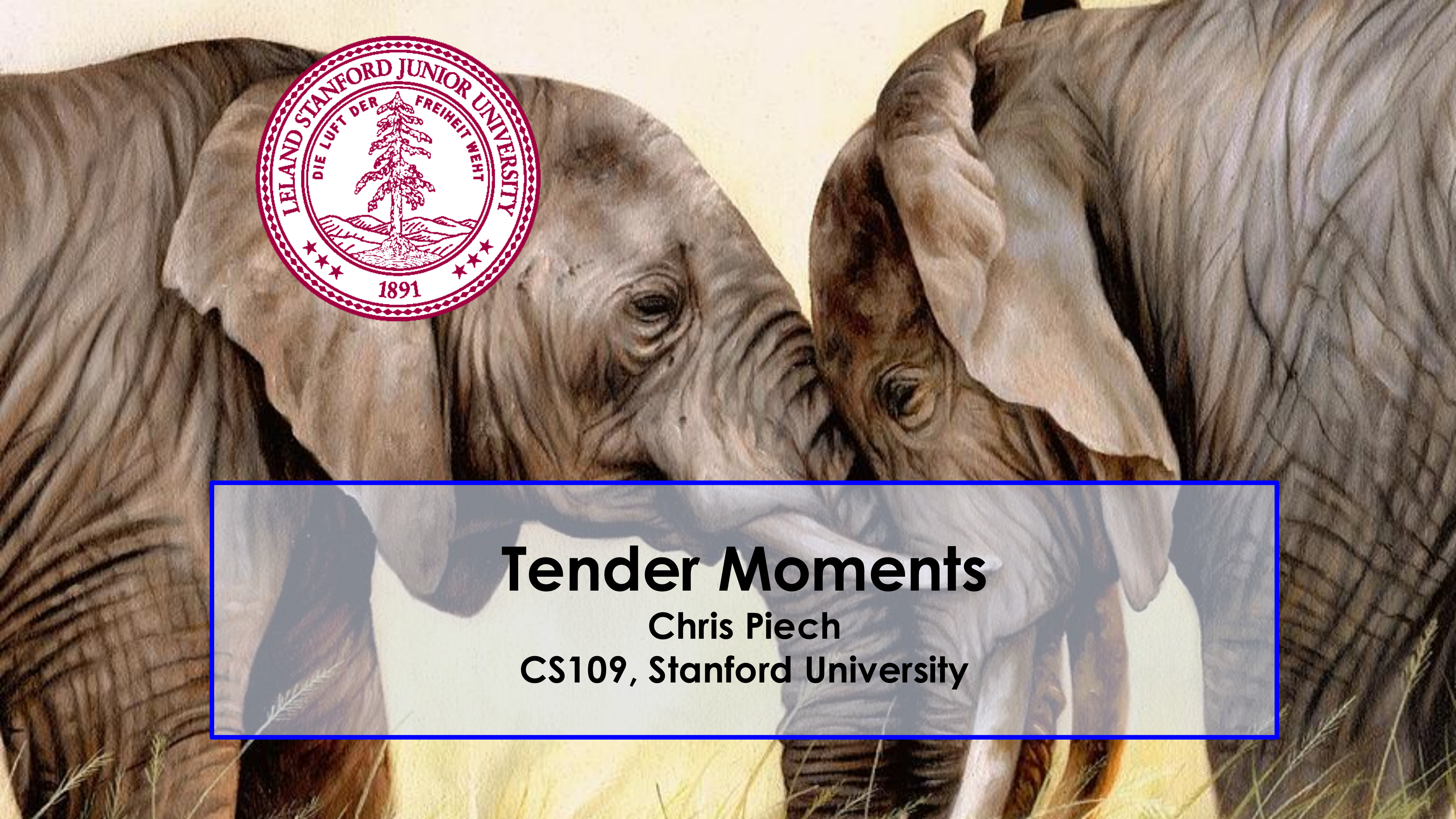

Tender Moments

Chris Piech
CS109, Stanford University

Learning Goals

1. Use New Random Variables!
2. Calculate Expectation of a Random Variable

Quick Announcements

Python review session 2 – after class today !! (will be recorded)

Pset 2 goes out today !! Due Jan 23rd (next Friday)

With MLK holiday – we made Pset 2 a bit shorter than normal so that you don't have to work over the holiday if you don't want to.

That said – get started early ☺

No lecture on Monday

Review

A **random variable** is a number which takes on values probabilistically.

A discrete random variable is fully described by a **probability mass function**.

Let Y be a random variable

$$Y$$

For example Y is the number of heads in 5 coin flips

Let Y be a random variable

$$Y = 2$$

*note: here equals means == in coding

It is an event when
Y takes on a value

For example Y is the number of heads in 5 coin flips

Let Y be a random variable

$$Y < 3$$

It is an event when
you ask any comparison question

For example Y is the number of heads in 5 coin flips

If this is a number

$$P(Y = 2)$$

Then this is a probability (between 0 and 1)

For example Y is the number of heads in 5 coin flips

If this is a variable

$$P(Y = k)$$

Then this is a function

For example Y is the number of heads in 5 coin flips

This is a function

$$P(Y = k)$$

$$k = 5$$

$$0.03125$$

For example Y is the number of heads in 5 coin flips

Random Variables are a big deal, because they allow other people to give you a PMF (and other helpful equations)

Classics

Declare a Random Variable to be Binomial

Our random variable

Num trials

Probability of success on each trial

$$X \sim \text{Bin}(n, p)$$

Is distributed as a

Binomial

With these parameters

Exactly k heads in n coin flips. Probability of exactly k heads:

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
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(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Automatically Know the PMF

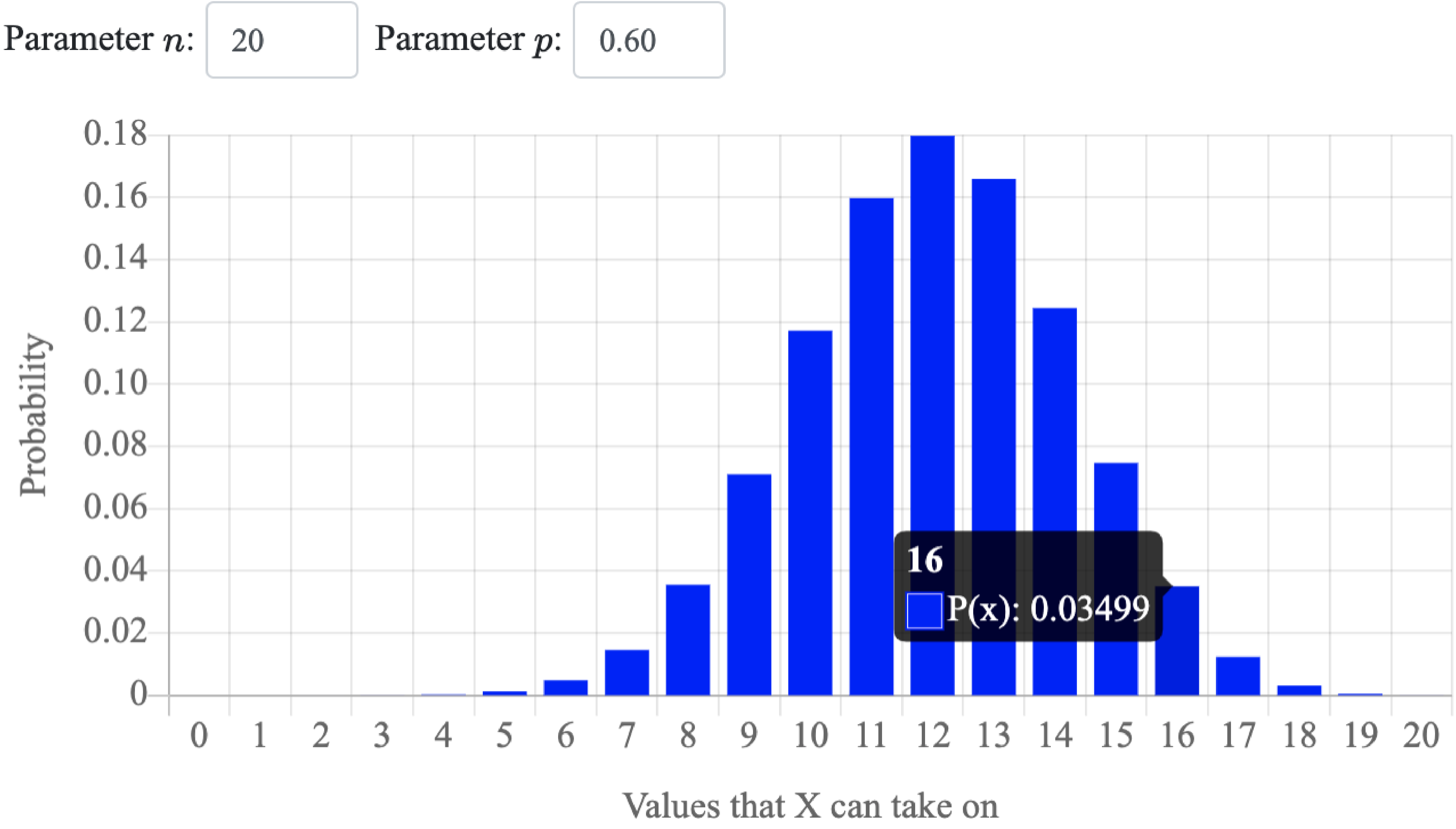
Probability Mass Function for a Binomial

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

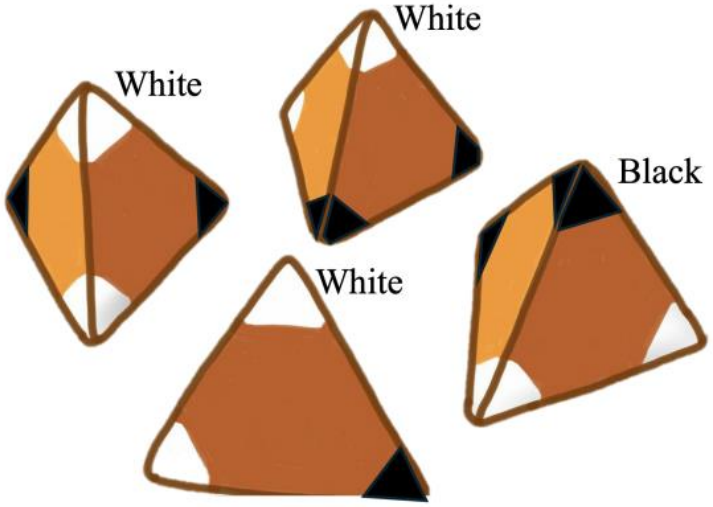
Probability that our variable takes on the value k

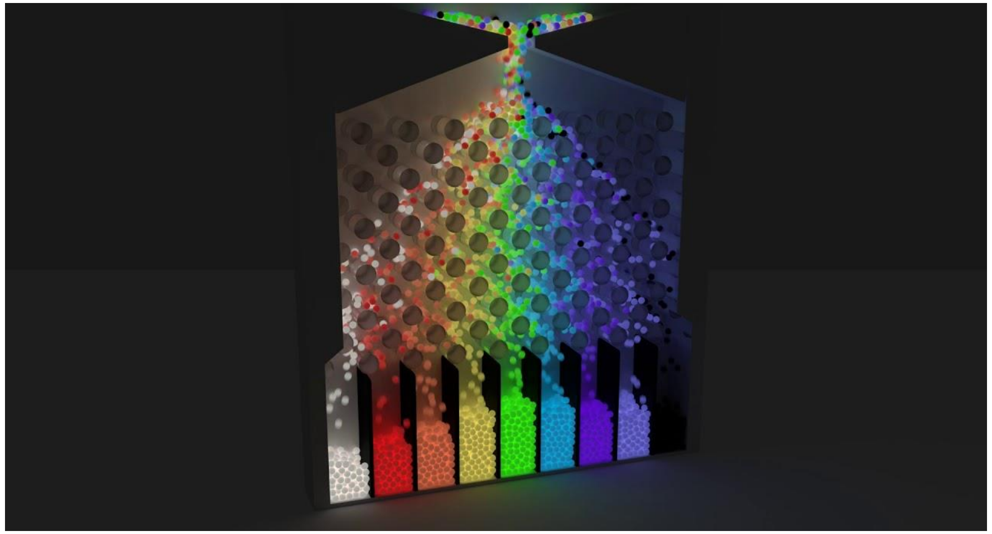
* This is also called the binomial term

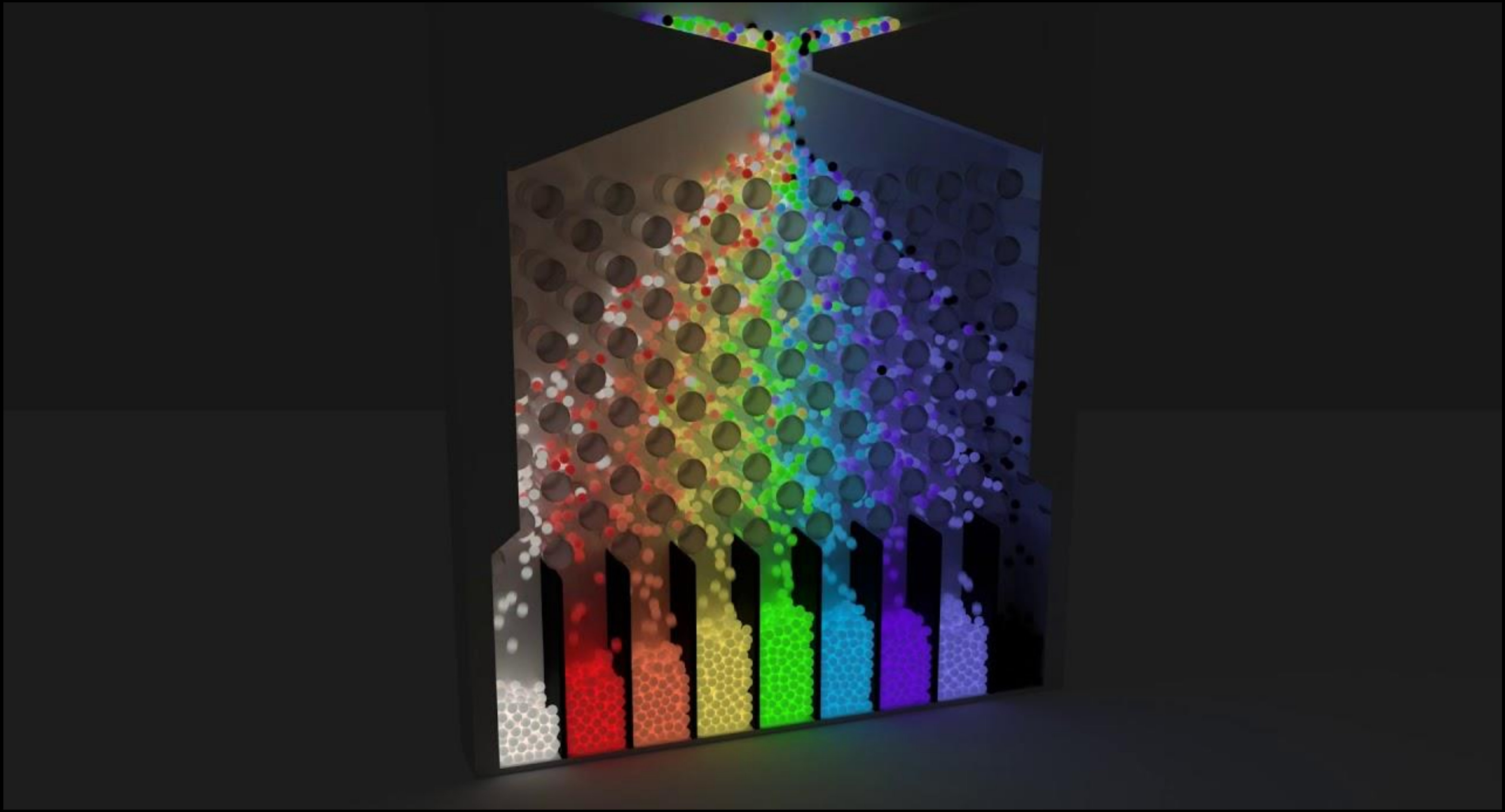
The PMF as a Graph: $X \sim \text{Bin}(n = 20, p = 0.6)$



Many Stories fit the Binomial

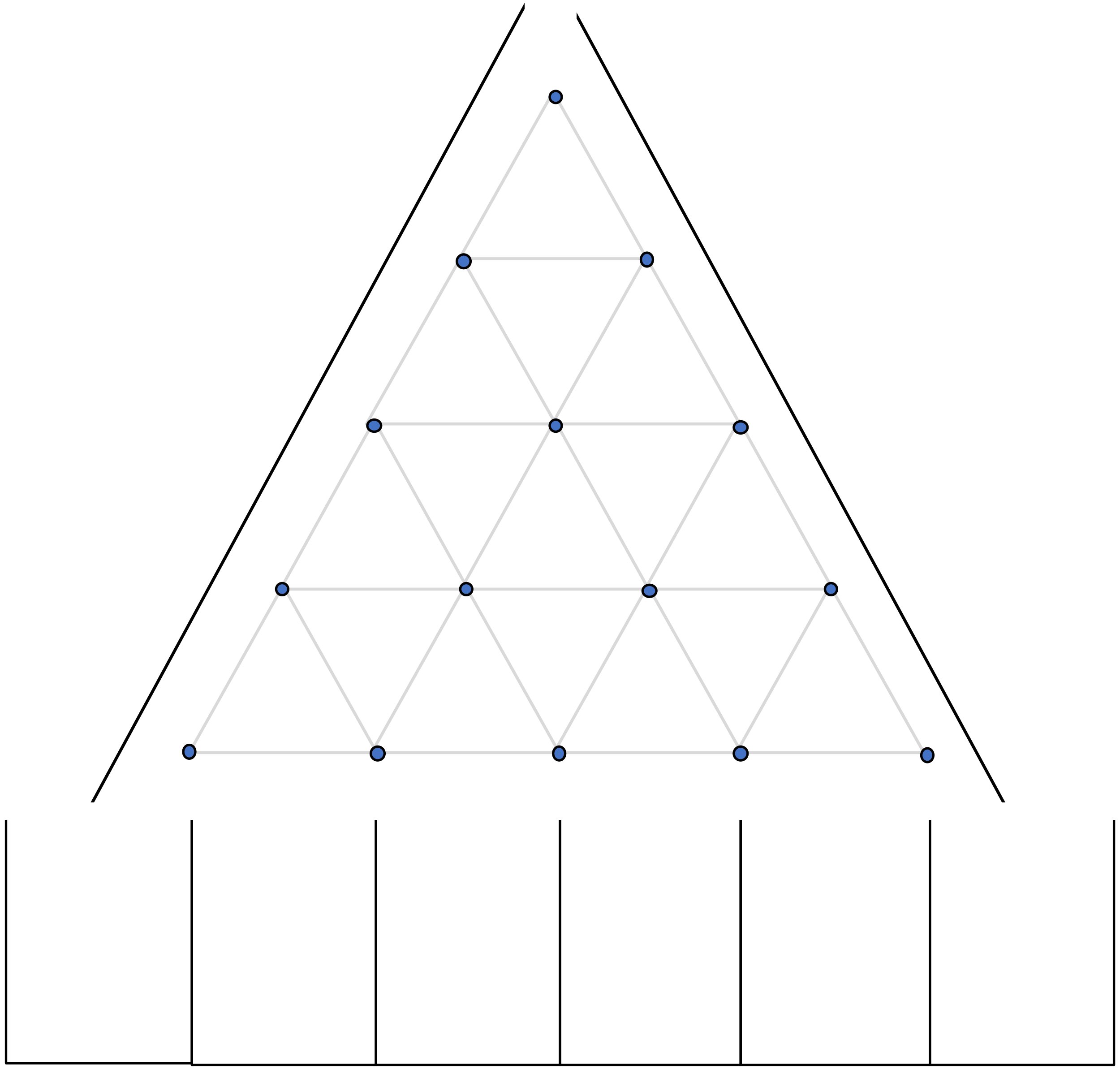




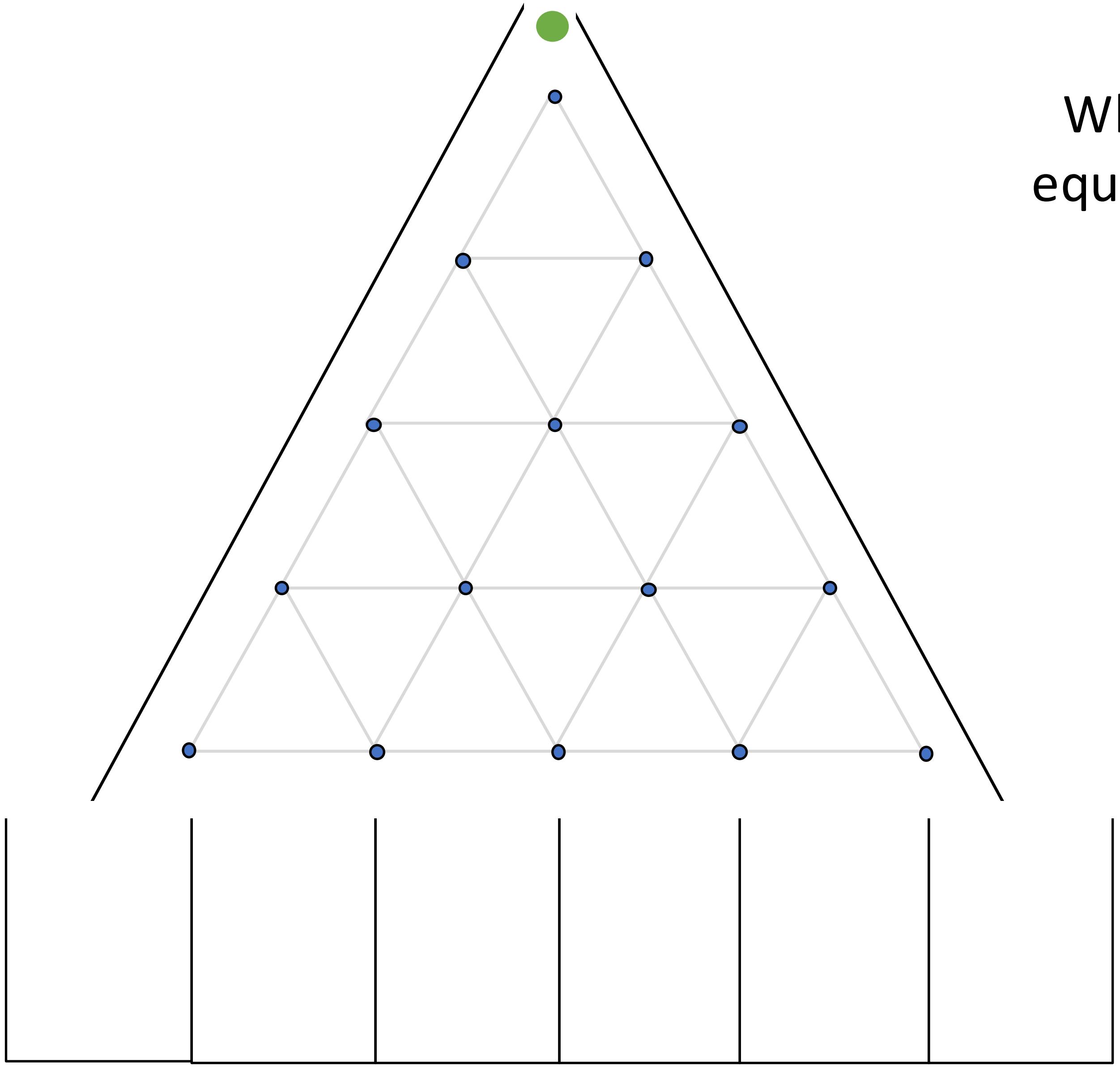


Galton Board Time!

Galton Board Fun

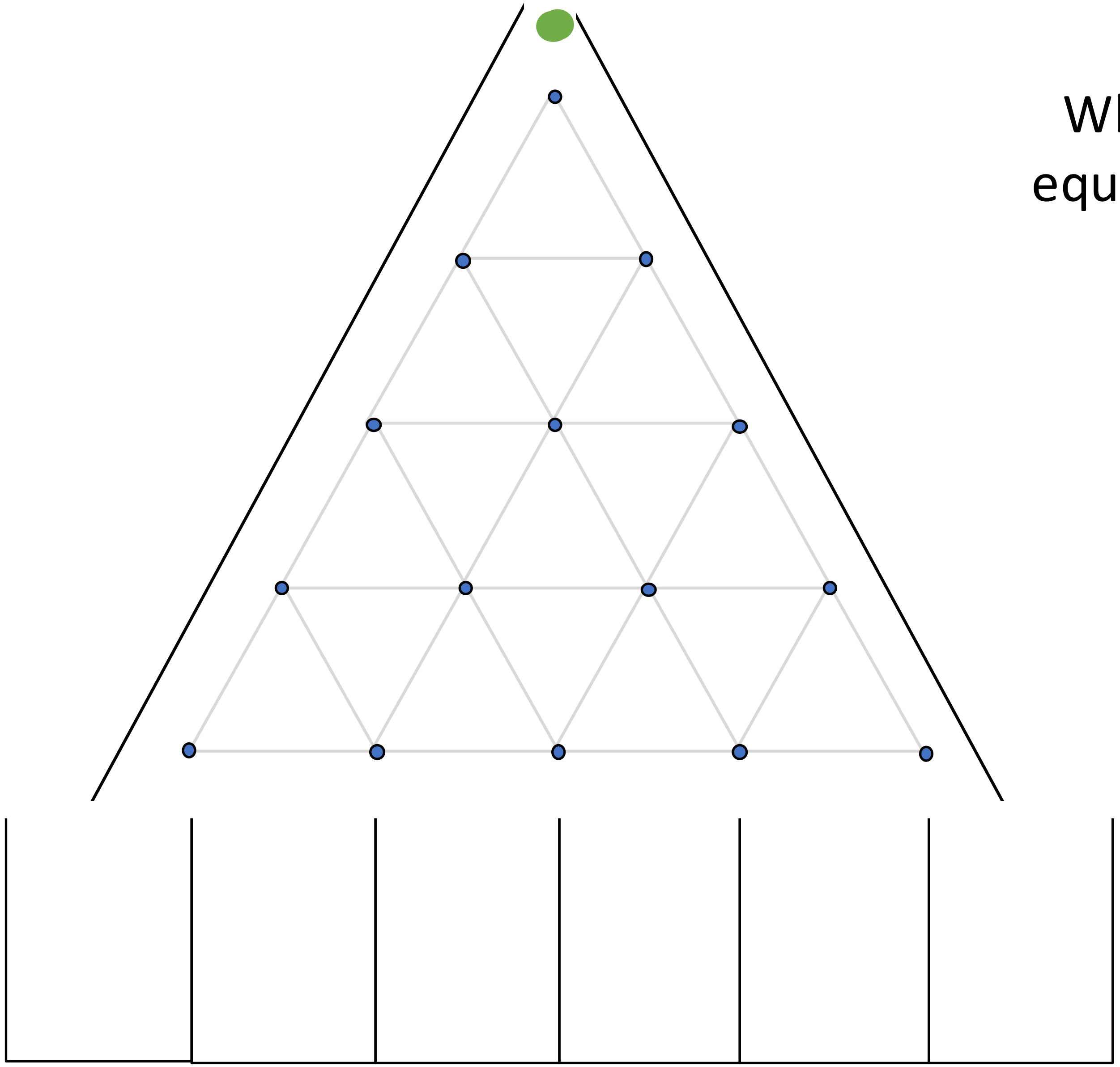


Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

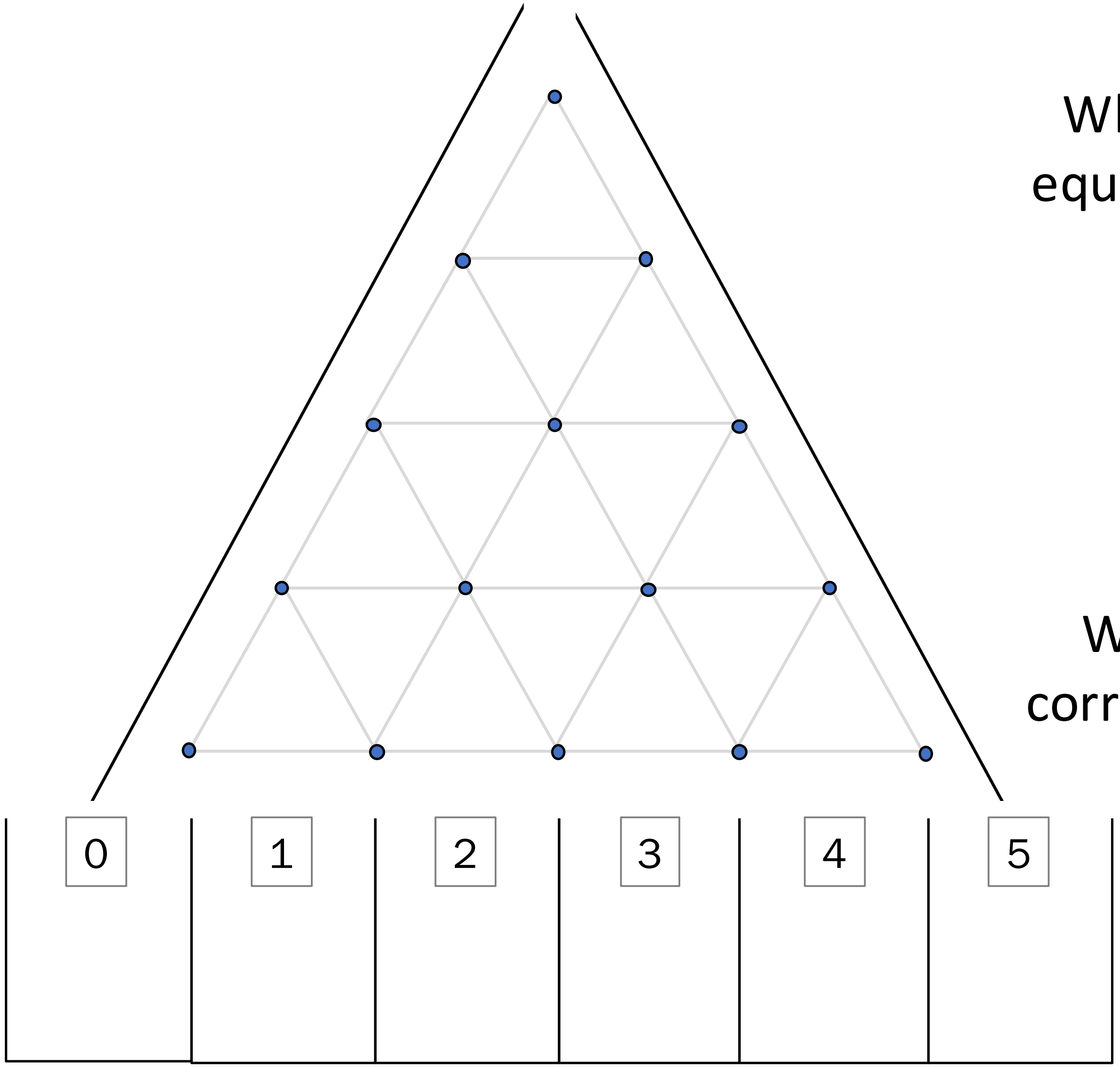
Galton Board Fun



When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Galton Board Fun

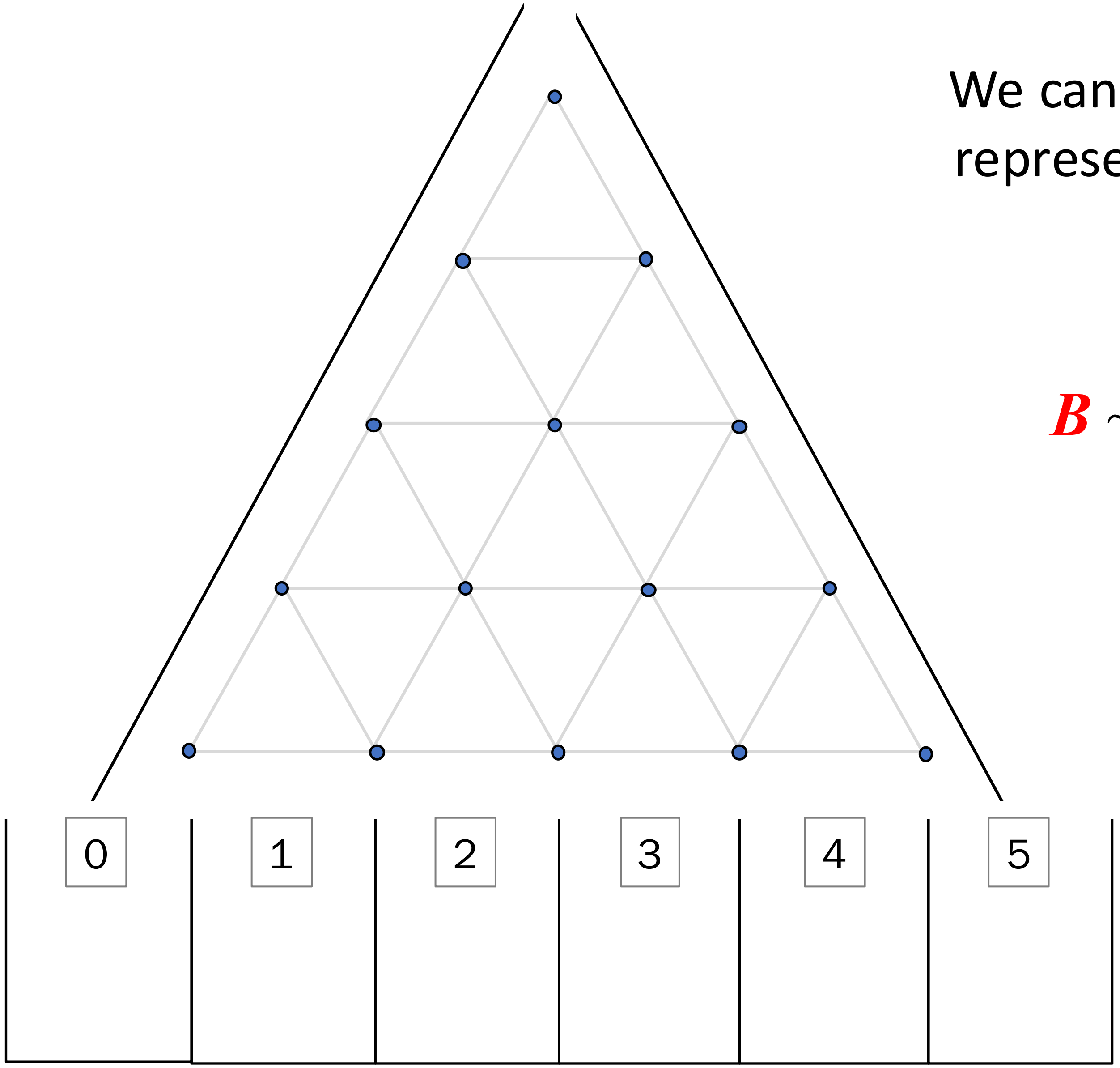


When a marble hits a pin, it has equal chance of going left or right.

Each pin represents an independent event.

Which bucket a marble lands in corresponds to the number of times the marble went right.

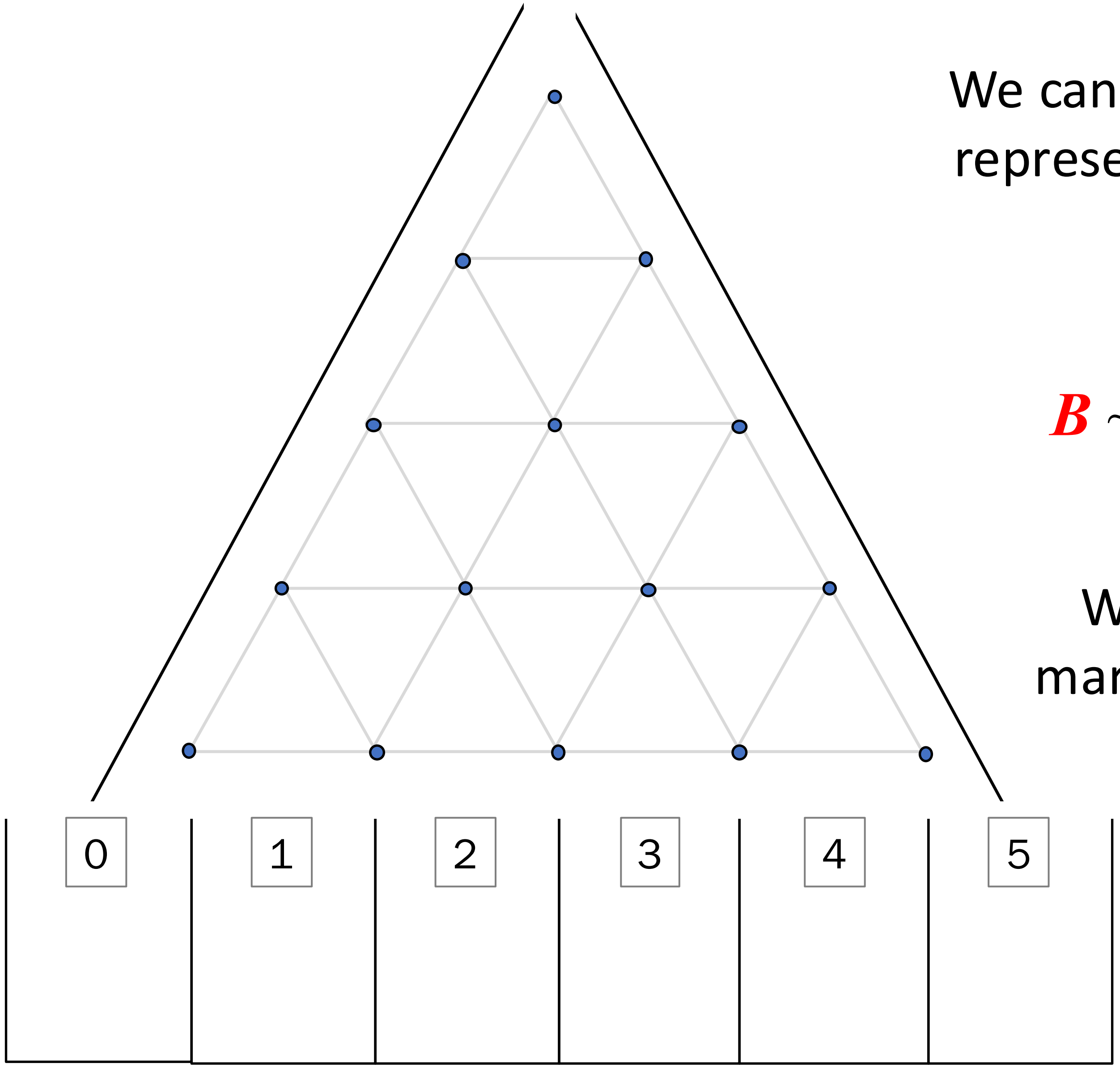
Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

Galton Board Fun

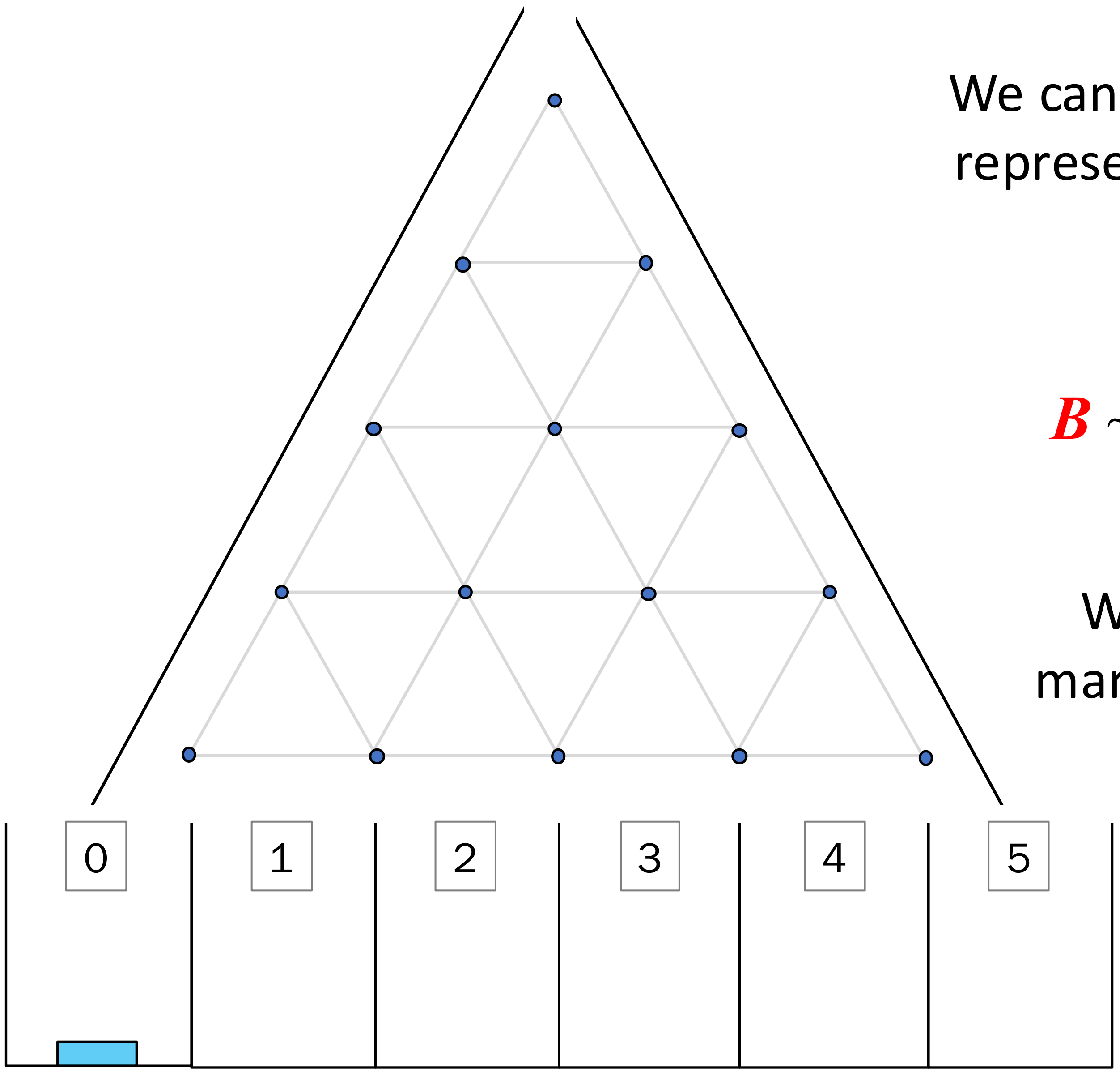


We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

Galton Board Fun



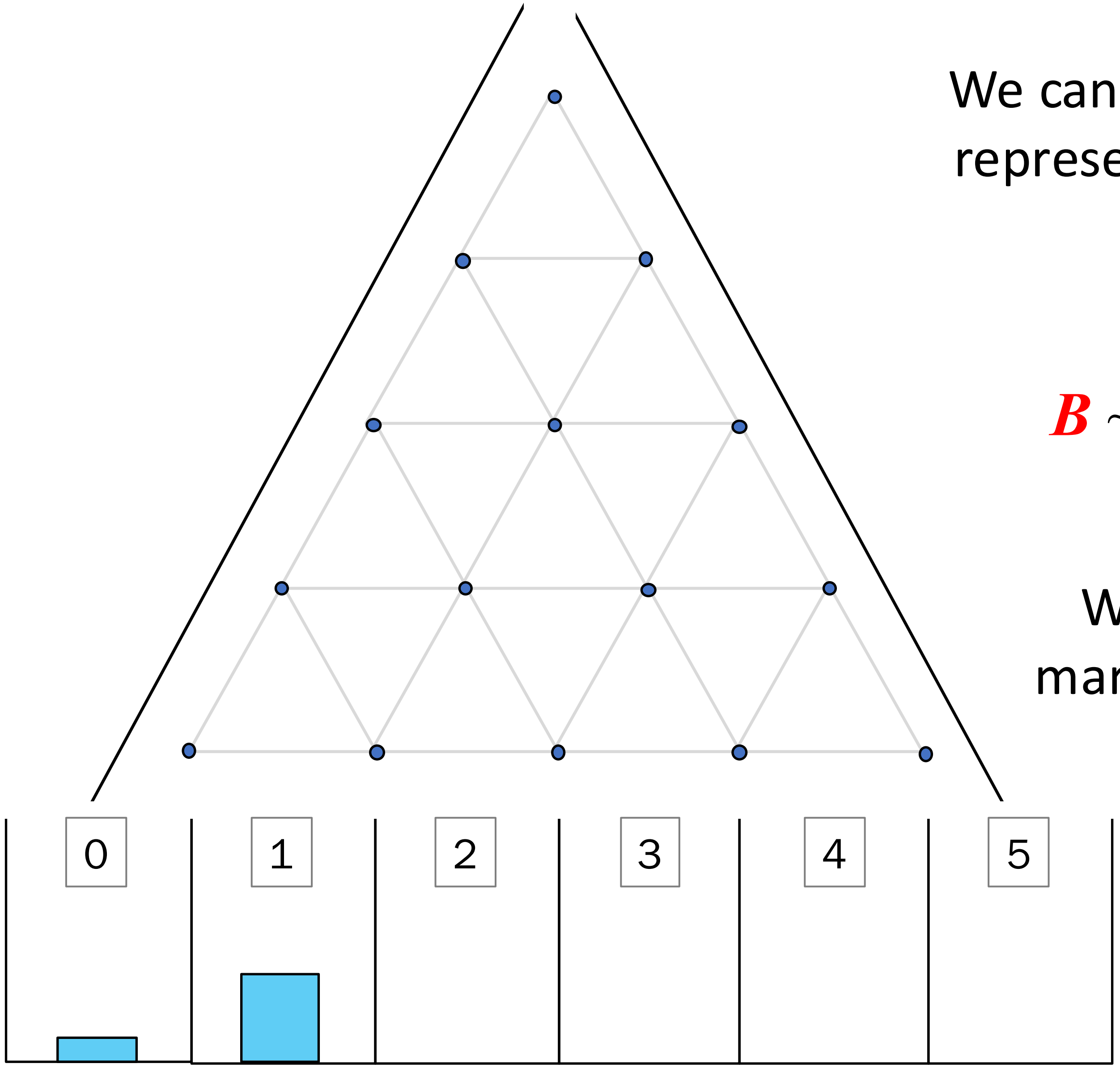
We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

$$P(B = 0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

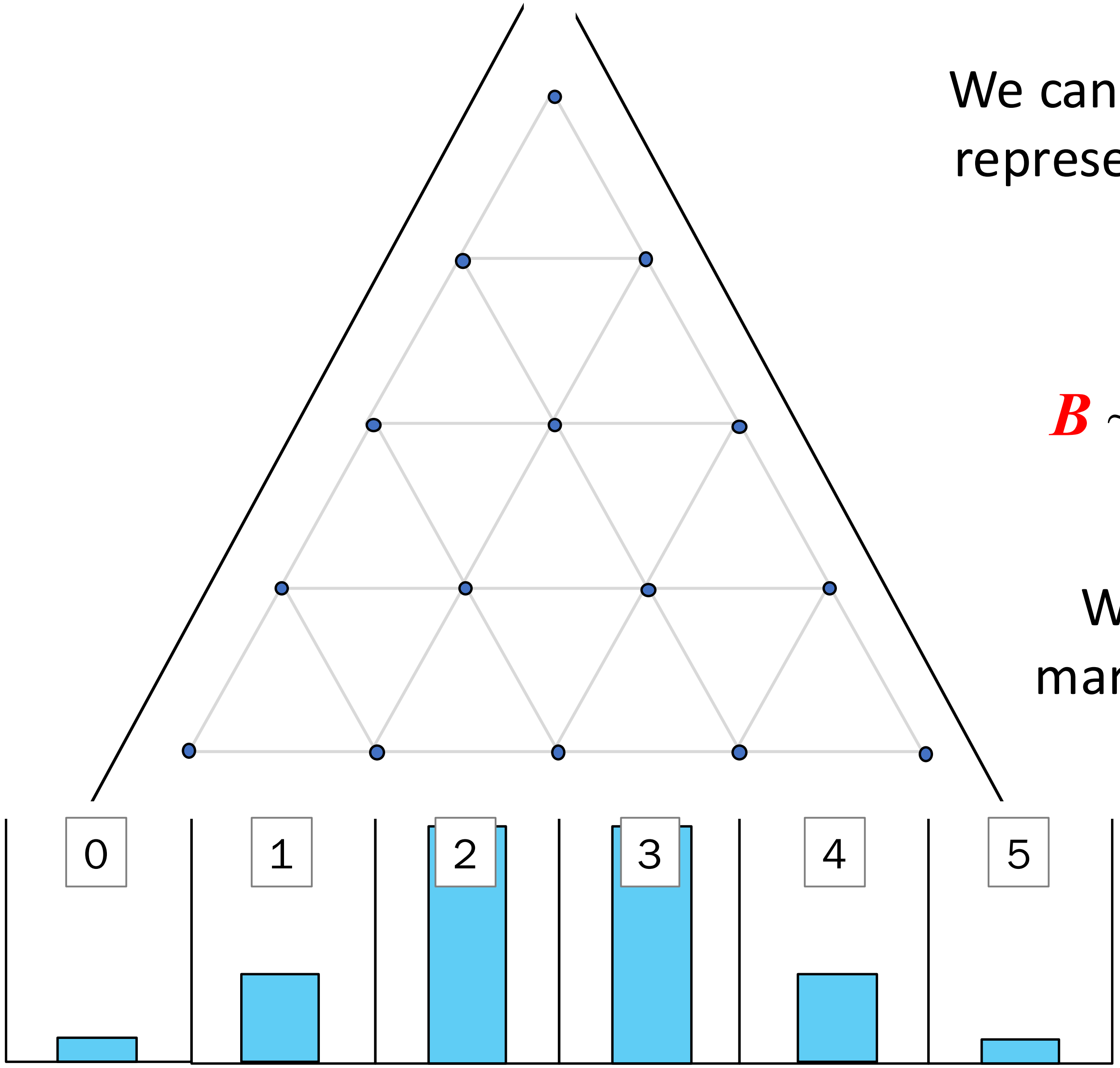
$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

$$P(B = 0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

$$P(B = 1) = \binom{5}{1} \frac{1}{2}^5 \approx 0.16$$

Galton Board Fun



We can define a random variable (***B***) representing which bucket a marble lands in.

$$B \sim \text{Bin}(n = \text{levels}, p = 0.5)$$

What is the probability of a marble landing in each bucket?

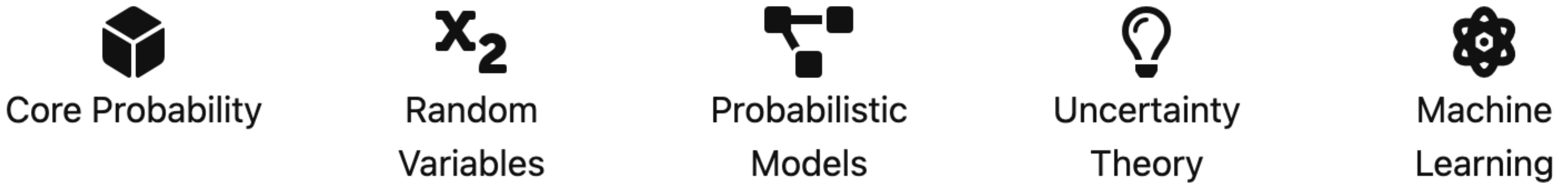
This is the PMF of the binomial!

End Review

Where are We in CS109?

You are here

↓



Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

Successes in one trial

$X \sim \text{Geo}(p)$

Trials until one success

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$Y \sim \text{NegBin}(r, p)$

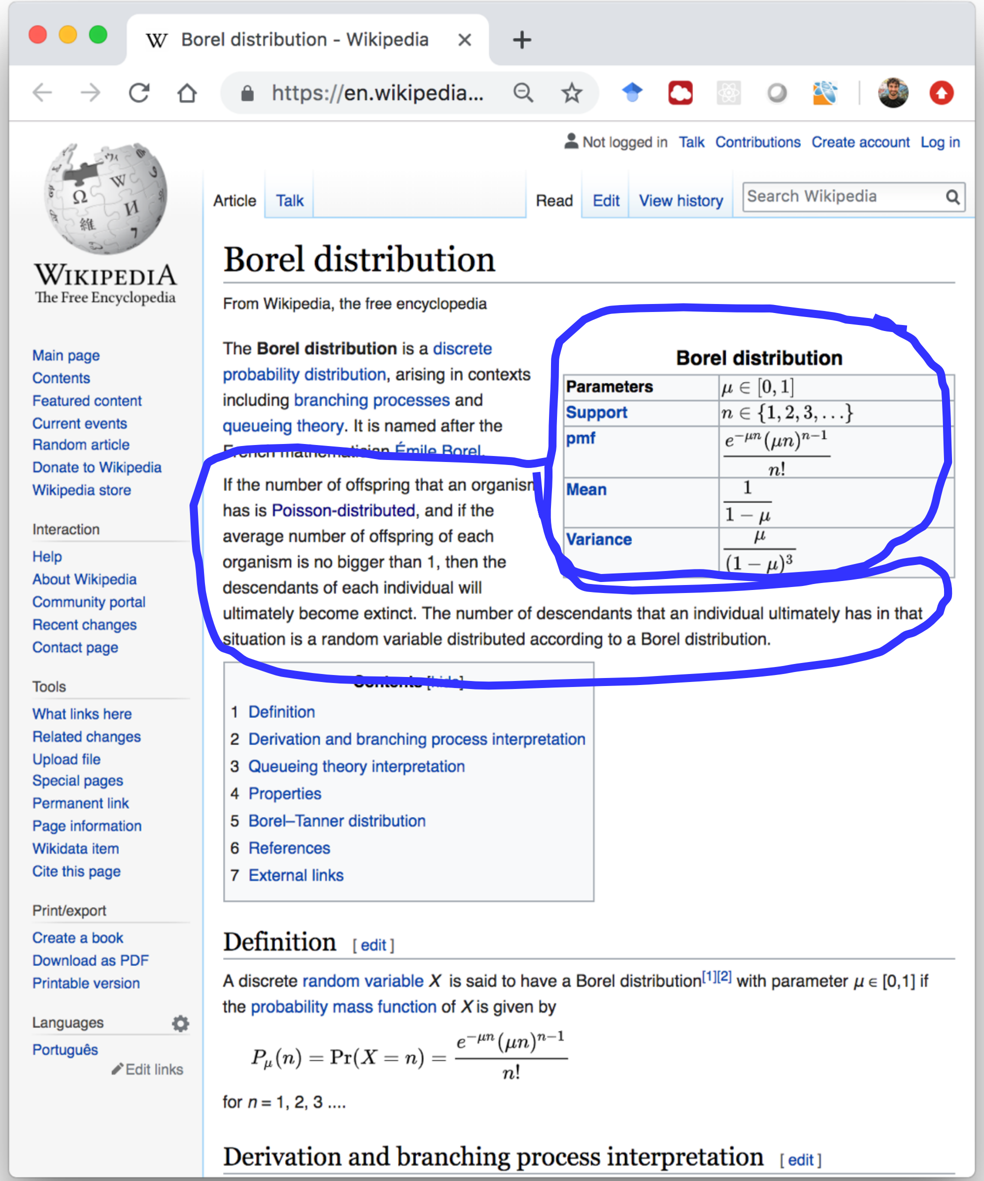
Trials until *r* success

Goal: Be Able to Use a New Random Variable

You are learning about servers…

You read about the MD1 queue…

You find a paper that says the length of a server "busy period" is distributed as a Borel with parameter μ = 0.2 …



Geometric Random Variable

X is **Geometric** Random Variable: X ~ Geo(*p*)

- X is number of independent trials until first success
- *p* is probability of success on each trial
- Assumes p does not change
- X takes on values 1, 2, 3, …, with probability:

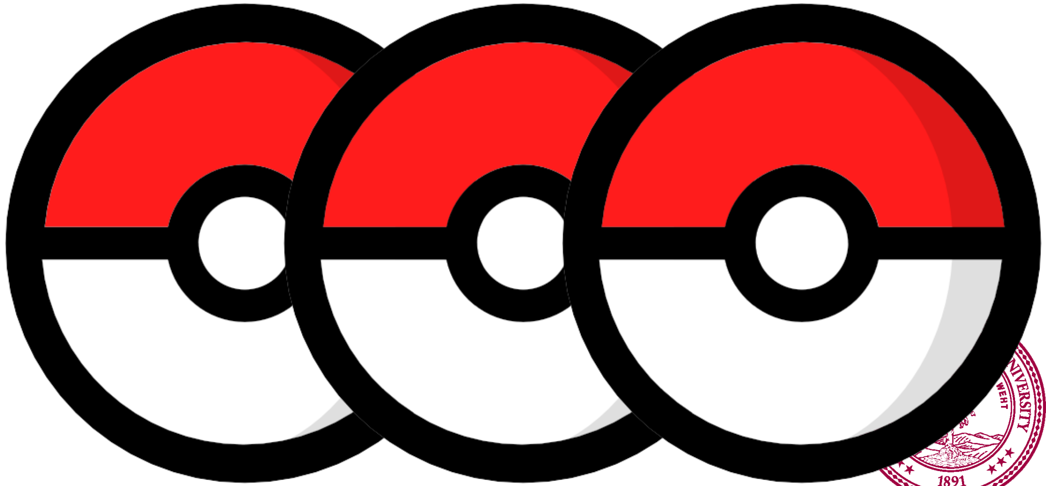
$$P(X = n) = (1 - p)^{n-1} p$$

Negative Binomial Random Variable

X is **Negative Binomial** RV: X ~ NegBin(r, p)

- X is number of independent trials until r successes
- p is probability of success on each trial
- Assumes p does not change.
- X takes on value n with probability:

$$P(X=n)=\binom{n-1}{r-1}p^r(1-p)^{n-r}, \text{ where } n=r, r+1, \ldots$$



Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

$X \sim \text{Geo}(p)$

Trials until one success

$$P(X = x) = (1-p)^{x-1}p$$

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$Y \sim \text{NegBin}(r, p)$

Trials until *r* success

$$P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Recipe For Solving Problems:

1. Recognize a classic random variable type

2. Define a random variable to be that type, with parameters

$$X \sim \mathrm{Bin}(n, p)$$

3. Profit off the PMF

Dating at Stanford

Each person you date has a 0.2 probability of being someone you spend your life with. What is the probability you need to date more than 5 people? **Your meta goal: what steps would you take to answer this question?**

Equity in the Courts

Berghuis v. Smith

If a group is underrepresented in a jury pool, how do you tell?

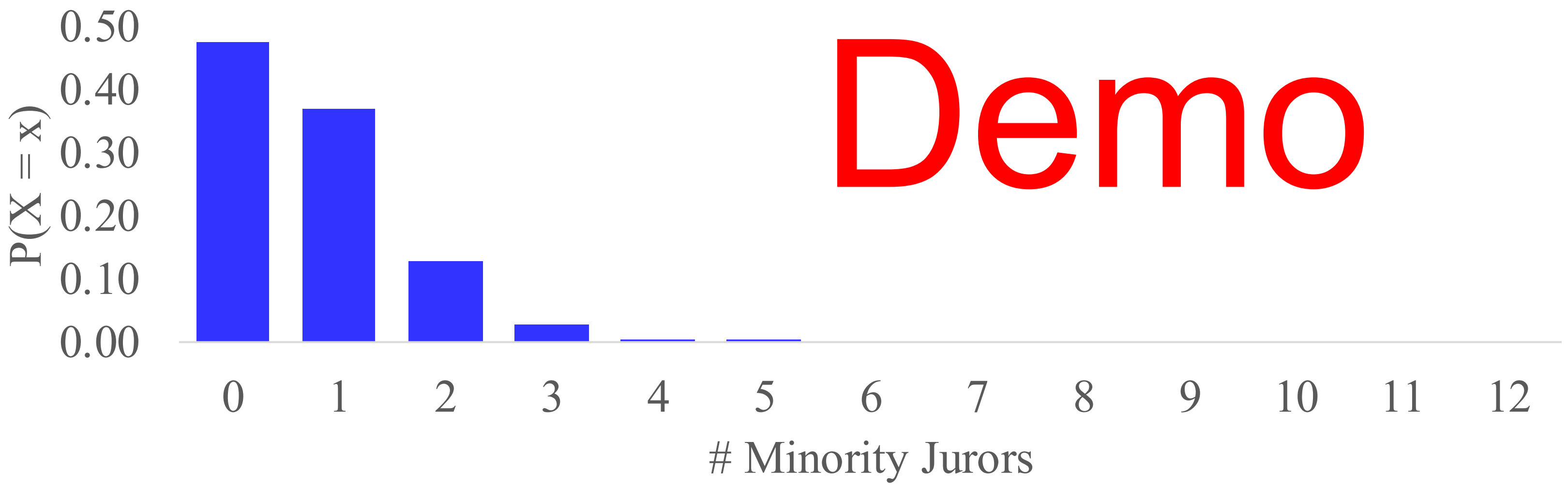
Justice Breyer [Stanford Alum] opened the questioning by invoking the binomial theorem. He hypothesized a scenario involving **"an urn with a thousand balls, and sixty are yellow, and nine hundred forty are navy-blue, and then you select them at random… twelve at a time."** According to Justice Breyer and the binomial theorem, if the purple balls were under represented jurors then **"you would expect… something like <u>a third to a half</u> of juries would have at least one yellow ball"** on them.

Equity in the Courts

Approximation using Binomial distribution

- Assume P(blue ball) constant for every draw = 60/1000
- X = # blue balls drawn. $X \sim Bin(12, 60/1000 = 0.06)$
- $P(X \geq 1) = 1 - P(X = 0) \approx 1 - 0.4759 = 0.5240$

In Breyer's description, should actually expect just ***over half*** *of juries to have at least one non-white person on them*



Demo

Bitcoin Mining

SHA-256 Hash(,)

Data
Fixed

Salt
Choice

Number that looks like random bits

You "mine a bitcoin" if, for given data *D*, you find a salt number *N* such that Hash(*D*, *N*) produces a string that starts with *g* zeroes.

You "mine a bitcoin" if, for given data D, you find a number N such that Hash(D, N) produces a string that starts with g zeroes.

(a) What is the probability that Hash outputs a bit string which starts with g zeroes (in other words you mine a bitcoin)?

Let X be the number of zeros in the first g bits. $X \sim \text{Bin}(n = g, p = 0.5)$

$$P(X = g) = \binom{g}{g} \frac{1}{2}^g = \frac{1}{2}^g$$

Call this answer p_a

(b) What is the probability that you will need under 100 attempts to mine 2 bit coins?

Let Y be the number of tries until you mine 2 bitcoins. $Y \sim \text{NegBin}(r = 2, p = p_a)$

$$P(Y < 100) = \sum_{x=2}^{99} P(Y = x)$$

$$= \sum_{x=2}^{99} \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

$X \sim \text{Geo}(p)$

Trials until one success

$$P(X = x) = (1-p)^{x-1}p$$

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$Y \sim \text{NegBin}(r, p)$

Trials until *r* success

$$P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Can Jacob Bernoulli Have a Variable Named After Him?

Here yee. I want to have a random variable named after myself. Huzzah.

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Yes - the Bernoulli random variable: $X \sim \text{Bern}(p)$

Can Jacob Bernoulli Have a Variable Named After Him?

Here yee. I want to have a random variable named after myself. Huzzah.

Yes - the Bernoulli random variable: $X \sim \text{Bern}(p)$

- The Bernoulli is an **indicator** random variable (value is either 0 or 1).
- $P(X = 1) = p$
- $P(X = 0) = 1 - p$
- Examples: a single coin flip, one ad click, any binary event

(this is the whole PMF)

Random Variable Sums

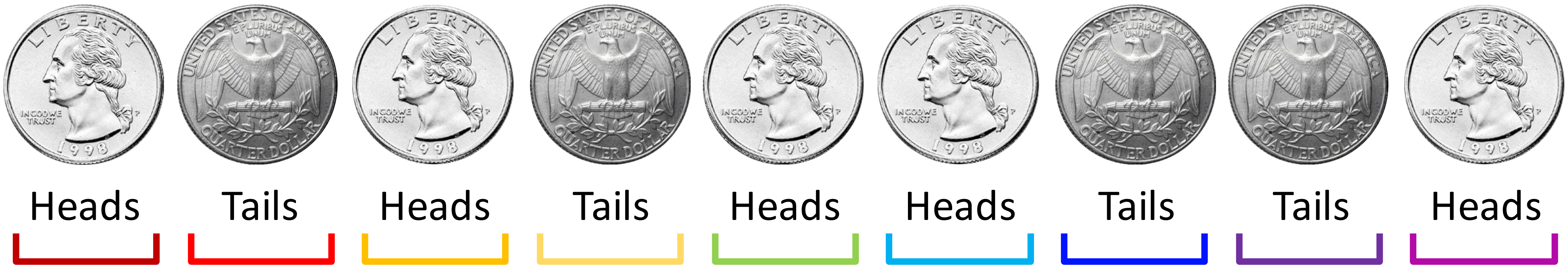
The Binomial

Heads Tails Heads Tails Heads Heads Tails Tails Heads

Random Variable Sums

The Binomial

…is a sum of Bernoulli random variables



Random Variable Sums

The Binomial ...is a sum of Bernoulli random variables

Heads Tails Heads Tails Heads Heads Tails Tails Heads

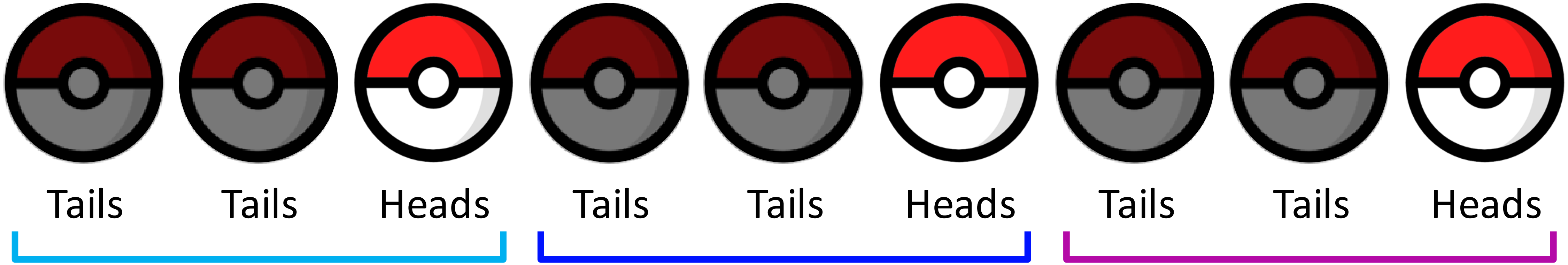
Let $X_1 \sim \text{Bern}(p = 1/2)$ and $X_2 \sim \text{Bern}(p = 1/2)$.

$$Y \sim \text{Bin}(n = 2, p = 1/2)$$

$$Y = X_1 + X_2$$

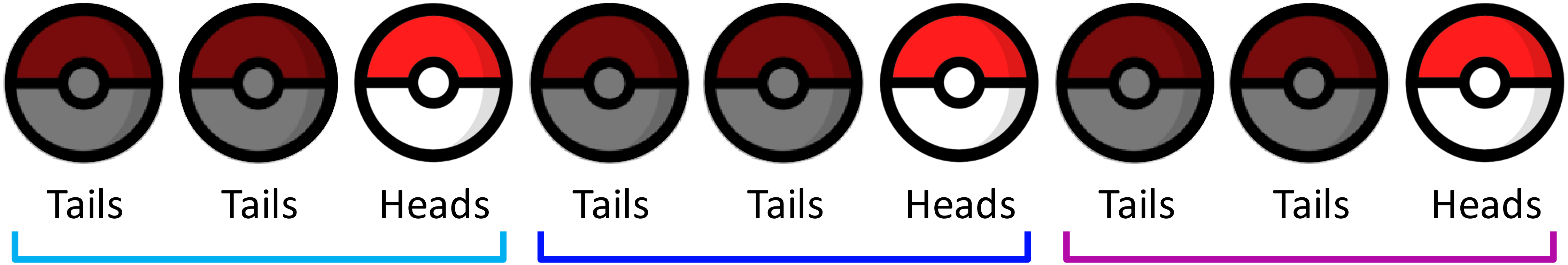
Random Variable Sums

The Negative Binomial



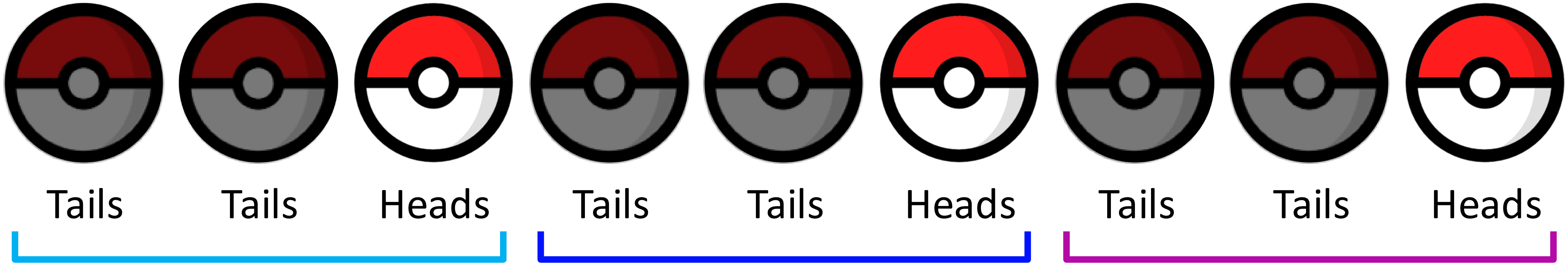
Random Variable Sums

The Negative Binomial ...is a sum of Geometric random variables



Random Variable Sums

The Negative Binomial ...is a sum of Geometric random variables



Let $X_1 \sim \text{Geo}(p = 1/3)$, $X_2 \sim \text{Geo}(p = 1/3)$, and $X_3 \sim \text{Geo}(p = 1/3)$.

$$Y \sim \text{NegBin}(r = 3, p = 1/3)$$

$$Y = X_1 + X_2 + X_3$$

Classic Random Variables (with PMFs)

$X \sim \text{Bern}(p)$

Successes in one trial

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

$X \sim \text{Geo}(p)$

Trials until one success

$$P(X = x) = (1 - p)^{x-1} p$$

$Y \sim \text{Bin}(n, p)$

Successes in *n* trials

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$Y \sim \text{NegBin}(r, p)$

Trials until *r* success

$$P(X = x) = \binom{x - 1}{r - 1} p^r (1 - p)^{x-r}$$

[Short Pedagogical Pause]

Time for some tender moments*

*Moments: numbers that summarize different aspects of a random variable

Expectation

Expected Value

The probability of that value

The value

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Loop over all values x that X can take on

Expected Value

Expected value answers the question:

What is the average value we could expect some random variable to be?

Also called: ***Mean***, *Expectation*, ***Weighted Average***, ***Center of Mass***, *1st Moment*

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Example: Expected Value of Dice Roll

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

Example: Expected Value of Dice Roll

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$E[X] = \sum_{x=1}^{6} x \cdot P(X = x)$$

Example: Expected Value of Dice Roll

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Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$E[X] = \sum_{x=1}^{6} x \cdot P(X = x)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= 3.5$$

Example: Expected Value of Dice Roll

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

Let X be the result of rolling a 6-sided dice.

$$P(X = x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

What is the expectation of X?

$$E[X] = \sum_{x=1}^{6} x \cdot P(X = x)$$
$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$
$$= 3.5$$

E[X] is not always an actual possible outcome for X

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **class** with equal probability.

Let X be the chosen class's size. What is $\mathrm{E}[X]$?

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **class** with equal probability.

Let X be the chosen class's size. What is $\mathrm{E}[X]$?

$$P(X = 5) = 1/3$$
$$P(X = 10) = 1/3$$
$$P(X = 150) = 1/3$$

$$E[X] = \sum_{x \in \{5,10,150\}} x \cdot P(X = x)$$
$$= 5 \cdot \frac{1}{3} + 10 \cdot \frac{1}{3} + 150 \cdot \frac{1}{3}$$
$$= 55$$

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **student** with equal probability.

Let X be the chosen student's class size. What is $\mathrm{E}[X]$?

Lying With Statistics

Imagine a university has 3 classes, with 5, 10, and 150 students in each class.

We randomly choose a **student** with equal probability.

Let X be the chosen student's class size. What is $\mathrm{E}[X]$?

$$P(X = 5) = 5/165$$
$$P(X = 10) = 10/165$$
$$P(X = 150) = 150/165$$

$$E[X] = \sum_{x \in \{5,10,150\}} x \cdot P(X = x)$$
$$= 5 \cdot \frac{5}{165} + 10 \cdot \frac{10}{165} + 150 \cdot \frac{150}{165}$$
$$= 137$$

Expectation from Data

List called data

X
3
2
6
10
1
1
5
4
...

$$E[X] = \sum_x x \cdot \mathrm{P}(X = x)$$

$$\approx \sum_x x \cdot \frac{\text{count}(X = x)}{N}$$

Length of data

$$\approx \frac{1}{N} \sum_x x \cdot \text{count}(X = x)$$

$$\approx \frac{1}{N} \sum_{v \in \text{data}} v$$

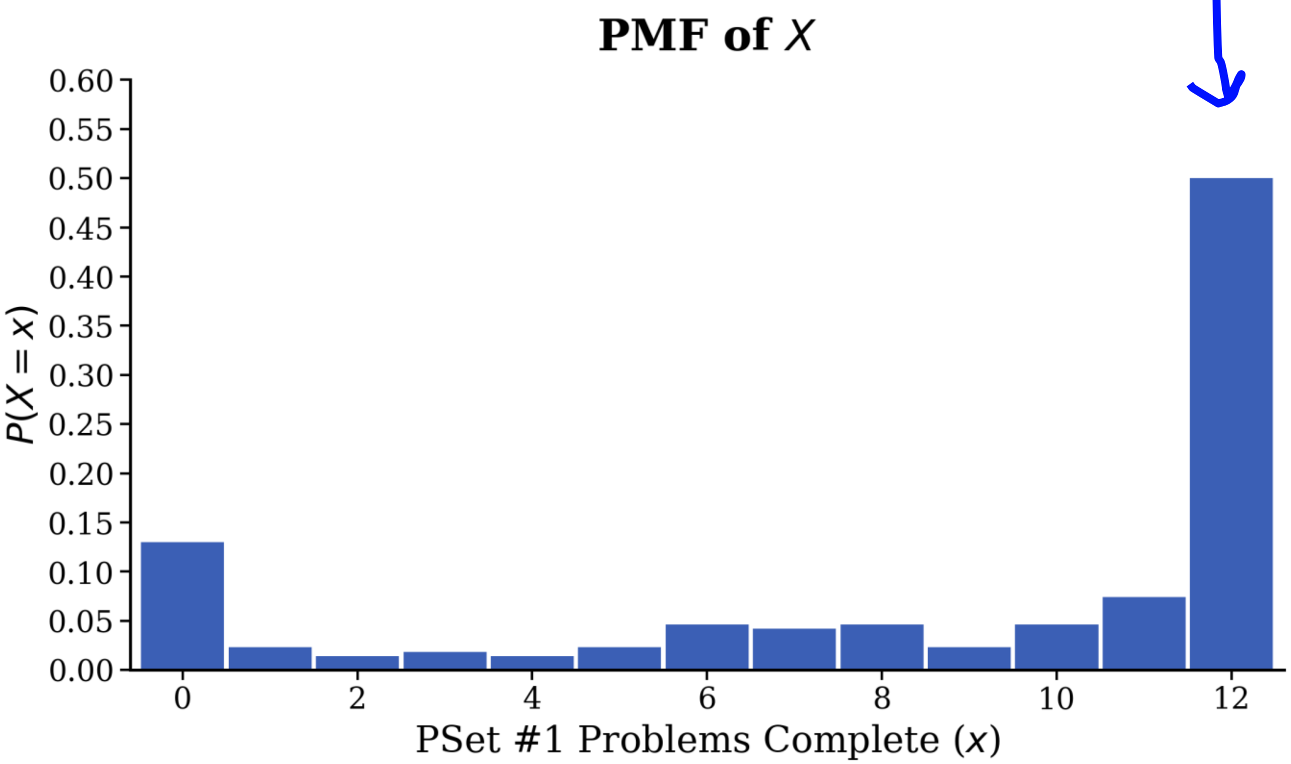
Expectation is a single number summary…

Expectation leaves much to be desired…

Expectation vs PMF

Let X be the number of problems that a randomly selected student has completed, as of 11a today.

X takes on values, with uncertainty. X is a random variable.



$$P(X = 12) \approx \frac{\text{Count}(X = 12)}{N}$$

$$E[X] = 8.7$$

Why People Care?

Properties of Expectation (proof later)

Linearity:

$$E[aX + b] = aE[X] + b$$

- Consider X = 6-sided die roll,   Winnings = 2X – 1.
- E[X] = 3.5   E[2X-1] = 6

Expectation of a sum is the sum of expectations

$$E[X + Y] = E[X] + E[Y]$$

Unconscious statistician:

$$E[g(X)] = \sum_{x \in X} g(x)\, P(X = x)$$

Law of the Unconscious Statistician (LOTUS)

$$E[g(X)] = \sum_x g(x)P(X = x)$$

This lets you get the expectation of **any** function of a random variable.

Examples:

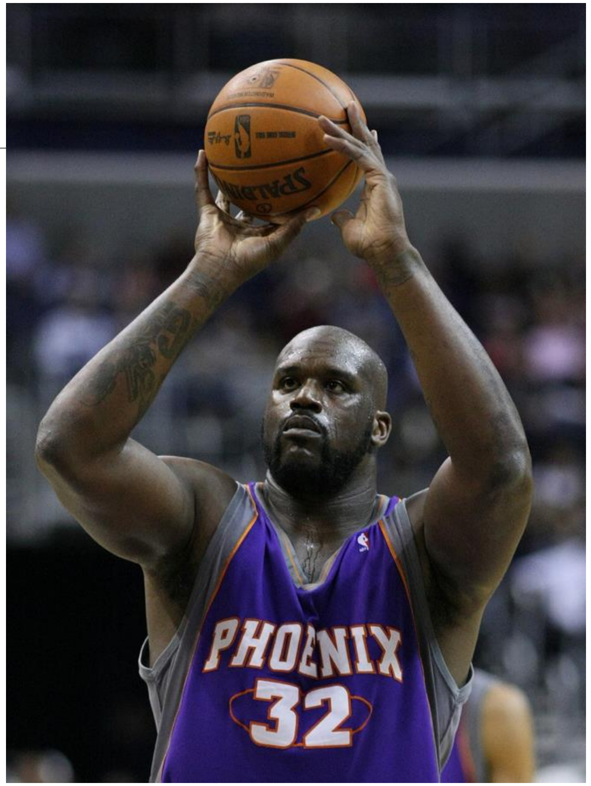
$$E[X^2] = \sum_x x^2 \cdot P(X = x)$$

$$E[\sin(X)] = \sum_x \sin(x) \cdot P(X = x)$$

$$E[\sqrt{(}X)] = \sum_x \sqrt{x} \cdot P(X = x)$$

Expectation of Classic Random Variables

Expected Value of Free Throws



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Some players are not very good at free throws, such as Shaq. While in the NBA, Shaq made only 53% of his free throws.

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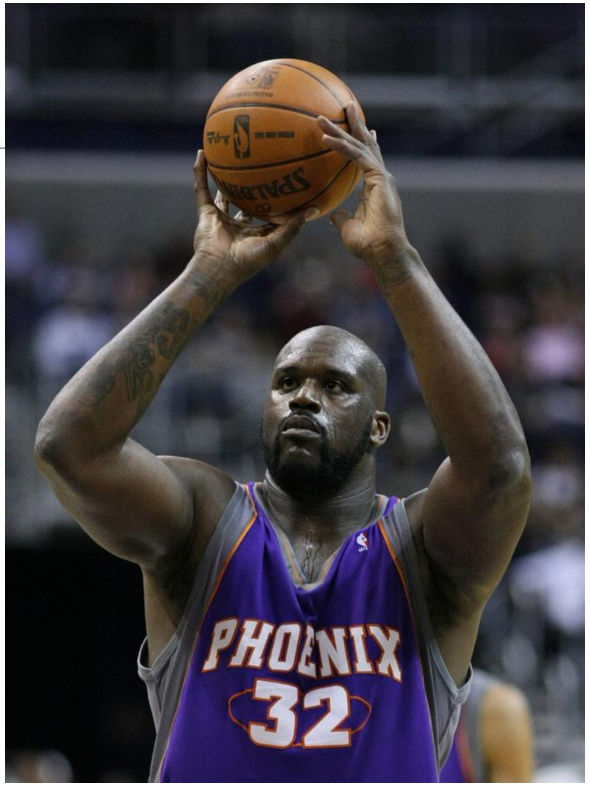

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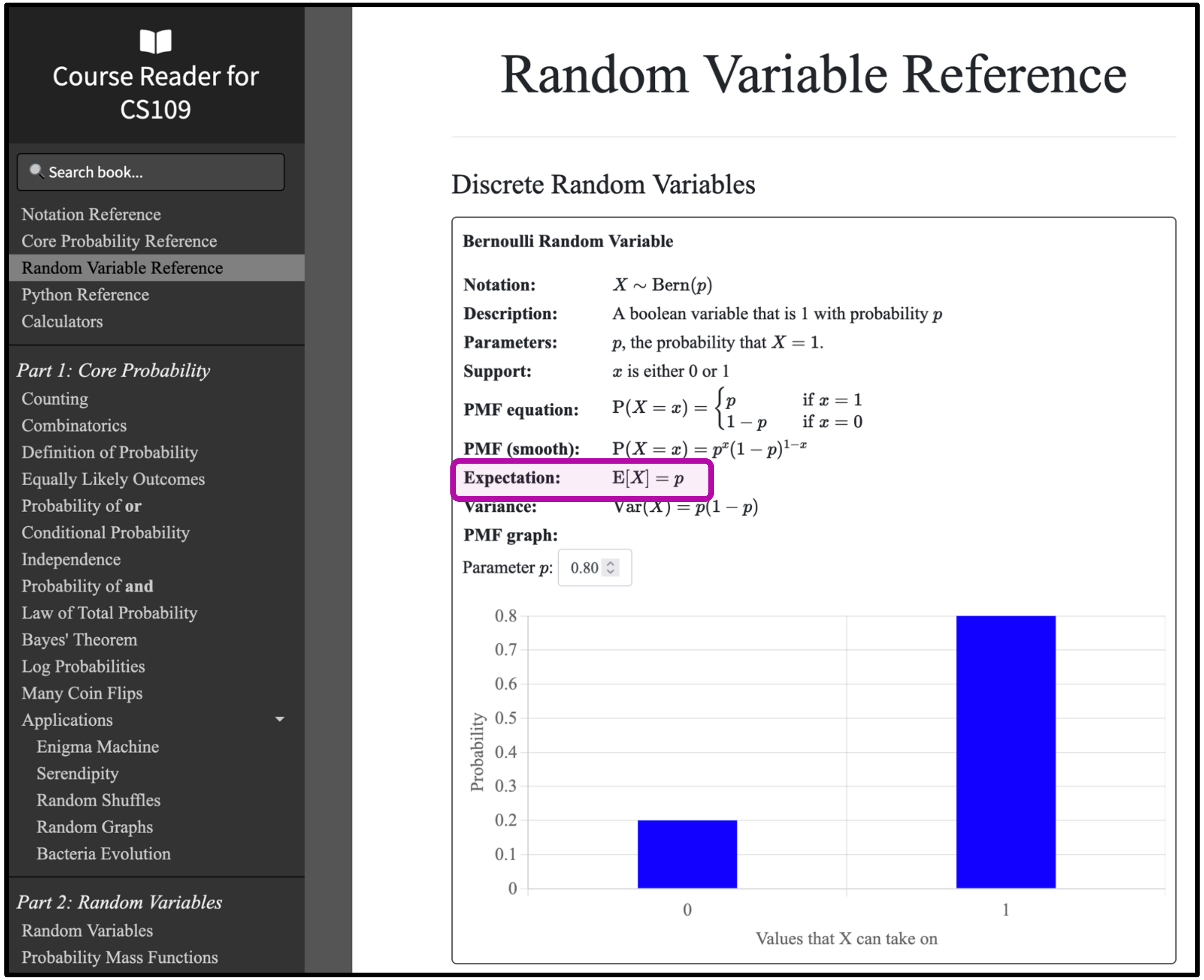
$$X \sim \text{Bern}(p = 0.53)$$

$$E[X] = 0 \cdot P(X = 0) + 1 \cdot P(X = 1)$$
$$= 0 \cdot 0.47 + 1 \cdot 0.53 = 0.53$$

For Bernoulli random variables, $\mathrm{E}[X] = p$ (always)

With Classic RVs, You Get Expectations For Free Too!



We Can Now Calculate Expectation of Binomial

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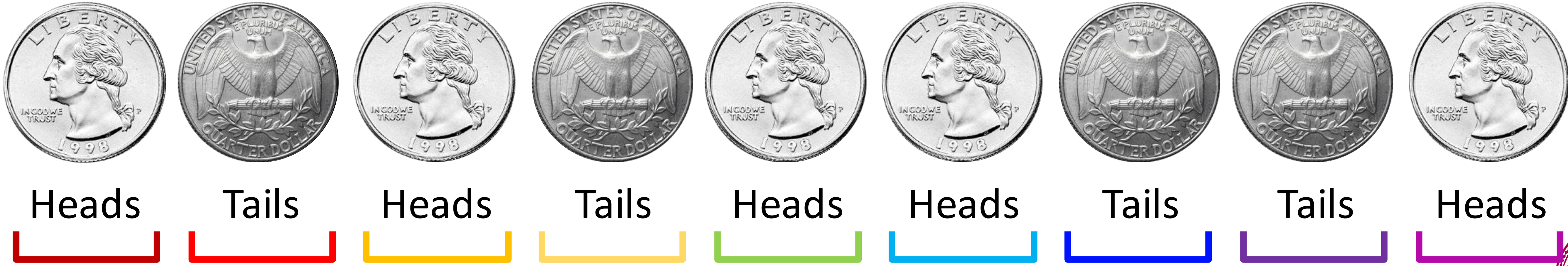
The Binomial ...is a sum of Bernoulli random variables



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$$= \sum_{i=1}^{n} \text{E}[Y_i] \qquad \text{Expectation of sum}$$

Expectation of a sum is the sum of expectations: $E[X+Y] = E[X] + E[Y]$

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$$\begin{aligned} \mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n} Y_i\right] && \text{Since } X = \sum_{i=1}^{n} Y_i \\ &= \sum_{i=1}^{n} \mathrm{E}[Y_i] && \text{Expectation of sum} \\ &= \sum_{i=1}^{n} p && \text{Expectation of Bernoulli} \\ &= n \cdot p && \text{Sum } n \text{ times} \end{aligned}$$

True for every binomial ever

You Get So Much For Free!

Binomial Random Variable

Notation: $X \sim \text{Bin}(n, p)$

Description: Number of "successes" in n identical, independent experiments each with probability of success p.

Parameters: $n \in \{0, 1, \ldots\}$, the number of experiments.
$p \in [0, 1]$, the probability that a single experiment gives a "success".

Support: $x \in \{0, 1, \ldots, n\}$

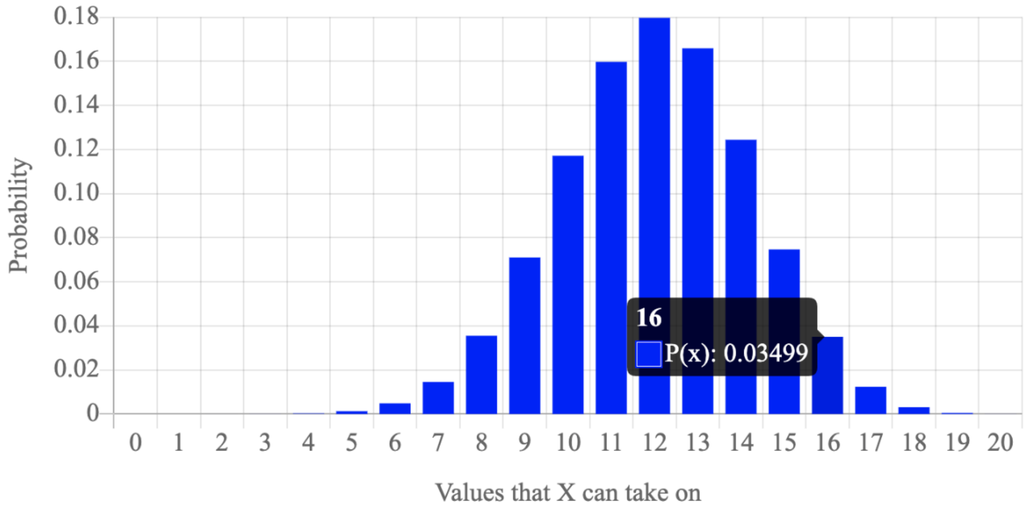
PMF equation: $\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

Expectation: $\text{E}[X] = n \cdot p$

Variance: $\text{Var}(X) = n \cdot p \cdot (1 - p)$

PMF graph:

Parameter n: 20 Parameter p: 0.60



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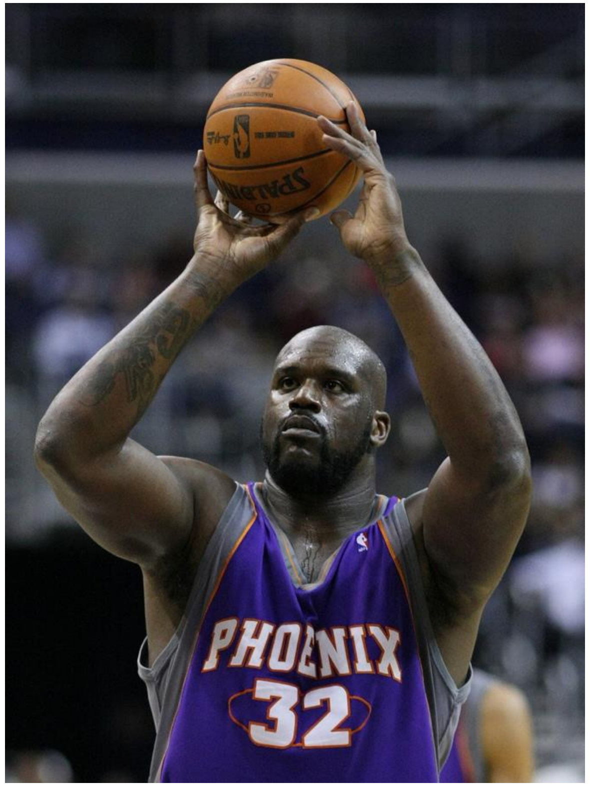
Let Y be the points gained from Shaq attempting **500** free throws. What is $\mathrm{E}[Y]$?



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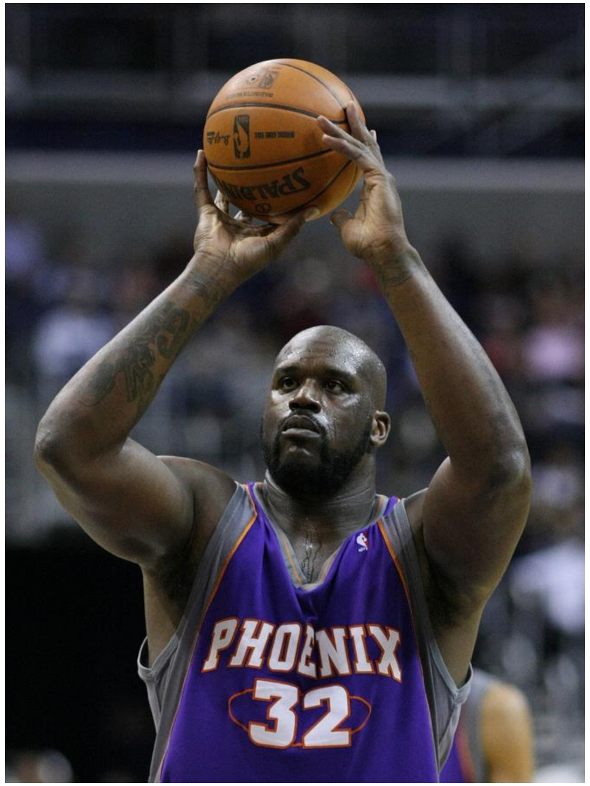

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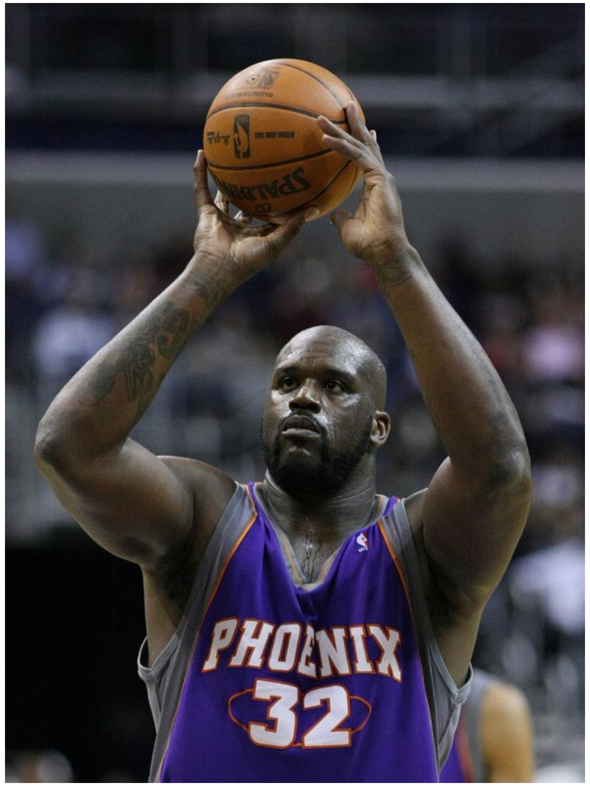

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$$E[Y] = n \cdot p = 500 \cdot 0.53 = 265$$

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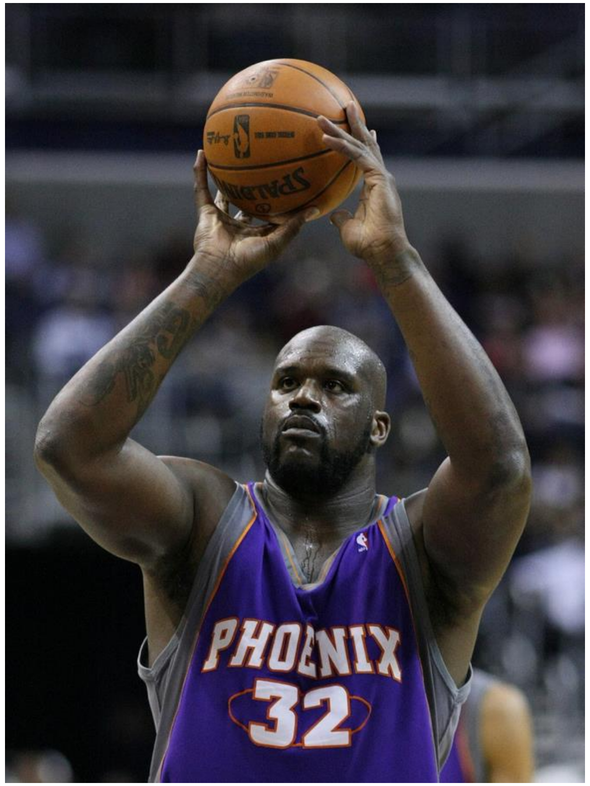

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Challenge: If Shaq was 10% better at shooting free throws, how many *more* free throws would you expect him to make, out of 500?

Expected Value of The Geometric

If $X \sim \text{Geo}(p)$, then $E[X] = \frac{1}{p}$

This definition has intuition built in:

- If Shaq makes about half his free throws, then on average, it will take him two shots to make one free throw. $E[X] = (1/2)^{-1} = 2$.

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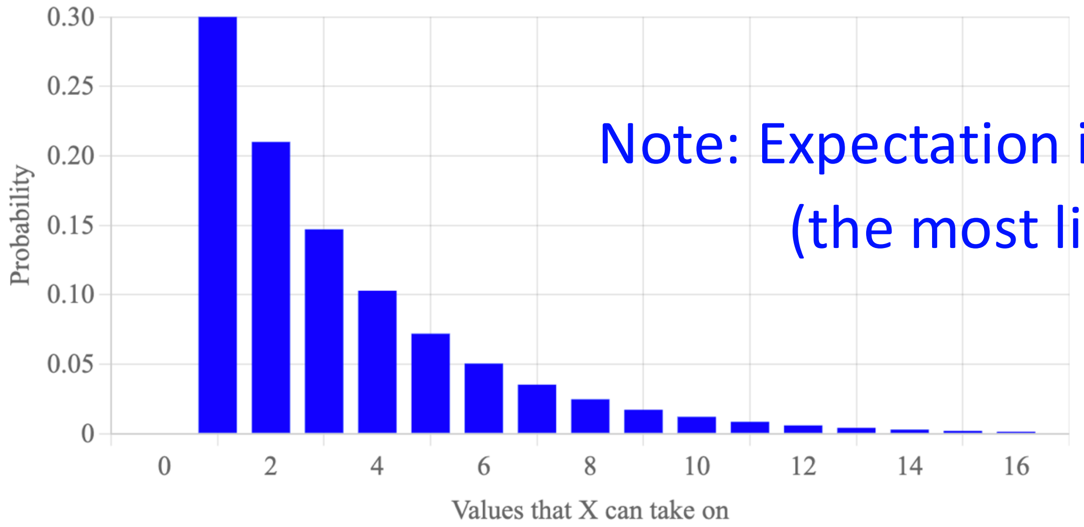
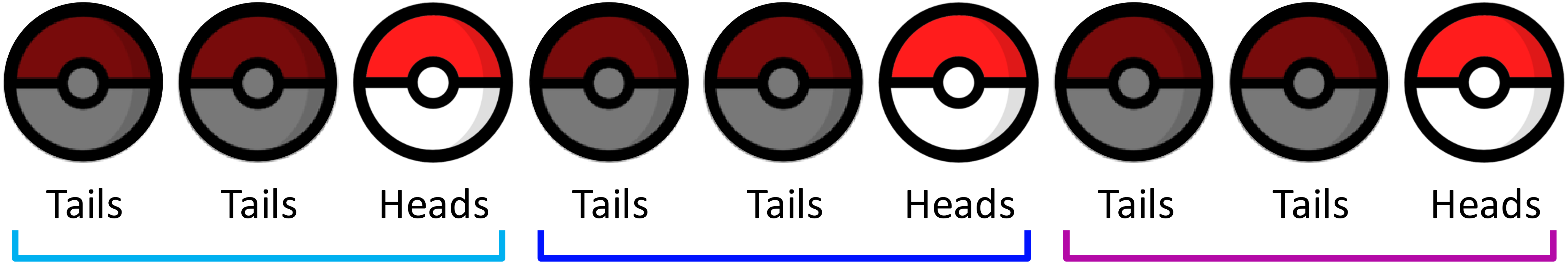


Note: Expectation is often **not** the mode (the most likely outcome)

Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

The Negative Binomial ...is a sum of Geometric random variables



Expected Value of The Negative Binomial

We can derive using the **sum of expectations** property, similar to binomials.

Let $X_i \sim \text{Geo}(p)$, for each i from 1 to r.

$$E[X_i] = \frac{1}{p}$$

Let $Y \sim \text{NegBin}(r, p)$.

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Expectations of Classic Random Variables

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$$E[X] = \frac{1}{p}$$

$X \sim \text{Bern}(p)$

$$E[X] = p$$

$Y \sim \text{NegBin}(r, p)$

$$E[Y] = \frac{r}{p}$$

$Y \sim \text{Bin}(n, p)$

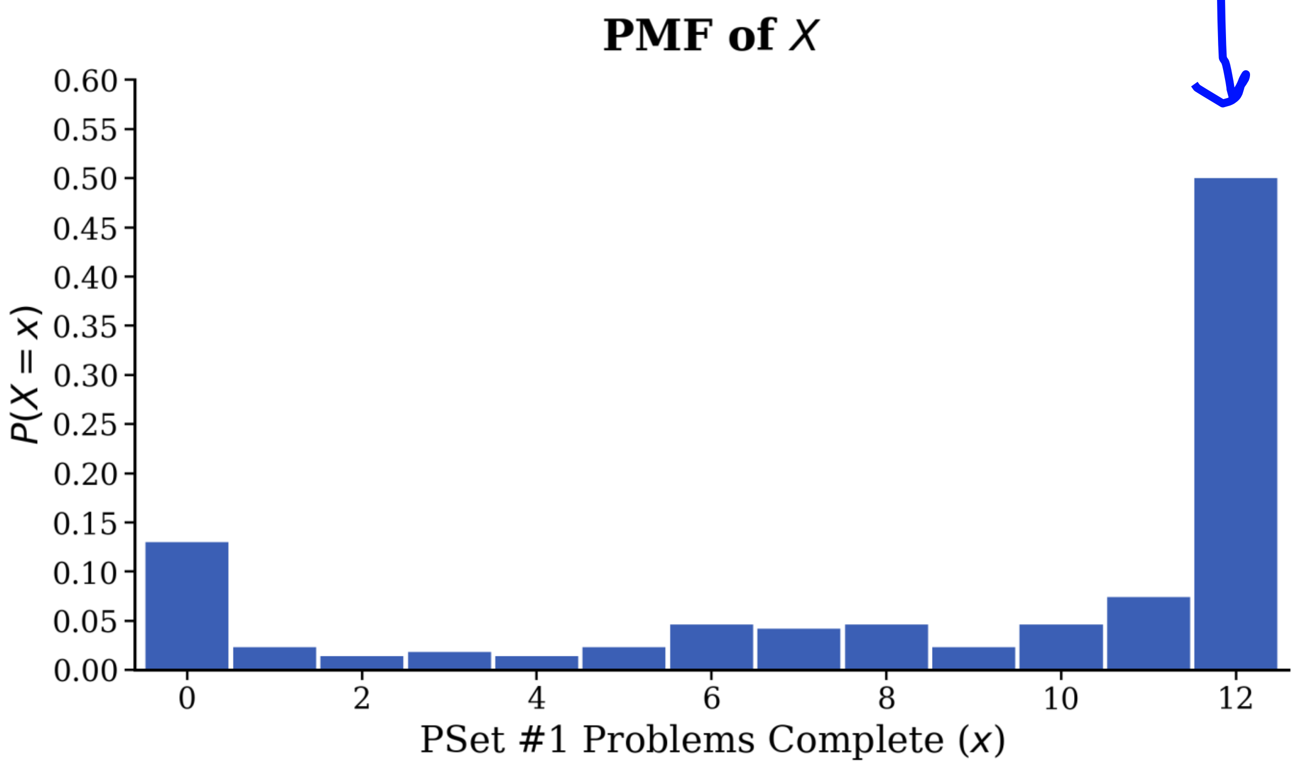
$$E[Y] = n \cdot p$$

Expectation is easy to work with, but still leaves much to be desired

Expectation vs PMF

Let X be the number of problems that a randomly selected student has completed, as of 11a today.

X takes on values, with uncertainty. X is a random variable.



$$P(X = 12) \approx \frac{\text{Count}(X = 12)}{N}$$

$$E[X] = 8.7$$

If extra time…

St. Petersburg Paradox

The Game:

- We have a fair coin (lands on heads with $p = 0.5$)
- Let n = number of coin flips (tails) to get the first heads
- You will win: $\$2^n$

How much would you pay to play?

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Let X be your winnings.

$$E[X] = \left(\frac{1}{2}\right)^1 2^1 + \left(\frac{1}{2}\right)^2 2^2 + \left(\frac{1}{2}\right)^3 2^3 + \ldots = \sum_{i=0}^{\infty} 1 = \infty$$

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What if you could play this game for only $1000…but just once?