Randomized Algorithms

Part Four

Announcements

- Problem Set Three due right now.
  - Due Wednesday using a late day.
- Problem Set Four out, due next Monday, July 29.
  - Play around with randomized algorithms!
  - Approximate **NP**-hard problems!
  - Explore a recent algorithm and why hashing matters!
- Handout: “Guide to Randomized Algorithms” also released.

Outline for Today

- **Chained Hash Tables**
  - How can you compactly store a small subset of a large set of elements?
- **Universal Hash Functions**
  - Groups of functions that distribute elements nicely.

Associative Structures

- The data structures we've seen so far are linear:
  - Stacks, queues, priority queues, lists, etc.
- In many cases, we want to store data in an unordered fashion.
- Queries like
  - Add element x.
  - Remove element x.
  - Is element x contained?

Bitvectors

- A **bitvector** is a data structure for storing a set of integers in the range {0, 1, 2, 3, …, Z – 1}.
- Store as an array of Z bits.
- If bit at position x is 0, x does not appear in the set.
- If bit at position x is 1, x appears in the set.

Analyzing Bitvectors

- What is the runtime for
  - Inserting an element?
  - Removing an element?
  - Checking if an element is present?
- How much space is used if the bitvector contains all Z possible elements?
- How much space is used if the bitvector contains n of the Z possible elements?

Another Idea

- Store elements in an unsorted array.
- To determine whether x is contained, scan over the array elements and return whether x is found.
- To add x, check to see x is contained and, if not, append x.
- To remove x, check to see if x is contained and, if so, remove x.

Analyzing this Approach

- How much space is used if the array contains all Z possible elements?
- How much space is used if the array contains n of the Z possible elements?
- What is the runtime for
  - Inserting an element?
  - Removing an element?
  - Checking if an element is present?

The Tradeoff

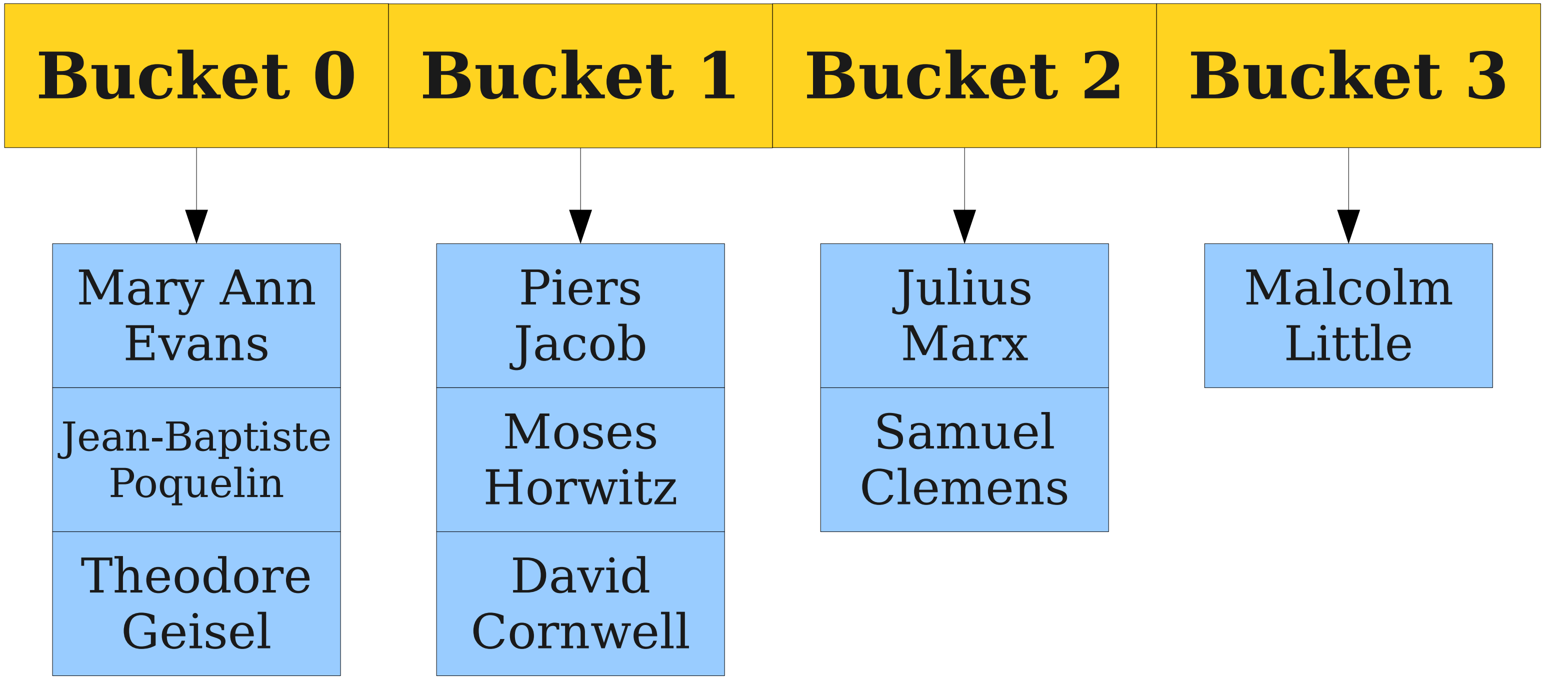
- Bitvectors are fast because we know where to look to find each element.
- Bitvectors are space-inefficient because we store one bit per possible element.
- Unsorted arrays are slow because we have to scan every element.
- Unsorted arrays are space-efficient because we only store the elements we use.
- This is a **time-space tradeoff**: we can improve performance by using more space.

Combining the Approaches

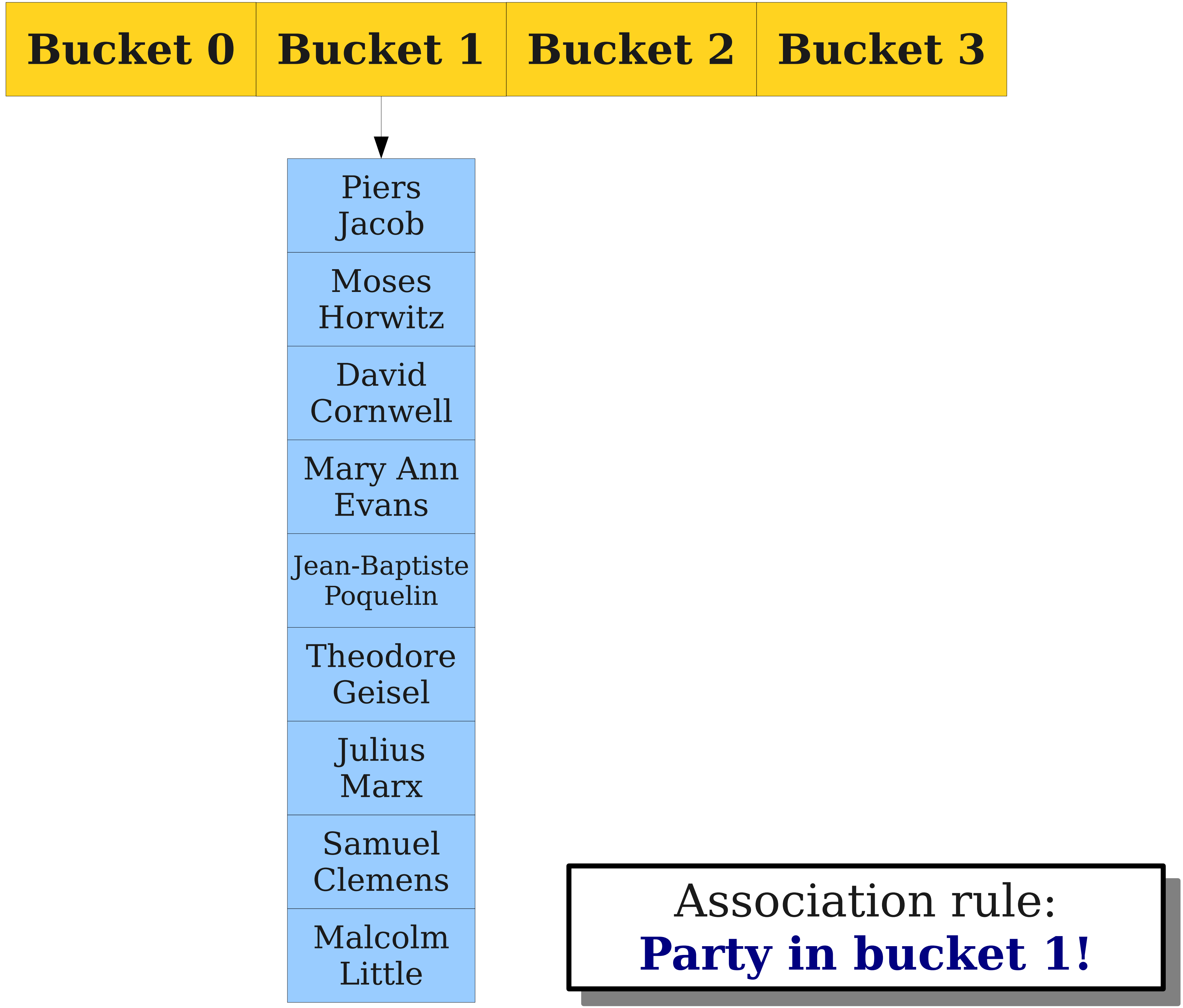
- Bitvectors always use a fixed amount of space and support fast lookups.
  - Good when number of possible elements is low, bad when number of possible elements is large.
- Unsorted arrays use variable space and don't support fast lookups.
  - Good when number of *used* elements is low, bad when number of *used* elements is large.

Chained Hash Tables

- Suppose we have a **universe** *U* consisting of all possible elements that we could want to store.
- Create *m* **buckets**, numbered {0, 1, 2, …, *m* – 1} as an array of length *m*. Each bucket is an unsorted array of elements.
- Find a rule associating each element in *U* with some bucket.
- To see if *x* is contained, look in the bucket *x* is associated with and see if *x* is there.
- To add *x*, see if *x* is contained and add it to the appropriate bucket if it's not.
- To remove *x*, see if *x* is contained and remove it from its bucket if it is.



Association rule:
***(length of first name)* mod 4**


Bucket 0
Bucket 1
Bucket 2
Bucket 3
Piers Jacob
Moses Horwitz
David Cornwell
Mary Ann Evans
Jean-Baptiste Poquelin
Theodore Geisel
Julius Marx
Samuel Clemens
Malcolm Little
Association rule:
Party in bucket 1!

Analyzing Runtime

- The three basic operations on a hash table (insert, remove, lookup) all run in time O(1 + X), where X is the total number of elements in the bucket visited.
  - *(Why is there a 1 here?)*
- Runtime depends on how well the elements are distributed.
- If n elements are distributed evenly across all the buckets, runtime is O(1 + n / m).
- If there are n elements distributed all into the same bucket, runtime is O(n).

Hash Functions

- Chained hash tables only work if we have a mechanism for associating elements of the universe with buckets.
- A **hash function** is a function

$$\boldsymbol{h : U \to \{0, 1, 2, \ldots, m - 1\}}$$

- In other words, for any $x \in U$, the value of $h(x)$ is the bucket that x belongs to.
- Since h is a mathematical function, it's defined for all inputs in U and always produces the same output given the same input.
- For simplicity, we'll assume hash functions can be computed in O(1) time.

Choosing Good Hash Functions

- The efficiency of a hash table depends on the choice of hash function.
- In the upcoming analysis, we will assume $|U| \gg m$ (that is, there are vastly more elements in the universe than there are buckets in the hash table.)
  - Assume at least $|U| > mn$, but probably more.

A Problem

Theorem: For any hash function h, there is a series of n values that, if stored in the table, all hash to the same bucket.

Proof: Because there are m buckets, under the assumption that $|U| > mn$, by the pigeonhole principle there must be at least $n + 1$ elements that hash to the same bucket. Inserting any n of those elements into the hash table places all those elements into the same bucket. ■

A Problem

- No matter how clever we are with our choice of hash function, there will always be an input that will degenerate operations to worst-case $\Omega(n)$ time.
- Theoretically, limits the worst-case effectiveness of chained hashing.
- Practically, leads to denial-of-service attacks.

Randomness to the Rescue

- For any *fixed* hash function, there is a degenerate series of inputs.
- The hash function itself cannot involve randomness.
  - *(Why?)*
- However, what if we choose ***which hash function to use*** at random?

A (Very Strong) Assumption

- Let's suppose that when we create our hash table, we choose a ***totally random function*** $h : U \to \{0, 1, 2, \ldots, m - 1\}$ as our hash function.
  - This has some issues; more on that later.
- Under this assumption, what would the expected cost of the three major hash table operations be?

Some Notation

- As before, let n be the number of elements in a hash table.
- Let those elements be x_1, x_2, …, x_n.
- Suppose that the element that we're looking up is the element z.
  - Perhaps z is in the list; perhaps it's not.

Analyzing Efficiency

- Suppose we perform an operation (insert, lookup, delete) on element z.
- The runtime is proportional to the number of elements in the same bucket as z.
- For any x_k, let C_k be an indicator variable that is 1 if x_k and z hash to the same bucket (i.e. $h(x_k) = h(z)$) and is 0 otherwise.
- Let random variable X be equal to the number of elements in the same bucket as z. Then

$$X = \sum_{x_i \neq z} C_i$$

Analyzing Efficiency

$$
\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\Big[\sum_{x_i \neq z} C_i\Big] \\
&= \sum_{x_i \neq z} \mathrm{E}[C_i] \\
&= \sum_{x_i \neq z} P\big(h(x_i) = h(z)\big) \\
&= \sum_{x_i \neq z} \frac{1}{m} \\
&\leq \frac{n}{m}
\end{aligned}
$$

So the expected cost of an operation is
O(1 + E[X]) = **O(1 + n / m)**

Analyzing Efficiency

- Assuming we choose a function uniformly at random from all functions, the expected cost of a hash table operation is $O(1 + n / m)$.
- What's the space usage?
  - $O(m)$ space for buckets.
  - $O(n)$ space for elements.
  - Some unknown amount of space to store the hash function.

A Problem

- We assume h is chosen uniformly at random from all functions from U to $\{0, 1, \ldots, m - 1\}$.
- There are $m^{|U|}$ possible functions from U to $\{0, 1, \ldots, m - 1\}$. *(Why?)*
- How much memory does it take to store h?
- If we assign k bits to store h, there are 2^k possible combinations of those bits.
- We need at least $\boldsymbol{|U| \log_2 m}$ bits to store h.
- **Question:** How can we get this performance without the huge space penalty?

Analyzing Efficiency

$$
\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\Big[\sum_{x_i \neq z} C_i\Big] \\
&= \sum_{x_i \neq z} \mathrm{E}[C_i] \\
&= \sum_{x_i \neq z} P\big(h(x_i) = h(z)\big) \\
&= \sum_{x_i \neq z} \frac{1}{m} \\
&\leq \frac{n}{m}
\end{aligned}
$$

So the expected cost of an operation is
O(1 + E[X]) = **O(1 + n / m)**

Universal Hash Functions

- A set $\mathscr{H}$ of hash functions from U to $\{0, 1, \ldots, m - 1\}$ is called a **universal family of hash functions** iff

  For any $x, y \in U$ where $x \neq y$, if h is drawn uniformly at random from $\mathcal{H}$, then

  $$P(h(x) = h(y)) \leq 1 / m$$

- In other words, the probability of a collision between two elements is at most $1 / m$ as long as we choose h from $\mathscr{H}$ uniformly at random.

Universal Hashing

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\Big[\sum_{x_i \neq z} C_i\Big] \\
&= \sum_{x_i \neq z} \mathrm{E}[C_i] \\
&= \sum_{x_i \neq z} P\big(h(x_i) = h(z)\big) \\
&\leq \sum_{x_i \neq z} \frac{1}{m} \\
&\leq \frac{n}{m}
\end{aligned}$$

So the expected cost of an operation is
O(1 + E[X]) = **O(1 + n / m)**

Universal Hash Functions

- The set of all possible functions from U to $\{0, 1, \ldots, m - 1\}$ is a universal family of hash functions.
  - However, requires $\Omega(|U| \log m)$ space.
- For certain types of elements, can find families of universal hash functions we can evaluate in O(1) time and store in O(1) space.
- **The Good News:** The intuitions behind these functions are quite nice.
- **The Bad News:** Formally proving that they're universal requires number theory and/or field theory, which is beyond the scope of this class.

Simple Universal Hash Functions

- We'll start with a simplifying assumption and generalize from there.
- Assume $U = \{0, 1, 2, \ldots, m - 1\}$ and that m is prime. (We'll relax this later.)
- Let $\mathscr{H}$ be the set of all functions of the form

$$\boldsymbol{h(x) = ax + b \pmod{m}}$$

- Where $a, b \in \{0, 1, 2, \ldots, m - 1\}$
- **Claim:** $\mathscr{H}$ is universal.

Showing Universality

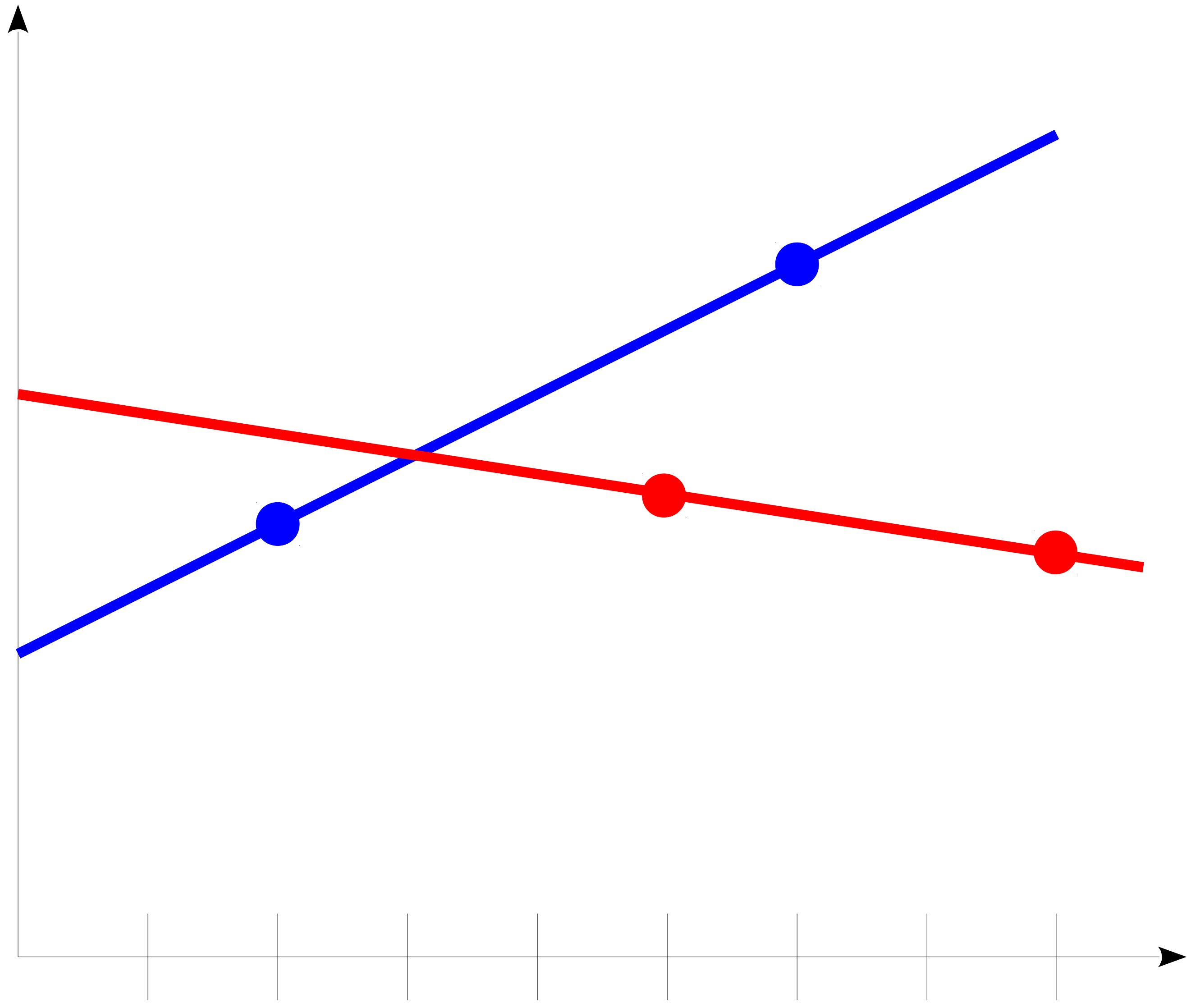
- We'll show $\mathscr{H}$ is universal by showing it obeys a stronger property called **2-independence**:

  For any $x_1, x_2 \in U$ where $x_1 \neq x_2$, if h is chosen uniformly at random from $\mathscr{H}$, then for any y_1 and y_2 we have

  $$\boldsymbol{P(h(x_1) = y_1 \wedge h(x_2) = y_2) = 1 / m^2.}$$

- (The probability that you can guess where any two distinct elements will be hashed is $1 / m^2$).
- ***Claim:*** Any 2-independent family of hash functions is universal.

$$h(x) = ax + b$$



Showing Universality

- If $h(x) = ax + b \pmod{m}$, knowing two points on the line determines the entire line.
- Can only guess the output at two points by guessing the coefficients: probability is $1 / m^2$!
- Need to use some more advanced math to formalize why this works; revolves around the fact that $\mathbb{F}_m$ is a finite field.

Generalizing the Result

- This hash function only works if m is prime and $|U| = m$.
- Suppose we can break apart any $x \in U$ into k integer "blocks" $x_1, x_2, \ldots, x_k$, where each block is between 0 and $m - 1$.
- Then the set $\mathscr{H}$ of all hash functions of the form

$$h(x) = a_1x_1 + a_2x_2 + \ldots + a_kx_k + b \pmod{m}$$

  is universal.
- Intuitively, after evaluating $k - 1$ of the products, you're left with a linear function in one remaining block and the same argument applies.

A Quick Aside

- Most programming languages associate “a” hash code with each object:
  - Java: **`Object.hashCode`**
  - Python: **`__hash__`**
  - C++: **`std::hash`**
- Unless special care is taken, there always exists the possibility of extensive hash collisions!

Looking Forward

- This is not the only type of hash table; others exist as well:
  - **Dynamic perfect hash tables** have *worst-case* O(1) lookup times and O(n) total storage space, but use a bit more memory.
  - Open addressing hash tables avoid chaining and have better locality, but require stronger guarantees on the hash function.
- Hash functions have *lots* of applications beyond hash tables; you'll see one in the problem set.

Next Time

- Greedy Algorithms
- Interval Scheduling