

1 Binary Search Trees

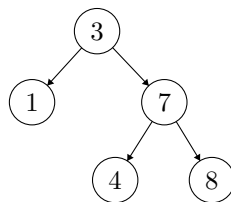
Definition 1.1. A binary search tree (BST) is a data structure that stores elements that have keys from a totally ordered universe (say, the integers). In this lecture, we will assume that each element has a unique key. A BST supports the following operations:

- $\text{search}(i)$: Returns an element in the data structure associated with key i
- $\text{insert}(i)$: Inserts an element with key i into the data structure
- $\text{delete}(i)$: Deletes an element with key i from the data structure, if such an element exists

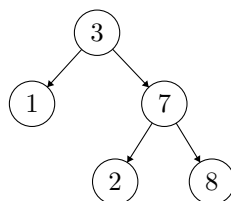
A BST stores the elements in a binary tree with a root r . Each node x has $\text{key}(x)$ (the key of the element stored in x), $\text{p}(x)$ (the parent of x , where $\text{p}(r) = \text{NIL}$), $\text{left}(x)$ (the left child of x), and $\text{right}(x)$ (the right child of x). The children of x are either other nodes or NIL.

The key BST property is that for every node x , the keys of all nodes under $\text{left}(x)$ are less than $\text{key}(x)$ and the keys of all nodes under $\text{right}(x)$ are greater than $\text{key}(x)$.

Example In the following example, the root node r stores 3 in it ($\text{key}(r) = 3$), its left child $\text{left}(x)$ stores 1, its right child $\text{right}(x)$ stores 7, and all leaf nodes (storing 1, 4, and 8, respectively) have NIL as their two children.



Example The following binary tree is not a BST since $2 > 3$ and 2 is a child of 7, which is the right child of 3:



Some properties. *Relationship to Quicksort:* We can think of each node x as a pivot for quicksort for the keys in its subtree. The left subtree contains $A_{<}$ for $\text{key}(x)$ and the right subtree contains $A_{>}$ for $\text{key}(x)$.

Sorting the keys: We can do an *inorder* traversal of the tree to recover the nodes in sorted order from left to right (the smallest element is in the leftmost node and the largest element is in the rightmost node). The *inorder* procedure takes a node x and returns the keys in the subtree under x in sorted order. We can recursively define $\text{Inorder}(x)$ as: (1) If $\text{left}(x) \neq \text{NIL}$, then run $\text{Inorder}(\text{left}(x))$, then: (2) Output $\text{key}(x)$ and

then: (3) If $\text{right}(x) \neq \text{NIL}$, run $\text{Inorder}(\text{right}(x))$. With this approach, for every x , all keys in its left subtree will be output before x , then x will be output and then every element in its right subtree.

Subtree property: If we have a subtree where x has y as a left child, y has z as a right child, and z is the root for the subtree T_z , then our BST property implies that all keys in T_z are $> y$ and $< x$. Similarly, if we have a subtree where x has y as a right child, y has z as a left child, and z is the root of the subtree T_z , then our BST property implies that all keys in T_z are between x and y .

1.1 Basic Operations on BSTs

The three core operations on a BST are **search**, **insert**, and **delete**. For this lecture, we will assume that the BST stores distinct numbers, i.e. we will identify the objects with their names and we will have each name be represented by a number.

1.1.1 search

To **search** for an element, we start at the root and compare the key of the node we are looking at to the element we are searching for. If the node's key matches, then we are done. If not, we recursively search in the left or right subtree of our node depending on whether this node was too large or too small, respectively. If we ever reach **NIL**, we know the element does not exist in our BST. In the following algorithm in this case, we simply return the node that would be the parent of this node if we inserted it into our tree.

Algorithm 1: $\text{search}(i)$

return $\text{search}(\text{root}, i)$;

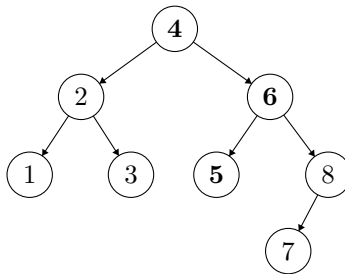
Algorithm 2: $\text{search}(x, i)$

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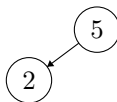
if  $\text{key}(x) == i$  then
  | return  $x$ 
else if  $i < \text{key}(x)$  then
  | if  $\text{left}(x) == \text{NIL}$  then
  | | return  $x$ 
  | else
  | | return  $\text{search}(\text{left}(x), i)$ 
else if  $i > \text{key}(x)$  then
  | if  $\text{right}(x) == \text{NIL}$  then
  | | return  $x$ 
  | else
  | | return  $\text{search}(\text{right}(x), i)$ 

```

Example If we call $\text{search}(5.5)$ on the following BST, we will return the node storing 5 (by taking the path in bold). This also corresponds to the path taken when calling $\text{search}(4.5)$ or $\text{search}(5)$.



Claim 1. $\text{search}(i)$ returns a node containing i if $i \in \text{BST}$, otherwise a node x such that either $\text{key}(x)$ is the smallest in $\text{BST} > i$ or the largest in $\text{BST} < i$, where node x happens to be the parent of the new node if it were to be inserted into BST . This follows directly from the BST property. Try to formally prove this claim as an exercise.



Remark 1. $\text{search}(i)$ does **not** necessarily return the node in BST that is closest in value. Consider the tree above. If we search for an element with a key of 4, the node with key 2 is returned, whereas the element with key 5 is the closest element by value.

1.1.2 insert

As before, we will assume that all keys are distinct. We will $\text{search}(i)$ for a node x to be the parent and create a new node y , placing it as a child of x where it would logically go according to the BST property.

Algorithm 3: $\text{insert}(i)$

```

 $x \leftarrow \text{search}(i);$ 
 $y \leftarrow$  new node with  $\text{key}(y) \leftarrow i, \text{left}(y) \leftarrow \text{NIL}, \text{right}(y) \leftarrow \text{NIL}, \text{p}(y) \leftarrow x;$ 
if  $i < \text{key}(x)$  then
  |  $\text{left}(x) \leftarrow y;$ 
else
  |  $\text{right}(x) \leftarrow y;$ 

```

Remark 2. Notice that x needed to have NIL as a child where we want to put y by the properties of our search algorithm.

1.1.3 delete

Deletion is a bit more complicated. To delete a node x that exists in our tree, we consider several cases:

1. If x has no children, we simply remove it by modifying its parent to replace x with NIL .
2. If x has only one child c , either left or right, then we elevate c to take x 's position in the tree by modifying the appropriate pointer of x 's parent to replace x with c , and also fixing c 's parent pointer to be x 's parent.
3. If x has two children, a left child c_1 and right child c_2 , then we find x 's immediate successor z and have z take x 's position in the tree. Notice that z is in the subtree under x 's right child c_2 and we can find it by running $z \leftarrow \text{search}(c_2, \text{key}(x))$. Note that since z is x 's successor, it doesn't have a left child, but it might have a right child. If z has a right child, then we make z 's parent point to that child instead of z (also fixing the child's parent pointer). Then we replace x with z , fixing up all relevant pointers: the rest of x 's original right subtree becomes z 's new right subtree, and x 's left subtree becomes z 's new left subtree.

(Note that alternatively, we could have used x 's immediate predecessor y and followed the same analysis in a mirrored fashion.)

In the following algorithm, if p is the parent of x , $\text{child}(p)$ refers to $\text{left}(p)$ if x was the left child of p and to $\text{right}(p)$ otherwise.

Algorithm 4: delete(i)

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 $x \leftarrow \text{search}(i);$ 
if  $\text{key}(x) \neq i$  then return;
if  $\text{NIL} = \text{left}(x)$  and  $\text{NIL} = \text{right}(x)$  then
   $\text{child}(\text{p}(y)) \leftarrow \text{NIL};$ 
   $\text{delete-node}(x);$ 
if  $\text{NIL} = \text{left}(x)$  then
   $y \leftarrow \text{right}(x);$ 
   $\text{p}(y) \leftarrow \text{p}(x);$ 
   $\text{child}(\text{p}(y)) \leftarrow y;$ 
   $\text{delete-node}(x);$ 
else if  $\text{NIL} = \text{right}(x)$  then
   $y \leftarrow \text{left}(x);$ 
   $\text{p}(y) \leftarrow \text{p}(x);$ 
   $\text{child}(\text{p}(y)) \leftarrow y;$ 
   $\text{delete-node}(x);$ 
else  $x$  has two children
   $z \leftarrow \text{search}(\text{right}(x), \text{key}(x));$ 
   $z' \leftarrow \text{right}(z);$ 
   $\text{left}(\text{p}(z)) \leftarrow z';$ 
   $\text{p}(z') \leftarrow \text{p}(z);$ 
  replace  $x$  with  $z;$ 
   $\text{delete-node}(x);$ 
```

1.1.4 Runtimes

The worst-case runtime for `search` is $O(\text{height of tree})$. As both `insert` and `delete` call `search` a constant number of times (once or twice) and otherwise perform $O(1)$ work on top of that, their runtimes are also $O(\text{height of tree})$.

In the best case, the height of the tree is $O(\log n)$, e.g., when the tree is completely balanced. However, in the worst case it can be $O(n)$ (a long rightward path, for example). This could happen because `insert` can increase the height of the tree by 1 every time it is called. Currently our operations do not guarantee logarithmic runtimes. To get $O(\log n)$ height we would need to rebalance our tree. There are many examples of self-balancing BSTs, including AVL trees, red-black trees, splay trees (somewhat different but super cool!), etc. Today, we will talk about red-black trees.

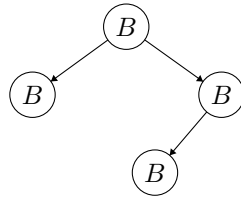
2 Red-Black Trees

One of the most popular balanced BST is the red-black tree developed by Guibas and Sedgwick in 1978. In a red-black tree, all leaves are assumed to have NILs as children.

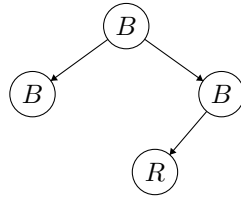
Definition 2.1. A red-black tree is a BST with the following additional properties:

1. Every node is red or black
2. The root is black
3. NILs are black
4. The children of a red node are black
5. For every node x , all x to NIL paths have the same number of black nodes on them

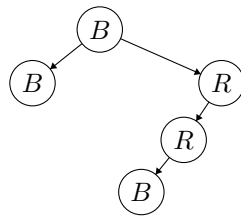
Example (B means a node is black, R means a node is red.) The following tree is *not* a red-black tree since property 5 is not satisfied:



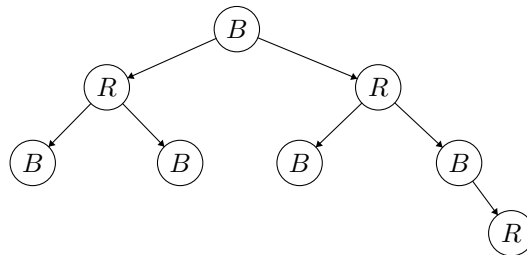
Example (B means a node is black, R means a node is red.) The following tree *is* a red-black tree since all the properties are satisfied:



Example (B means a node is black, R means a node is red.) The following tree is *not* a red-black tree since property 4 is not satisfied:



Example (B means a node is black, R means a node is red.) The following tree *is* a red-black tree:



Remark 3. *Intuitively, red nodes represent when a path is becoming too long.*

Claim 2. *Any valid red-black tree on n nodes (non-NIL) has height $\leq 2 \log_2(n + 1) = O(\log n)$.*

Proof. For some node x , let $b(x)$ be the “black height” of x , which is the number of black nodes on a $x \rightarrow \text{NIL}$ path excluding x . We first show that the number of non-NIL descendants of x is at least $2^{b(x)} - 1$ (including x) via induction on the height of x .

Base case: NIL node has $b(x) = 0$ and $2^0 - 1 = 0$ non-NIL descendants. \checkmark

For our inductive step, let $d(x)$ be the number of non-NIL descendants of x . Then

$$\begin{aligned}
 d(x) &= 1 + d(\text{left}(x)) + d(\text{right}(x)) \\
 &\geq 1 + (2^{b(x)-1} - 1) + (2^{b(x)-1} - 1) \text{ (by induction)} \\
 &= 2^{b(x)} - 1 \checkmark
 \end{aligned}$$

Notice that $b(x) \geq \frac{h(x)}{2}$ (where $h(x)$ is the height of x) since on any root to NIL path there are no two consecutive red nodes, so the number of black nodes is at least the number of red nodes, and hence the black height is at least half of the height. We apply this and the above inequality to the root r (letting $h = h(r)$) to obtain $n \geq 2^{b(r)} - 1 \geq 2^{\frac{h}{2}} - 1$, and hence $h \leq 2 \log(n + 1)$. \square

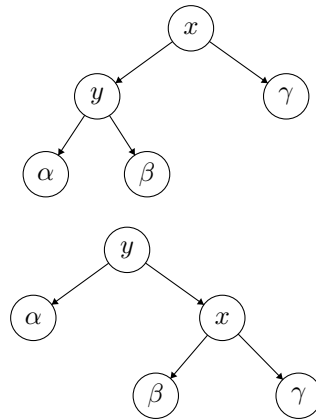
Here is some intuition on why the tree is roughly balanced.

Intuition: By Property (5) of a red-black tree, all $r \rightarrow \text{NIL}$ paths have $b(r)$ black nodes (excluding the root). Therefore, all these paths have length $\geq b(r)$. However, they also have length $\leq 2 \cdot b(r)$: by Property (4), the number of red nodes is limited to half of the path, since every red node must be followed by a black node, and hence the number of black nodes is at least half of the length of the path. Hence, the lengths of all paths from r to a NIL are within a factor of 2 of each other, and the tree must be reasonably balanced.

Today we will take a brief look at how the red-black tree properties are maintained. Our coverage here is detailed, but not comprehensive, and meant as a case study. For complete coverage, please refer to Ch. 13 of CLRS.

2.1 Rotations

Red-black trees, as do other balanced BSTs, use a concept called rotation. A tree rotation restructures the tree shape locally, usually for the purpose of balancing the tree better. A rotation preserves the BST property (as shown in the following two diagrams). Notably, tree rotations can be performed in $O(1)$ time.



Moving from the first tree to the second is known as a *right rotation of x* . The other direction (from the second tree to the first) is a *left rotation of y* . Notice that we only move the β subtree, which is why we preserve the BST property.

2.2 Insertion in a Red-Black Tree

Let's see how we can perform $\text{Insert}(i)$ on a red-black tree while still maintaining all of its properties. The process for inserting a new node is initially similar to that of insertion into any BST.

Algorithm 5: insert_rb(i)

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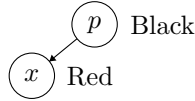
 $p \leftarrow \text{search}(i)$ ;
 $x \leftarrow$  new node with  $\text{key}(x) \leftarrow i$ ,  $\text{left}(x) \leftarrow \text{NIL}$ ,  $\text{right}(x) \leftarrow \text{NIL}$ ,  $\text{p}(x) \leftarrow p$ ;
if  $i < \text{key}(p)$  then
  |  $\text{left}(p) \leftarrow x$ ;
else
  |  $\text{right}(p) \leftarrow x$ ;
 $\text{color}(x) \leftarrow \text{red}$ ;
recolor if needed;

```

Note that when x is inserted as a red node: Property (1) is satisfied, as we colored the new node red; Property (2) is satisfied, as we did not touch the root; Property (3) is satisfied, as we can color the new NILs black; and Property (5) is satisfied, as we did not change the number of black nodes in the tree. Thus, the only invariant we have to worry about is Property (4), that red nodes have black children.

The recoloring step is broken down into multiple cases. We consider each of them:

Case 1: p is black. In this case, Property (4) is also maintained. So, we simply add x as a new red child of p , and the red-black tree properties are maintained.



Case 2: p is red, and x 's uncle u is red. In this case, we insert a red x , change p and u to black, and change p' to red. Because we switched the colors of two nodes on each of these paths (one red \rightarrow black and one black \rightarrow red), the number of black nodes on each path is unchanged, so Property (5) remains unchanged. If the parent of p' , p'' , is black, then Property (4) is maintained. Otherwise, if p'' is red and breaks Property (4) by introducing a "double-red" pair of nodes (p' and p''), then we have to recolor recursively starting at p' .



Case 3: p is red, and u is black. There are two possibilities here:

1. We are inserting x as a leaf node. Then, u must be NIL for the red-black tree to have been valid before inserting x . We insert a red x .
2. We are not inserting x ; rather, we are recoloring the tree at x from the recursive call in Case 2. We aim to recolor the tree and maintain the number of black nodes on each path from the root to NIL. Note that in this case, x actually has nodes under it.

In both cases, x is red, so we make p black, make p' red, and do a right rotation at p' . We can see that this also maintains the same number of black nodes on each path from the root to NIL, and satisfies Property (4) below p because the original tree was a red-black tree.



If we end up in Case 2 and recursively call `recolor`, then in the worst case the recursion will bottom out when we hit the root, with a constant number of relabelings and rotations at each level. So, it will be an $O(h)$ operation overall, where h is the height of the tree.

In the analysis above, we considered the cases where x is a left child of p and u is a right child of its parent p' . These cases are representative, showing most of the machinery that we'll need to insert an arbitrary element into an arbitrary red-black tree. (Within Case 2 and Case 3, there are actually a total of four cases each, where p 's tree and p' 's children could each be swapped, but the recoloring procedure is similar. You are encouraged to read the text for details.)

To summarize, the following is the algorithm for recoloring, in the case where x is a left child and u is a right child.

Algorithm 6: `recolor(x)` // x is a left child, u is a right child

```

p ← parent(x);
if black = color(p) then
  return;
p' ← parent(p);
u ← right(p');
if red = color(u) then
  color(p) ← black;
  color(u) ← black;
  color(p') ← red;
  recolor(p');
else if black = color(u) then
  color(p) ← black;
  color(p') ← red;
  right_rotate(p');

```

Based on our analysis above, we can update our red-black trees in $O(h)$ time upon insertion, where h is the height of the tree. The other operations are similar, and also give the guarantee of worst-case performance of $O(h)$ search, insertion, and deletion. Together with Claim 2, which states that $h = O(\log n)$, we get:

Claim 3. *Red-black trees support insert, delete, and search in $O(\log n)$ time.*

As we have seen, BSTs are very nice – they allow us to maintain a set and report membership, insert, and delete in $O(\log n)$ time. In addition to these basic underlying operations, we can also support other types of queries efficiently. Because the elements are stored maintaining the binary search tree property, we can search for the next largest element or the elements on a range very efficiently. But what if we don't care about these properties? What if we only need to support membership queries? Can we improve our performance of $O(\log n)$ time to nearly constant time? This question motivates our discussion of hash tables, which we will cover in the next lecture.