Greedy Algorithms
Part Three
Announcements

• Problem Set Four due right now.
  • Due on Wednesday with a late day.

• Problem Set Five out, due Monday, August 5.
  • Explore greedy algorithms, exchange arguments, “greedy stays ahead,” and more!
  • *Start early*. Greedy algorithms are tricky to design and the correctness proofs are challenging.

• Handout: “Guide to Greedy Algorithms” also available.

• Problem Set Three graded; will be returned at the end of lecture.
  • Sorry for the mixup from last time!
Outline for Today

- **Implementing Prim's Algorithm**
  - Efficiently finding MSTs.
- **Kruskal's Algorithm**
  - A different algorithm for finding MSTs.
- **Disjoint-Set Forests**
  - A specialized data structure for speeding up Kruskal's algorithm.
Recap: **Prim's Algorithm**
Prim's Algorithm

- **Prim's Algorithm** is the following:
  - Choose some $v \in V$ and let $S = \{v\}$.
  - Let $T = \emptyset$.
  - While $S \neq V$:
    - Choose a least-cost edge $e$ with one endpoint in $S$ and one endpoint in $V - S$.
    - Add $e$ to $T$.
    - Add both endpoints of $e$ to $S$.
  - Naive implementation takes time $O(mn)$. 
A Faster Implementation

• Can speed up using binary heaps:
  • Create a priority queue initially holding all edges incident to $v$.
  • At each step, dequeue edges from the priority queue until we find an edge $(x, y)$ where $x \in S$ and $y \notin S$.
  • Add $(x, y)$ to $T$.
  • Add to the queue all edges incident to $y$ whose endpoints aren't in $S$.
• Each edge is enqueued and dequeued at most once. (Why?)
• Total runtime: $O(m \log m)$. 
A Note on Runtimes

• In any graph, $m = O(n^2)$.

• Therefore:

$$O(m \log m) = O(m \log (n^2)) = O(m \log n)$$

• This version is more common and we will use it going forward.
A Different Approach: Kruskal's Algorithm
Kruskal's Algorithm

- **Kruskal's Algorithm** is the following:
  - Let $T = \emptyset$.
  - For each edge $(u, v)$ sorted by cost:
    - If $u$ and $v$ are not already connected in $T$, add $(u, v)$ to $T$.

- Can prove by induction that the result is a spanning tree by showing that
  - Exactly $n - 1$ edges are added.
  - No edges are added that close a cycle.
Showing Correctness

- The correctness proof for Kruskal's algorithm uses an exchange argument similar to that for Prim's algorithm.

- **Recall**: Prove Prim's algorithm is correct by looking at cuts in the graph:
  - Can swap an edge added by Prim's for a specially-chosen edge crossing some cut.
  - Since that edge is the lowest-cost edge crossing the cut, this cannot increase the cost.
Correctness Proof Intuition

- **Claim:** Every edge added by Kruskal's algorithm is a least-cost edge crossing some cut \((S, V - S)\).
  - When the edge was chosen, it did not close a cycle.
  - Choose \(S\) to be the CC of nodes on one end of the edge to get cut \((S, V - S)\).
  - Edge must be cheapest edge crossing this cut, since otherwise we would have selected a different edge.
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**Proof:** Let \( T \) be the tree produced by Kruskal's algorithm and \( T^* \) be an MST. We will prove \( c(T) = c(T^*) \). If \( T = T^* \), we are done.
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Let $S$ be the CC containing $u$ at the time $(u, v)$ was added to $T$. We claim $(u, v)$ is a least-cost edge crossing cut $(S, V - S)$. 

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Note that $|T - T^*| = |T - T^*| - 1$. Therefore, if we repeat this process once for each edge in $|T - T^*|$, we will have converted $T^*$ into $T$ while preserving $c(T^*)$. Thus $c(T) = c(T^*)$. ■
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Implementing Kruskal's Algorithm
Kruskal's Algorithm

- High-level overview of Kruskal's algorithm:
  - Let $T = \emptyset$.
  - For each edge $(u, v)$ sorted by cost:
    - If $u$ and $v$ are not connected by $T$, add $(u, v)$ to $T$.
- Can visit edges in order by sorting them in time $O(m \log n)$.
- Can check whether $u$ and $v$ are connected in time $O(n)$ by doing DFS. (*Why?*)
- Total time required: $O(mn)$. 
The “bottleneck” in Kruskal's algorithm is checking whether a pair of nodes are connected to one another.

**Goal:** Optimize queries of the form “are $x$ and $y$ connected?”

To do this, we will introduce a new data structure called the disjoint-set forest.
Set Partitions

- A **partition** of a set $S$ is a family $X$ of nonempty sets where each element of $S$ belongs to exactly one set in $X$.

- **Goal:** Build a data structure (called a *disjoint-set data structure*) that efficiently supports three operations:
  - **make-set**($v$), which places $v$ into its own set,
  - **union**($u$, $v$), which combines the sets containing $u$ and $v$ into one set, and
  - **in-same**($u$, $v$), which returns whether $u$ and $v$ belong to the same set.
Kruskal's Algorithm

- Using our new data structure:
  - Let $T = \emptyset$.
  - Let $S$ be a disjoint-set data structure.
  - For each $v \in V$:
    - Call $S$.make-set($v$)
  - For each edge $(u, v)$ sorted by cost:
    - If $S$.in-same($u, v$) is false:
      - Add $(u, v)$ to $T$.
      - Call $S$.union($u, v$).
Representatives

- Given a partition of a set $S$, we can choose one **representative** from each of the sets in the partition.
- Representatives give a simple proxy for which set an element belongs to: two elements are in the same set in the partition iff their set has the same representative.
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• Representatives give a simple proxy for which set an element belongs to: two elements are in the same set in the partition iff their set has the same representative.
Data Structure Idea

- **Idea:** Associate each element in a set with a representative from that set.

- To determine if two nodes are in the same set, check if they have the same representative.

- To link two sets together, change all elements of the two sets so they reference a single representative.
Using Representatives
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- If there are $n$ total elements, what is the cost of looking up a representative?
  - $O(1)$
- If there are $n$ total elements, what is the cost of merging two sets together?
  - $O(n)$
- Can we improve this?
Hierarchical Representatives
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- If there are $n$ total elements, what is the cost of merging two sets together?
  - $O(n)$

- The inefficiency arises because the path from any node to its representative can be very large.

- Can we fix that?
Union by Size
Union by Size
Union by Size
Union by Size
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Union by Size
Union by Size

Diagram showing a Union by Size process with nodes labeled 1, 2, 4, and 1 connected in various ways.
Union by Size
Union by Size
Union by Size
Union by Size
Union by Size
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• **Idea:** Store in each node the number of nodes that count it as a representative.

• To merge the sets containing two nodes together:
  • Find the representatives of each.
  • Choose one of the representatives with the least number of nodes below it.
  • Set its representative to the other node.
  • Update the total number of nodes below the other node.
Analyzing Union by Size

- The runtime of these operations depends on the height of the trees formed this way.

- **Claim:** A tree with height $k$ contains at least $2^k$ nodes.

- **Proof Idea:** Use induction.
  - Trees with height 0 start with $2^0 = 1$ nodes.
  - Merging two trees of unequal heights always results in a single tree of the height of the larger of the two.
  - Merging two trees of height $k$ into a tree of height $k + 1$ results in a tree with at least $2 \cdot 2^k = 2^k + 1$ nodes.

- **Corollary:** If there are $n$ total nodes, all operations take $O(\log n)$ time.
Kruskal's Algorithm

• Using our new data structure:
  • Let $T = \emptyset$.
  • Let $S$ be a disjoint-set data structure.
  • For each $v \in V$:
    – Call $S$.make-set($v$)
  • For each edge $(u, v)$ sorted by cost:
    – If $S$.in-same($u$, $v$) is false:
      • Add $(u, v)$ to $T$.
      • Call $S$.union($u$, $v$).
• Total runtime: $O(m \log n)$. 
Looking Forward

- It is possible to speed up our data structure by using two modifications:
  - **Path Compression**: After looking up a representative, change the pointers of all visited nodes to directly point to the representative.
  - **Union-by-Rank**: Link trees based on *height* rather than number of nodes.

- New runtime: *m* total operations takes time $O(m \, \alpha(m))$, where $\alpha(m)$ is a *ridiculously slowly-growing* function.
Next Time

- Dynamic Programming
- Purchasing Cell Towers
- A Different Approach to Recursion