CS161 Homework 0 | **Due:** 8 April 2016, 12 noon
Submit on Gradescope | **Handed out:** 1 April 2016

Instructions: Please answer the following questions to the best of your ability. If you are asked to show your work, please include relevant calculations for deriving your answer. If you are asked to explain your answer, give a short ($\sim$ 1 sentence) intuitive description of your answer. If you are asked to prove a result, please write a complete proof at the level of detail and rigor expected in prior CS Theory classes (i.e. 103). When writing proofs, please strive for clarity and brevity (in that order). Cite any sources you reference.

1 (12 points)

What is the cardinality of each of the following sets?

(a) subsets of $\{1, 2, \ldots, n\}$ of size k

Solution: Choose k objects from n.

$$\binom{n}{k}$$

(b) simple paths of length k in a complete undirected graph with vertex set V, edge set E and $|V| = n$ (*Recall that a simple path of length k is a sequence of vertices $\langle v_0, v_1 \ldots, v_k\rangle$ such that $(v_i, v_{i+1}) \in E$ for $i \in \{0, k-1\}$ and $v_i \neq v_j$ for all $i \neq j$. You can consider $\langle v_0, v_1, \ldots, v_{k-1}, v_k\rangle \neq \langle v_k, v_{k-1}, \ldots, v_1, v_0\rangle$.*)

Solution: For a path of length k, we have k edges, which means $k+1$ vertices. Choose $k+1$ vertices, then permute them.

$$\binom{n}{k+1} \cdot (k+1)!$$

(c) bitstrings in $\{0, 1\}^n$ that have an even number of 1s

Solution: For all i from 0 to $\lfloor n/2 \rfloor$, choose $2i$ bits from all n. This is an even number of bits.

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} = 2^{n-1}.$$

Alternatively, we can prove this by induction. For the base case $n = 1$, there is only $1 = 2^0$ bit string with an even number of ones, i.e. 0. Let's assume for our induction hypothesis that for $n-1$ we have 2^{n-1} strings with even number of 1s. For n consider the first $n-1$ bits. There are 2^{n-2} strings with an odd number of 1s and adding a 1 as the last bit gives 2^{n-2} even strings of length n with a 1 at the end. Similarly, there are 2^{n-2} strings (for the first $n-1$ positions) with an even number of 1s and adding a 0 as the last bit gives 2^{n-2} even strings of length n with a 0 at the end. The total number of strings with an even number of 1s is hence exactly 2^{n-1}.

A third solution: Consider all bit strings of length n. We can pair them in 1:1 fashion as follows. Take any string x and pair it with string x' where x' is the same as x in the first $n-1$ positions and differs in the last position (i.e. if $x[n] = 1$, then $x'[n] = 0$ and vice versa). This ensures that x and x' have different parities- one has an even number of ones and the other an odd number. Because this matching partitions the strings, the number of even parity strings must be equal to the number of odd parity, so each of these numbers is the total number of strings divided by 2, i.e. 2^{n-1}.

For full-credit, show any work and explain where your answer comes from. You need not simplify your expressions (summations, factorials, "choose" notation, etc. are all fine).

2 (12 points)

Let `flip(p)` be a procedure which returns the result of a Bernoulli(p) trial – that is, it returns 1 with probability p and 0 with probability $1 - p$. Each call to `flip` is independent. Consider the two following functions:

```
iterativeF(n,p):
  tot = 0
  for i = 1 to n:
    tot = tot + flip(p)
  return tot
```

```
recursiveF(n,p):
  if n <= 1:
    return flip(p)
  return 2*recursiveF(n/2,p)
```

Suppose we're given some power of two $n = 2^k$ for $k \in \mathbb{N}$ and a legal probability $p \in [0, 1]$.

(a) What is the expected value of `iterativeF(n,p)`?

Solution: We know the expected value of a Bernoulli RV is its probability of success. The return value $X = \sum_{i=1}^{n} F_p$, namely it is the sum of n independent calls to `flip(p)`. By linearity of expectation, $E[f(n, p)] = np$.

(b) What is the expected value of `recursiveF(n,p)`?

Solution: In this case, the recursive function makes 1 call to `flip(p)` and multiplies the result by $2^k = 2^{\log n} = n$. The resulting value will be n with probability p and 0 with probability $1 - p$. Thus, the expected value is $E[f(n, p)] = np$.

(c) What is the variance of `iterativeF(n,p)`?

Solution: The variance of a Bernoulli RV is $p(1 - p)$. Because the n calls to `flip(p)` are independent, the variance of the sum is the sum of the variances. Thus, $\text{Var}(f(n, p)) = np(1 - p)$. Alternatively, one can recognize that this is sampling from a Binomial distribution and arrive at the same answer.

(d) What is the variance of `recursiveF(n,p)`?

Solution: The result is a single random variable multiplied by a scalar, n. $\text{Var}(aX) = a^2\text{Var}(X)$, so $\text{Var}(f(n, p)) = \text{Var}(n * \text{Bernoulli}(p)) = n^2 p(1 - p)$.

(e) Which properties of `recursiveF` change, if any, if we return `recursiveF(n/2,p)+recursiveF(n/2,p)` rather than `2*recursiveF(n/2,p)`?

Solution: If we call `recursiveF` twice independently, we will get a call tree with n leaves. This will be equivalent to the iterative version, where we will make n independent calls to `flip(p)`. The variance changes by a factor of n to $np(1 - p)$.

For full-credit, show your work.

3 (6 points)

A *tripartite graph* T is a graph where the vertices can be partitioned into three groups, U, V, and W, such that no edge in T runs within U, V, or W.

Consider a graph G on $k \geq 3$ vertices. We call G the *induced k-cycle* if we can label the vertices in G with $\{1, \ldots, k\}$ such that an edge (i, j) is in the graph if and only if $i = (j \mod k) + 1$. Prove that the induced k-cycle graph is tripartite for all $k \in \mathbb{N}$.

Solution: Suppose the induced k-cycle has its nodes labeled $1, \ldots, k$ according to its definition. Consider the following partition of the first $k - 1$ nodes: $U = \{i | i = 0 \mod 2\}$, $V = \{i | i = 1 \mod 2\}$. Let $W = \{k\}$. We claim this is a legal tripartition. To see this, note that the first $k - 1$ edges run from $(i, i + 1)$, so it is impossible for nodes with the same parity to share an edge. The final edge, which runs from k to 1 may run between nodes of the same parity (if k is odd). Thus, by placing k in W by itself, we know that no edges run within a partition.

(In fact, elaborating on this proof you can show that a graph is bipartite if and only if it has no odd cycles.)

4 (16 points)

For each of the following functions, indicate which of the following asymptotic bounds hold for $f(n)$.

(i) $O(g(n))$

(ii) $\Omega(g(n))$

(iii) Both (i.e. $\Theta(g(n))$)

For full-credit, if you believe that $f(n)$ is $O(g(n))$, then exhibit constants c and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. Similarly, if you believe that $f(n)$ is $\Omega(g(n))$, exhibit c and n_0 such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

Every log below is base 2.

(a)	$f(n) = 3n^2$	$g(n) = n^2$
(b)	$f(n) = 2n^4 - 3n^2 + 7$	$g(n) = n^5$
(c)	$f(n) = \frac{\log n}{n}$	$g(n) = \frac{1}{n}$
(d)	$f(n) = \log n$	$g(n) = \log n + \frac{1}{n}$
(e)	$f(n) = 2^{k \log n}$	$g(n) = n^k$
(f)	$f(n) = 2^n$	$g(n) = 2^{2n}$
(g)	$f(n) = \begin{cases} 4^n & \text{if } n < 2^{1000} \\ 2^{1000}n^2 & \text{if } n \geq 2^{1000} \end{cases}$	$g(n) = \dfrac{n^2}{2^{1000}}$
(h)	$f(n) = 2^{\sqrt{\log n}}$	$(\log n)^{100}$

Solution:

(a)	(iii)	$c_\Omega = c_O = 3, n_0 = 0$
(b)	(i)	$c_O = 2, n_0 = 7$
(c)	(ii)	$c_\Omega = 1, n_0 = 4$
(d)	(iii)	$c_\Omega = 1/3, c_O = 1, n_0 = 2$
(e)	(iii)	$c_\Omega = c_O = 1, n_0 = 0$
(f)	(i)	$c_O = 1, n_0 = 0$
(g)	(iii)	$c_\Omega = c_O = 2^{2000}, n_0 = 2^{1000}$
(h)	(ii)	$c_\Omega = 1, n_0 = 2^{2^{24}}$