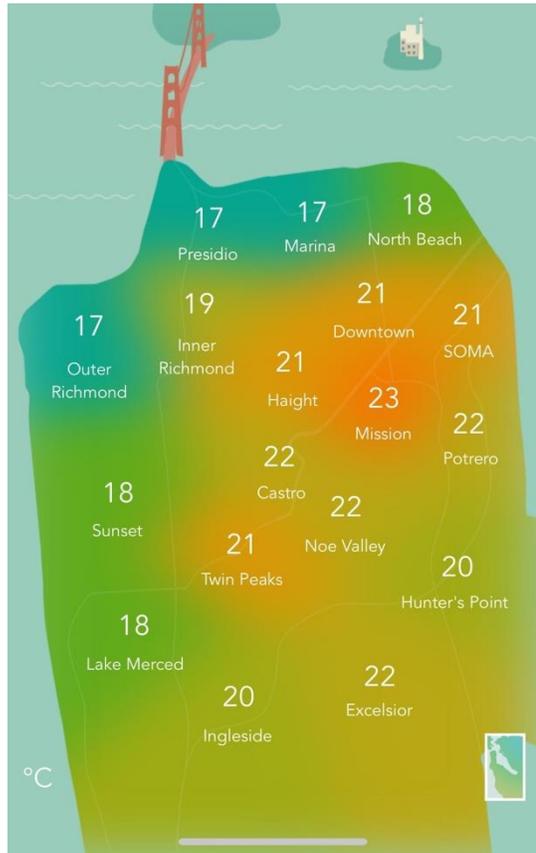
Wind: 8 mph

Weather

Monday

Rain showers

Background event.
Where is the person in
San Francisco?



From Google's Perspective:
There are 18 different "districts" in San Francisco.

Know:

	Mission District	Presidio	...	SOMA
$P(R D_i)$	0.23	0.84	...	0.52
$P(D_i)$	0.15	0.02		0.24

Want:

$$P(R) = \sum_{\text{district } i} P(R \text{ and } D_i) = \sum_{\text{district } i} P(R|D_i) \cdot P(D_i)$$

Monty Hall Problem

Monty Hall Problem



Monty Hall Problem from Let's Make a Deal

Behind one door is a prize (equally likely to be any door).

Behind the other two doors is nothing

1. We choose a door
2. Host opens 1 of other 2 doors, revealing nothing
3. We are given an option to change to the other door.

Should we switch?



Doors A,B,C

Note: If we don't switch,

$$P(\text{Win}) = 1/3$$



In the world where we switch

Without loss of generality, say we pick A (out of Doors A,B,C).

1/3



1/3



1/3



A: Prize in Door A

- Host opens B or C
- We switch
- We always lose

$$P(\text{Win} | A) = 0$$

B: Prize in Door B

- Host must open C
- We switch to B
- We always win

$$P(\text{Win} | B) = 1$$

C: Prize in Door C

- Host must open B
- We switch to C
- We always win

$$P(\text{Win} | C) = 1$$

$$\begin{aligned} P(\text{Win}) &= P(\text{Win}|A)P(A) + P(\text{Win}|B)P(B) + P(\text{Win}|C)P(C) \\ &= \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 = \frac{2}{3} \end{aligned}$$

You should switch!



Some Intuition

https://psetapp.stanford.edu/win26/lecture3/monty_hall

The screenshot shows a web browser window with the URL `psetapp.stanford.edu`. The page is titled "L3 Monty Hall". On the left sidebar, there are navigation icons and a list of questions, with question 1 selected. The main content area contains the following text:

Play the Monty Hall game.

Without loss of generality assume you chose door 1 to start. Monty will show you a door with no prize. Your agent will have to decide to 'stay' or 'switch'

On the right side, there is an "Answer Editor" tab. It contains a code editor with the following Python code:

```
1 """
2 monty_agent.py
3 This file defines an agent "monty_agent" which plays the game of
4 Monty Hall. The function gets called each time it is the agents turn.
5 The agent always choses door 1 to start.
6 The agent can either stay or switch.
7 """
8
9 def monty_agent(door_opened, actions):
10     # default strategy: always stay
11     return 'stay'
```

Below the code editor, there are two buttons: "Run One Game" and "Test Agent". The "Test Agent" button is highlighted in blue. Below these buttons, there is a text box containing the output of the test:

```
playing 1,000 games
Number of rounds won 328.0
```

At the bottom of the page, there are "Previous Question" and "Next Question" buttons. In the bottom left corner, there is a notification icon and the number "43".



Marilyn Vos Savant



Ask Marilyn™

BY MARILYN VOS SAVANT



Some Intuition

https://probabilitycoders.stanford.edu/win26/monty_hall

probabilitycoders.stanford.edu

Envelopes/doors N :

1. Pick a door.
2. The host opens $N - 2$ losing doors.
3. Should you switch?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

$$\left\{ \begin{array}{l} \frac{1}{1000} = P(\text{envelope is prize}) \\ \frac{999}{1000} = P(\text{other 999 envelopes have prize}) \end{array} \right.$$

2. I open 998 of remaining 999 (showing they are empty).

3. Should you switch?

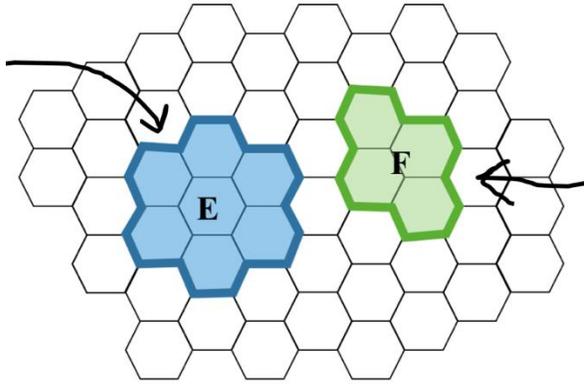
No: $P(\text{win without switching}) =$

$$\frac{1}{\text{original \# envelopes}}$$

Yes: $P(\text{win with new knowledge}) =$

$$\frac{\text{original \# envelopes} - 1}{\text{original \# envelopes}}$$

Learning Goals for Rest of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

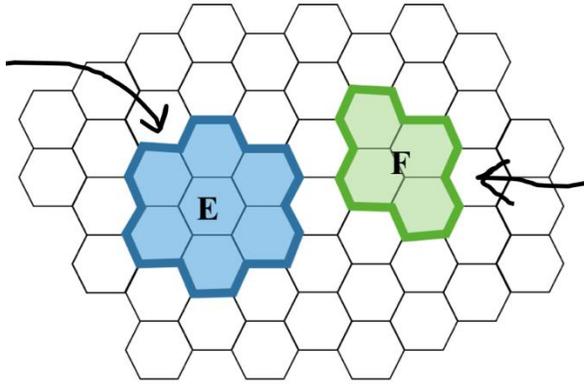
$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Pedagogical Pause

Learning Goals for Rest of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

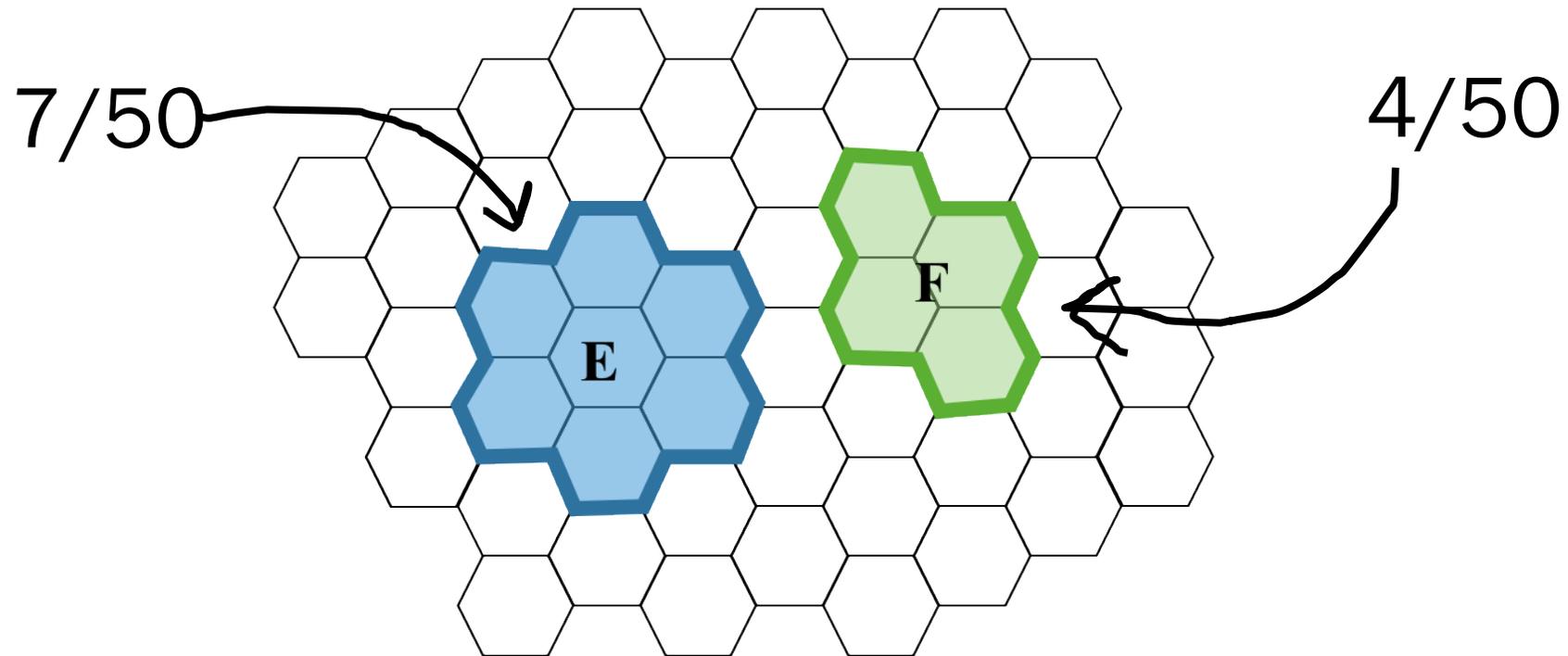
$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Probability of “OR”

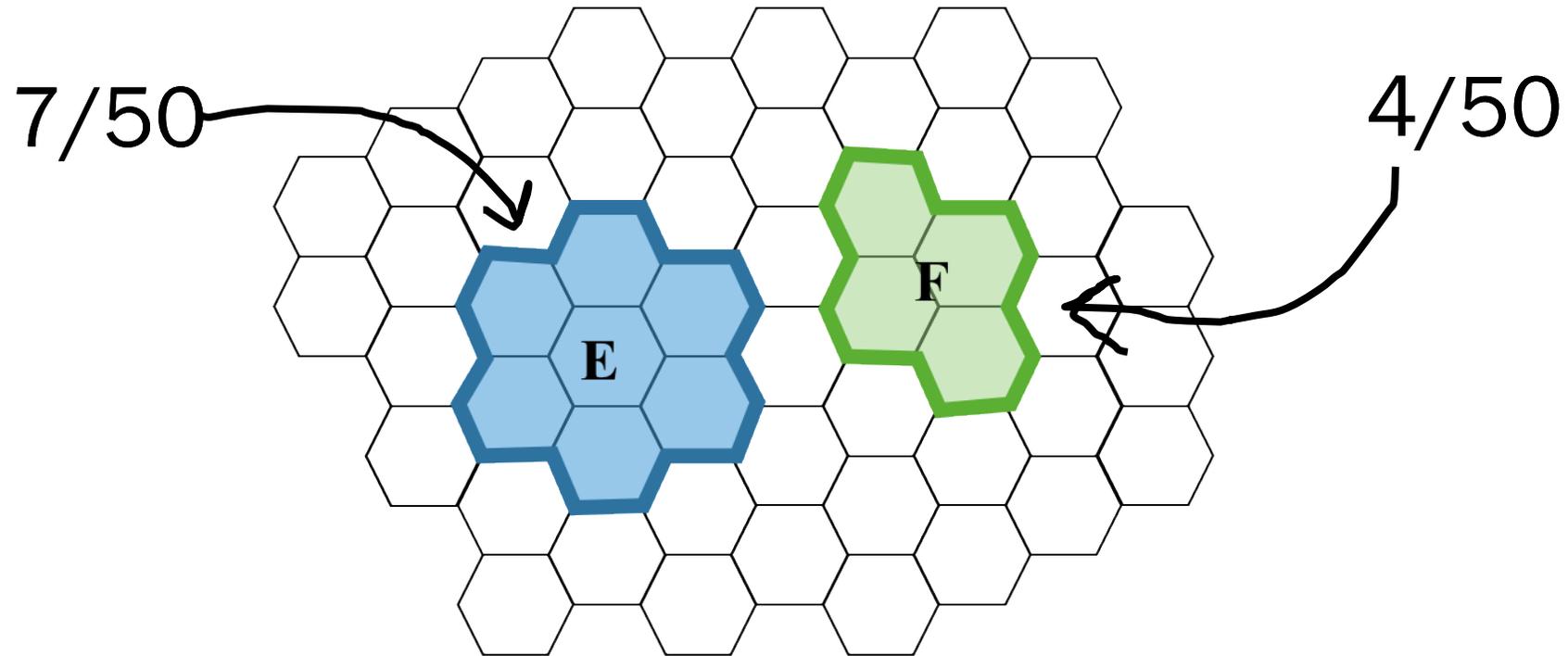
Review: OR with Mutually Exclusive Events



If events are mutually exclusive, probability of OR is simple:

$$P(E \text{ or } F) = P(E) + P(F)$$

Review: OR with Mutually Exclusive Events

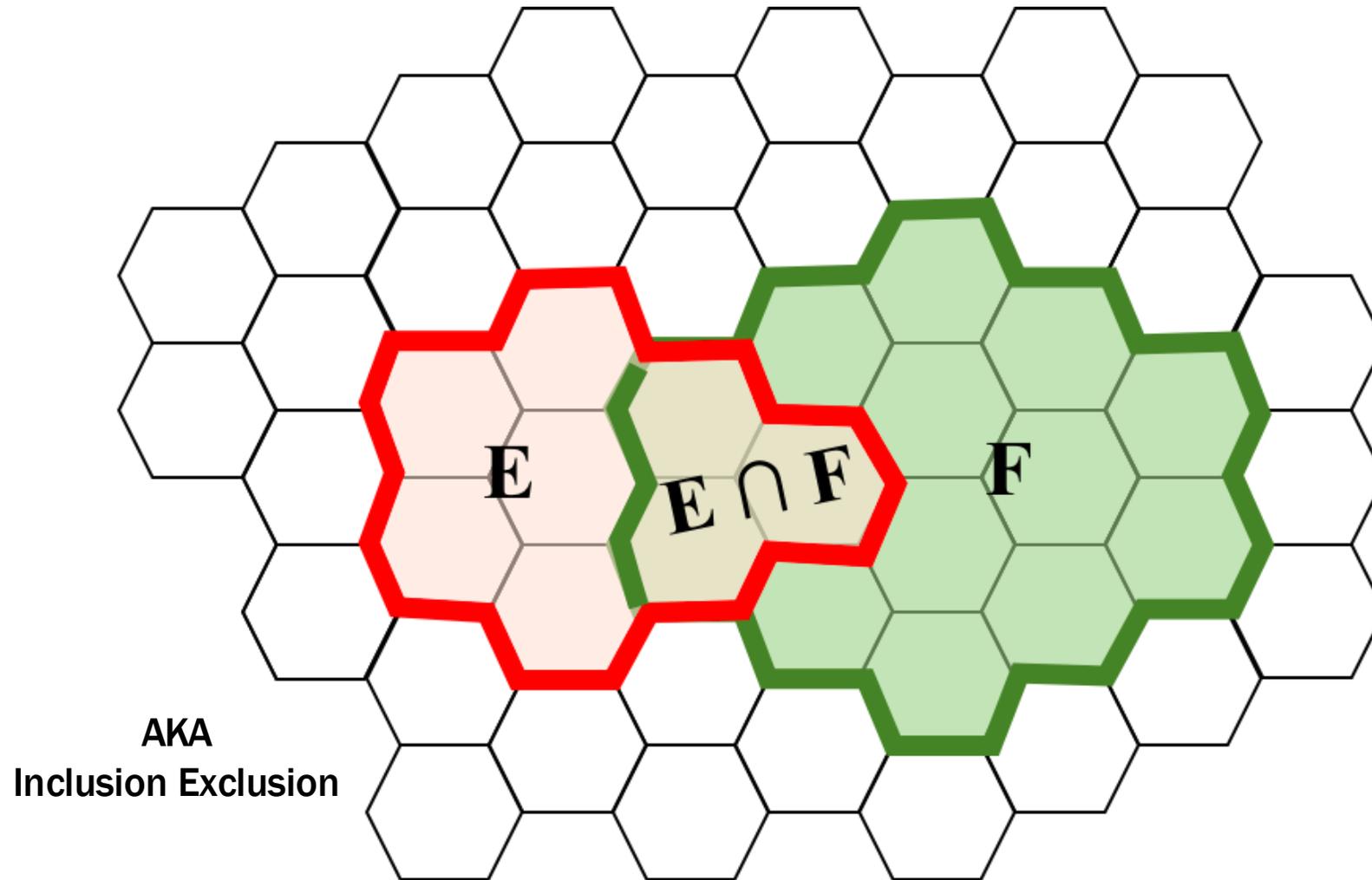


If events are mutually exclusive, probability of OR is simple:

$$P(E \text{ or } F) = \frac{7}{50} + \frac{4}{50} = \frac{11}{50}$$

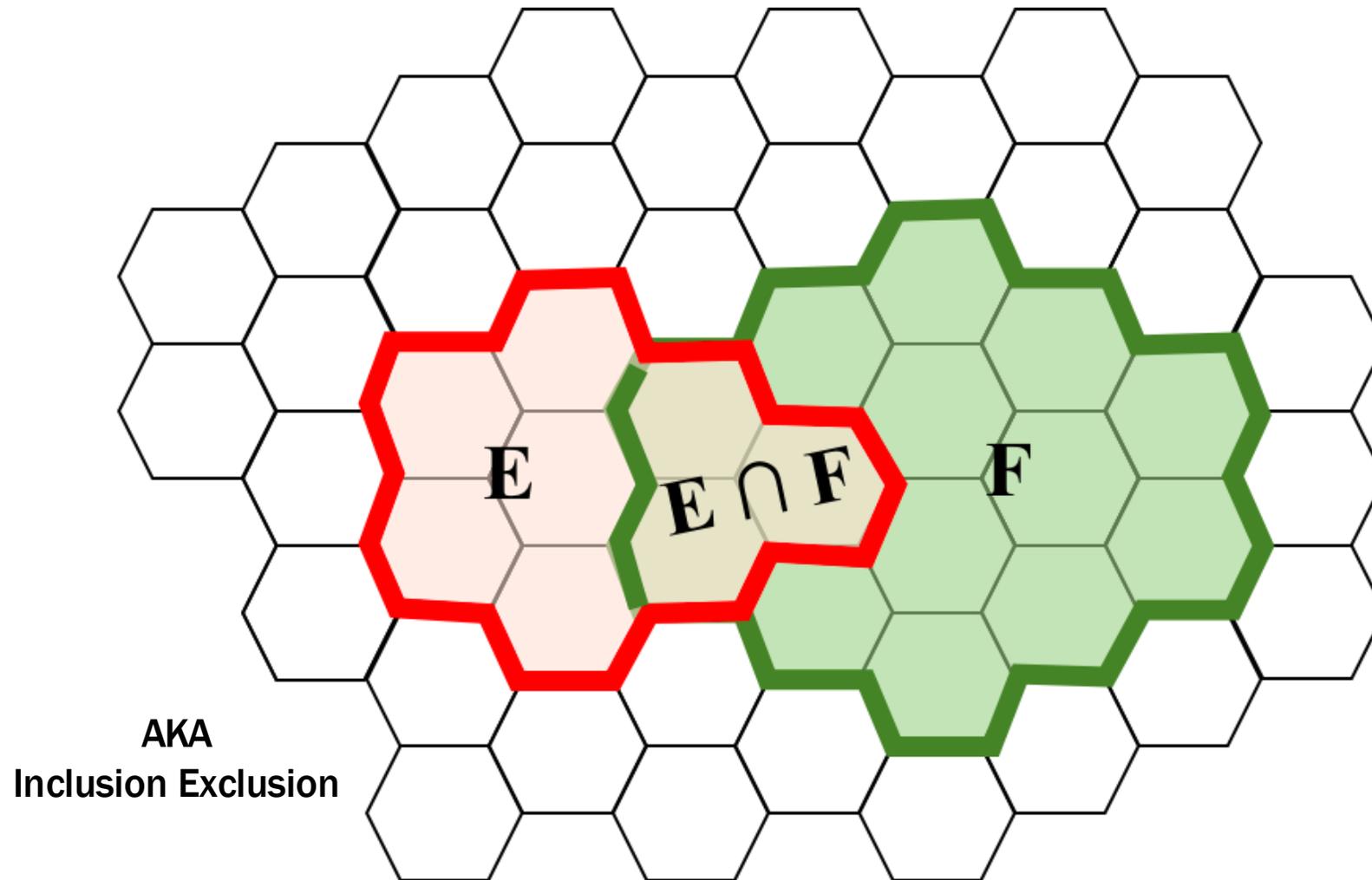
What about when they are not
Mutually exclusive?

OR *without* Mutually Exclusive Events



$$P(E \text{ or } F) = P(E) + P(F) - P(EF)$$

OR *without* Mutually Exclusive Events

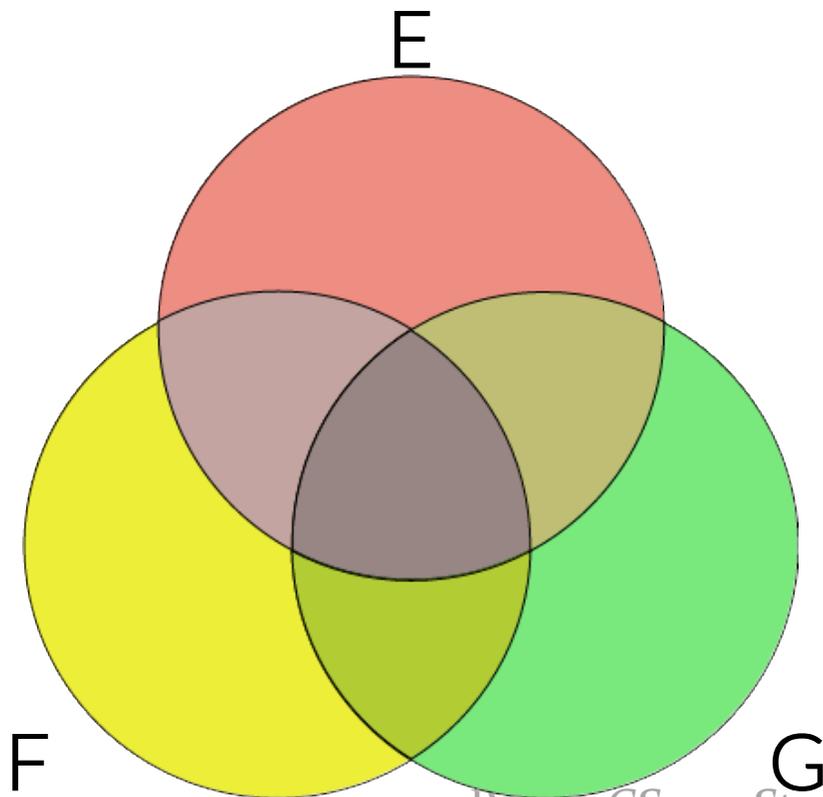


$$P(E \text{ or } F) = \frac{8}{50} + \frac{14}{50} - \frac{3}{50}$$

More than two sets?

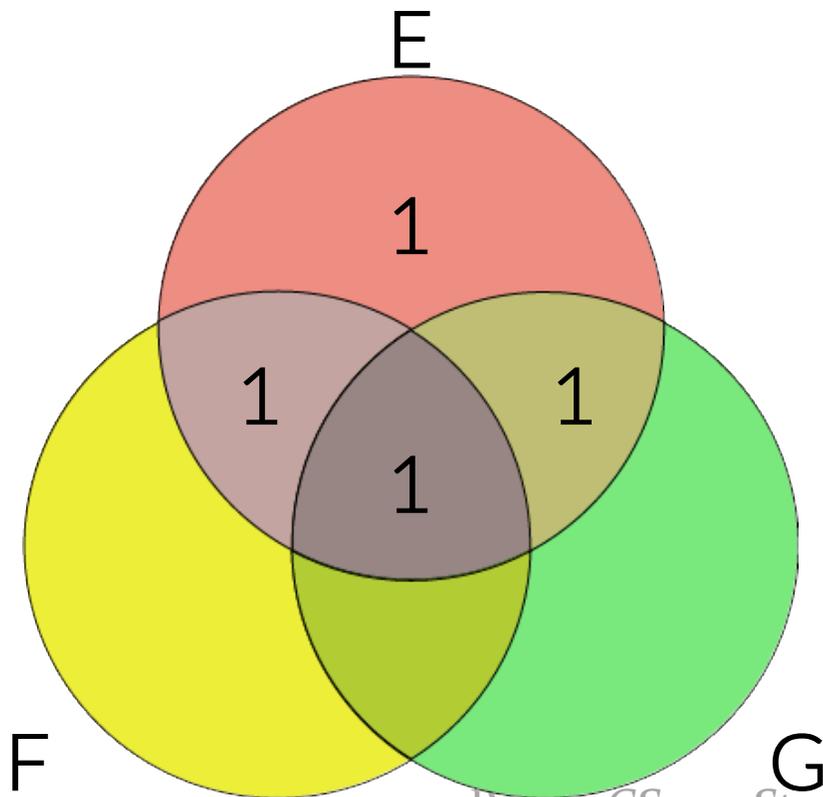
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) =$$



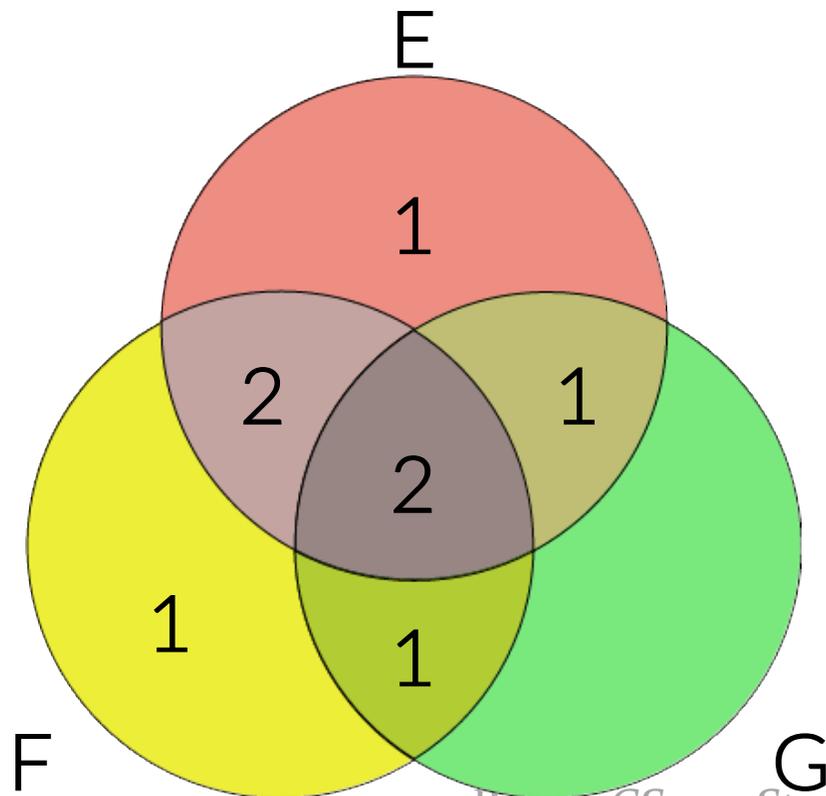
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E)$$



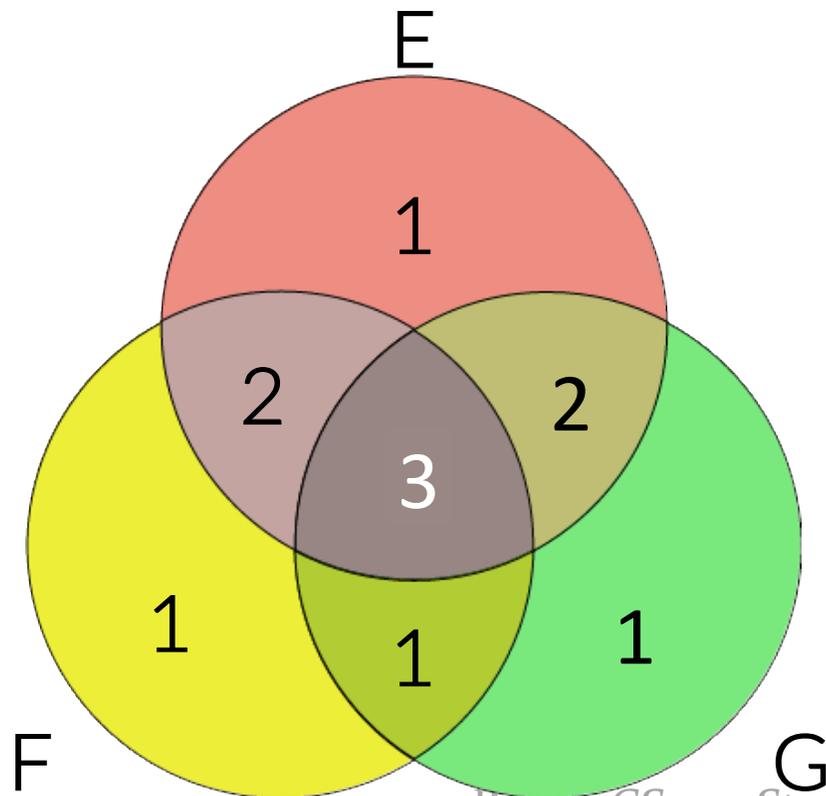
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$$



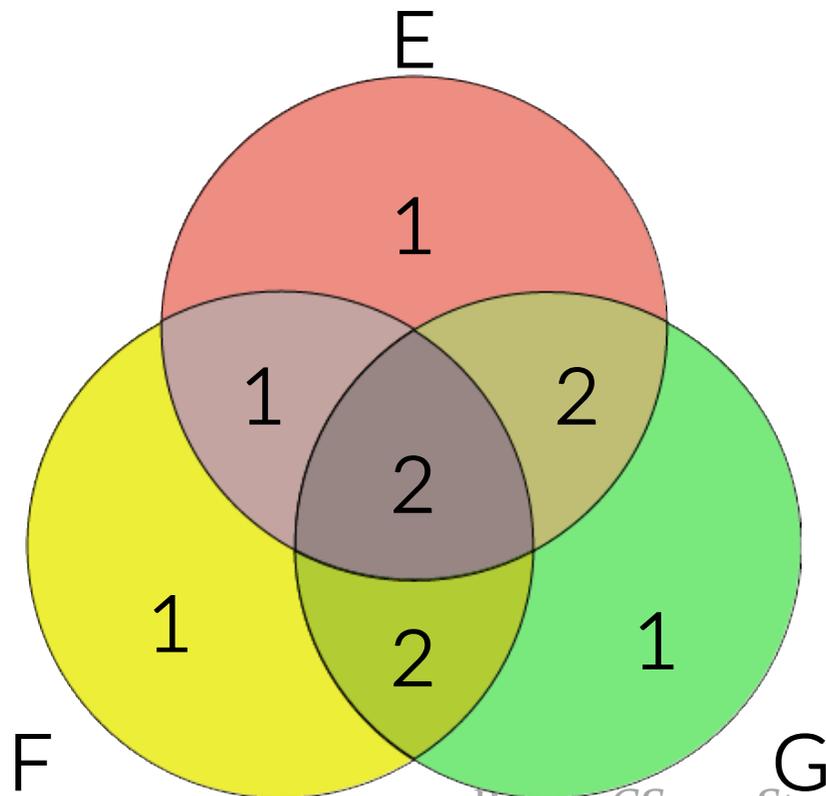
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G)$$



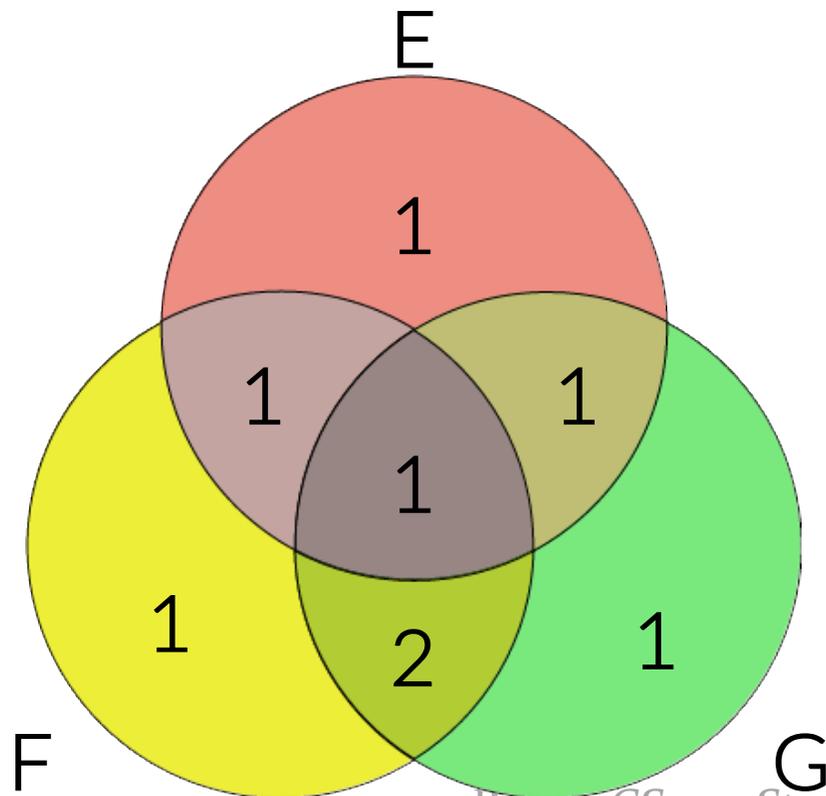
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) - P(EF)$$



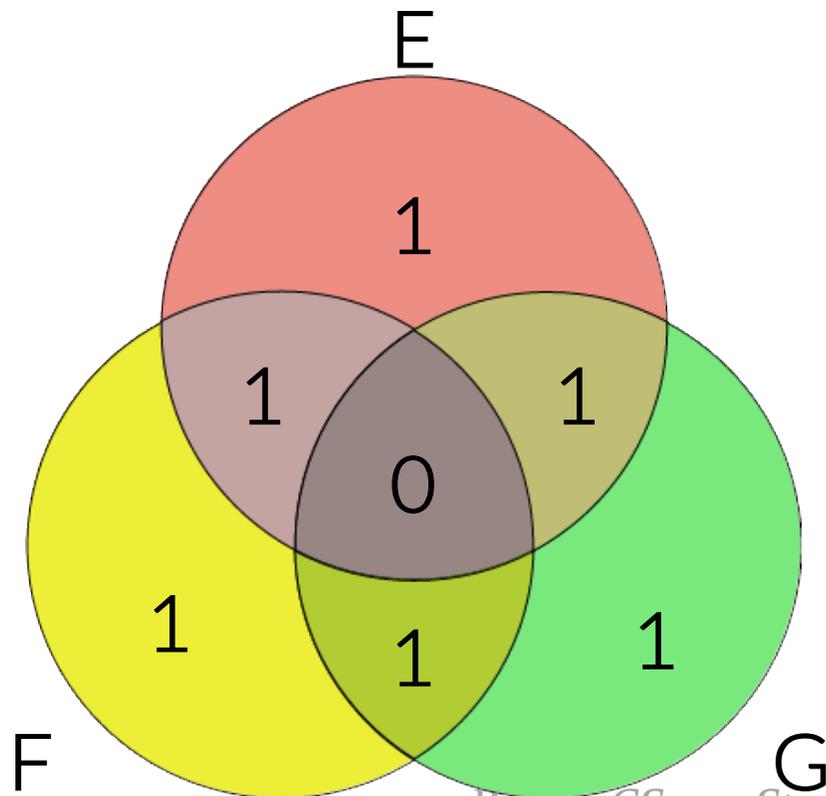
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) \\ - P(EF) - P(EG)$$



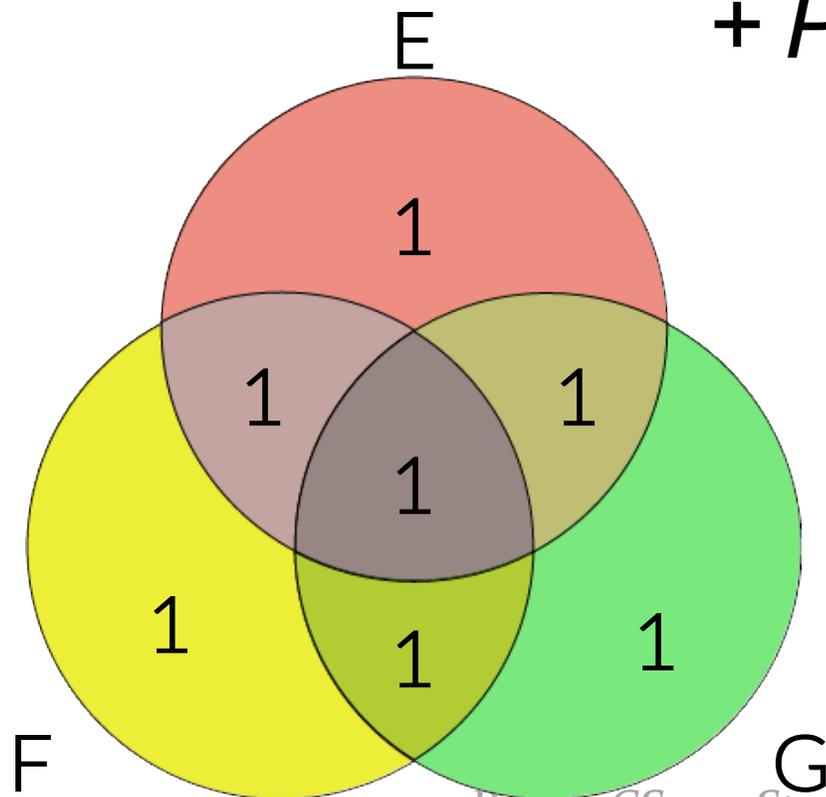
Inclusion / Exclusion with Three Events

$$P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) \\ - P(EF) - P(EG) - P(FG)$$



Inclusion / Exclusion with Three Events

$$\begin{aligned} P(E \text{ or } F \text{ or } G) &= P(E) + P(F) + P(G) \\ &\quad - P(EF) - P(EG) - P(FG) \\ &\quad + P(EFG) \end{aligned}$$



Inclusion / Exclusion with 3 Events

$$\begin{aligned} P(E_1 \text{ or } E_2 \text{ or } E_3) = & \\ & + P(E_1) \\ & + P(E_2) \\ & + P(E_3) \\ & - P(E_1 \text{ and } E_2) \\ & - P(E_1 \text{ and } E_3) \\ & - P(E_2 \text{ and } E_3) \\ & + P(E_1 \text{ and } E_2 \text{ and } E_3) \end{aligned}$$

Inclusion / Exclusion with 4 Events

$$\begin{aligned} P(E_1 \text{ or } E_2 \text{ or } E_3 \text{ or } E_4) = & \\ & + P(E_1) \\ & + P(E_2) \\ & + P(E_3) \\ & + P(E_4) \\ & - P(E_1 \text{ and } E_2) \\ & - P(E_1 \text{ and } E_3) \\ & - P(E_1 \text{ and } E_4) \\ & - P(E_2 \text{ and } E_3) \\ & - P(E_2 \text{ and } E_4) \\ & - P(E_3 \text{ and } E_4) \\ & + P(E_1 \text{ and } E_2 \text{ and } E_3) \\ & + P(E_1 \text{ and } E_2 \text{ and } E_4) \\ & + P(E_1 \text{ and } E_3 \text{ and } E_4) \\ & + P(E_2 \text{ and } E_3 \text{ and } E_4) \\ & - P(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } E_4) \end{aligned}$$

General Inclusion / Exclusion

$$P(E_1 \text{ or } E_2 \text{ or } \dots \text{ or } E_n) = \sum_{r=1}^n (-1)^{r+1} Y_r$$

Y_1 = Sum of all events on their own

$$\sum_i P(E_i)$$

Y_2 = Sum of all pairs of events

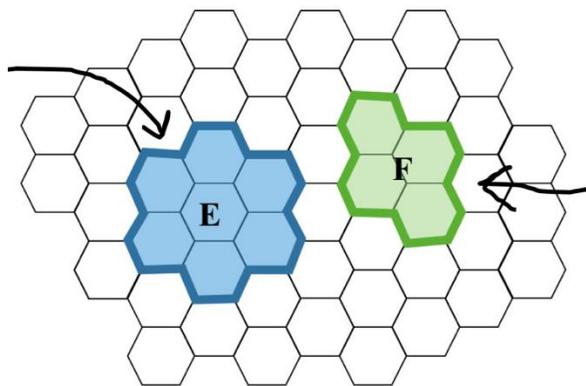
$$\sum_{i,j} P(E_i \text{ and } E_j) \quad \text{s.t. } i \neq j$$

Y_3 = Sum of all triples of events

$$\sum_{i,j,k} P(E_i \text{ and } E_j \text{ and } E_k) \quad \text{s.t. } i \neq j, j \neq k, i \neq k$$

Where Y_r is the sum, for all combinations of r events, of the probability of the intersection of those events

Learning Goals of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$



Probability of “AND”

WE THE PEOPLE
insure domestic Tranquility, provide for the common defence,
and our Posterity, do ordain and establish this Constitution

Article I
Section 1
All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.
Section 2
The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.
Section 3
The Senate of the United States shall be composed of two Senators from each State, chosen by the Legislature thereof, for six Years; and each Senator shall have the Qualifications requisite for Senators of the most numerous Branch of the State Legislature.

Independence

Two events A and B are called **independent** if:

$$P(A) = P(A|B)$$

Knowing that event B happened, doesn't change our belief that A will happen.

Otherwise, they are called **dependent** events

Alternative Definition of Independence

Notation for *and*

$$\begin{aligned} P(A, B) &= P(A) \cdot P(B|A) \\ &= P(A) \cdot P(B) \end{aligned}$$

Chain rule

Since B is independent of A

If you show this is true, you have proved the two events are independent!



If events are *independent*
probability of AND is easy!

*You will need to use this “trick” with high probability

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2

- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 1$
- Let G be event: $D_1 + D_2 = 5$

What is $P(E)$, $P(F)$, and $P(E F)$?

- $P(E) = 1/6$, $P(F) = 1/6$, $P(E F) = 1/36$
- $P(E F) = P(E) P(F) \rightarrow E$ and F independent

What is $P(E)$, $P(G)$, and $P(E G)$? $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$

- $P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(E G) = 1/36$
- $P(E G) \neq P(E) P(G) \rightarrow E$ and G dependent

Intuition through proofs:

Independence is reciprocal

If A is independent of B, then B is independent of A

$$P(A) = P(A|B)$$

$$P(B|A) = P(B)$$

Proof:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Bayes' Thm.

$$= \frac{P(A)P(B)}{P(A)}$$

Because A is independent of B

$$= P(B)$$

Independence of a complement

Given independent events A and B , prove that A and B^C are independent

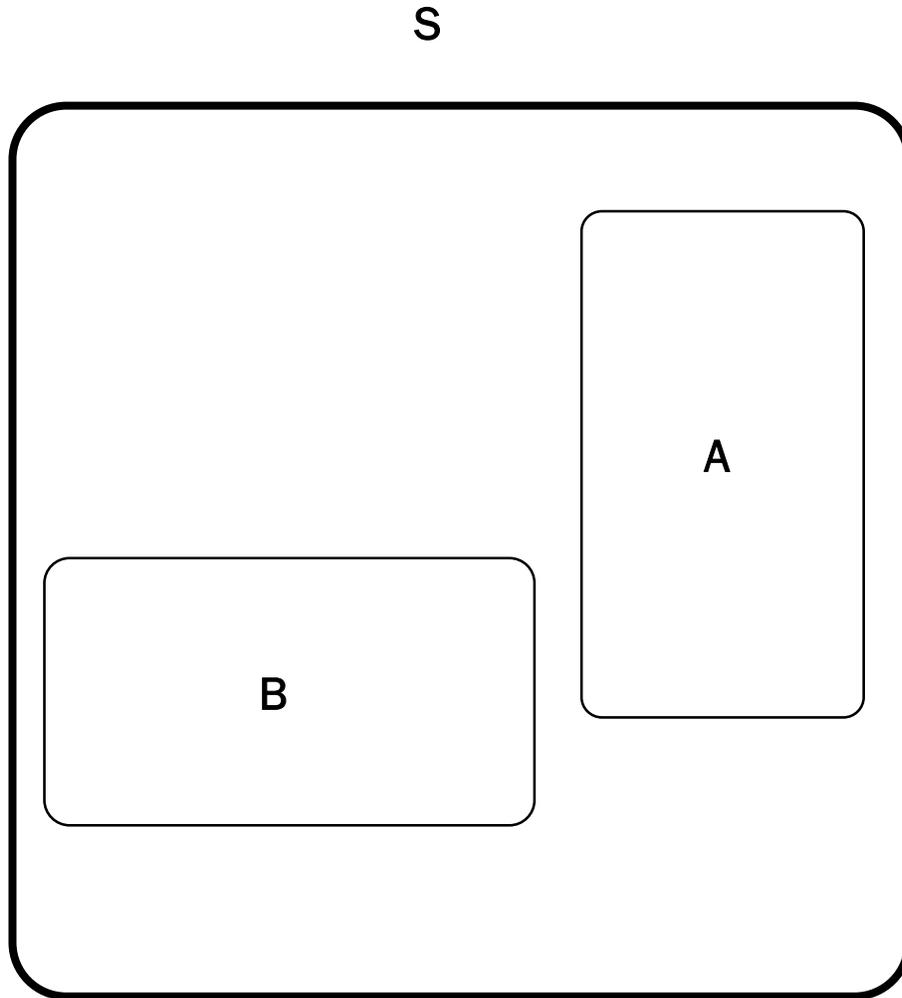
We want to show that $P(AB^C) = P(A)P(B^C)$

$$\begin{aligned}P(AB^C) &= P(A) - P(AB) && \text{By Total Law of Prob.} \\ &= P(A) - P(A)P(B) && \text{By independence} \\ &= P(A)[1 - P(B)] && \text{Factoring} \\ &= P(A)P(B^C) && \text{Since } P(B) + P(B^C) = 1\end{aligned}$$

So if A and B are independent A and B^C are also independent

What does independence look like?

Independence ???



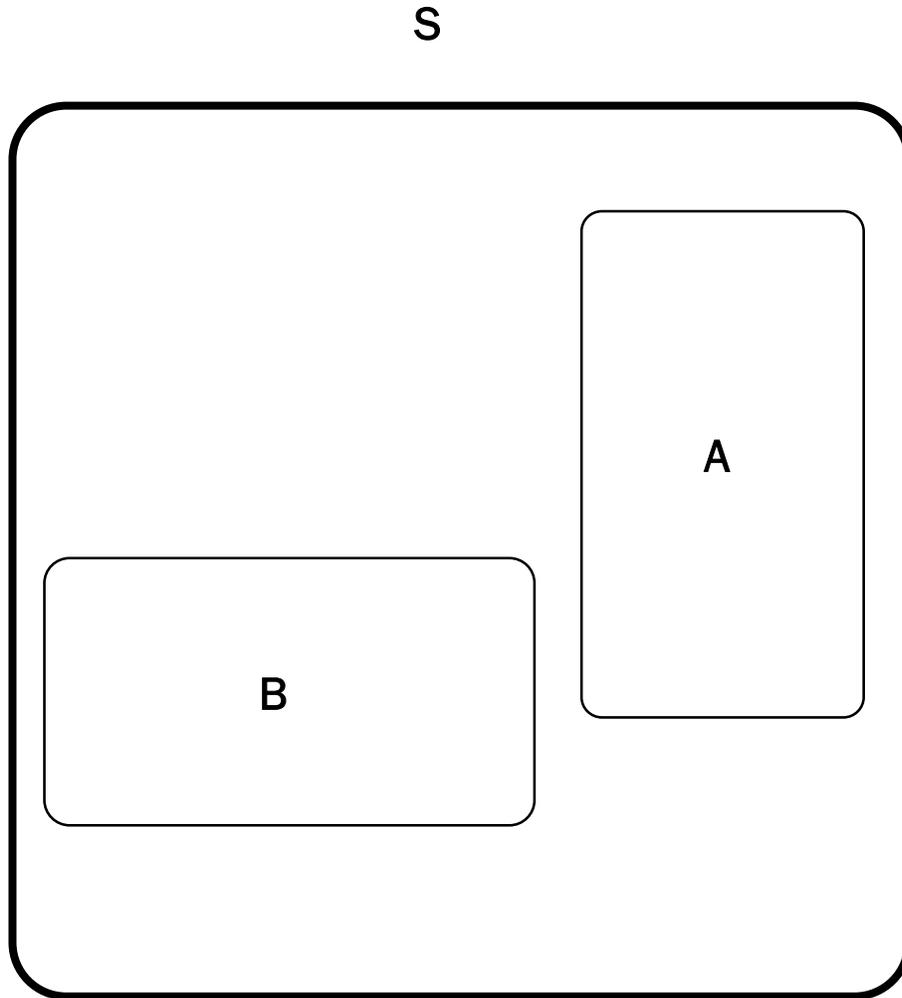
Independence Definition 1:

$$P(A \cap B) = P(A)P(B)$$
$$\frac{|A \cap B|}{|S|} = \frac{|A|}{|S|} \cdot \frac{|B|}{|S|}$$

Note: In the original image, a black arrow points to the intersection term $|A \cap B|$ in the second equation, and a small '0' is written above the arrow.

Independence ???

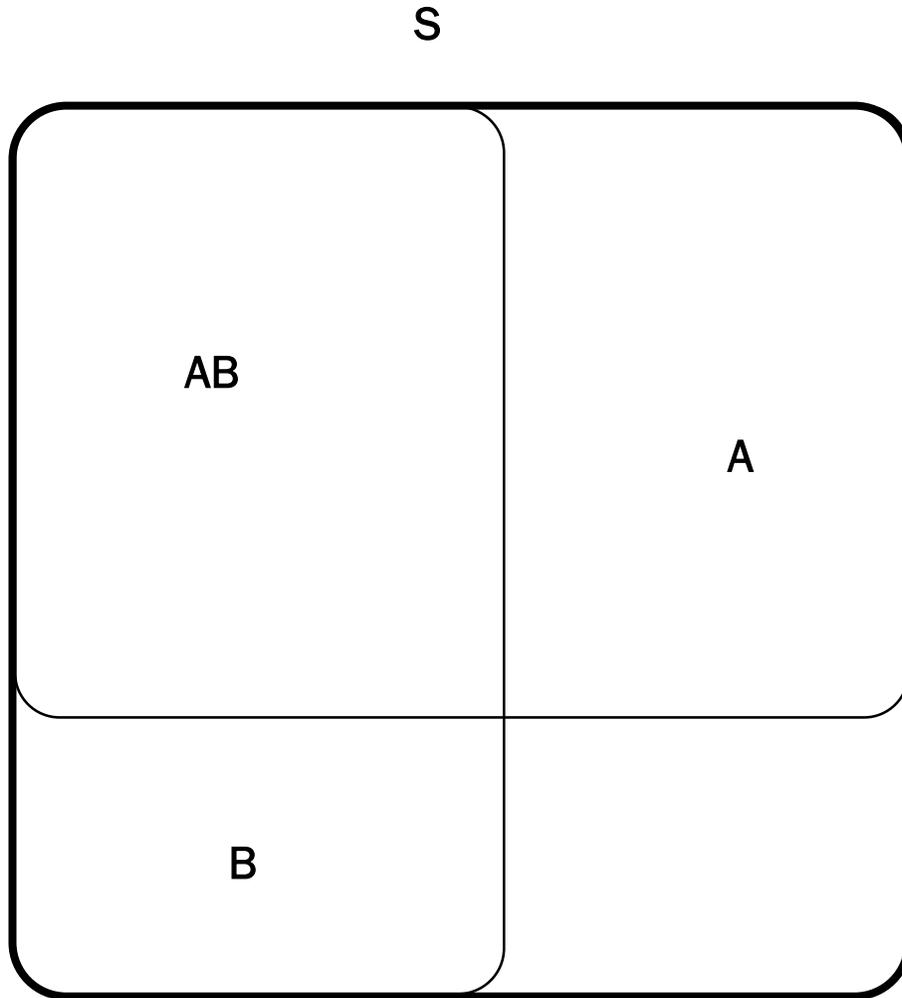
Mutually Exclusive events are NOT independent.



Independence Definition 1:

$$P(AB) = P(A)P(B)$$
$$\frac{|AB|^0}{|S|} = \frac{|A|}{|S|} \rightarrow \frac{|B|}{|S|}$$

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \rightarrow \frac{|B|}{|S|}$$

Independence Definition 2:

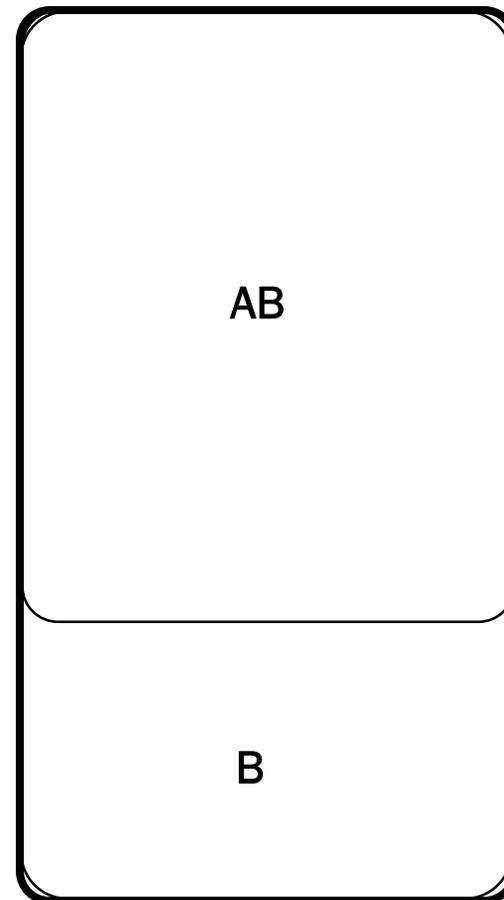
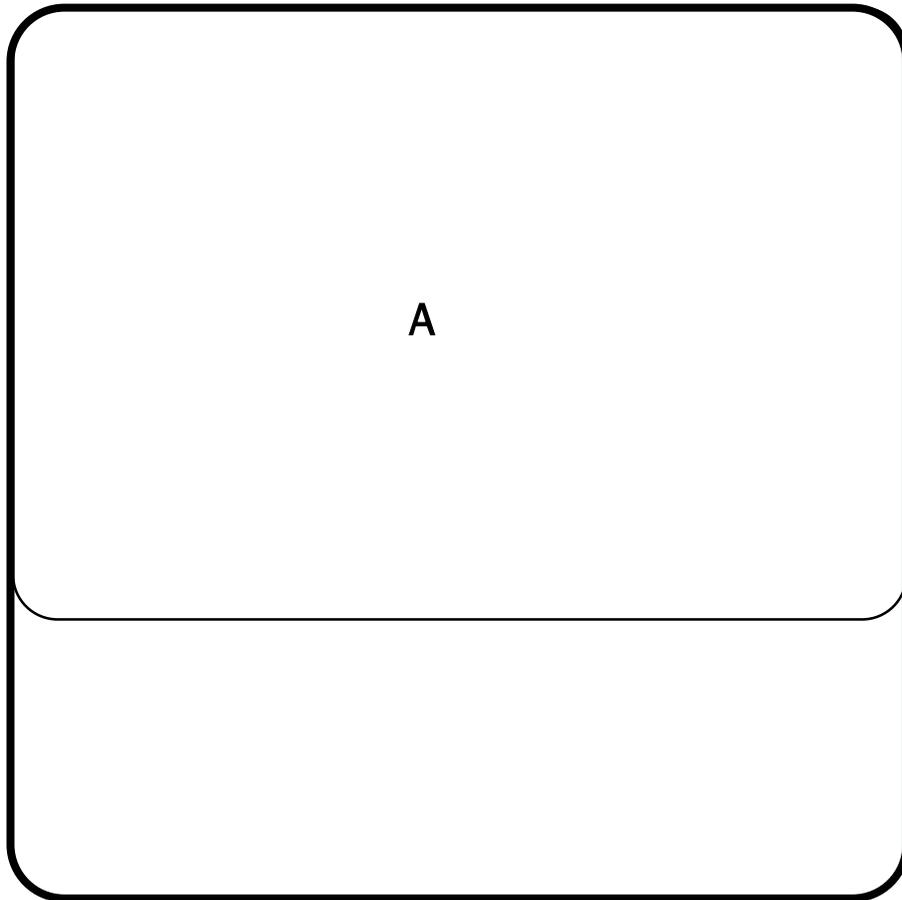
$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Independence

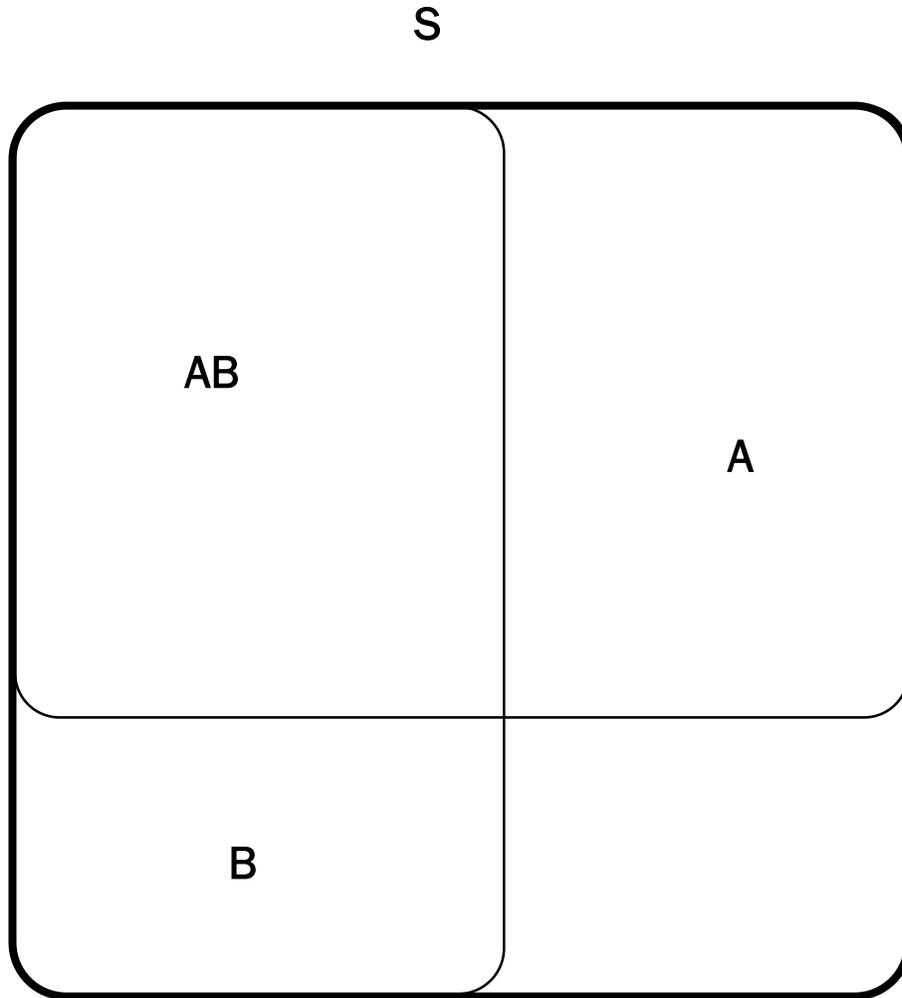
This ratio, $P(A)$...

... is the same as this one, $P(A|B)$



S

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

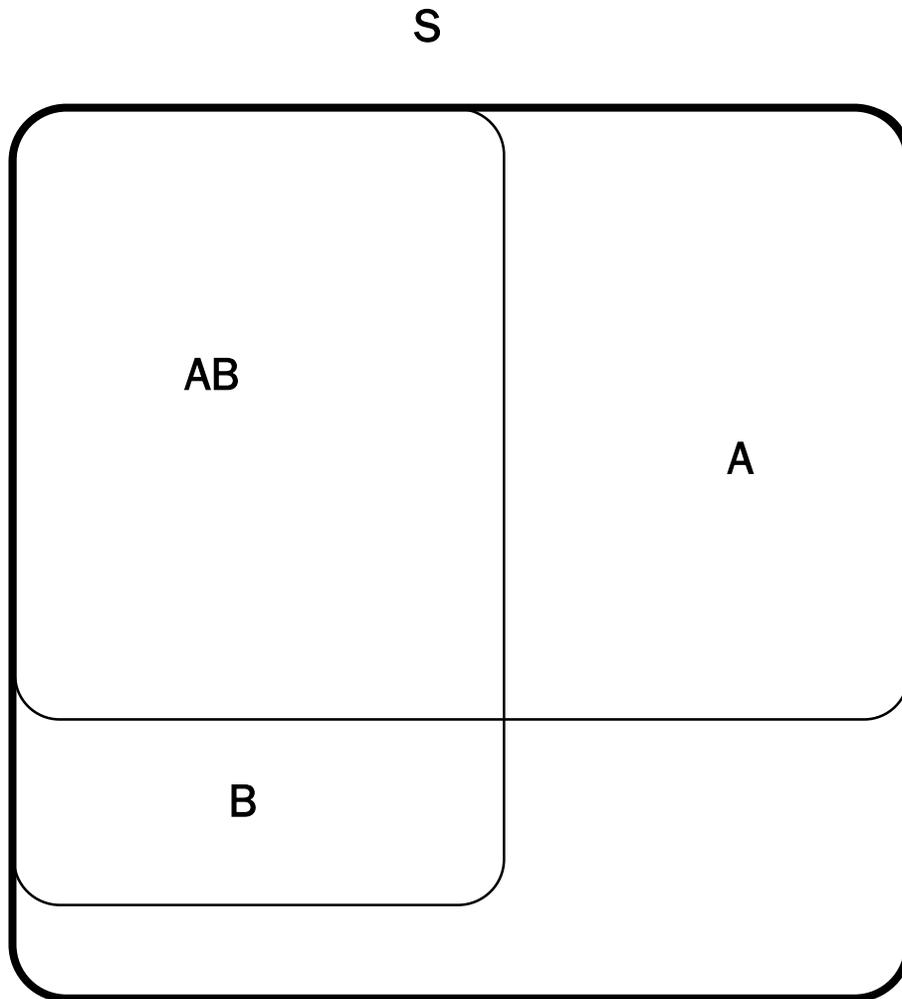
$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \not\rightarrow \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Dependence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \not\rightarrow \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Generalized Independence

General definition of Independence:

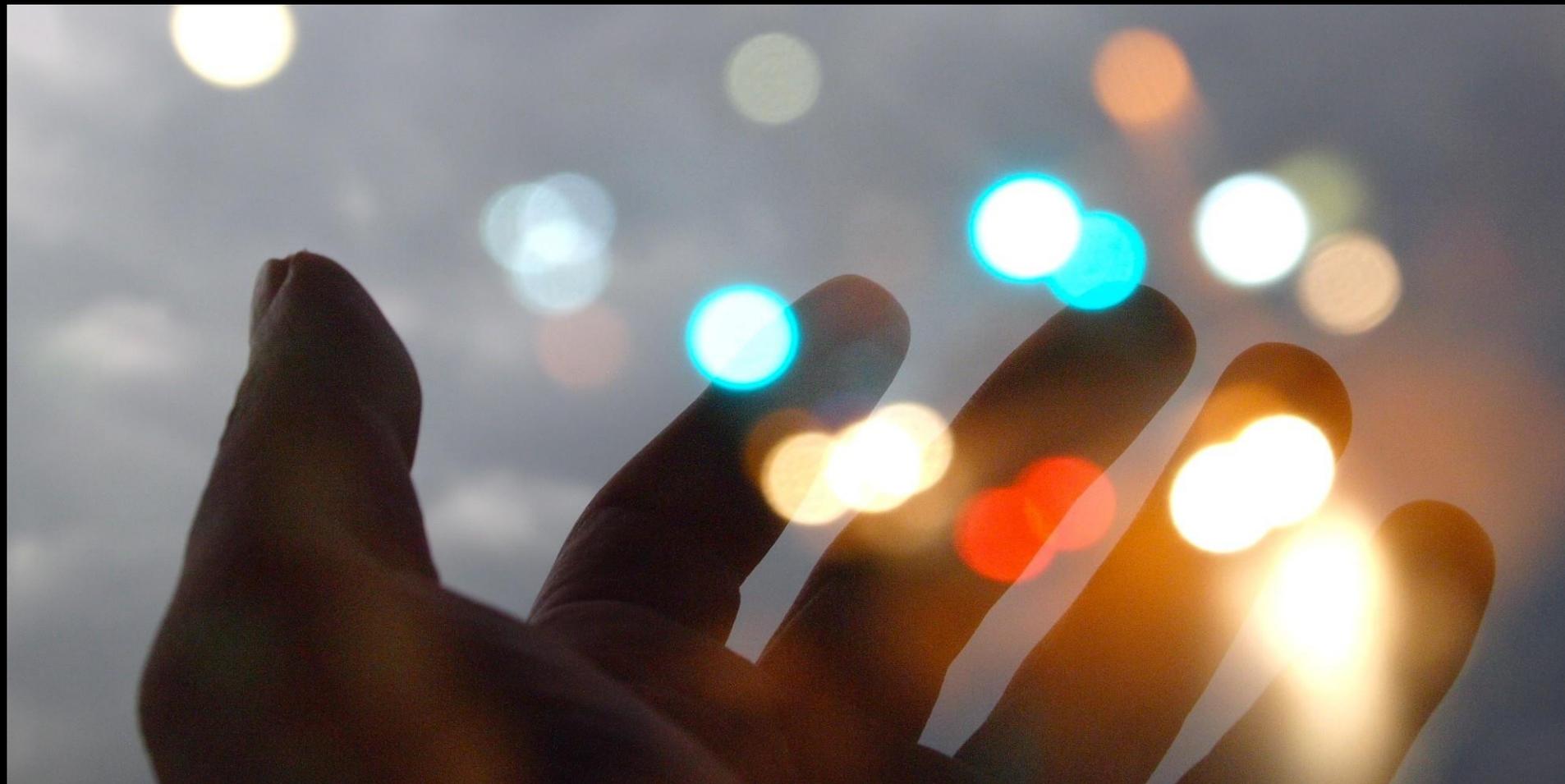
Events E_1, E_2, \dots, E_n are independent if **for every subset** with r elements (where $r \leq n$) it holds that:

$$P(E_1, E_2, E_3, \dots, E_r) = \prod_i P(E_i)$$

Example: outcomes of n separate flips of a coin are all independent of one another

- Each flip in this case is called a “trial” of the experiment

Math > Intuition



Two Dice

Roll two 6-sided dice, yielding values D_1 and D_2

- Let E be event: $D_1 = 1$
- Let F be event: $D_2 = 6$
- Let G be event: $D_1 + D_2 = 7$

1. Are E and F independent? **Yes!**

2. Are E and G independent? **Yes!**

- $P(E) = 1/6$, $P(G) = 1/6$, $P(E \cap G) = 1/36$ [roll (1, 6)]

3. Are F and G independent? **Yes!**

- $P(F) = 1/6$, $P(G) = 1/6$, $P(F \cap G) = 1/36$ [roll (1, 6)]

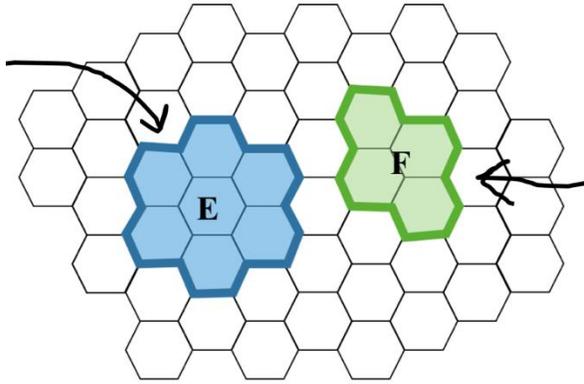
4. Are E , F and G independent? **No!**

- $P(E \cap F \cap G) = 1/36 \neq 1/216 = (1/6)(1/6)(1/6)$

New Ability



Properties of Pairs of Events



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

also:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

also:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$



Story: Ultimate Probability



Ultimate Probability

3,290 views • 1 Dec 2018

👍 14 💬 0 ➦ SHARE ⌵ SAVE ...



Maika Isogawa
21 subscribers

SUBSCRIBE

<https://www.maikaisogawa.com/ultimate-frisbee-probability/>

<https://www.youtube.com/watch?v=H2IfTwGisOg>



Practice: Lets do the Frisbee Problem.

You flip **two frisbees**. For each frisbee, the probability that it lands “**heads**” is **0.6**. The two frisbees are considered “**even**” if both frisbees are heads **or** both frisbees are tails.

What is the probability that the frisbees are even?



Same Problem!

Let p be the probability of a 1 from `unknown_random`.

What is the probability of a True from `fair_random` if $p = .45$?

```
def fair_random():  
    """  
    There are four outcomes for assignments to r1 and r2:  
    [0, 0], [0, 1], [1, 0], [1, 1]. Return 1 if the  
    outcomes are [0, 0] or [1, 1]  
    """  
    r1 = unknown_random()  
    r2 = unknown_random()  
    return r1 == r2
```

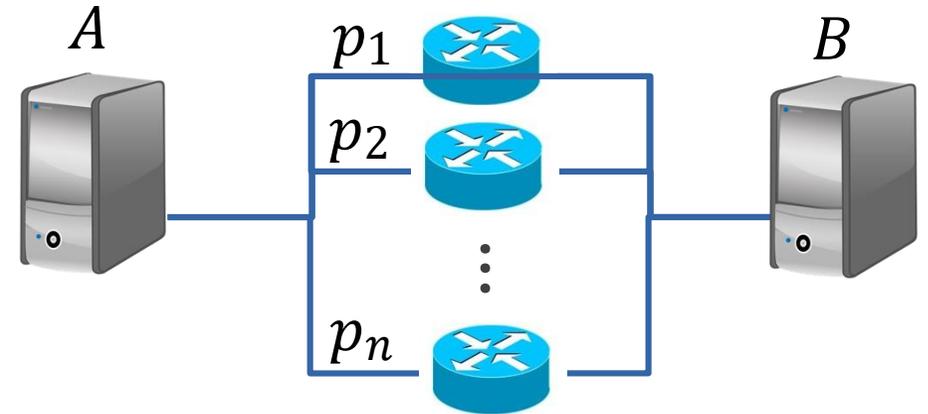


Network reliability

Consider the following parallel network:

- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.

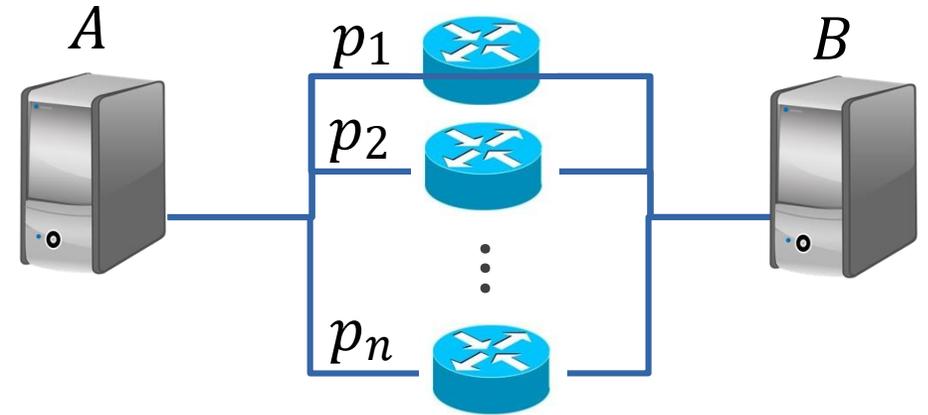
What is $P(E)$?



Network reliability

Consider the following parallel network:

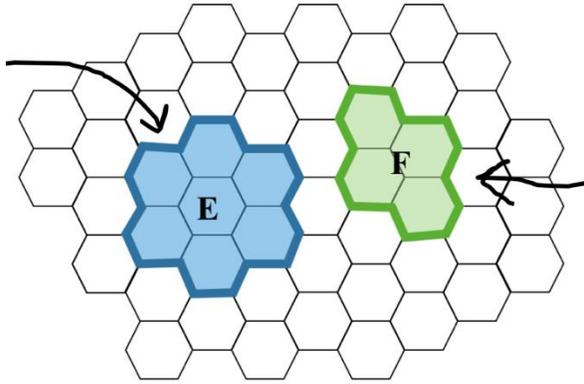
- n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- E = functional path from A to B exists.



What is $P(E)$?

$$\begin{aligned} P(E) &= P(\geq 1 \text{ one router works}) \\ &= 1 - P(\text{all routers fail}) \\ &= 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n) \\ &= 1 - \prod_{i=1}^n (1 - p_i) \end{aligned}$$

Learning Goals of Today



Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$



Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Here we are



Source: The Hobbit

G_1

G_2

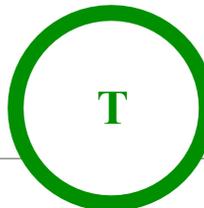
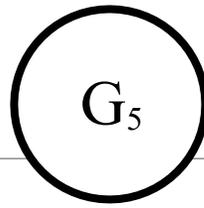
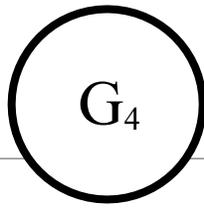
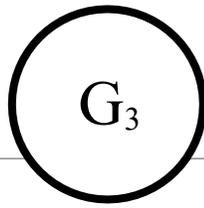
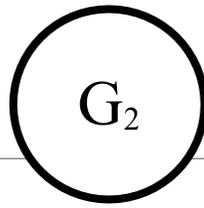
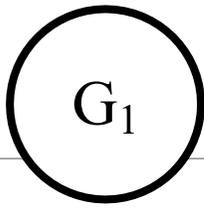
G_3

G_4

G_5

T





```

dna.txt
1 False,True,False,False,True,False
2 True,True,False,True,True,False
3 True,True,False,True,True,True
4 False,True,False,True,True,False
5 False,True,False,False,True,False
6 True,True,False,True,True,True
7 False,False,True,False,False,False
8 False,False,True,False,True,False
9 True,False,False,True,False,False
10 False,True,False,True,True,False
11 True,False,False,True,False,False
12 True,False,True,True,False,False
13 False,True,False,False,True,False
14 False,False,True,True,False,False
15 True,True,False,False,True,True
16 True,False,True,True,False,False
17 True,True,True,True,True,True |
18 True,False,True,False,False,True
19 False,True,False,True,True,True
20 False,False,True,False,False,False
21 False,False,False,True,True,False
22 False,True,False,False,True,False
23 True,True,False,True,True,True
24 False,True,False,True,True,False
25 True,False,False,False,False,True
26 False,False,True,True,False,True
27 False,False,False,True,False,False
28 False,True,True,False,False,True
29 False,True,False,False,True,True
30 False,False,False,False,False,True
31 False,True,False,True,True,False
32 True,False,False,True,False,False
33 True,True,False,True,True,True
34 True,True,False,False,True,True
35 True,True,False,True,True,True
36 False,False,False,True,False,False
..

```



100,000 samples

6 observations per sample



Discovered Pattern

```
woodrow/win26_cs109  $ python findStructure.py
size data = 100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

■ ■ ■

```
p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
```

Discovered Pattern

```
woodrow/win26_cs109 $ python findStructure.py
```

```
size data = 100000
```

```
p(G1) = 0.500
```

```
p(G2) = 0.545
```

```
p(G3) = 0.299
```

```
p(G4) = 0.701
```

```
p(G5) = 0.600
```

```
p(T) = 0.390
```

```
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
```

```
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
```

```
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
```

```
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
```

```
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

■■■

```
p(T and G5 | G2) = 0.450
```

```
p(T | G2)p(G5 | G2) = 0.450
```

Discovered Pattern

```
woodrow/win26_cs109 $ python findStructure.py
```

```
size data = 100000
```

```
p(G1) = 0.500
```

```
p(G2) = 0.545
```

```
p(G3) = 0.299
```

```
p(G4) = 0.701
```

```
p(G5) = 0.600
```

```
p(T) = 0.390
```

```
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
```

```
p(T and G2) = 0.200 , P(T)p(G2) = 0.212
```

```
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
```

```
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
```

```
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

■■■

```
p(T and G5 | G2) = 0.450
```

```
p(T | G2)p(G5 | G2) = 0.450
```

Discovered Pattern

```
woodrow/win26_cs109 $ python findStructure.py
```

```
size data = 100000
```

```
p(G1) = 0.500
```

```
p(G2) = 0.545
```

```
p(G3) = 0.299
```

```
p(G4) = 0.701
```

```
p(G5) = 0.600
```

```
p(T) = 0.390
```

```
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
```

```
p(T and G2) = 0.200 , P(T)p(G2) = 0.213
```

```
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
```

```
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
```

```
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

■■■

```
p(T and G5 | G2) = 0.450
```

```
p(T | G2)p(G5 | G2) = 0.450
```

Discovered Pattern

```
woodrow/win26_cs109 $ python findStructure.py
size data = 100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

■ ■ ■

```
p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
```

T and G5 are NOT
independent !!

Discovered Pattern

```
woodrow/win26_cs109  $ python findStructure.py
size data = 100000
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```

■ ■ ■

```
p(T and G5 | G2) = 0.450
p(T | G2)p(G5 | G2) = 0.450
```

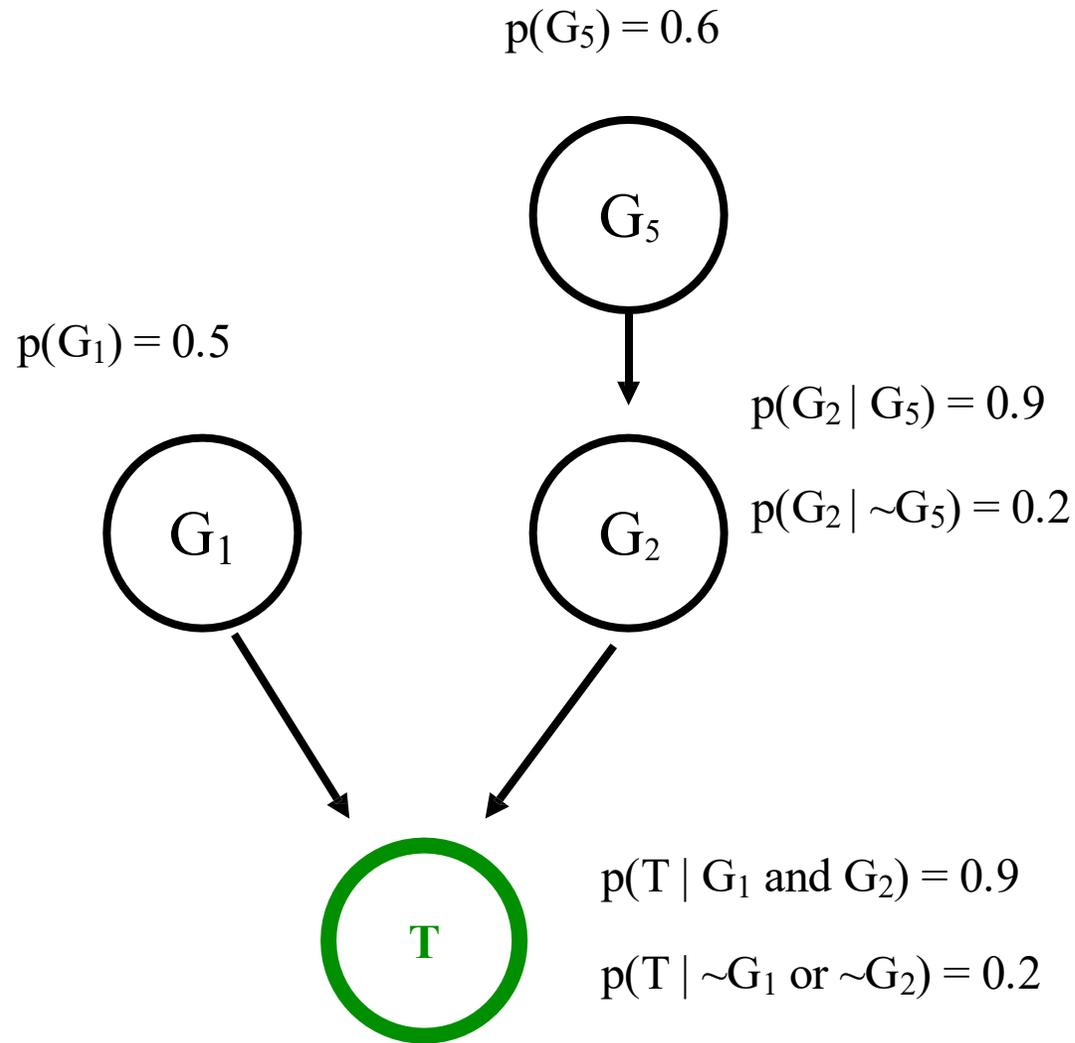


Independence
relationships can change
with conditioning.

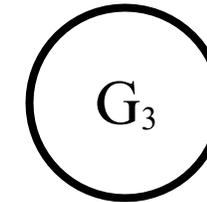
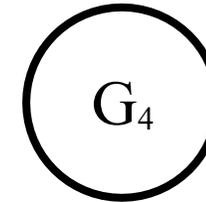
If E and F are independent, that does not mean they will still be independent given another event G .

Only Causal Structure That Fits

$$p(T \text{ and } G_5 \mid G_2) = 0.450$$
$$p(T \mid G_2)p(G_5 \mid G_2) = 0.450$$



These genes don't impact T



Example Conditional Independence

Example Conditional Independence

F is event that
sprinkler.



Sprinkler went on
this morning



Rained this
morning

R is event that it
rained.

Example Conditional Independence

F is event that
sprinkler.



Sprinkler went on
this morning



Rained this
morning

R is event that it
rained.

G is event that grass
is wet.



Grass is Wet

See you Monday