CS161 Homework 0 **Due:** 8 April 2016, 12 noon
Submit on Gradescope **Handed out:** 1 April 2016

Instructions: Please answer the following questions to the best of your ability. If you are asked to show your work, please include relevant calculations for deriving your answer. If you are asked to explain your answer, give a short ($\sim$ 1 sentence) intuitive description of your answer. If you are asked to prove a result, please write a complete proof at the level of detail and rigor expected in prior CS Theory classes (i.e. 103). When writing proofs, please strive for clarity and brevity (in that order). Cite any sources you reference.

1 (12 points)

What is the cardinality of each of the following sets?

(a) subsets of $\{1, 2, \ldots, n\}$ of size k

(b) simple paths of length k in a complete undirected graph with vertex set V, edge set E and $|V| = n$ (*Recall that a simple path of length k is a sequence of vertices $\langle v_0, v_1 \ldots, v_k\rangle$ such that $(v_i, v_{i+1}) \in E$ for $i \in \{0, k-1\}$ and $v_i \neq v_j$ for all $i \neq j$. You can consider $\langle v_0, v_1, \ldots, v_{k-1}, v_k\rangle \neq \langle v_k, v_{k-1}, \ldots, v_1, v_0\rangle$.*)

(c) bitstrings in $\{0, 1\}^n$ that have an even number of 1s

For full-credit, show any work and explain where your answer comes from. You need not simplify your expressions (summations, factorials, "choose" notation, etc. are all fine).

2 (12 points)

Let `flip(p)` be a procedure which returns the result of a Bernoulli(p) trial – that is, it returns 1 with probability p and 0 with probability $1 - p$. Each call to `flip` is independent. Consider the two following functions:

```
iterativeF(n,p):
  tot = 0
  for i = 1 to n:
    tot = tot + flip(p)
  return tot
```

```
recursiveF(n,p):
  if n <= 1:
    return flip(p)
  return 2*recursiveF(n/2,p)
```

Suppose we're given some power of two $n = 2^k$ for $k \in \mathbb{N}$ and a legal probability $p \in [0, 1]$.

(a) What is the expected value of `iterativeF(n,p)`?

(b) What is the expected value of `recursiveF(n,p)`?

(c) What is the variance of `iterativeF(n,p)`?

(d) What is the variance of `recursiveF(n,p)`?

(e) Which properties of `recursiveF` change, if any, if we return `recursiveF(n/2,p)+recursiveF(n/2,p)` rather than `2*recursiveF(n/2,p)`?

For full-credit, show your work.

3 (6 points)

A *tripartite graph* T is a graph where the vertices can be partitioned into three groups, U, V, and W, such that no edge in T runs within U, V, or W.

Consider a graph G on $k \geq 3$ vertices. We call G the *induced k-cycle* if we can label the vertices in G with $\{1, \ldots, k\}$ such that an edge (i, j) is in the graph if and only if $i = (j \mod k) + 1$. Prove that the induced k-cycle graph is tripartite for all $k \in \mathbb{N}$.

4 (16 points)

For each of the following functions, indicate which of the following asymptotic bounds hold for $f(n)$.

(i) $O(g(n))$

(ii) $\Omega(g(n))$

(iii) Both (i.e. $\Theta(g(n))$)

For full-credit, if you believe that $f(n)$ is $O(g(n))$, then exhibit constants c and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. Similarly, if you believe that $f(n)$ is $\Omega(g(n))$, exhibit c and n_0 such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

Every log below is base 2.

(a) $f(n) = 3n^2$ $\quad g(n) = n^2$

(b) $f(n) = 2n^4 - 3n^2 + 7$ $\quad g(n) = n^5$

(c) $f(n) = \frac{\log n}{n}$ $\quad g(n) = \frac{1}{n}$

(d) $f(n) = \log n$ $\quad g(n) = \log n + \frac{1}{n}$

(e) $f(n) = 2^{k \log n}$ $\quad g(n) = n^k$

(f) $f(n) = 2^n$ $\quad g(n) = 2^{2n}$

(g) $f(n) = \begin{cases} 4^n & \text{if } n < 2^{1000} \\ 2^{1000} n^2 & \text{if } n \geq 2^{1000} \end{cases}$ $\quad g(n) = \dfrac{n^2}{2^{1000}}$

(h) $f(n) = 2^{\sqrt{\log n}}$ $\quad (\log n)^{100}$