Lecture 9

Graphs, BFS and DFS

Announcements!

- HW4 due **MONDAY** (Notice the change)
  - But you might want to get it in on Friday anyway, so that you can study for the….
- **<u>MIDTERM</u>** Wednesday May 10.
  - In this room, during class (3-4:20)
  - Please show up
  - Any material through Wednesday 5/3 is fair game
  - You may bring one double-sided letter-size page of notes, that you have prepared yourself.

Thanks for filling out feedback 2!

- **Sorry** about typos on homework!
  - We have changed our review process to be more stringent.
- I still need to work on the pacing of my lectures
  - **Less review at the beginning, go slower over technical stuff.**
  - I **do** want to take time to cover the important basics carefully, since there are many different backgrounds in this class.
  - I view the stuff at the end of the lecture as less important.
  - But I will try to figure out a more satisfying was to present things.
- The point of lecture? (In my view)
  - High-level overview of the big conceptual ideas.
  - When it comes to proofs:
    - My intent is to emphasize the important structure/outline of the proofs, rather than technical details.
  - (But obviously this is not an excuse for typos!)
- To get all the details, do the suggested reading
  - Before or after class, depending on how you learn best.

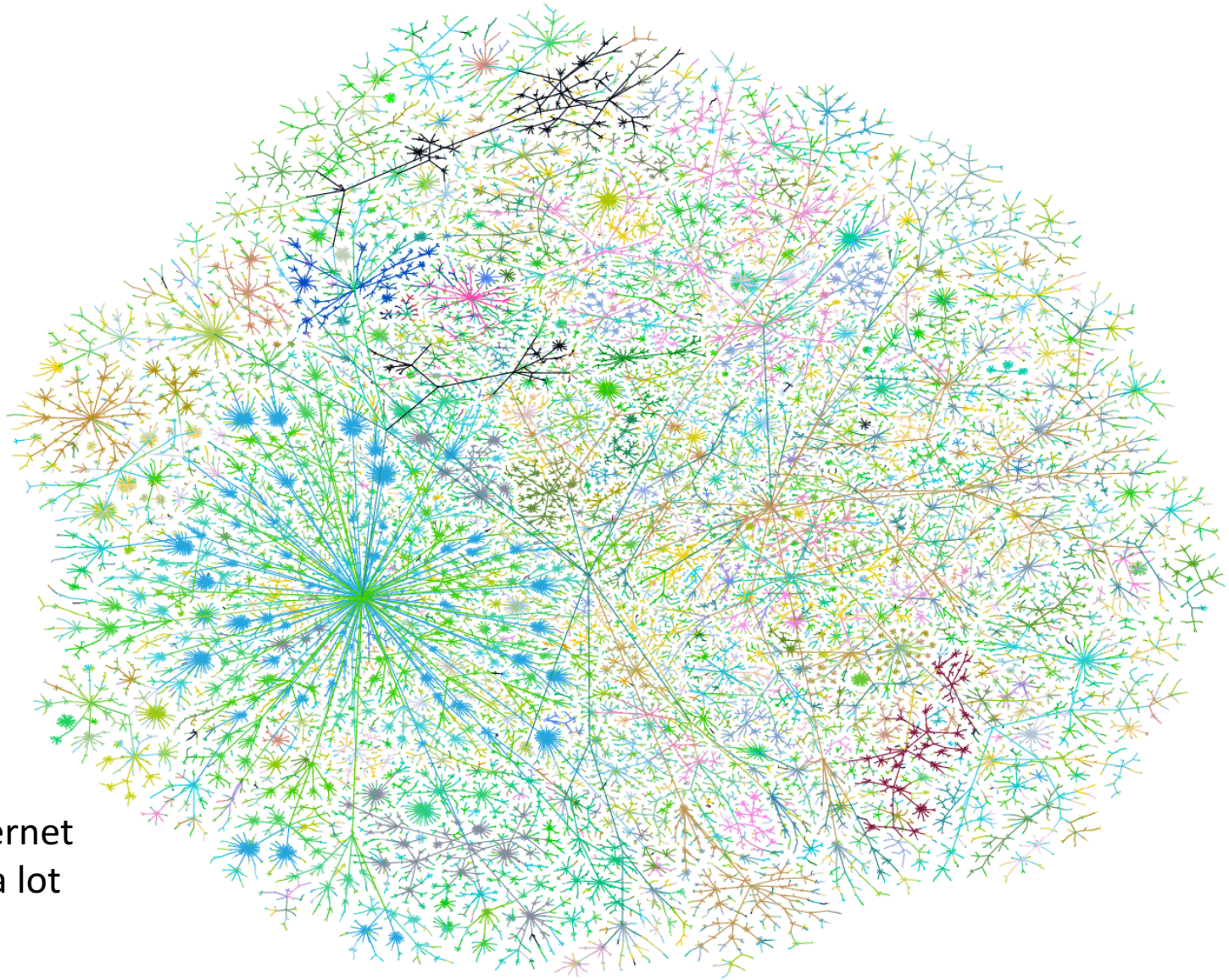
Outline

- Part 0: Graphs and terminology

- Part 1: Depth-first search
  - Application: topological sorting
  - Application: in-order traversal of BSTs
- Part 2: Breadth-first search
  - Application: shortest paths
  - Application (if time): is a graph bipartite?

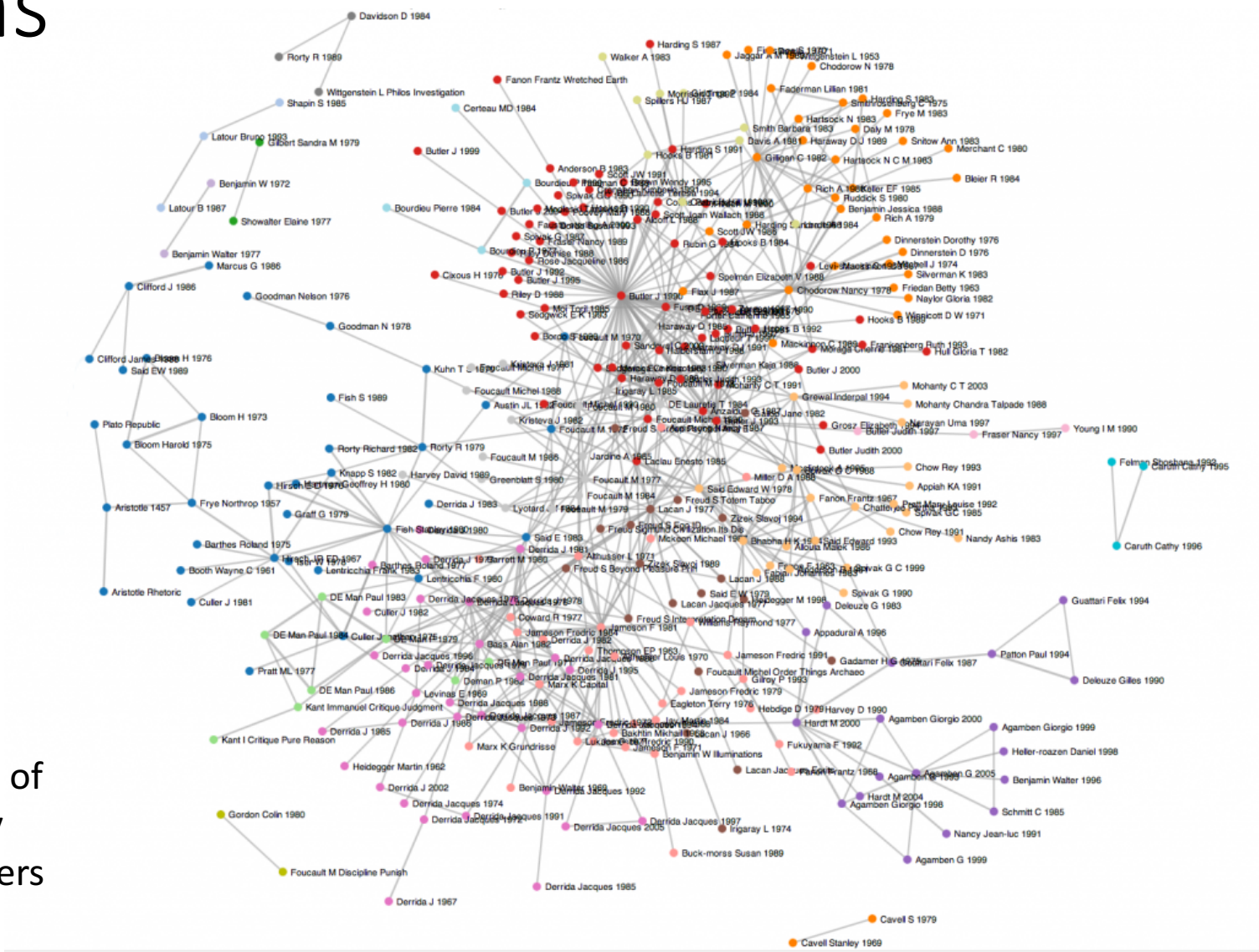
Part 0: Graphs

Graphs



Graph of the internet (circa 1999…it's a lot bigger now…)

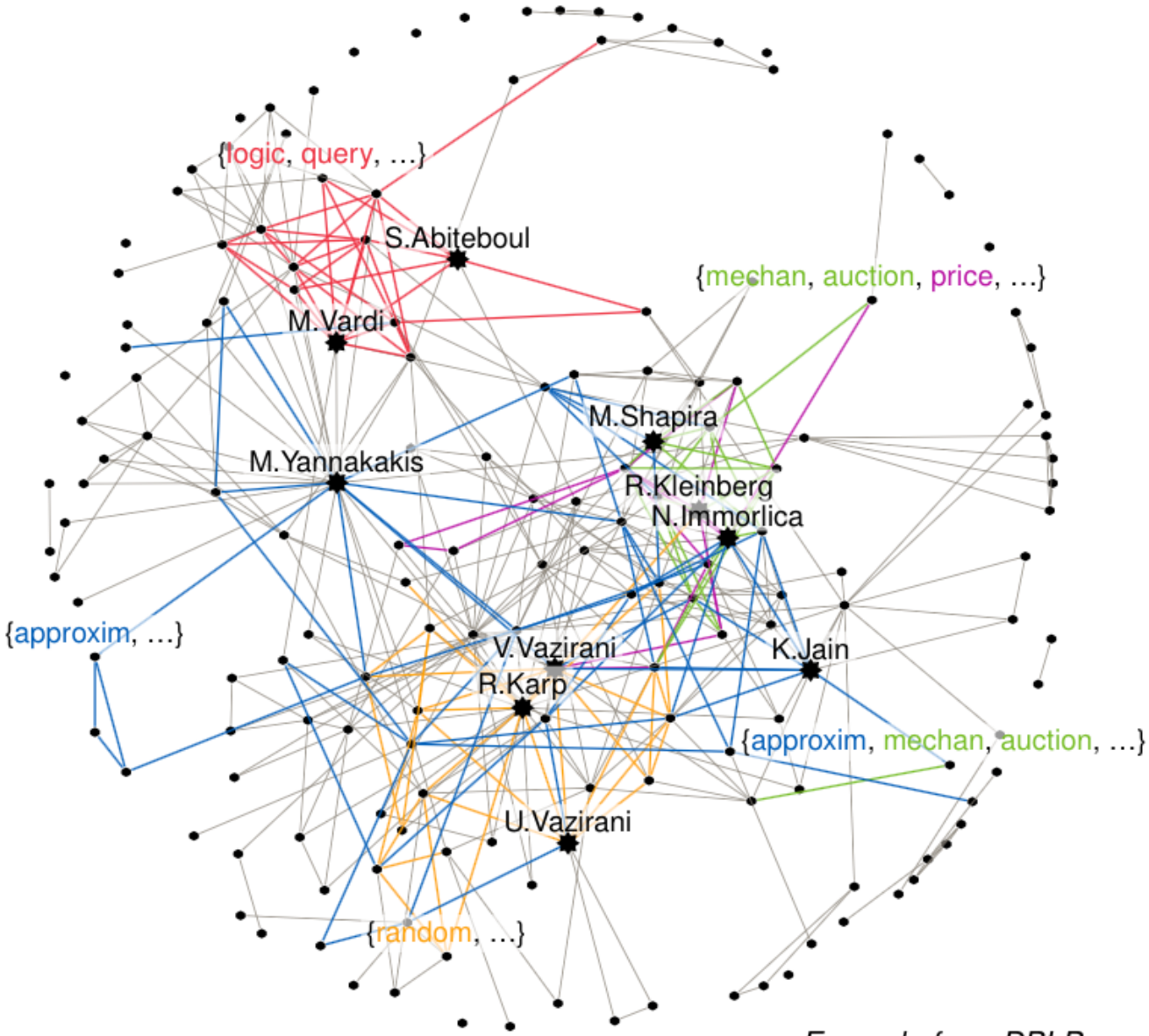
Graphs



Citation graph of literary theory academic papers

Graphs

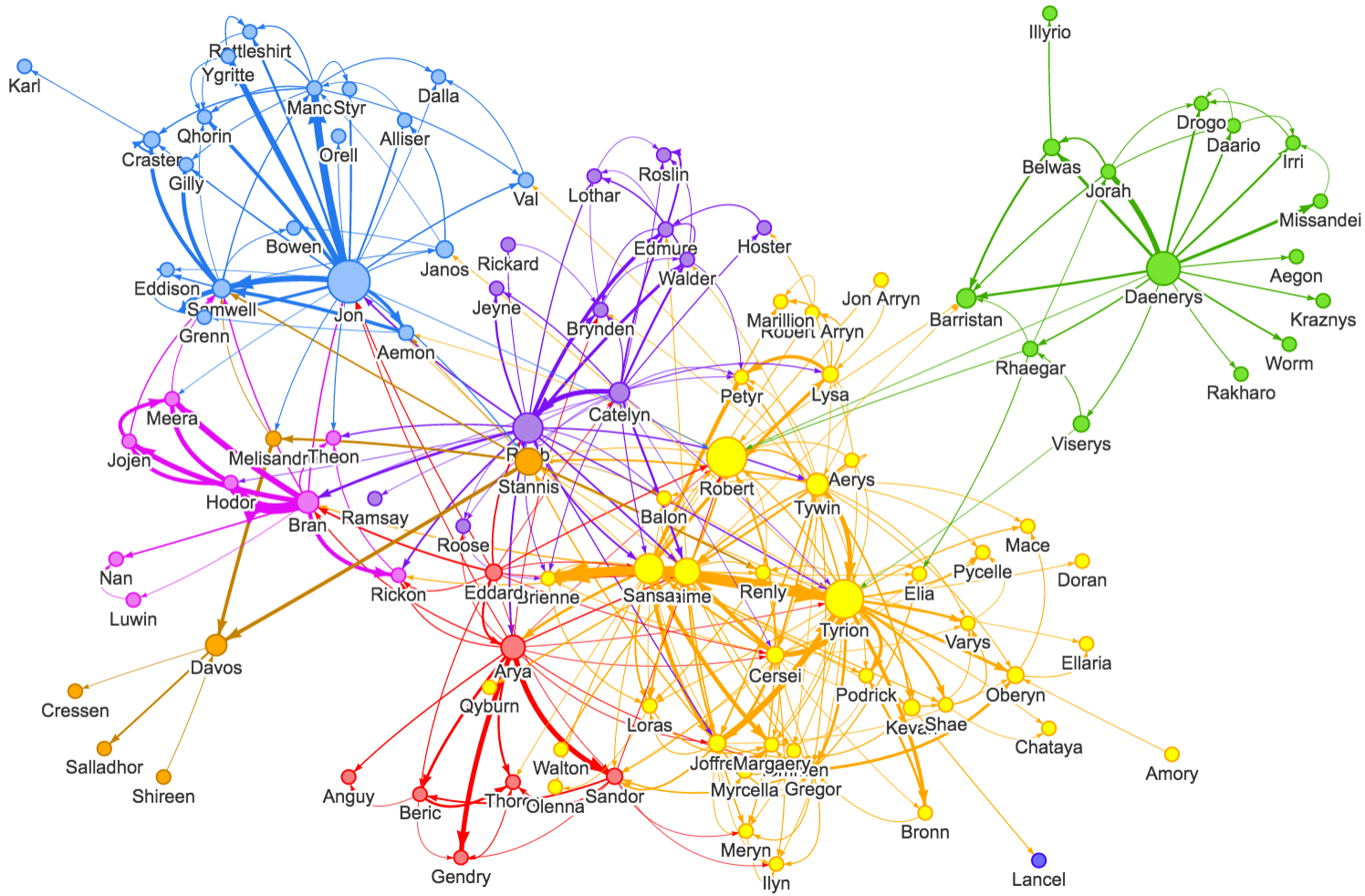
Theoretical Computer Science academic communities



Example from DBLP:
Communities within the co-authors of Christos H. Papadimitriou

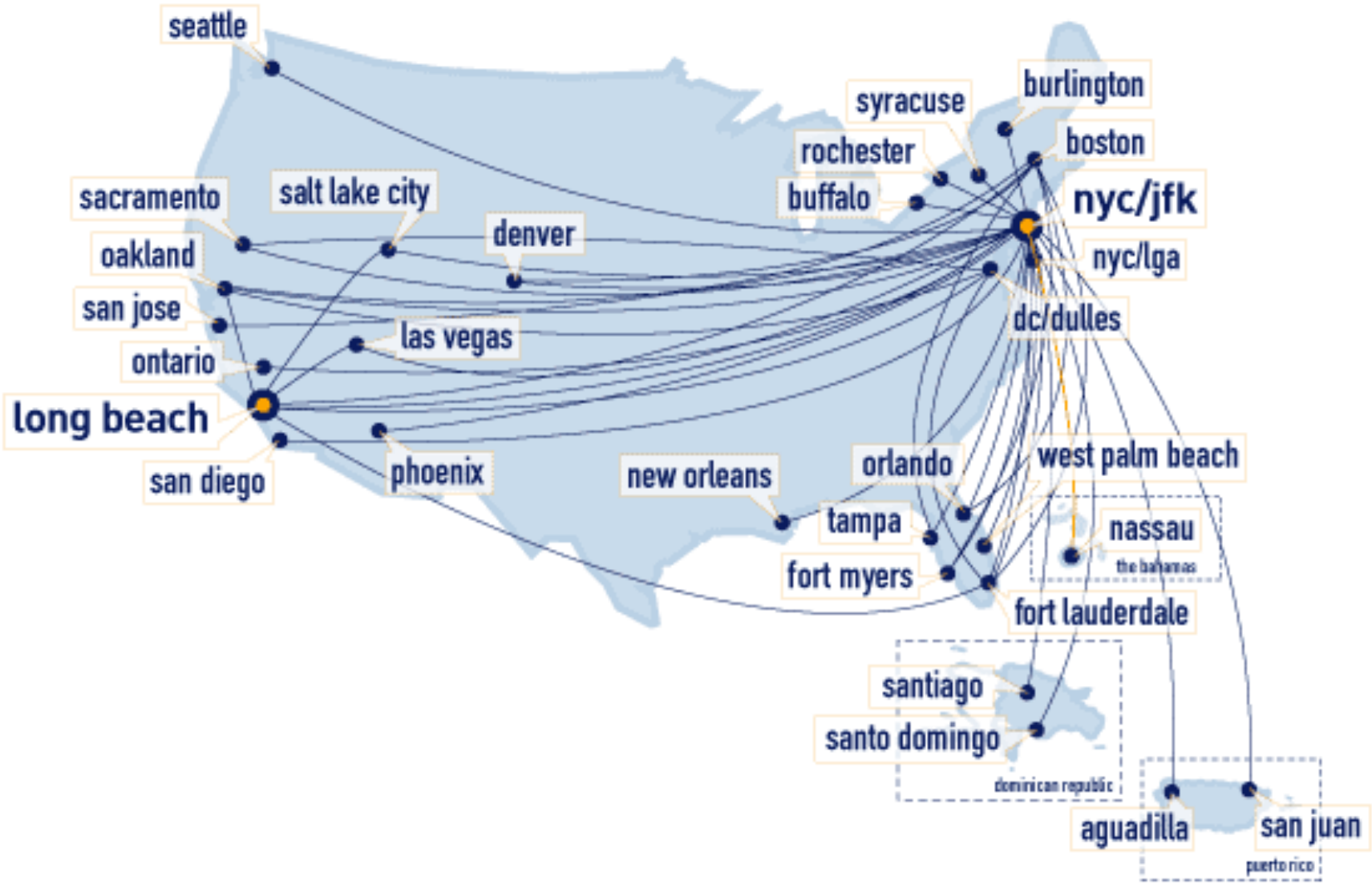
Graphs

Game of Thrones Character Interaction Network



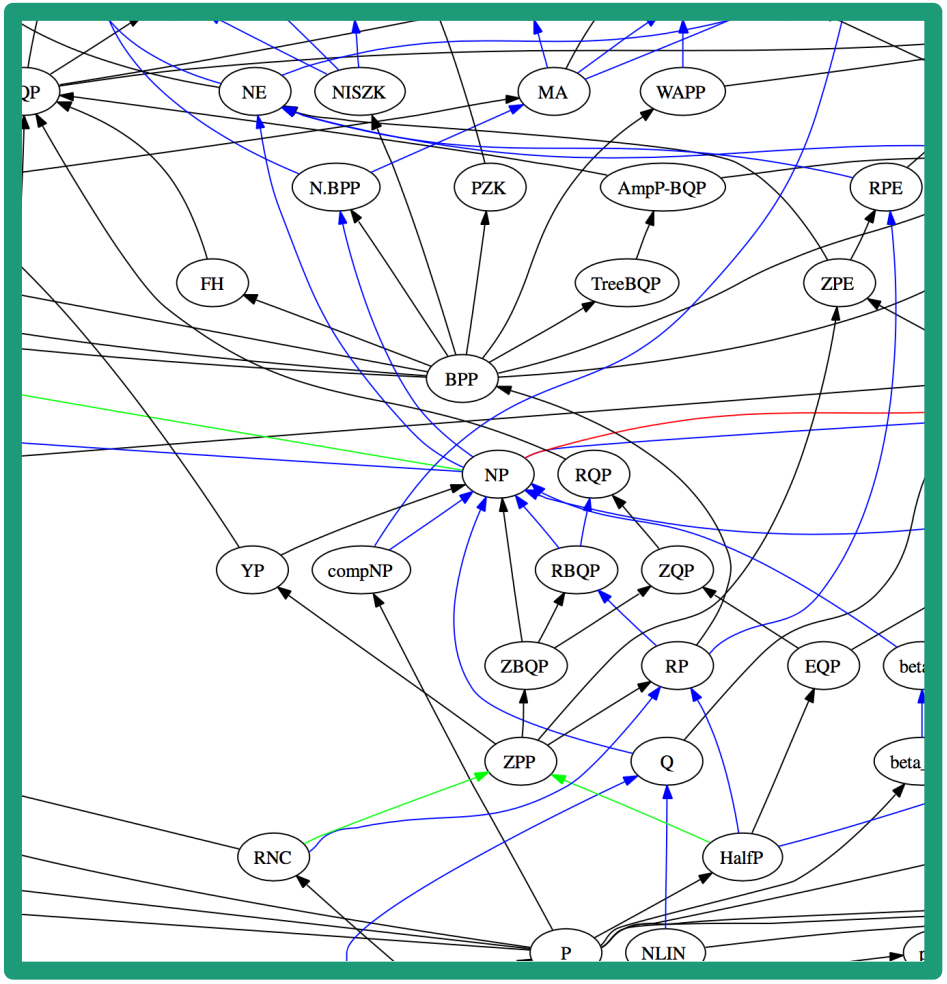
Graphs

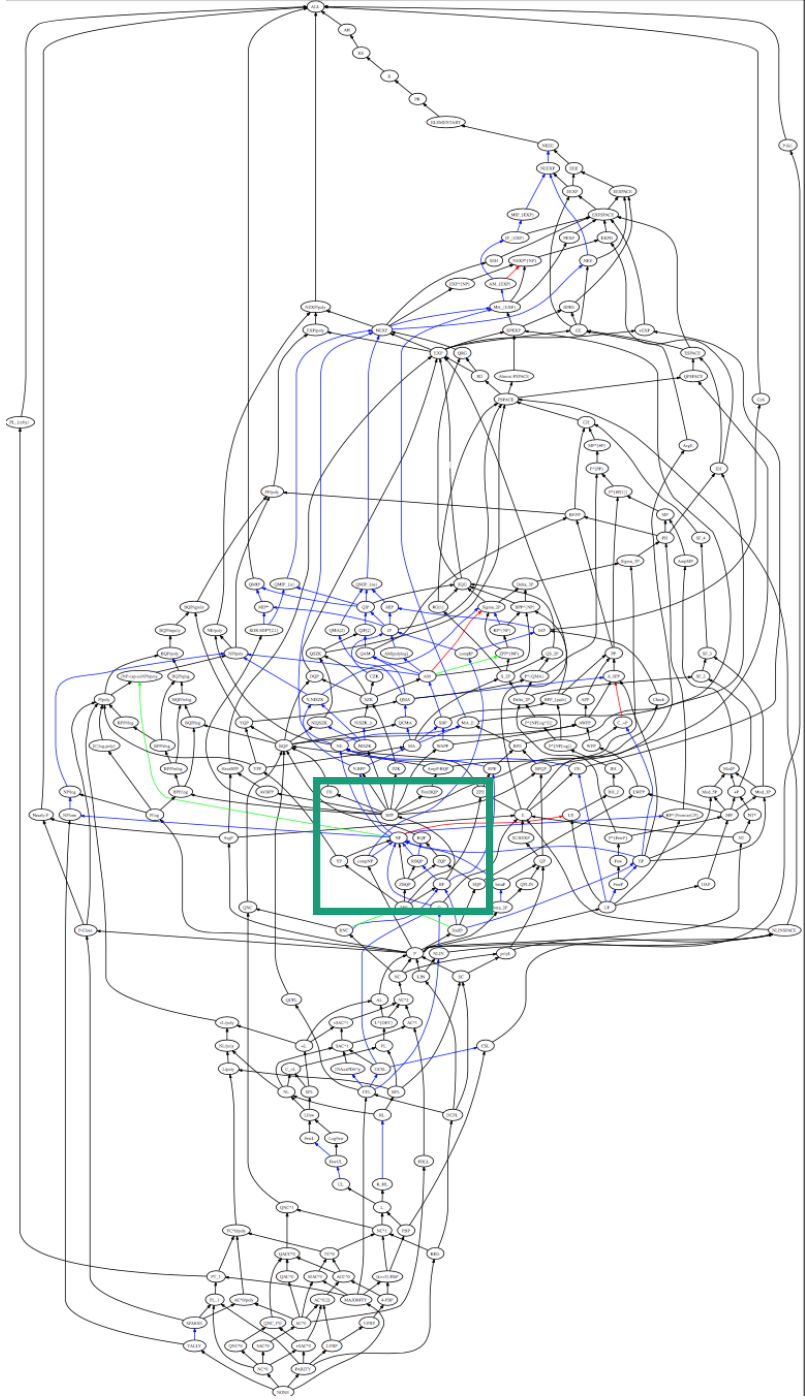
jetblue flights



Graphs

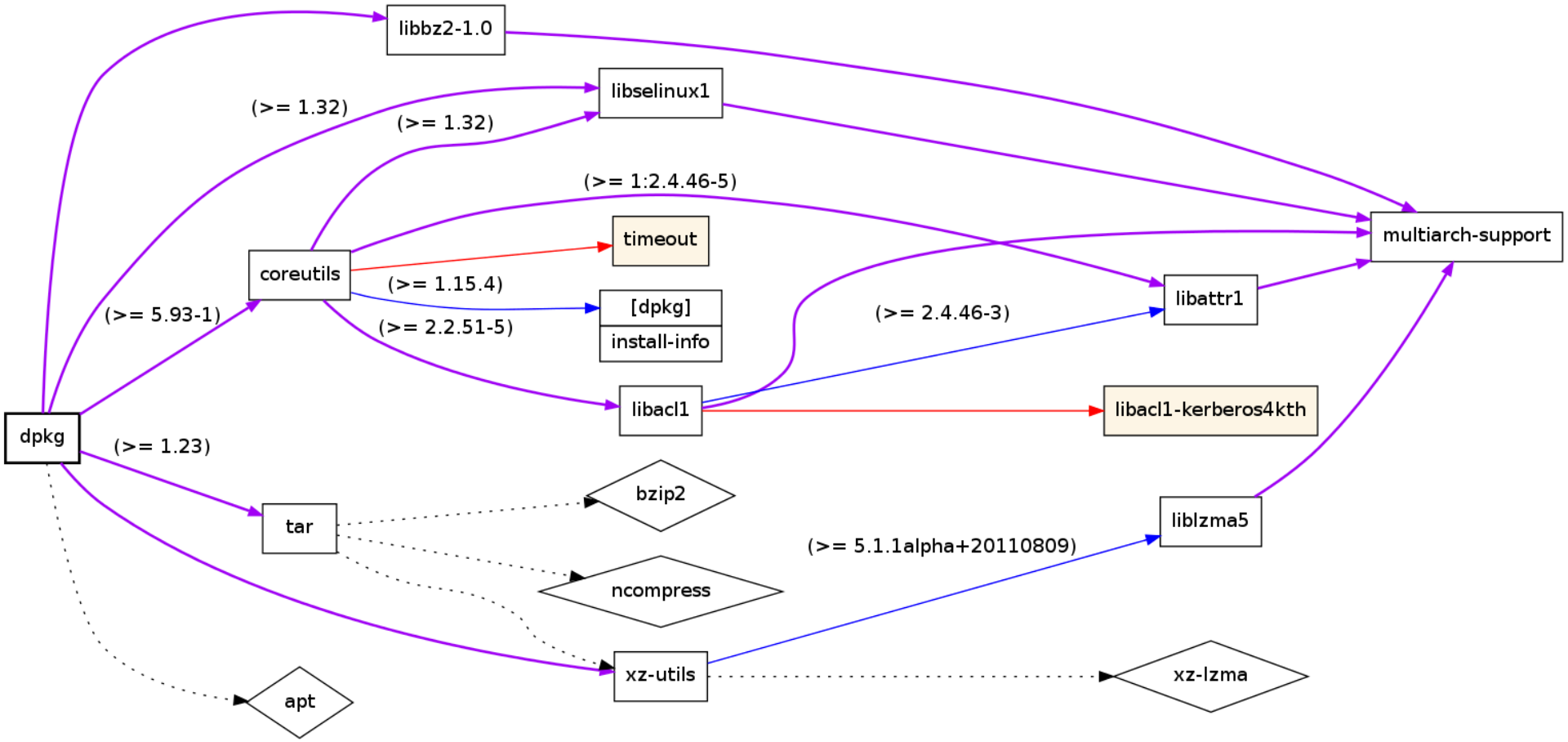
Complexity Zoo
containment graph





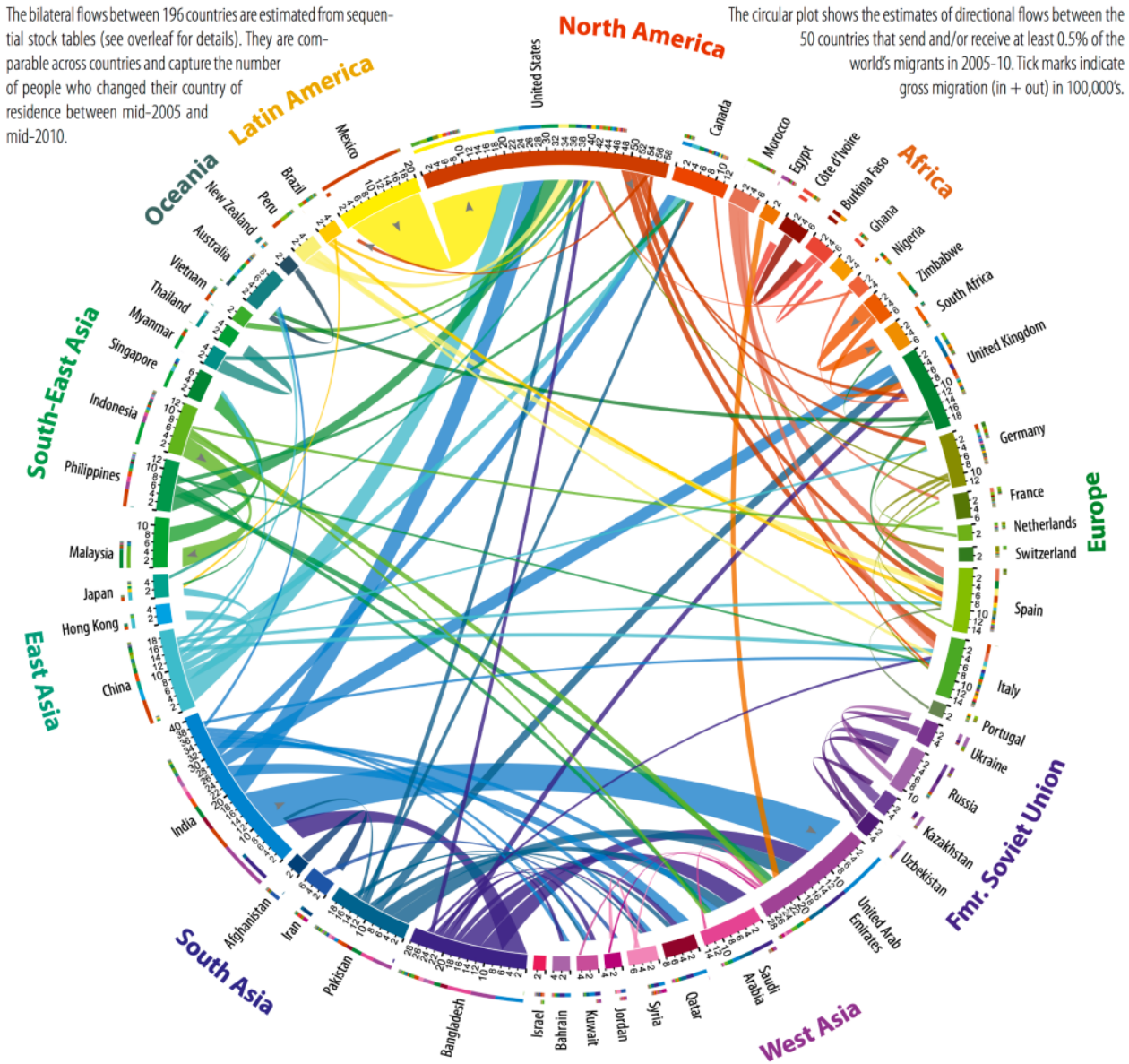
Graphs

debian dependency (sub)graph



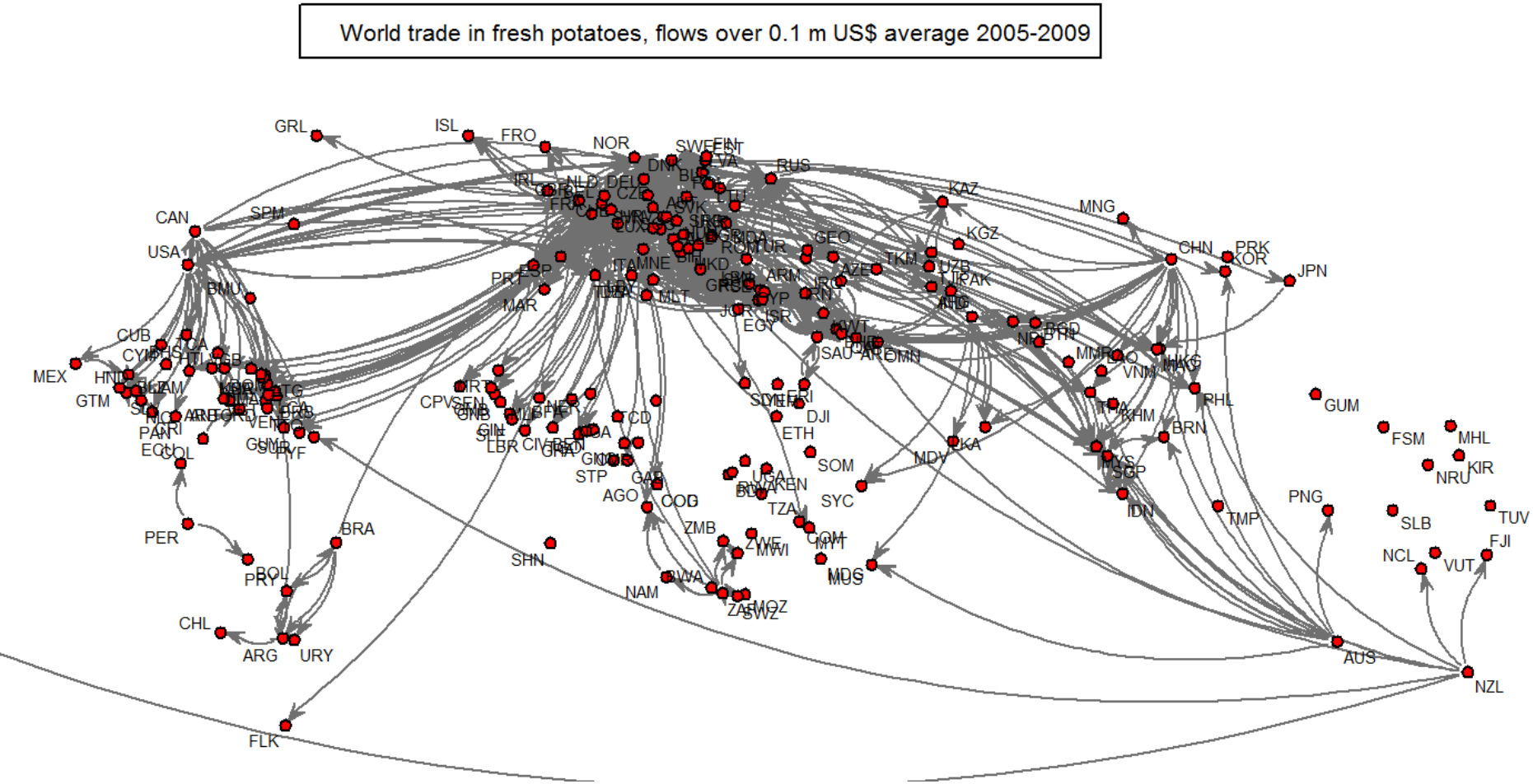
Graphs

Immigration flows



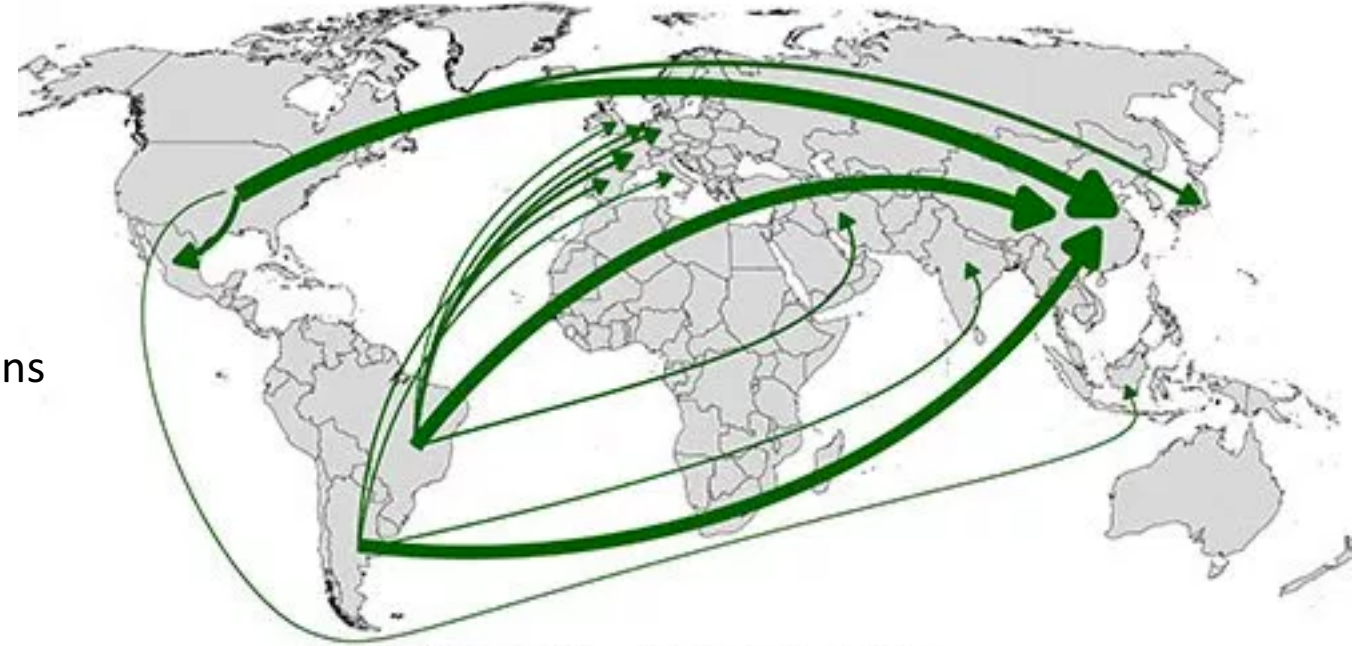
Graphs

Potato trade



Graphs

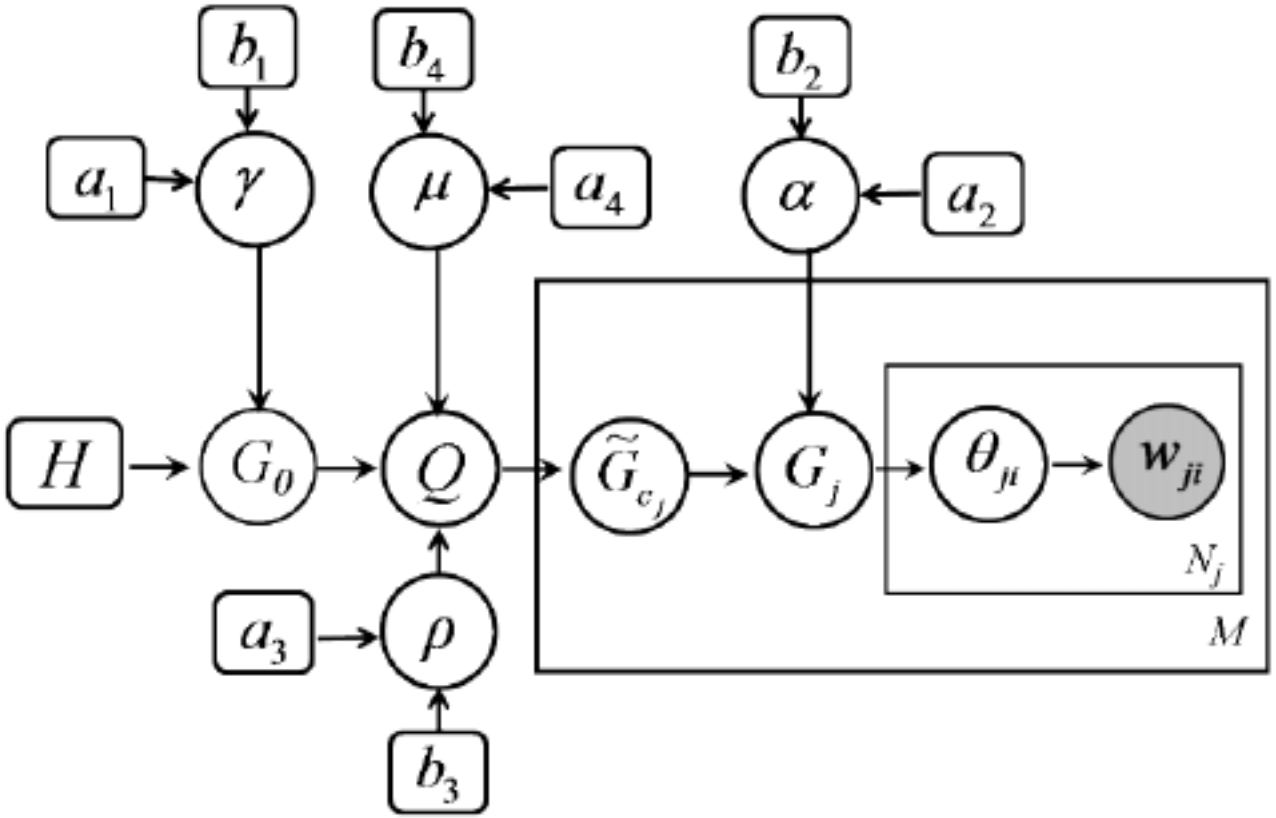
Soybeans



Water

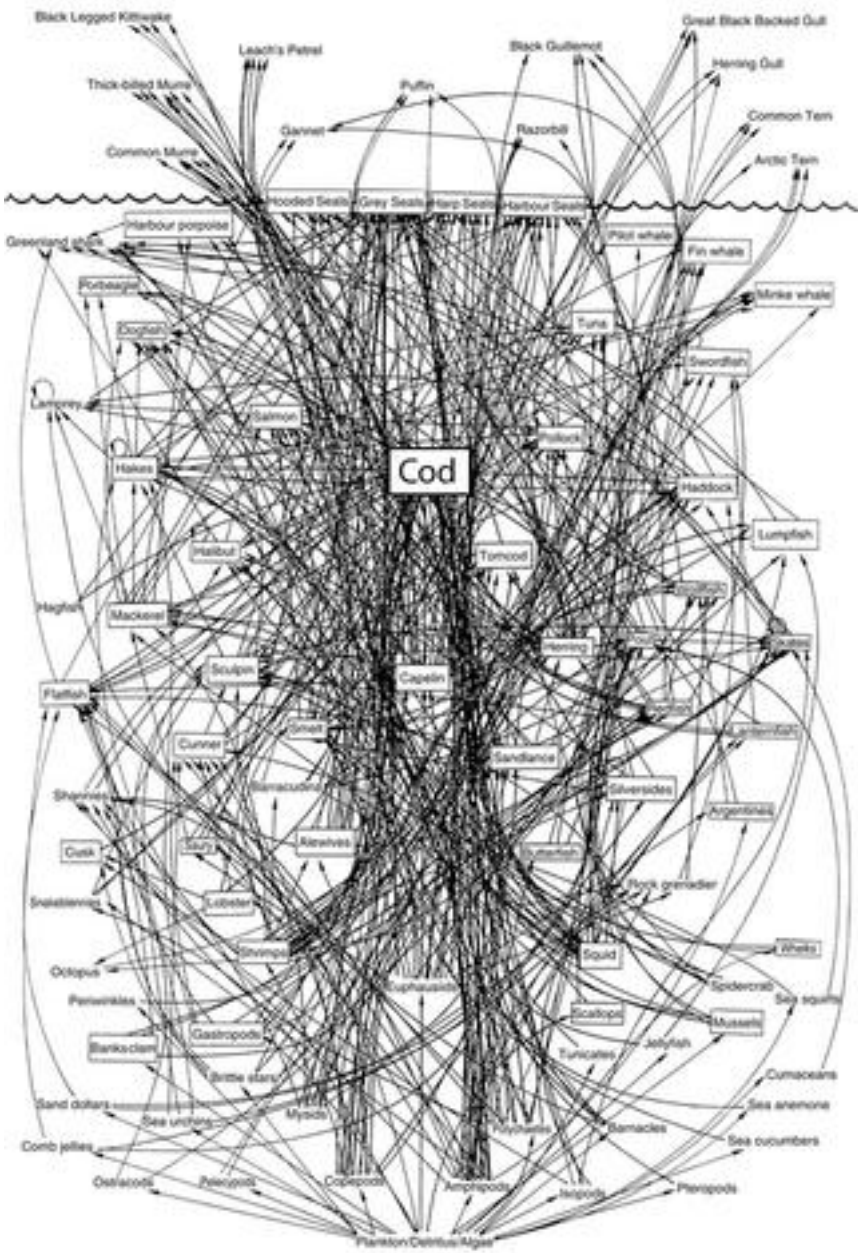
Graphs

Graphical models



Graphs

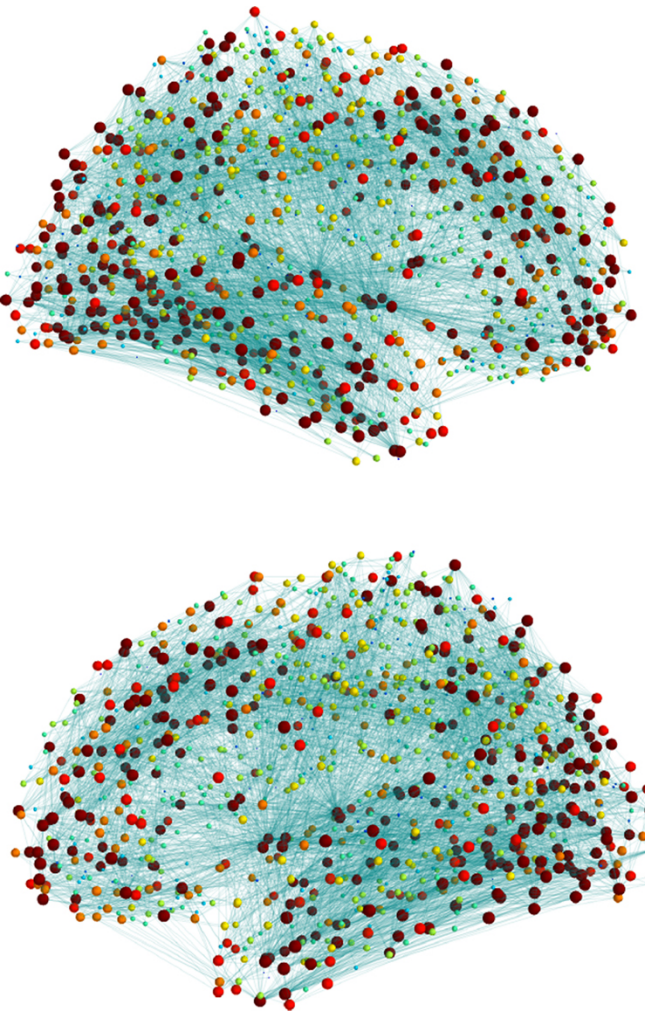
What eats what in
the Atlantic ocean?

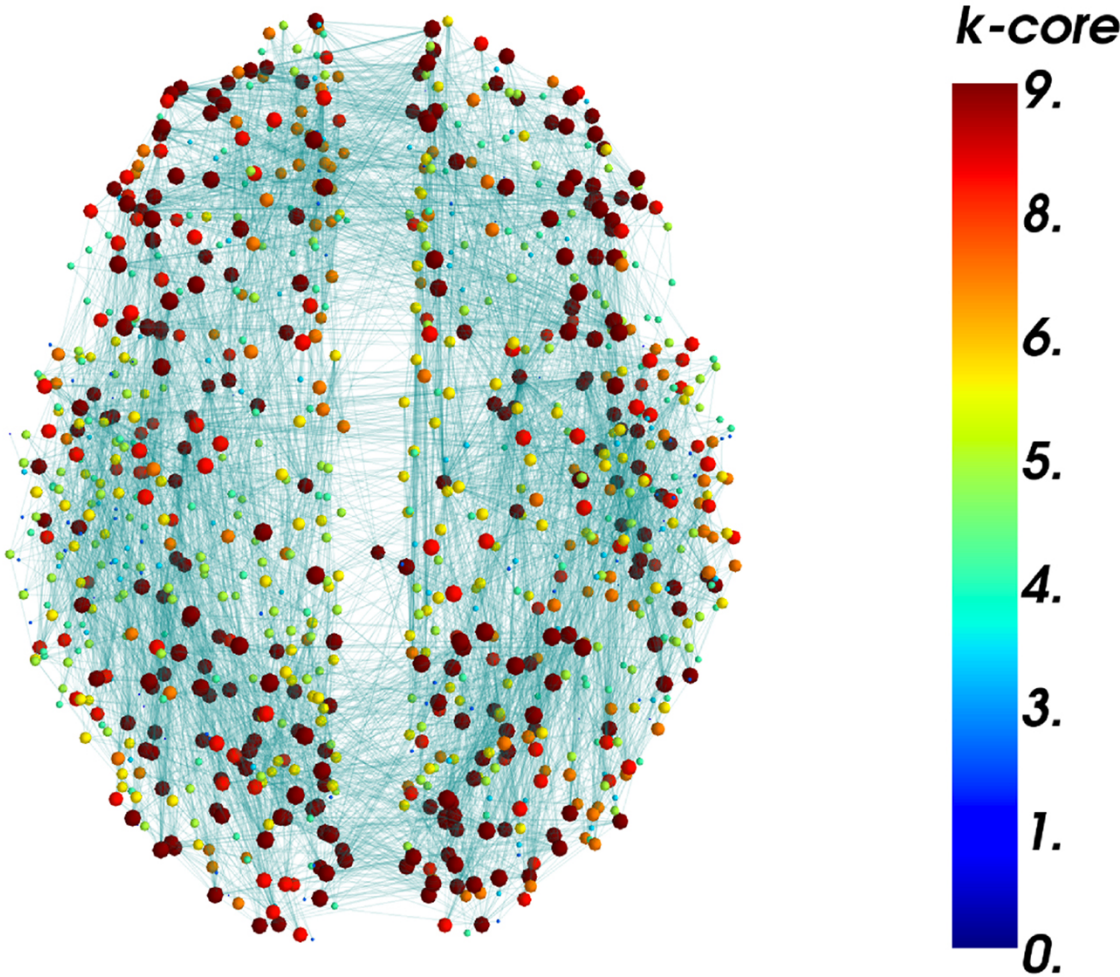


A simplified food web for the Northwest Atlantic. © IMMA

Graphs

Neural connections
in the brain





Graphs

- **There are a lot of graphs.**

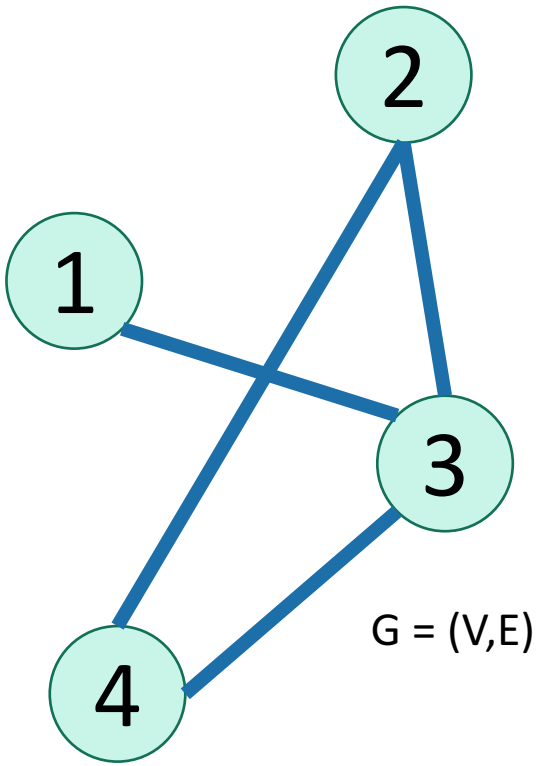
- We want to answer questions about them.
  - Efficient routing?
  - Community detection/clustering?
  - An ordering that respects dependencies?

- This is what we'll do for the next several lectures.

Undirected Graphs

- Has vertices and edges
  - V is the set of vertices
  - E is the set of edges
  - Formally, a graph is G = (V,E)

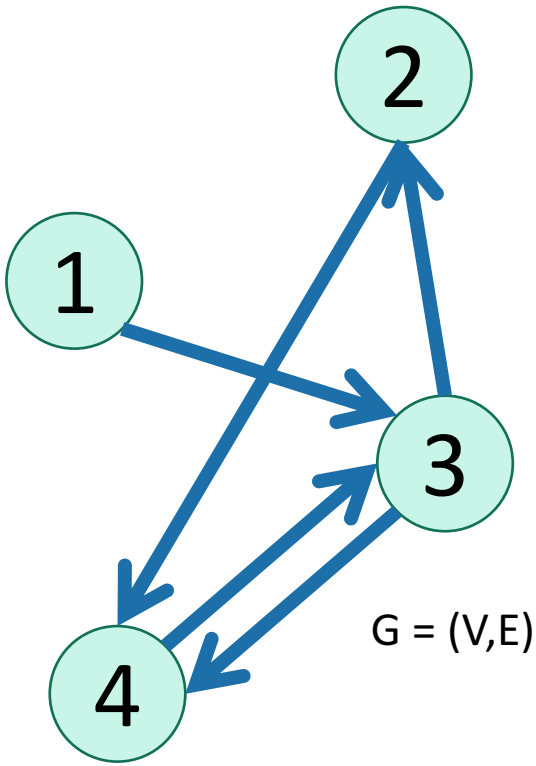
- Example
  - V = {1,2,3,4}
  - E = { {1,3}, {2,4}, {3,4}, {2,3} }



- The degree of vertex 4 is 2.
  - There are 2 edges coming out
- Vertex 4's neighbors are 2 and 3

Directed Graphs

- Has vertices and edges
  - V is the set of vertices
  - E is the set of **DIRECTED** edges
  - Formally, a graph is G = (V,E)



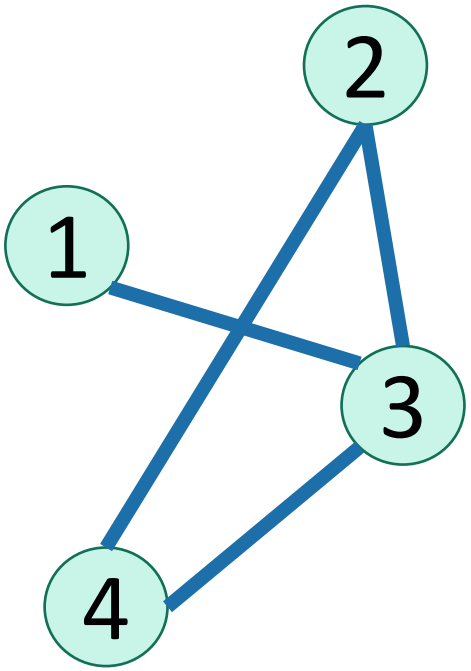
- Example
  - V = {1,2,3,4}
  - E = { (1,3), (2,4), (3,4), (4,3), (3,2) }

- The in-degree of vertex 4 is 1.
- The out-degree of vertex 4 is 1.
- Vertex 4's incoming neighbors are 2
- Vertex 4's outgoing neighbor is 3.

How do we represent graphs?

- Option 1: adjacency matrix

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$



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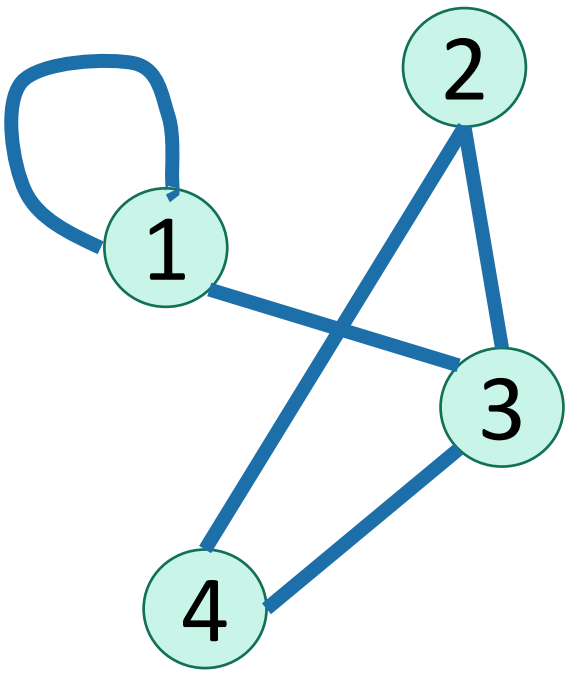
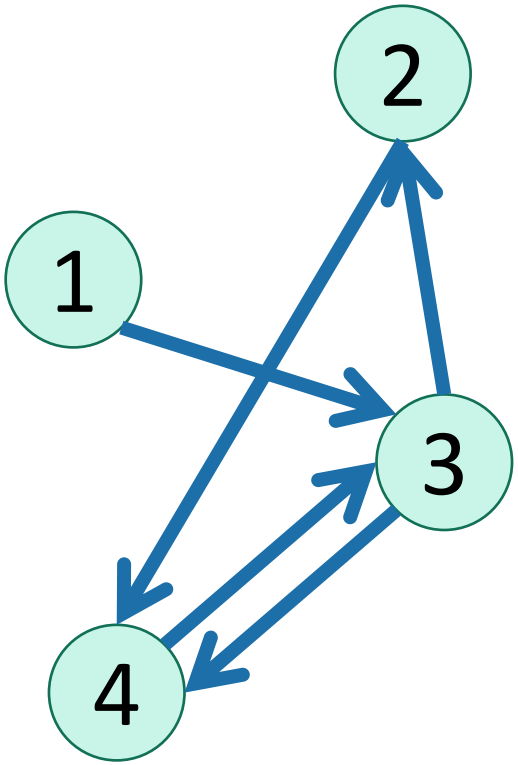


How do we represent graphs?

- Option 1: adjacency matrix

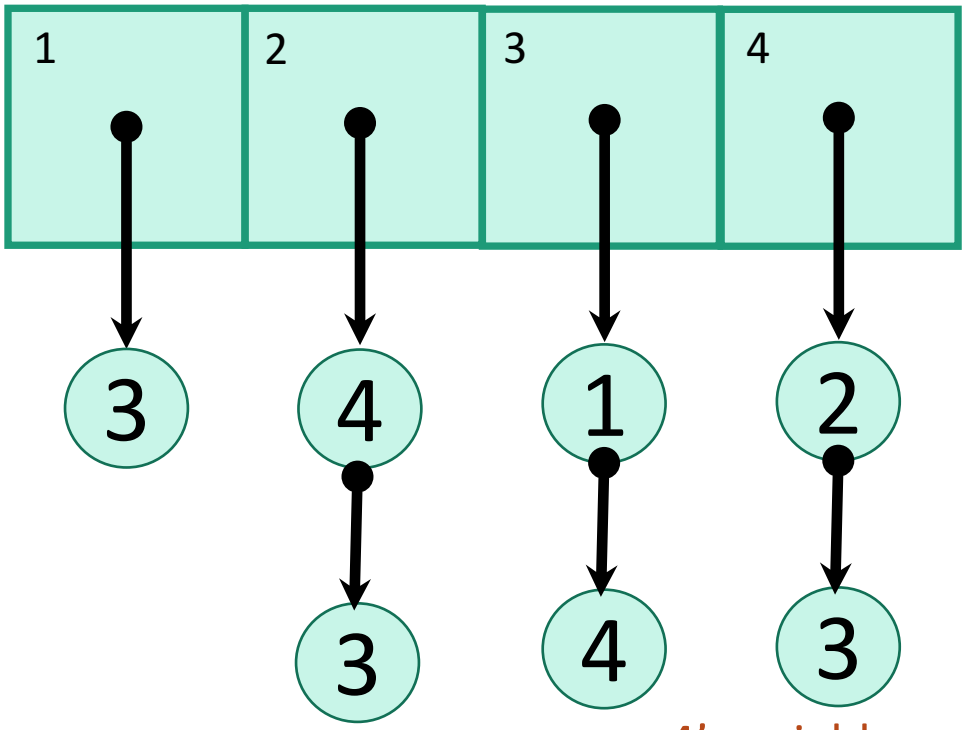
Destination

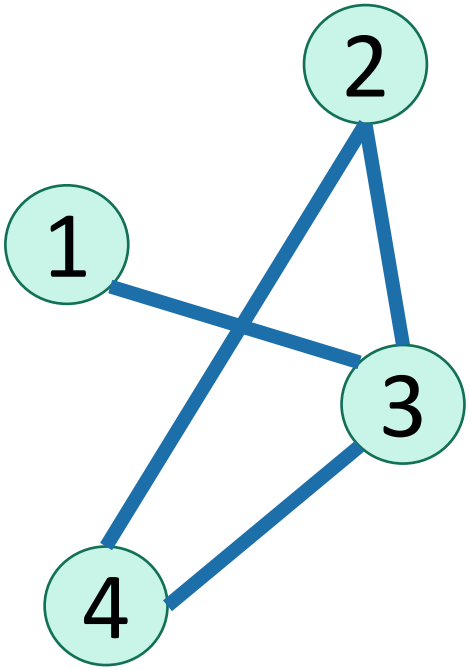
$$
\begin{array}{c|c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\text{Source} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{array}
$$



How do we represent graphs?

- Option 2: linked lists.





4's neighbors are 2 and 3

How would you modify this for directed graphs?

In either case

- May think of vertices storing other information
  - Attributes (name, IP address, …)
  - helper info for algorithms that we will perform on the graph

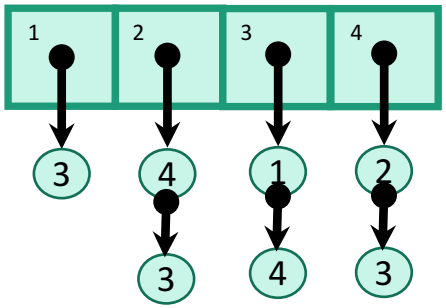
- We will want to be able to do the following ops:
  - Edge Membership: Is edge e in E?
  - Neighbor Query: What are the neighbors of vertex v?

Trade-offs

Say there are n vertices and m edges.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Generally better for **sparse** graphs



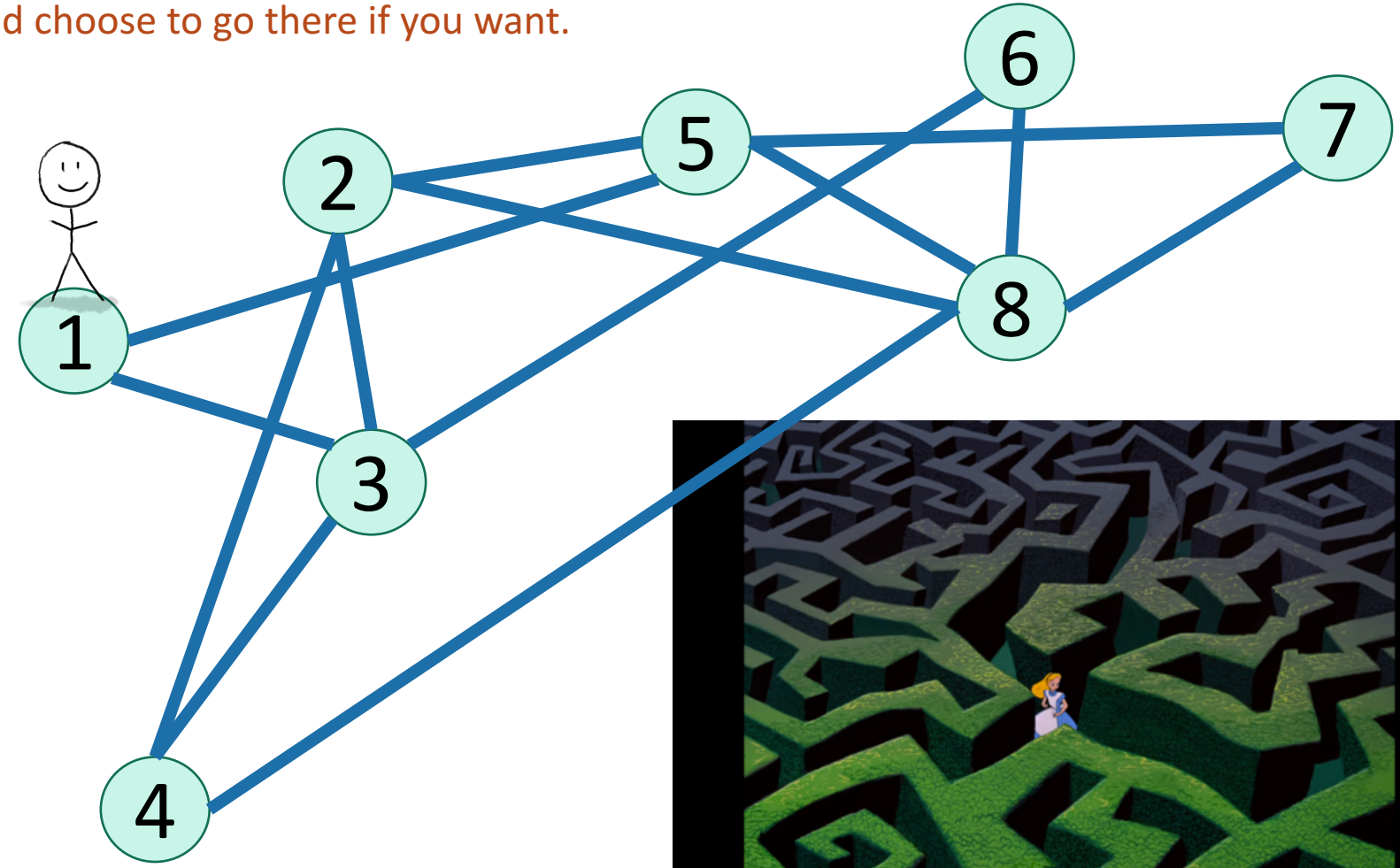
Edge membership Is e = {v,w} in E?	O(1)	O(deg(v)) or O(deg(w))
Neighbor query Give me v's neighbors.	O(n)	O(deg(v))
Space requirements	$O(n^2)$	O(n + m)

We'll assume this representation for the rest of the class

Part 1: Depth-first search

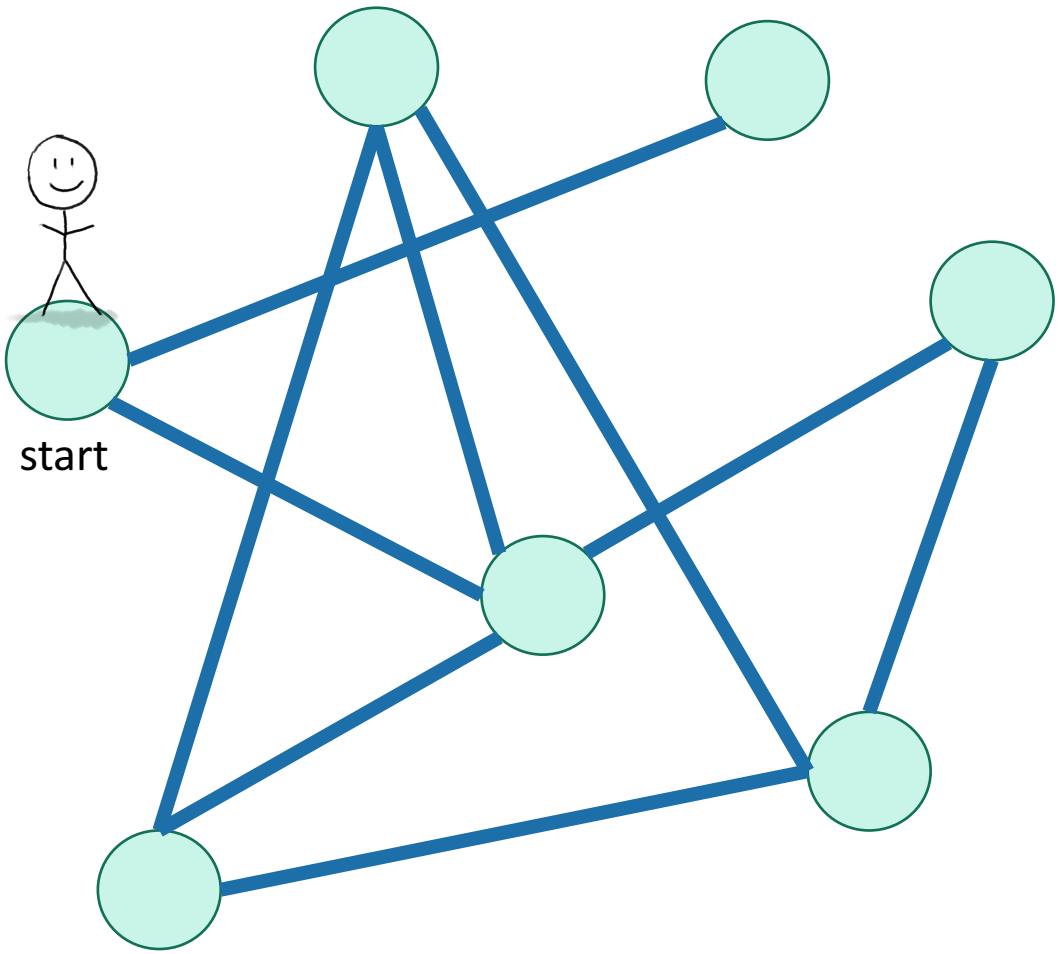
How do we explore a graph?

At each node, you can get a list of neighbors,
and choose to go there if you want.



Depth First Search

Exploring a labyrinth with chalk and a piece of string



Not been there yet

Been there, haven't explored all the paths out.

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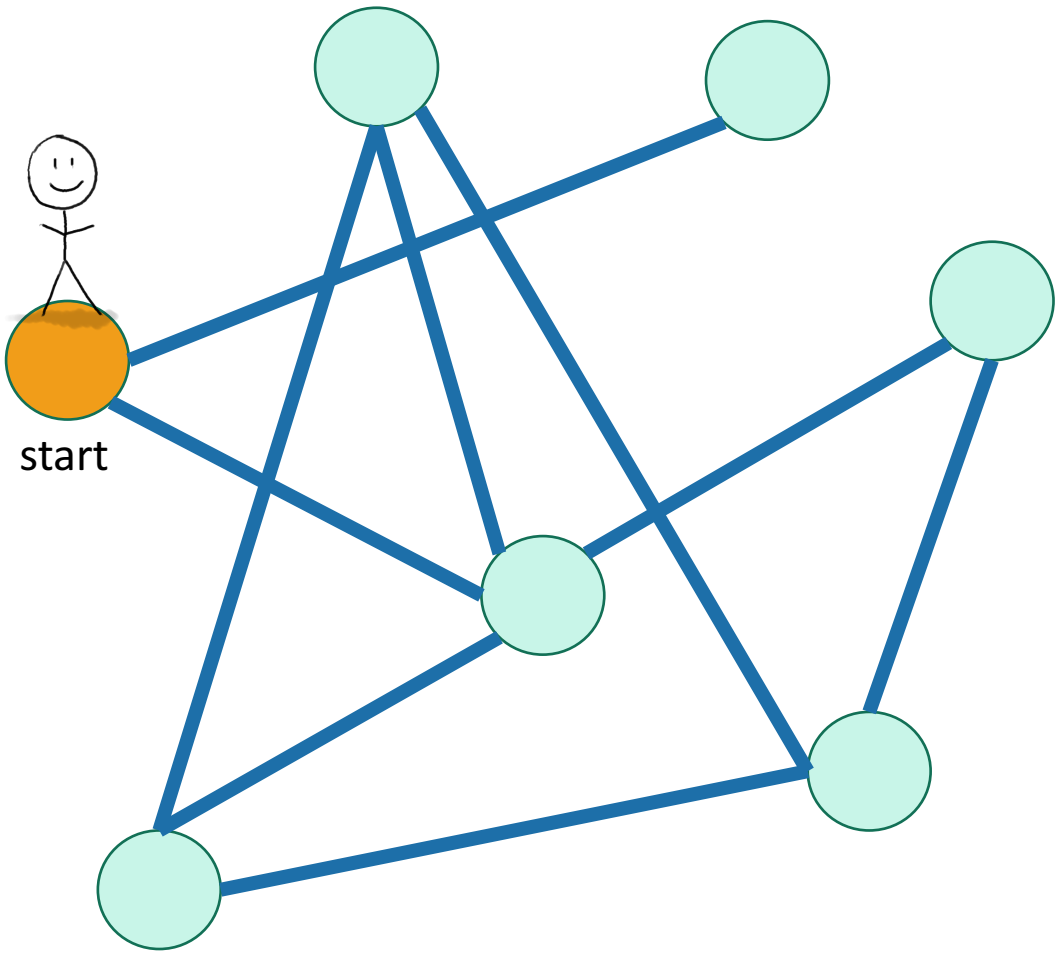

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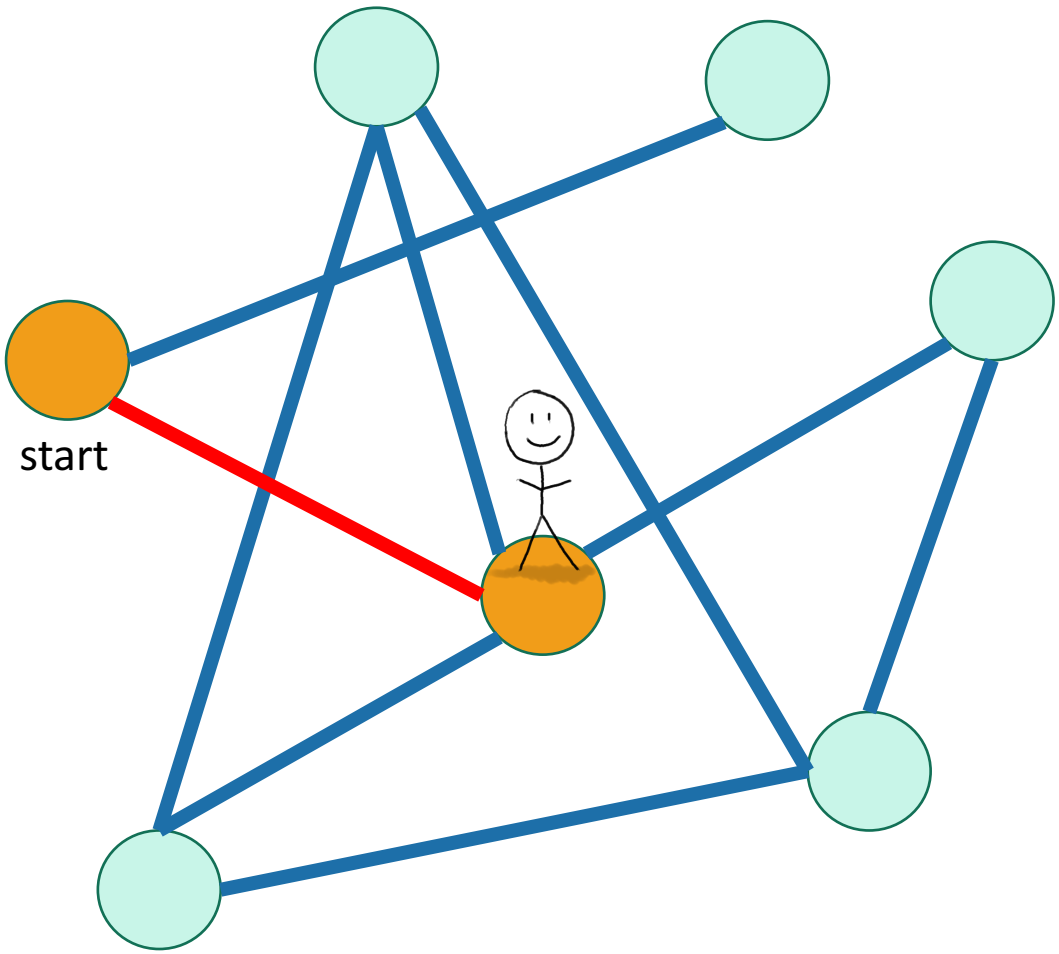

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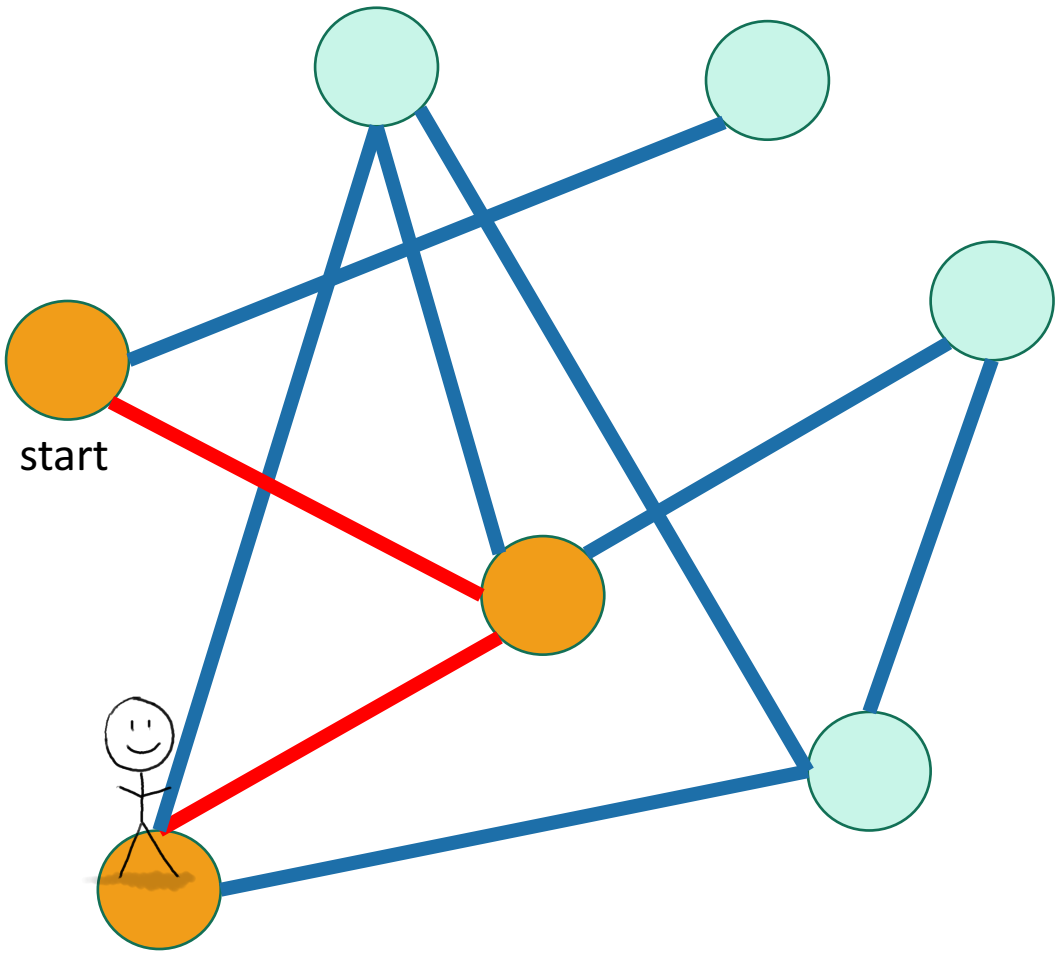

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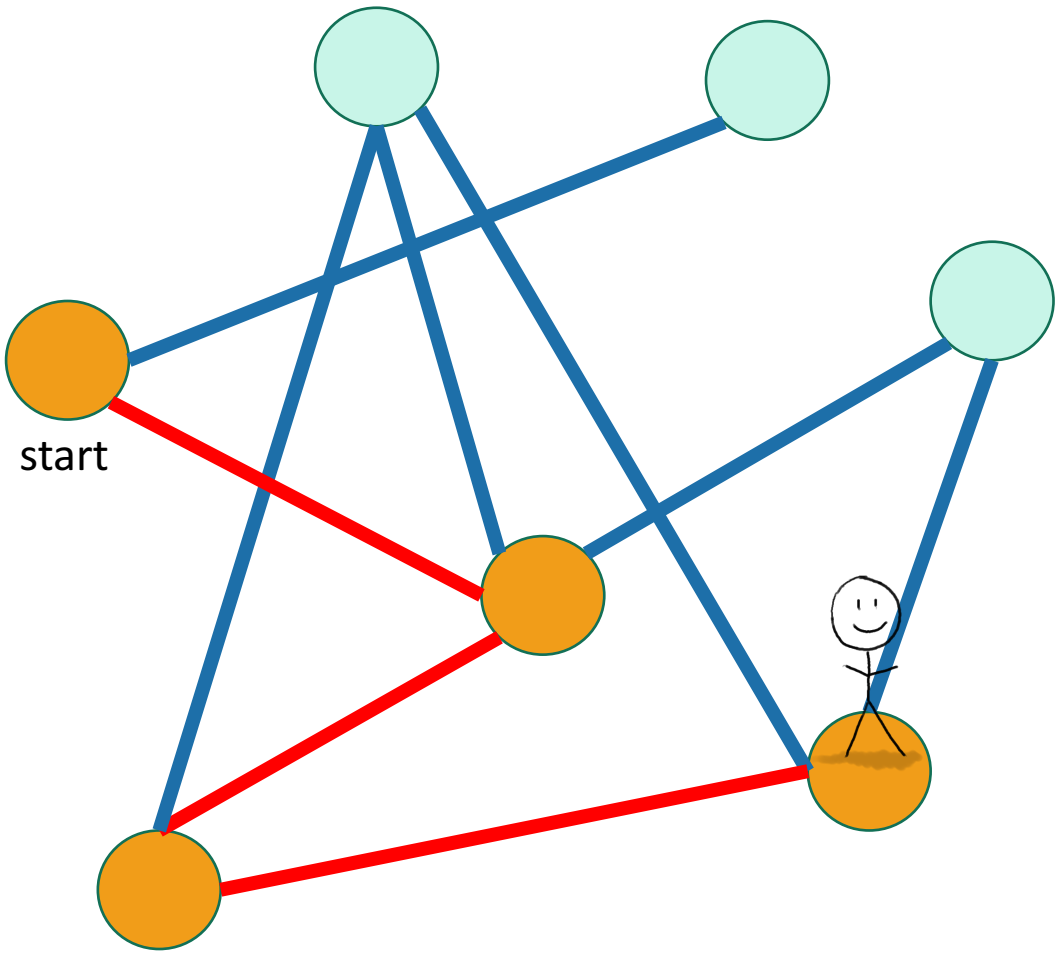

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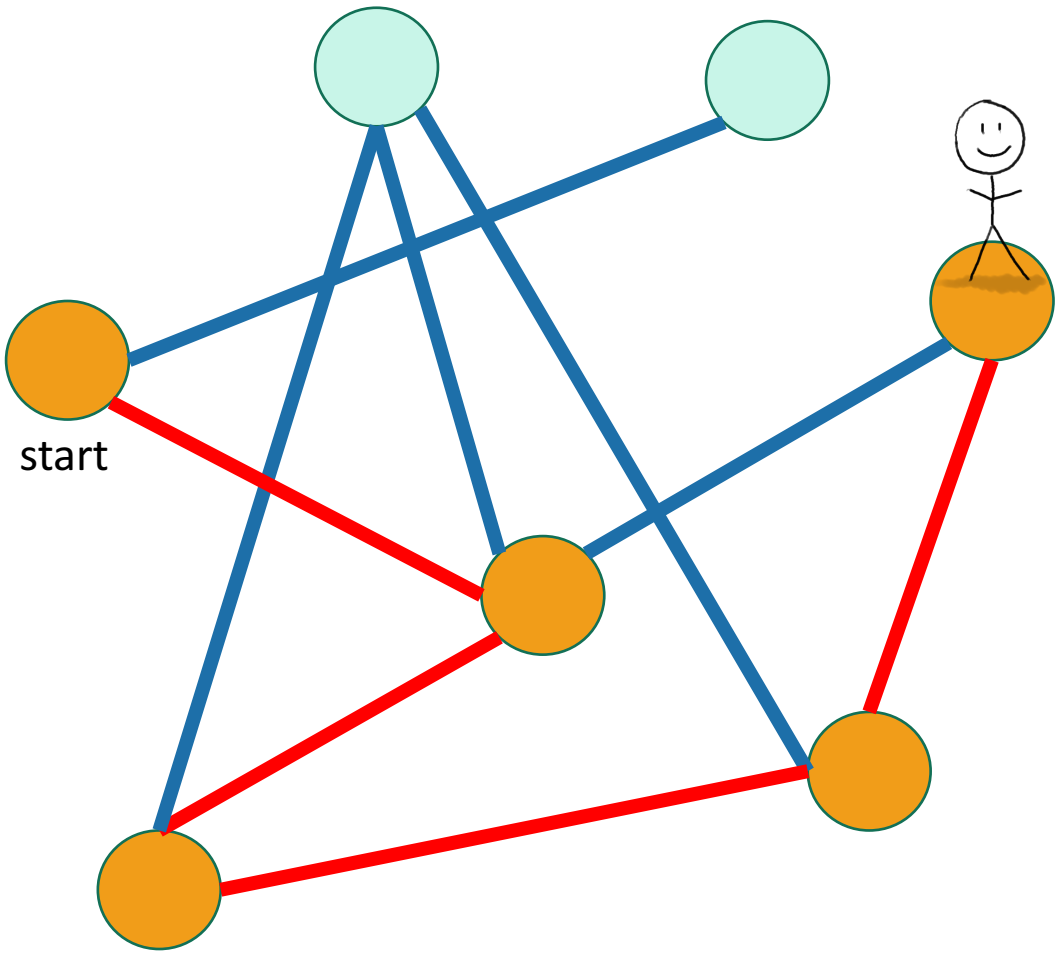

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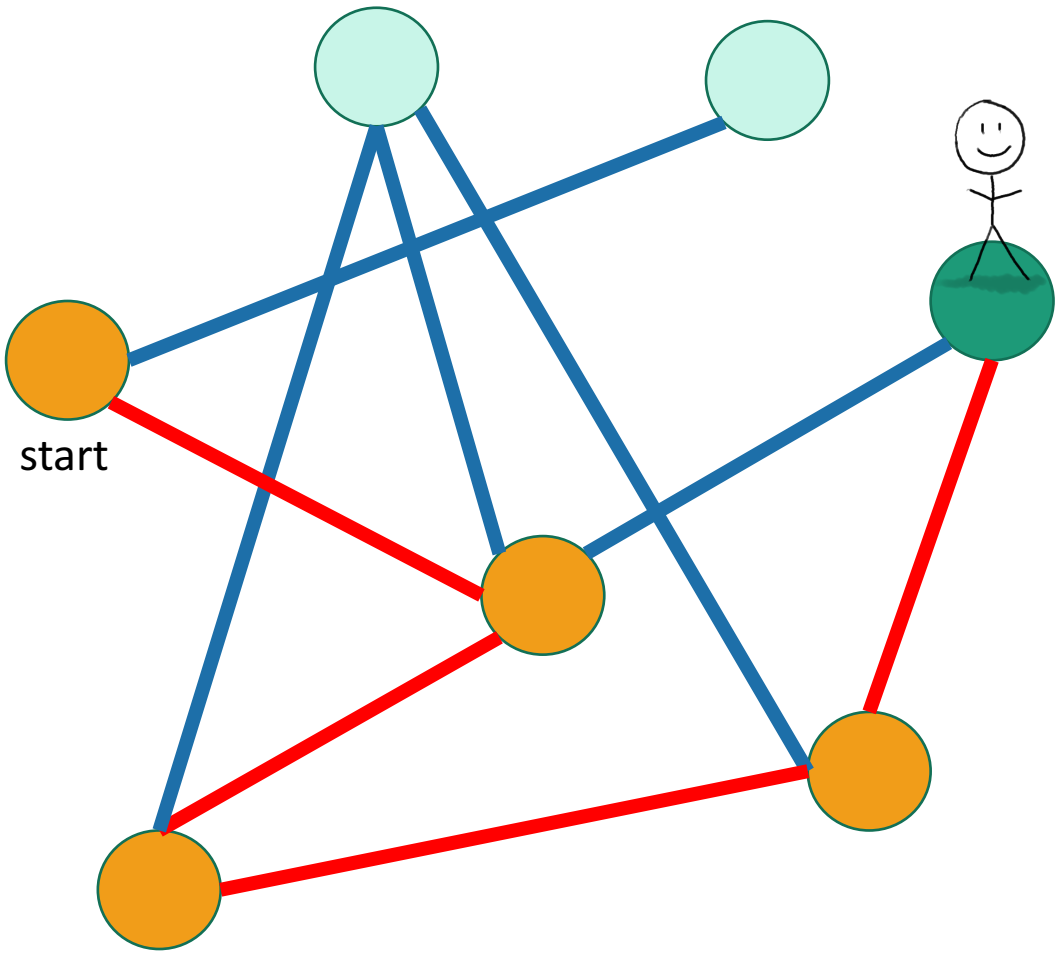

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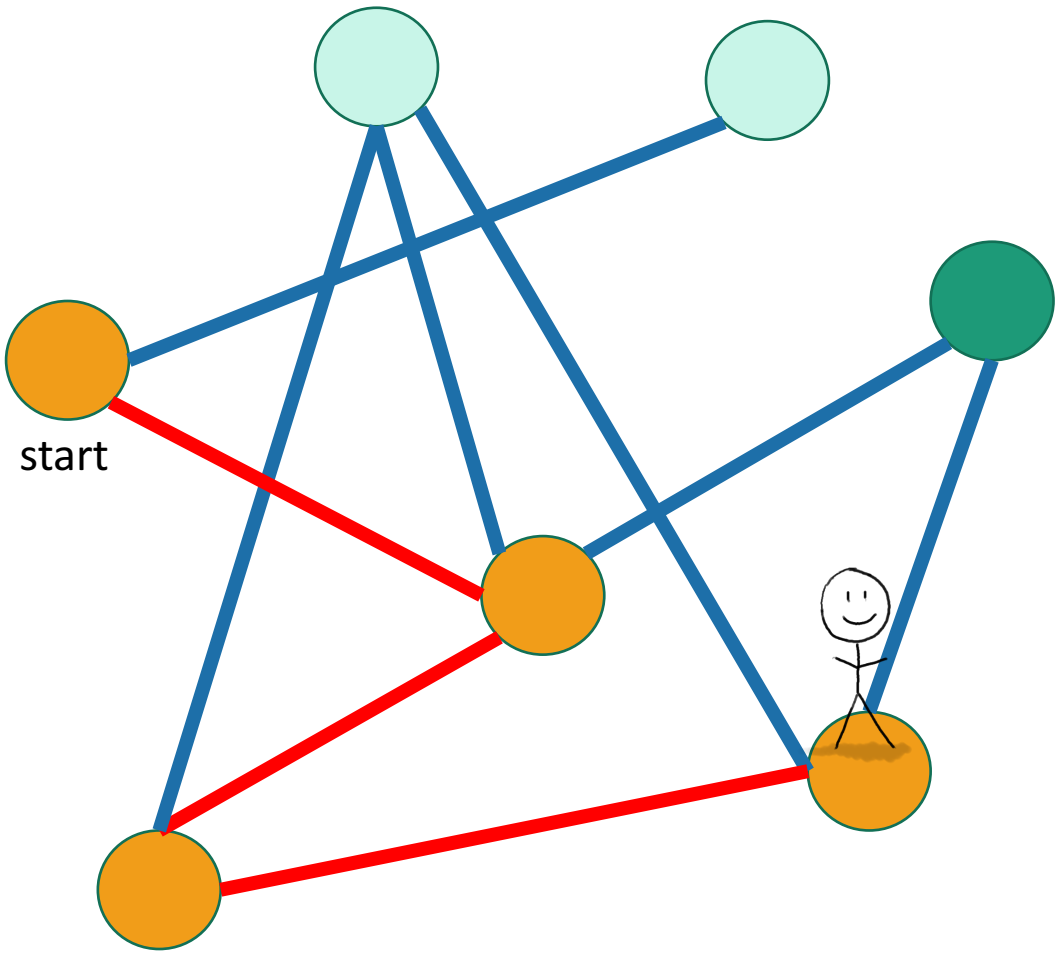

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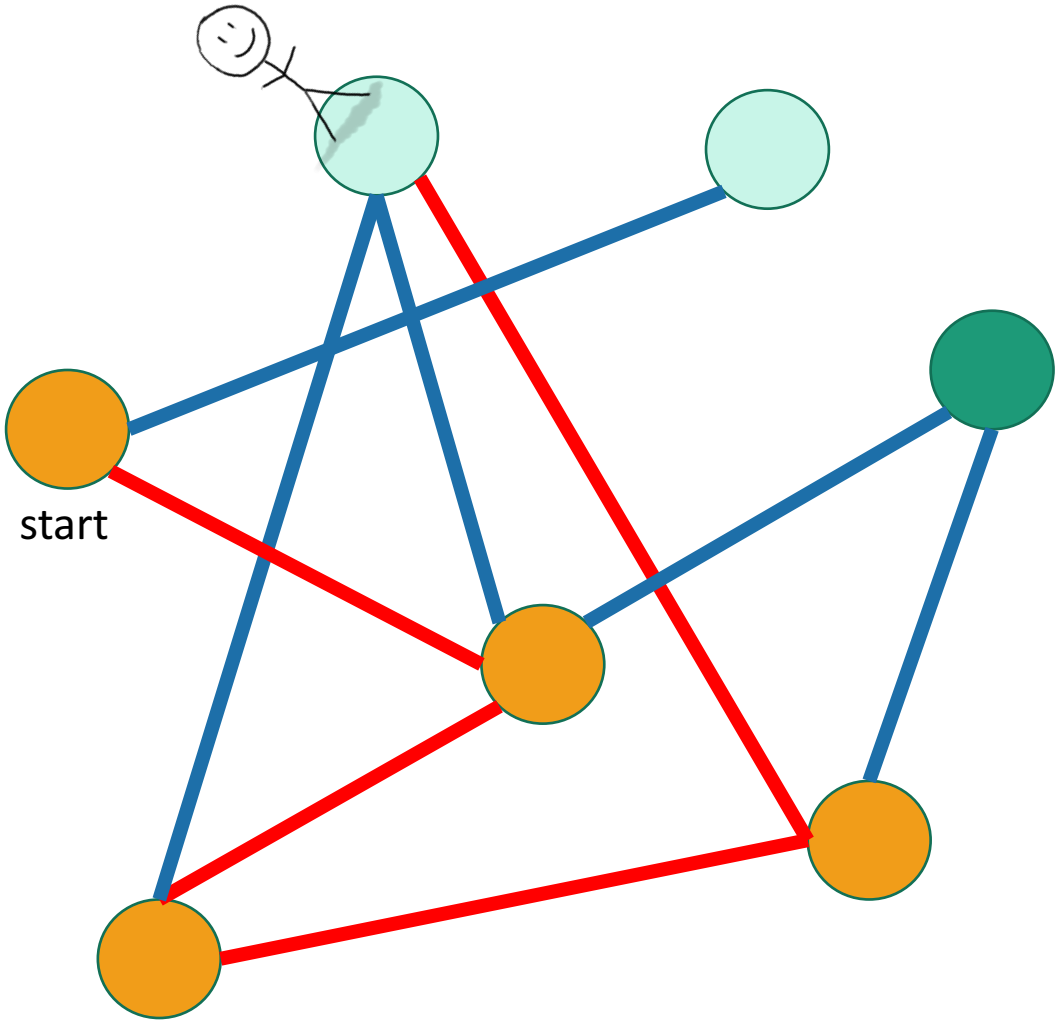

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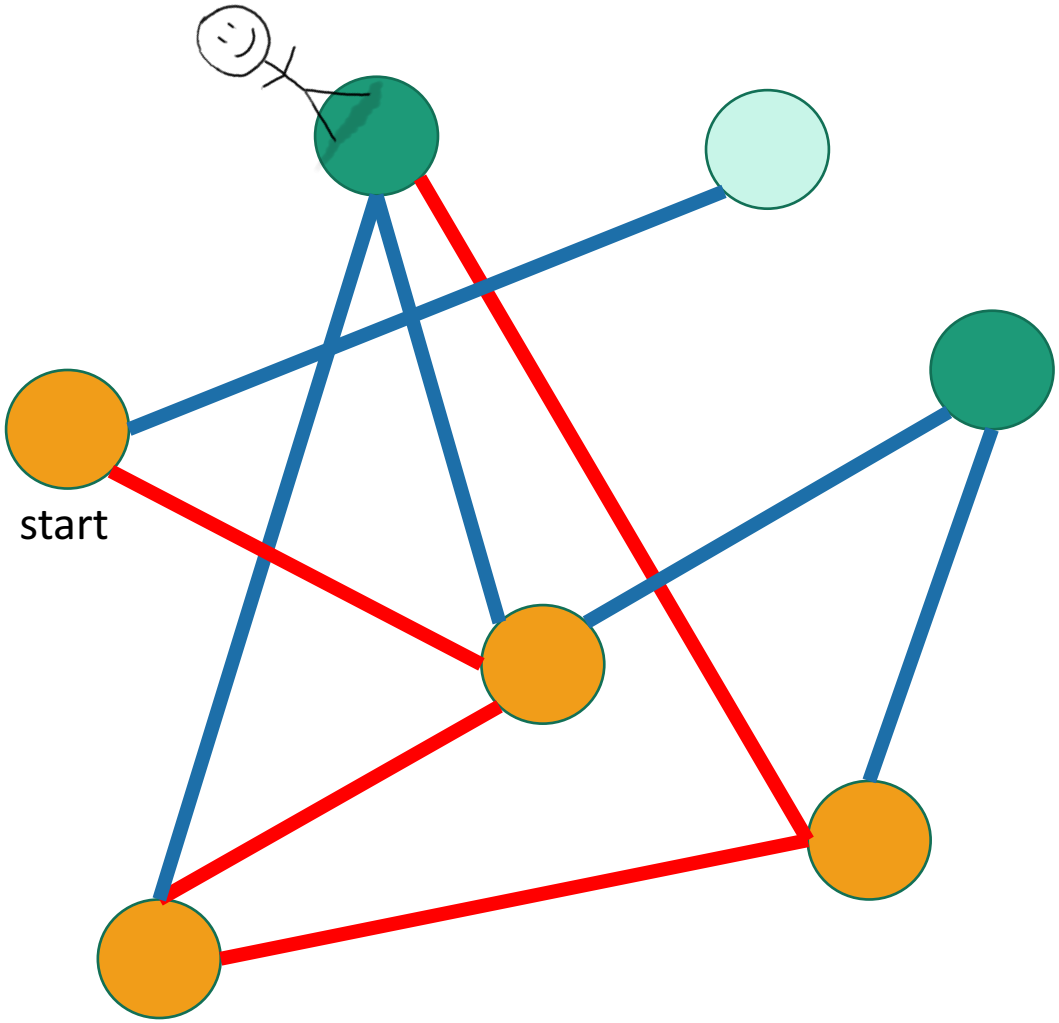

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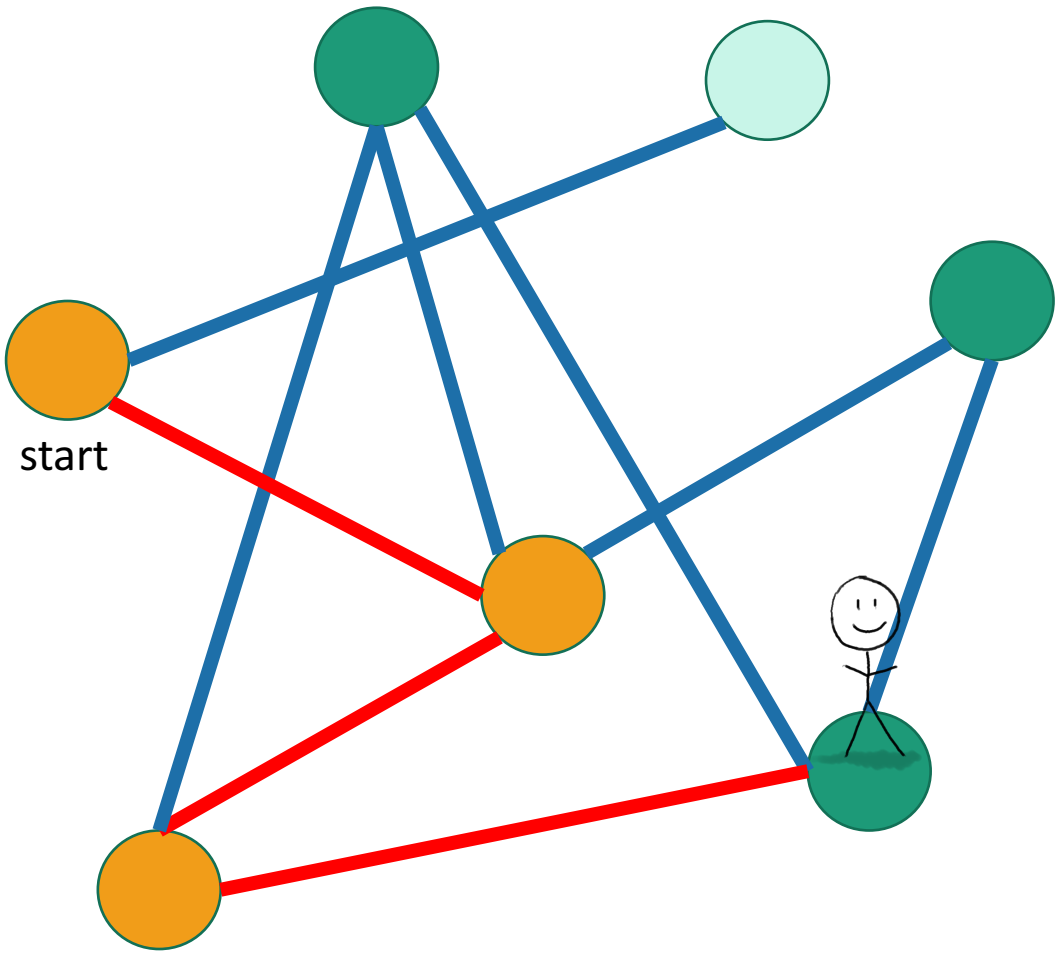

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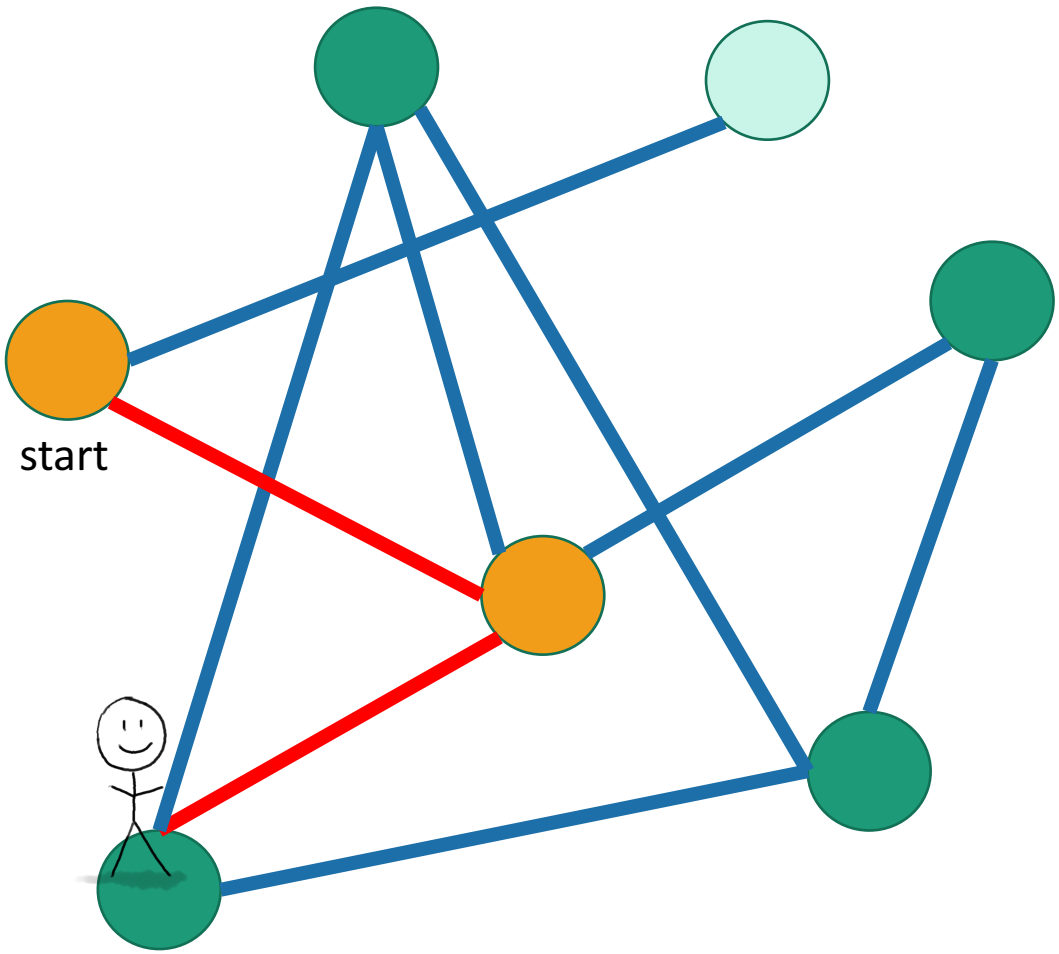

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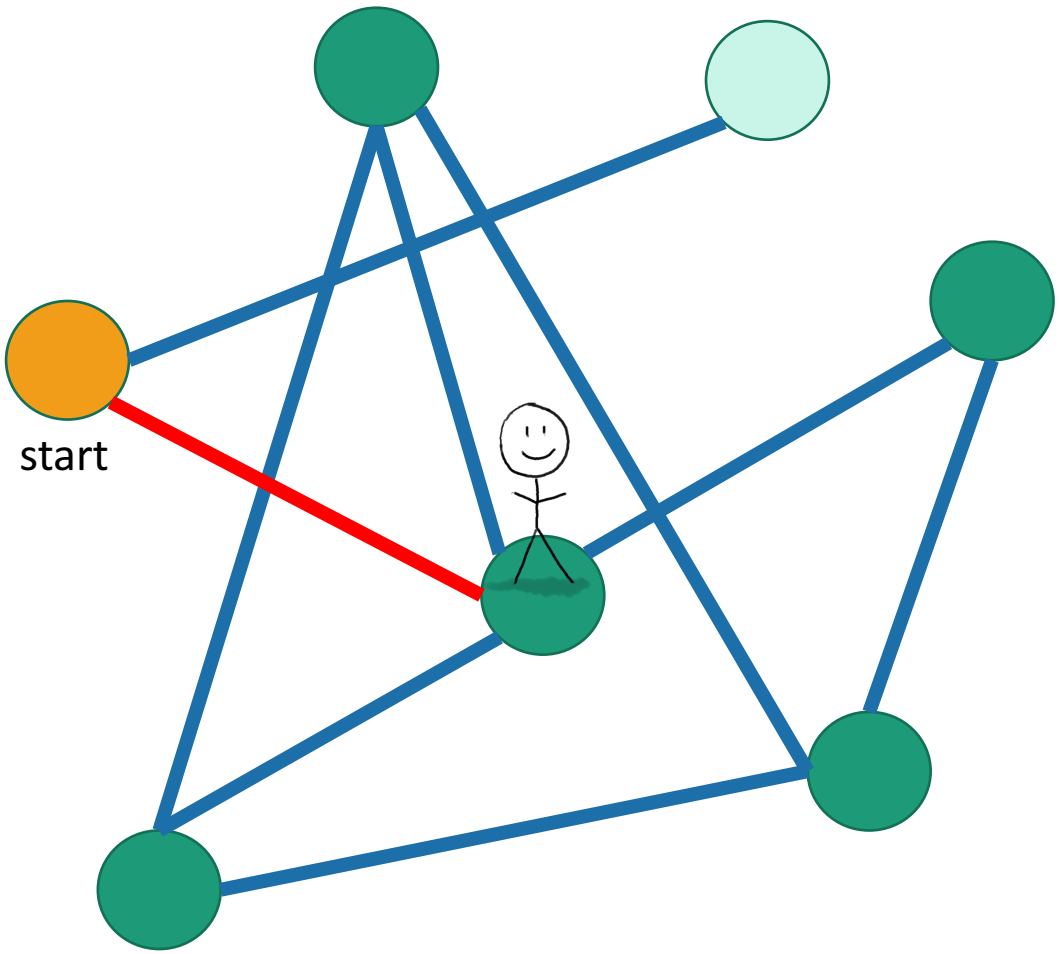

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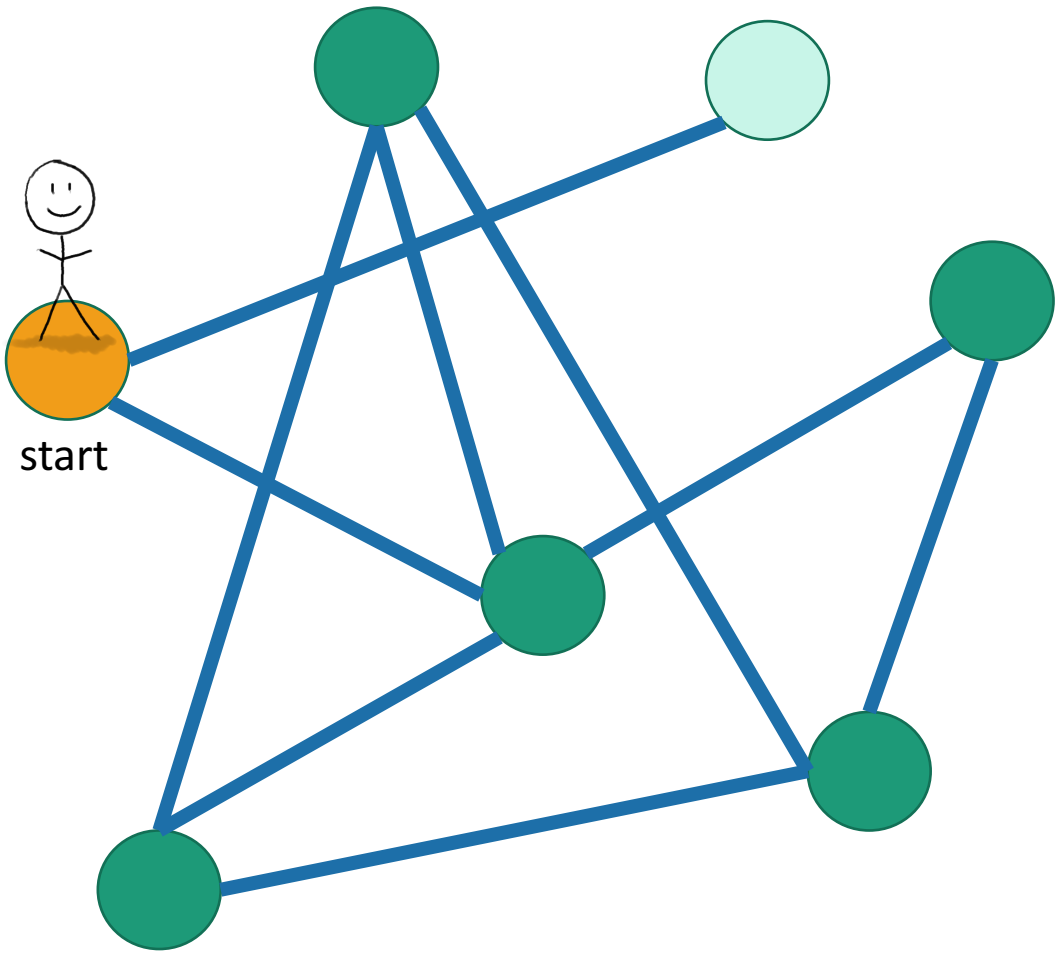

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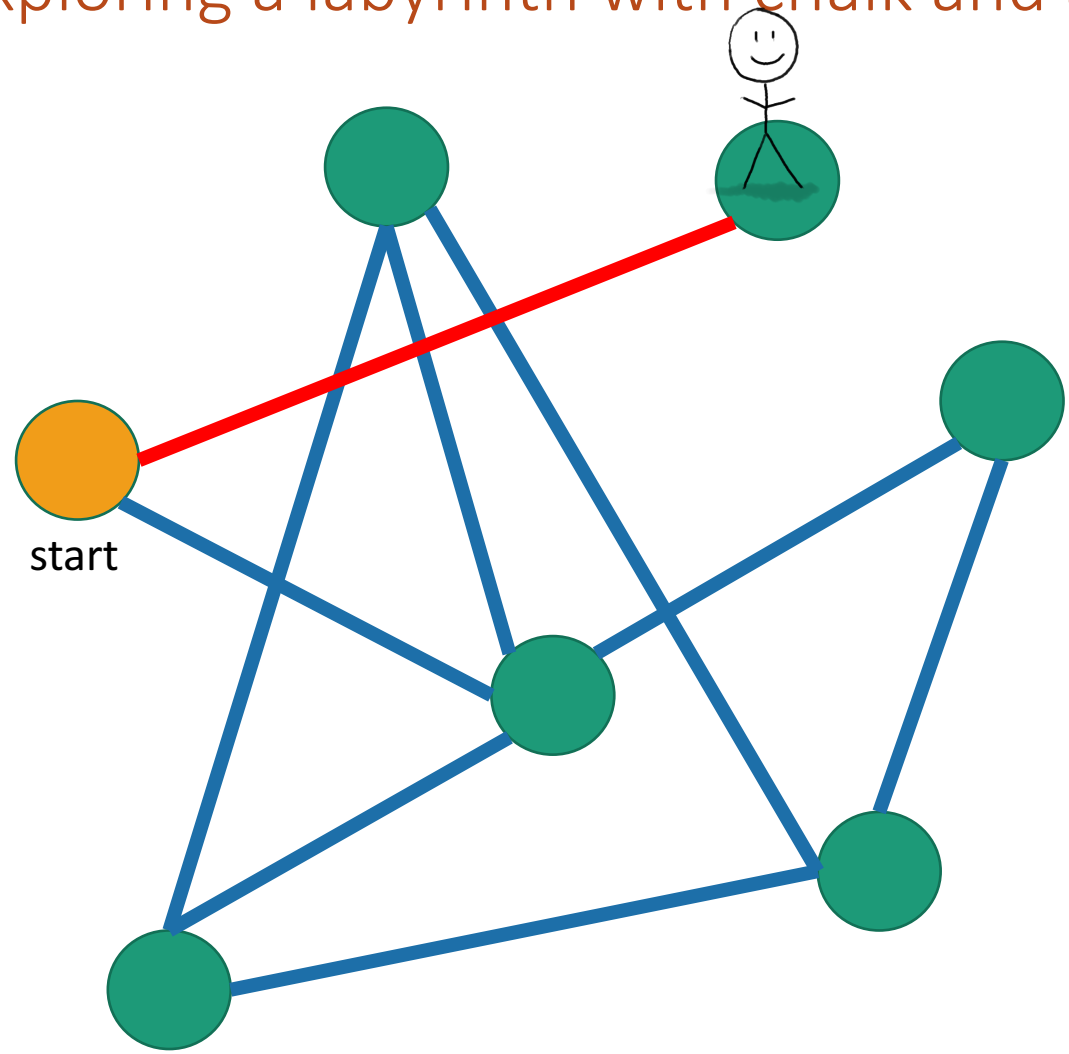

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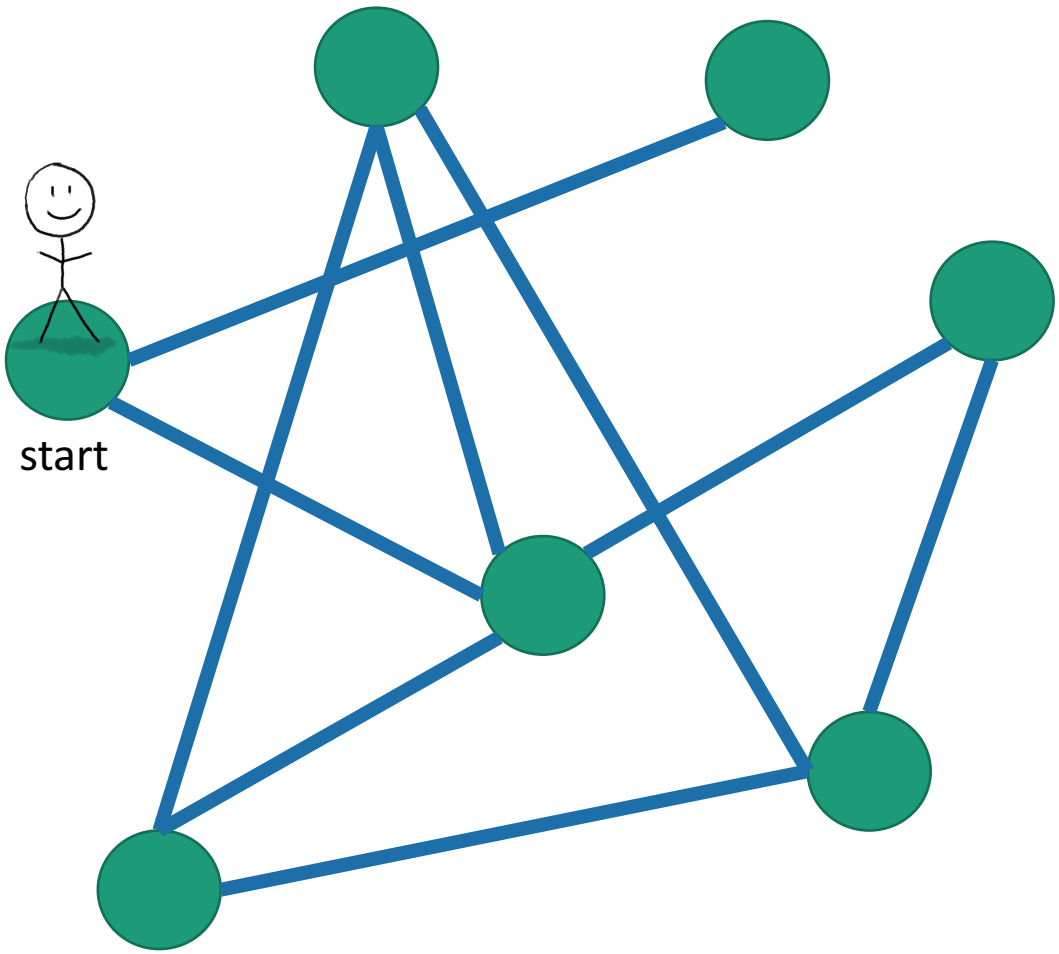

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Labyrinth:

EXPLORED!

Depth First Search

Exploring a labyrinth with pseudocode

- **DFS**(w, currentTime):
  - w.entryTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Each node is going to keep track of whether it's unvisited, in progress, or all done.

We'll also keep track of the time at which we started and finished with that node.

Plucky the pedantic penguin

Depth First Search

Exploring a labyrinth with pseudocode

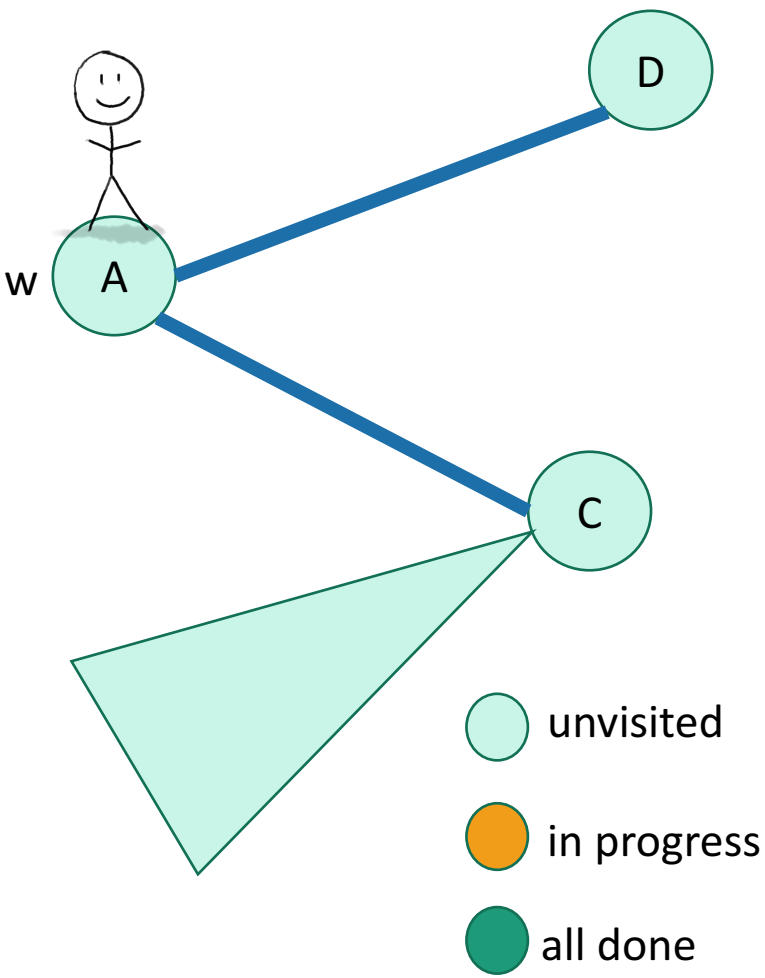
- Each vertex keeps track of whether it is:
  - Unvisited
  - In progress
  - All done

- Each vertex will also keep track of:
  - The time we **first enter it**.
  - The time we finish with it and mark it **all done**.

You might have seen other ways to implement DFS than what we are about to go through. This way has more bookkeeping, but more intuition – also, the bookkeeping will be useful later!

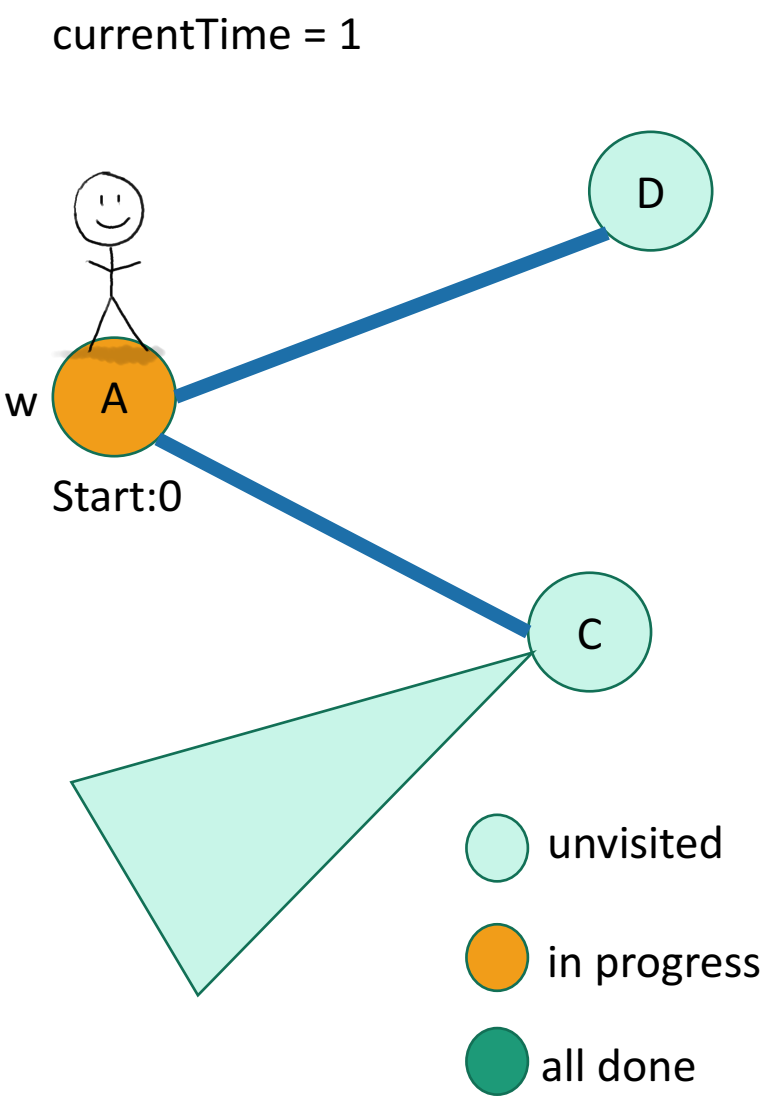
Depth First Search

currentTime = 0



- **DFS**(w, currentTime):
  - w.entryTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

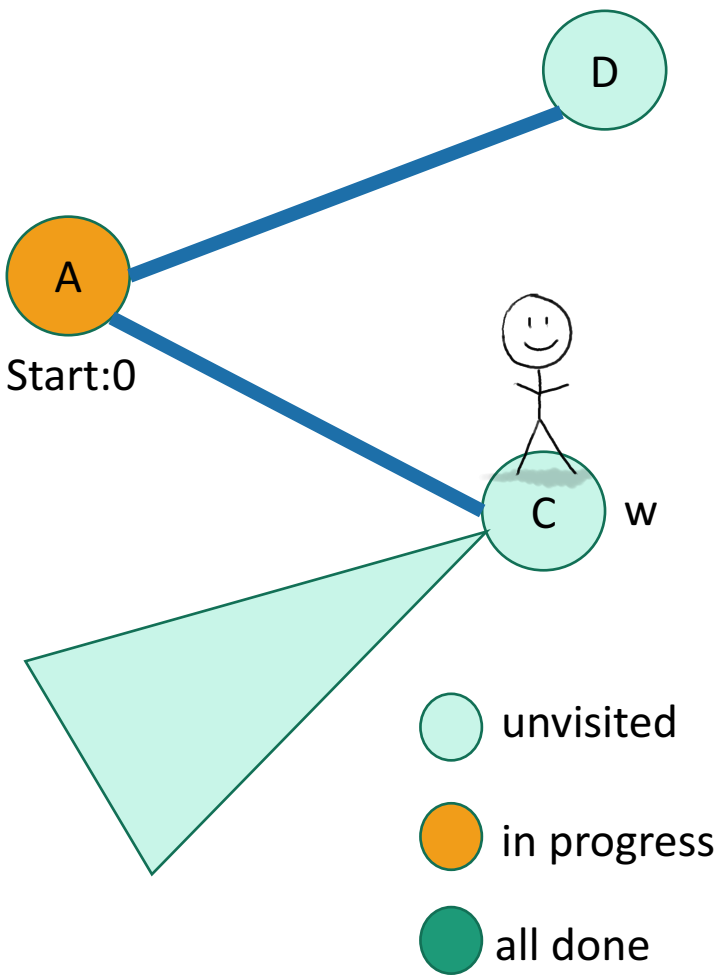
Depth First Search



- **DFS**(w, currentTime):
  - w.entryTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search

currentTime = 1



- **DFS**(w, currentTime):
  - w.entryTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime = **DFS**(v, currentTime)
      - currentTime ++
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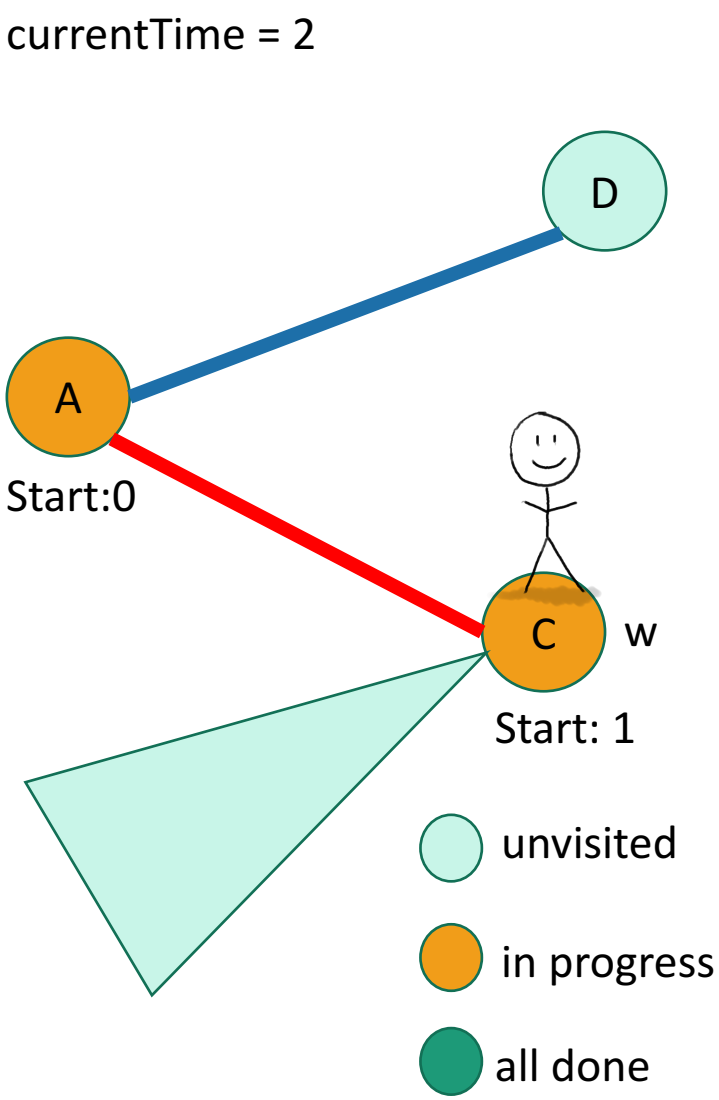
Depth First Search



- **DFS**(w, currentTime):
  - w.entryTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
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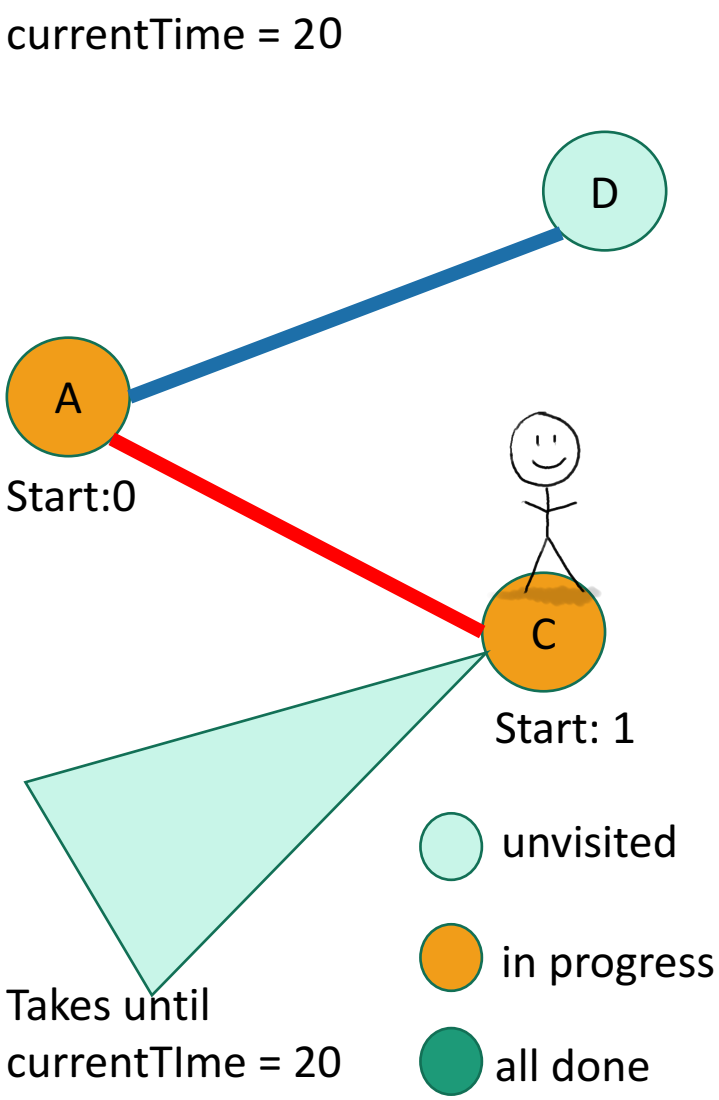
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



- **DFS**(w, currentTime):
  - w.entryTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
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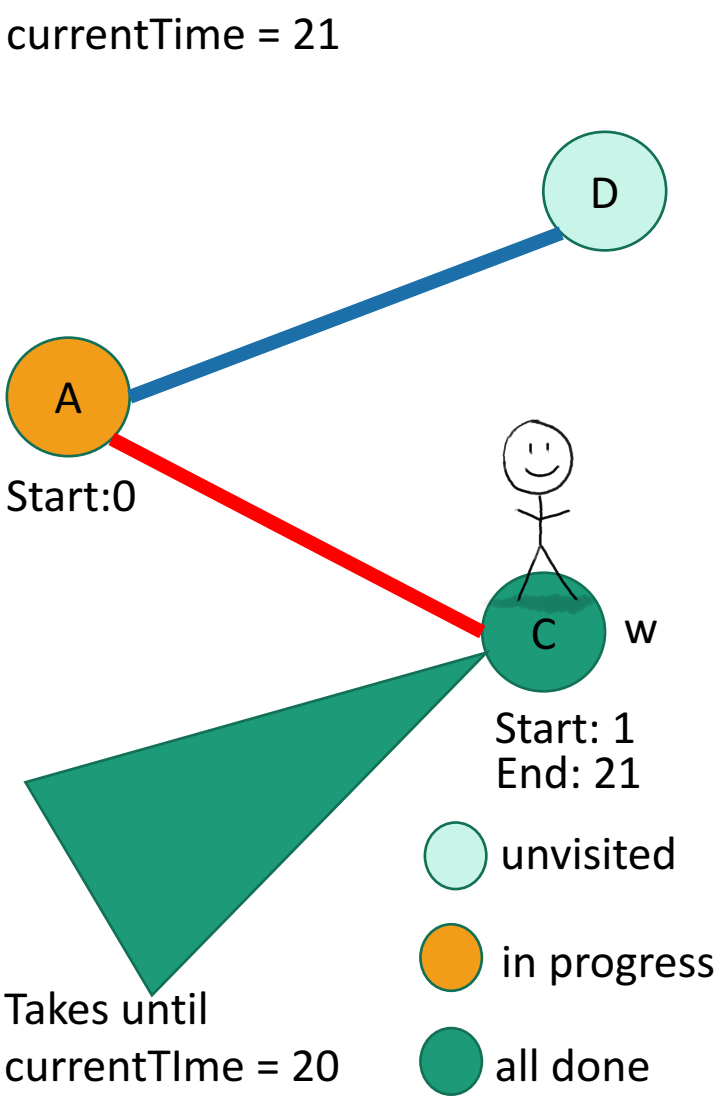
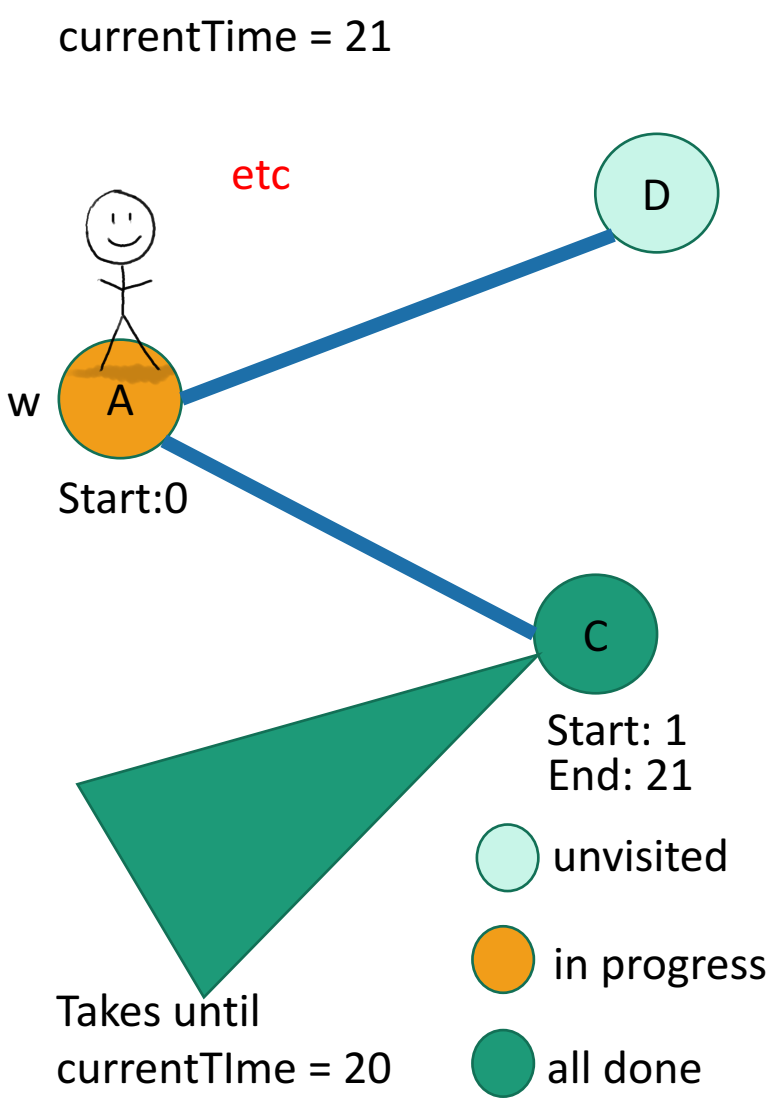
Depth First Search



- **DFS**(w, currentTime):
  - w.startTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
    - **if** v is **unvisited**:
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

Depth First Search



- **DFS**(w, currentTime):
  - w.startTime = currentTime
  - currentTime ++
  - Mark w as **in progress**.
  - **for** v in w.neighbors:
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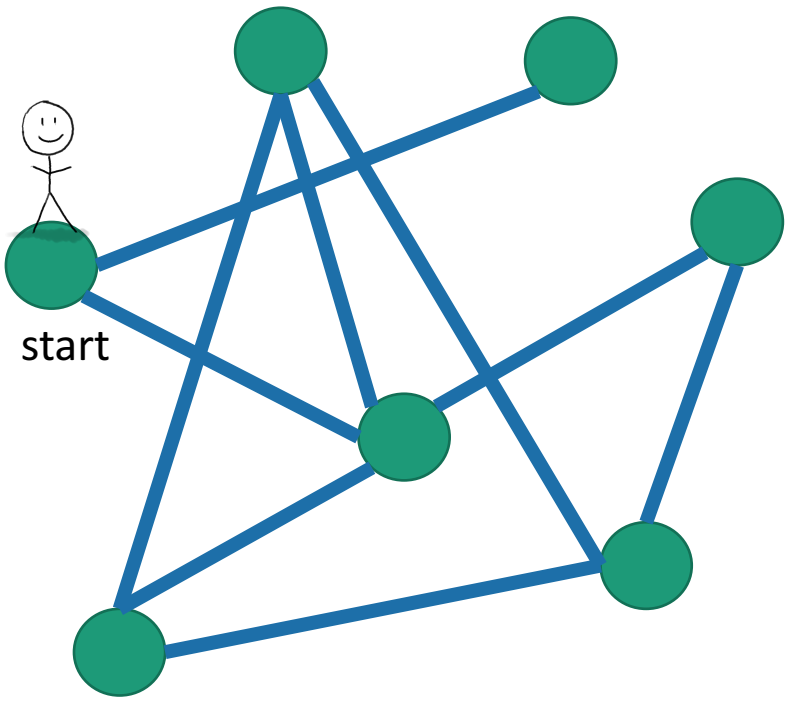
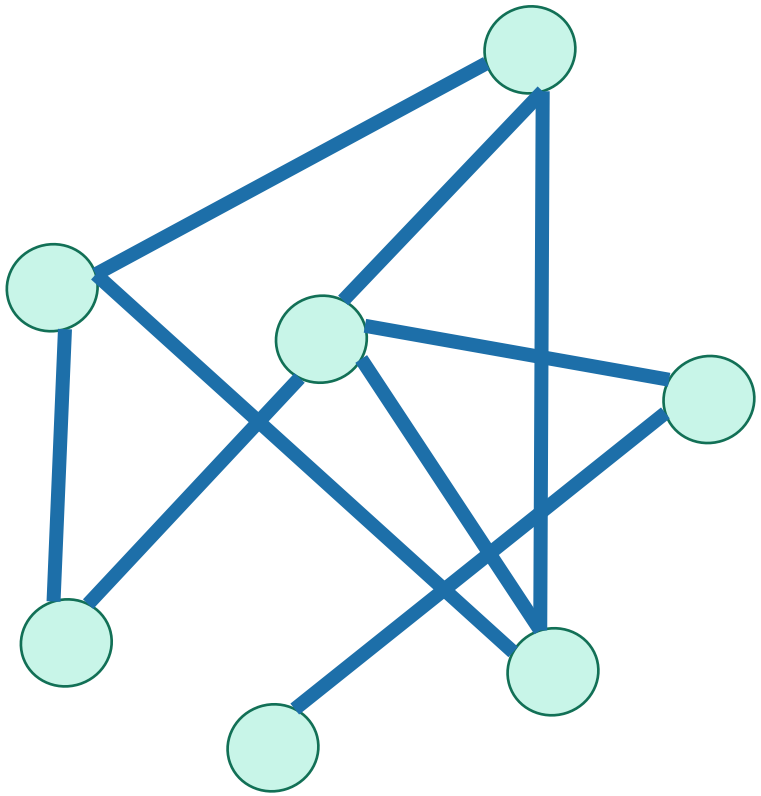
      - currentTime
        = **DFS**(v, currentTime)
      - currentTime ++
  - w.finishTime = currentTime
  - Mark w as **all done**
  - **return** currentTime

DFS finds all the nodes reachable from the starting point



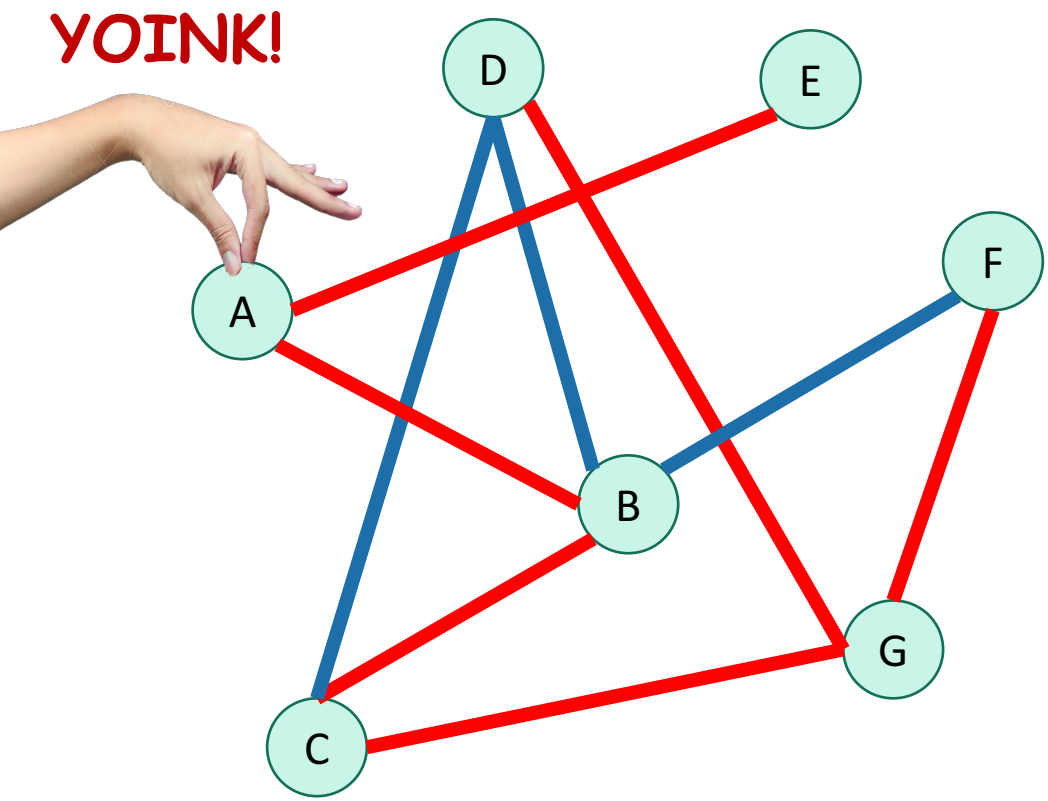
In an undirected graph, this is called a **connected component.**

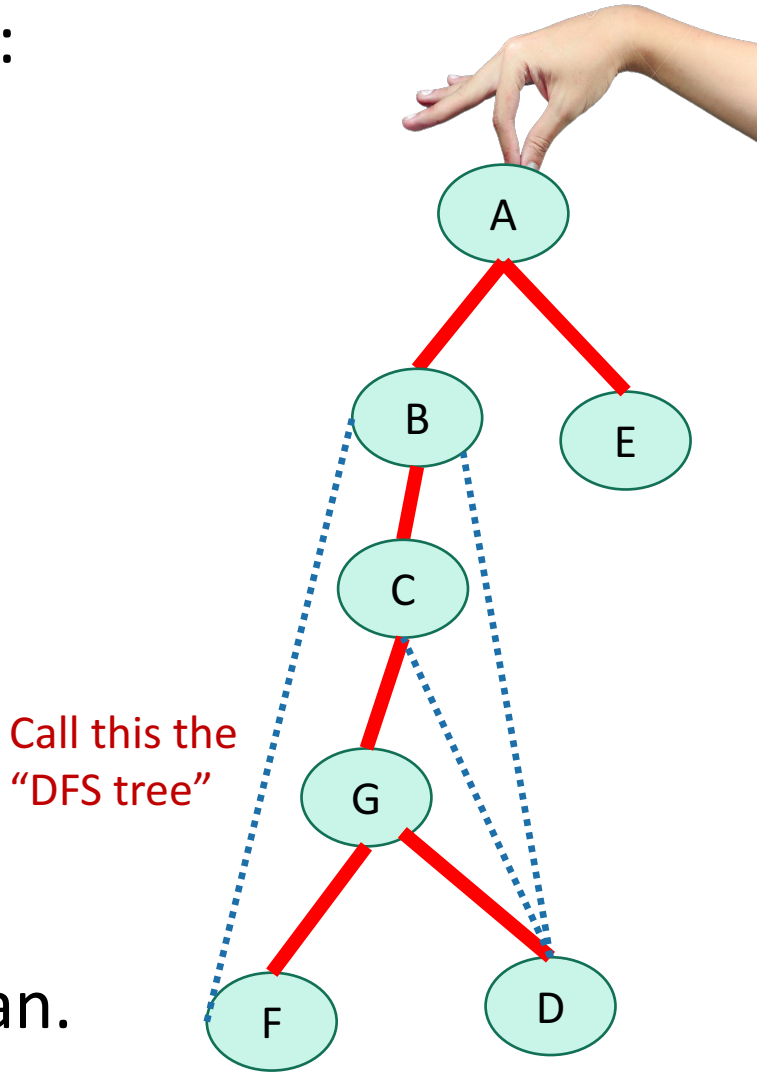


One application: finding connected components.

Why is it called depth-first?

- We are implicitly building a tree:





- And first we go as deep as we can.

Running time

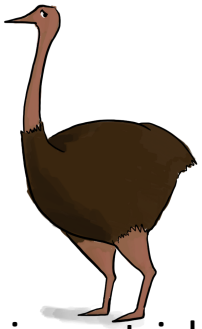
To explore just the connected component we started in

- We look at each edge only once.
- And basically don't do anything else.
- So…

$$O(m)$$

- (Assuming we are using the linked-list representation)

Verify this formally!



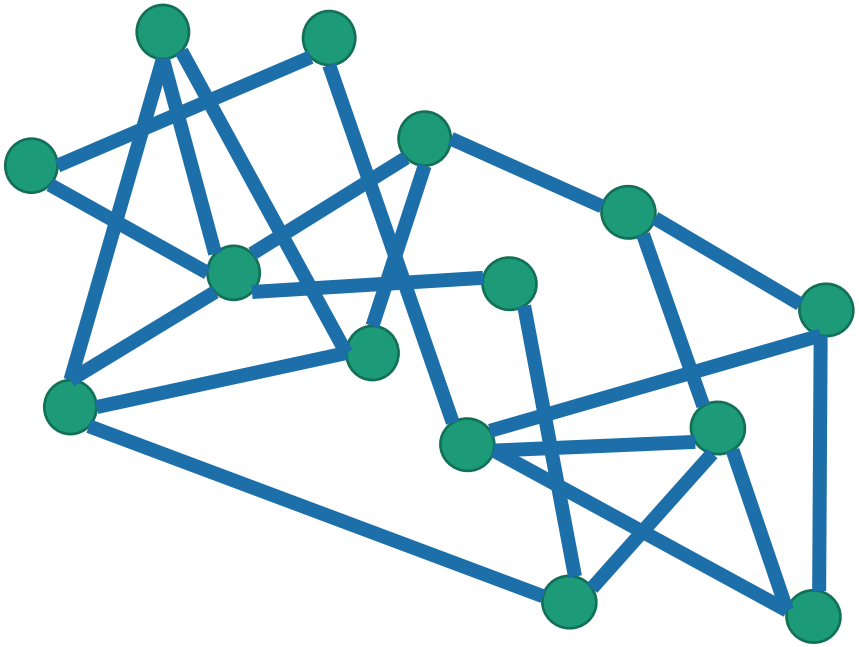
Ollie the over-achieving ostrich

Running time

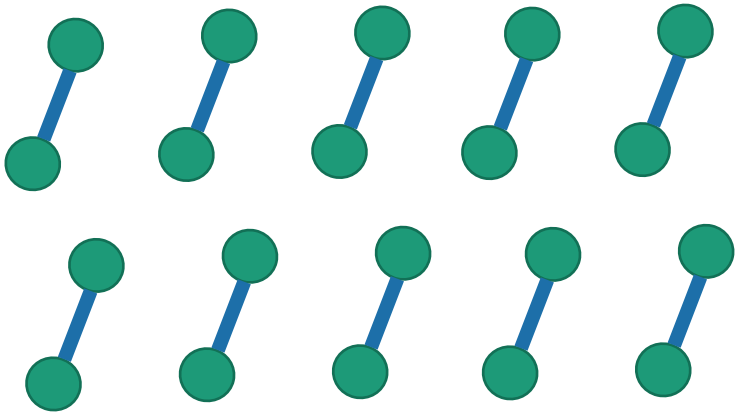
To explore the whole thing

- Explore the connected components one-by-one.
- This takes time

$$O(n + m)$$

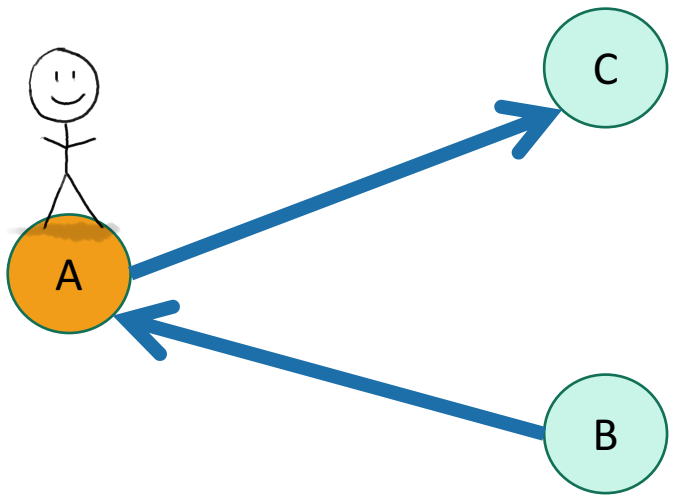


or

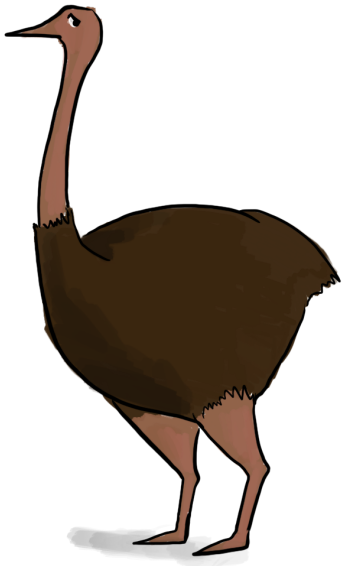


You check:

DFS works fine on directed graphs too!



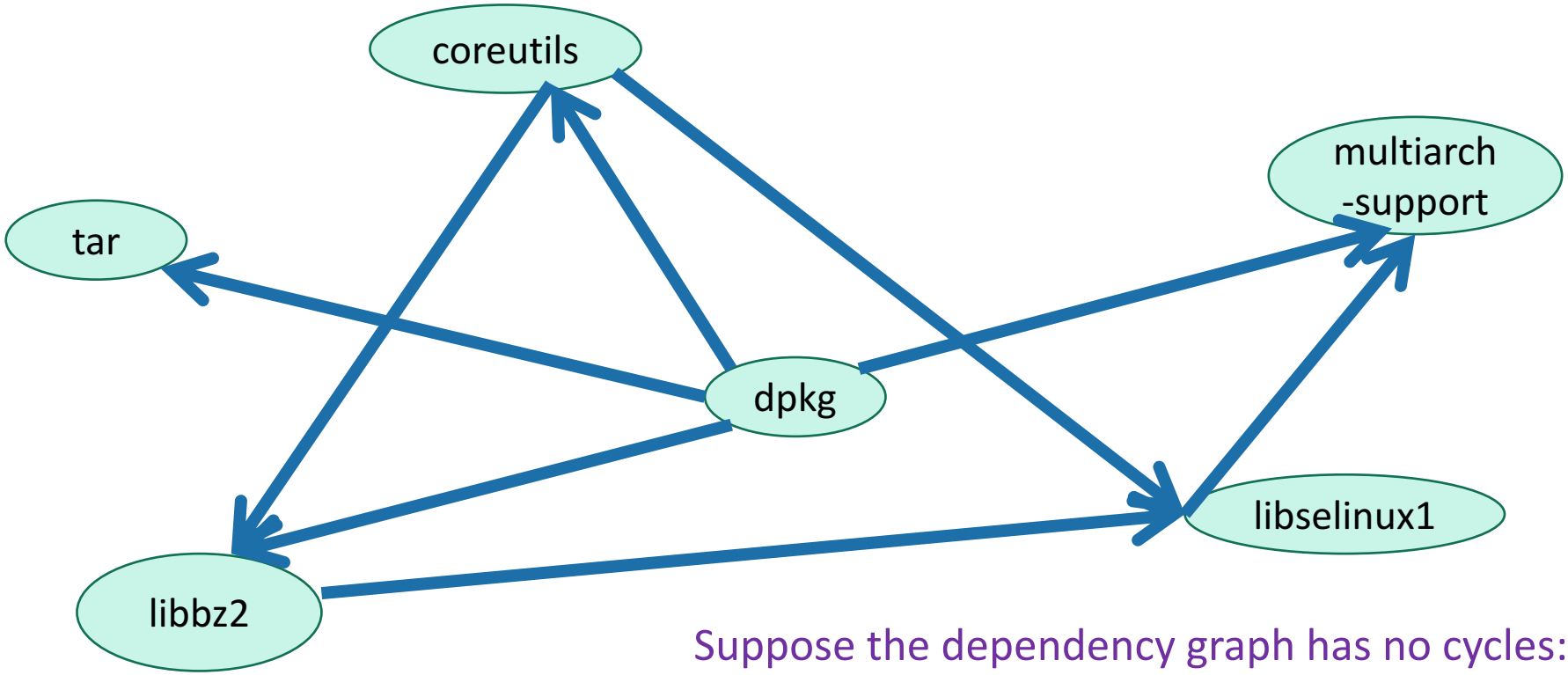
Only walk to C, not to B.



Ollie the over-achieving ostrich

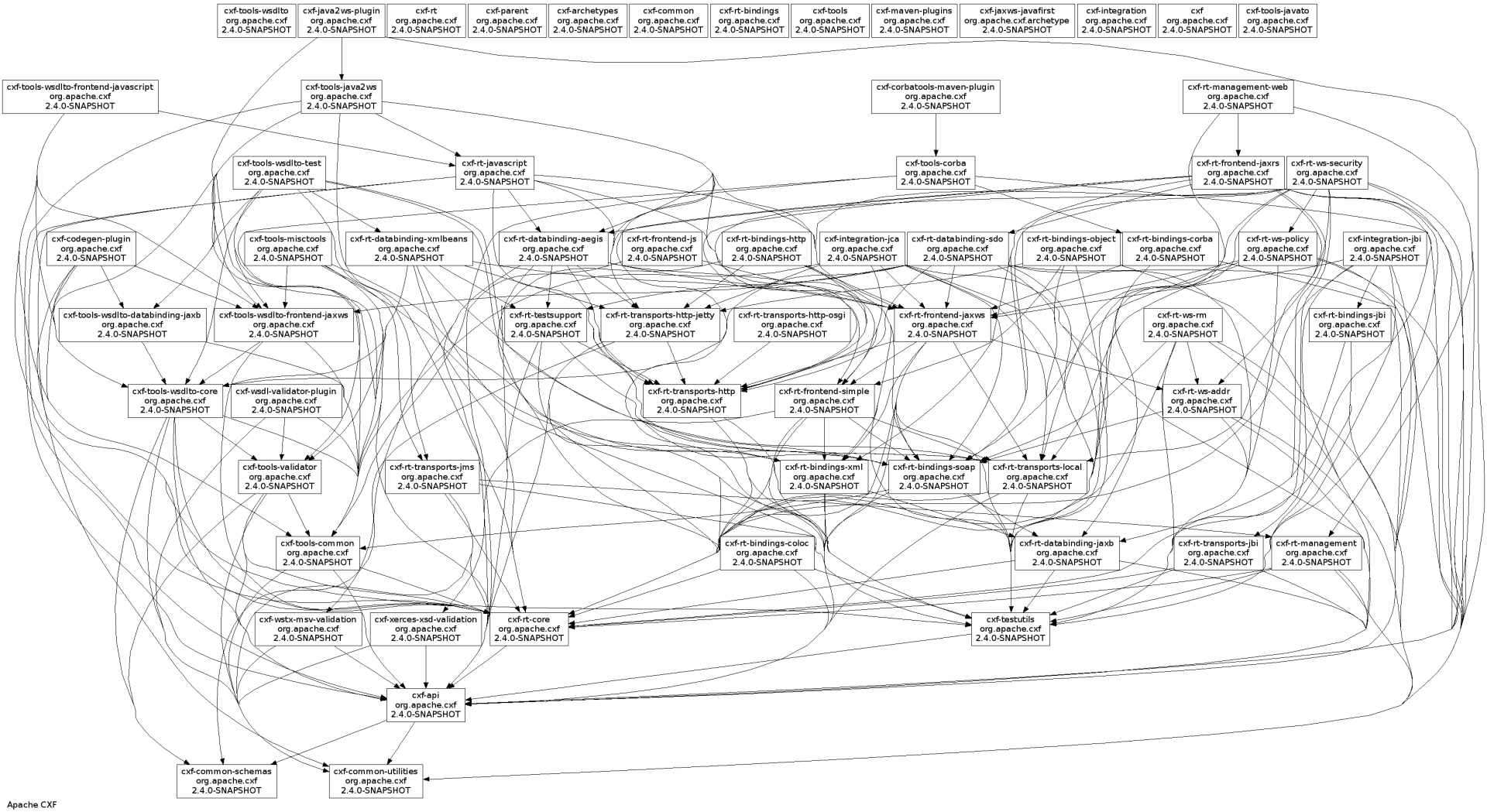
Application: topological sorting

- Example: package dependency graph
- Question: in what order should I install packages?



Suppose the dependency graph has no cycles:
it is a **Directed Acyclic Graph (DAG)**

Can’t always eyeball it.



Application: topological sorting

- Example: package dependency graph
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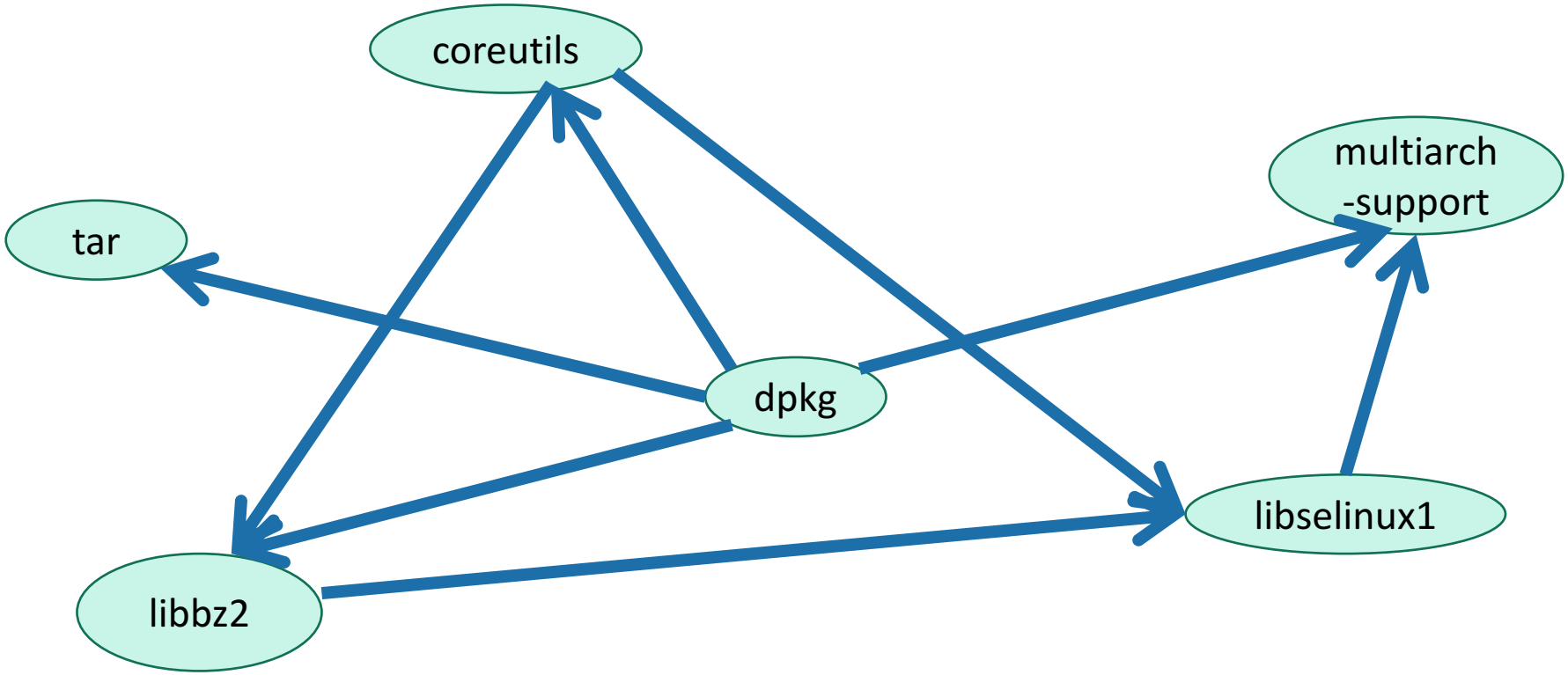
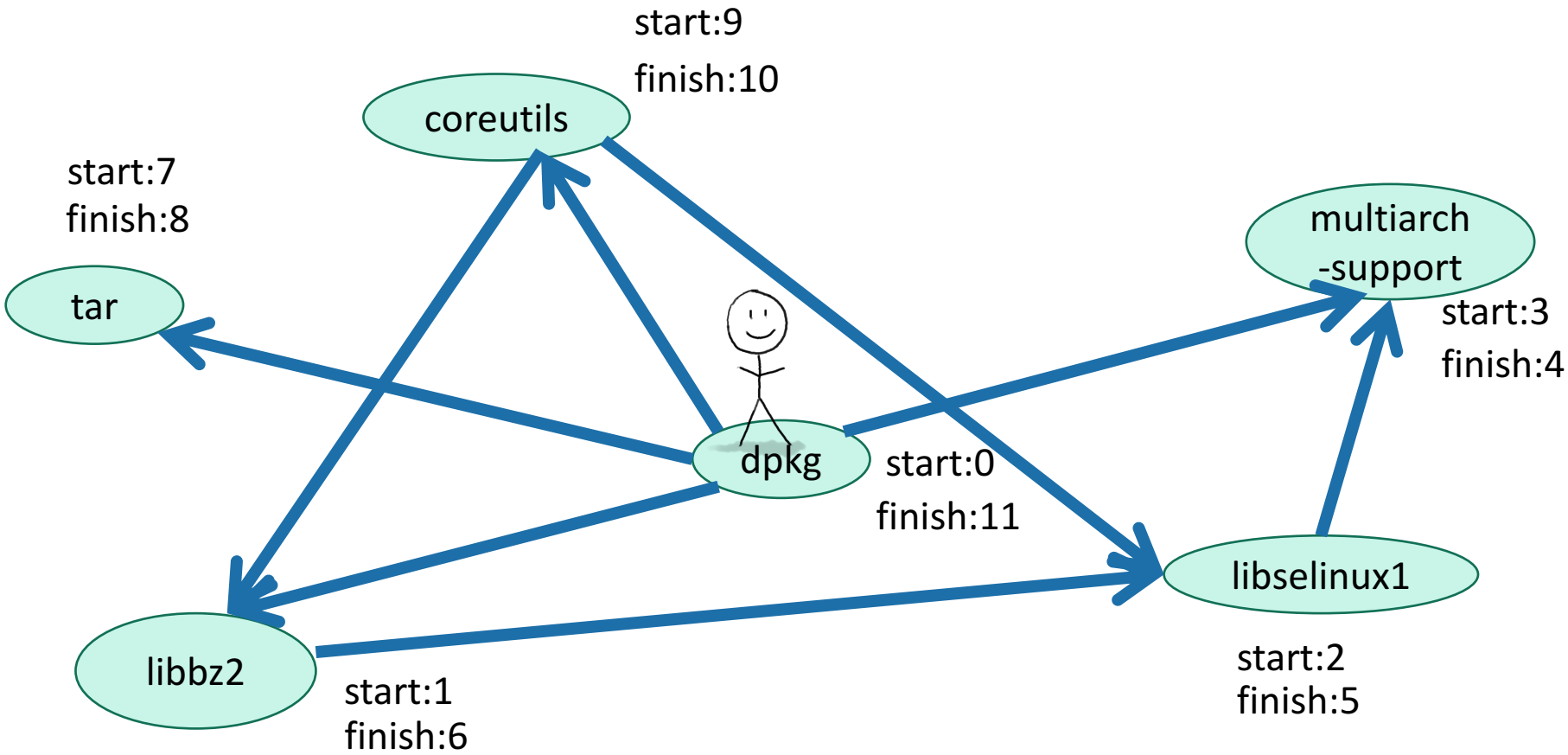


Let's do DFS

Observations:

- The start times don't seem that useful.
- But the packages we should include earlier have larger **finish times.**



Suppose the underlying graph has no cycles

This is not an accident

Claim: In general, we'll always have:

finish: [larger] finish: [smaller]

To understand why, let's go back to that DFS tree.

A more general statement

(this holds even if there are cycles)

This is called the "parentheses theorem" in CLRS

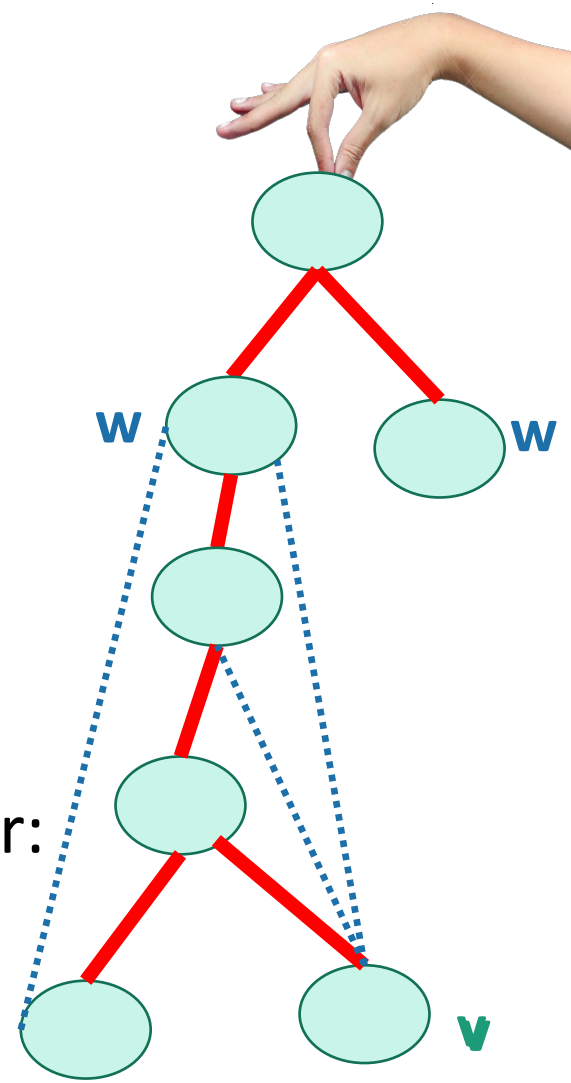
(check this statement carefully!)

- If v is a descendent of w in this tree:

- If w is a descendent of v in this tree:

- If neither are descendents of each other:

(or the other way around)

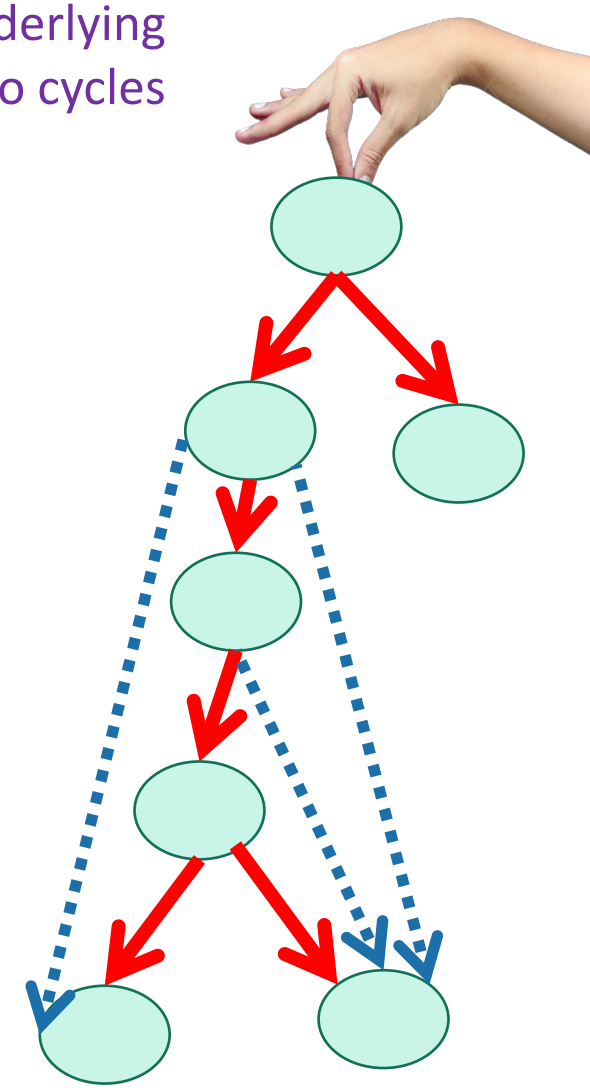


So to prove this ->

If $A \rightarrow B$

Then B.finishTime < A.finishTime

Suppose the underlying graph has no cycles



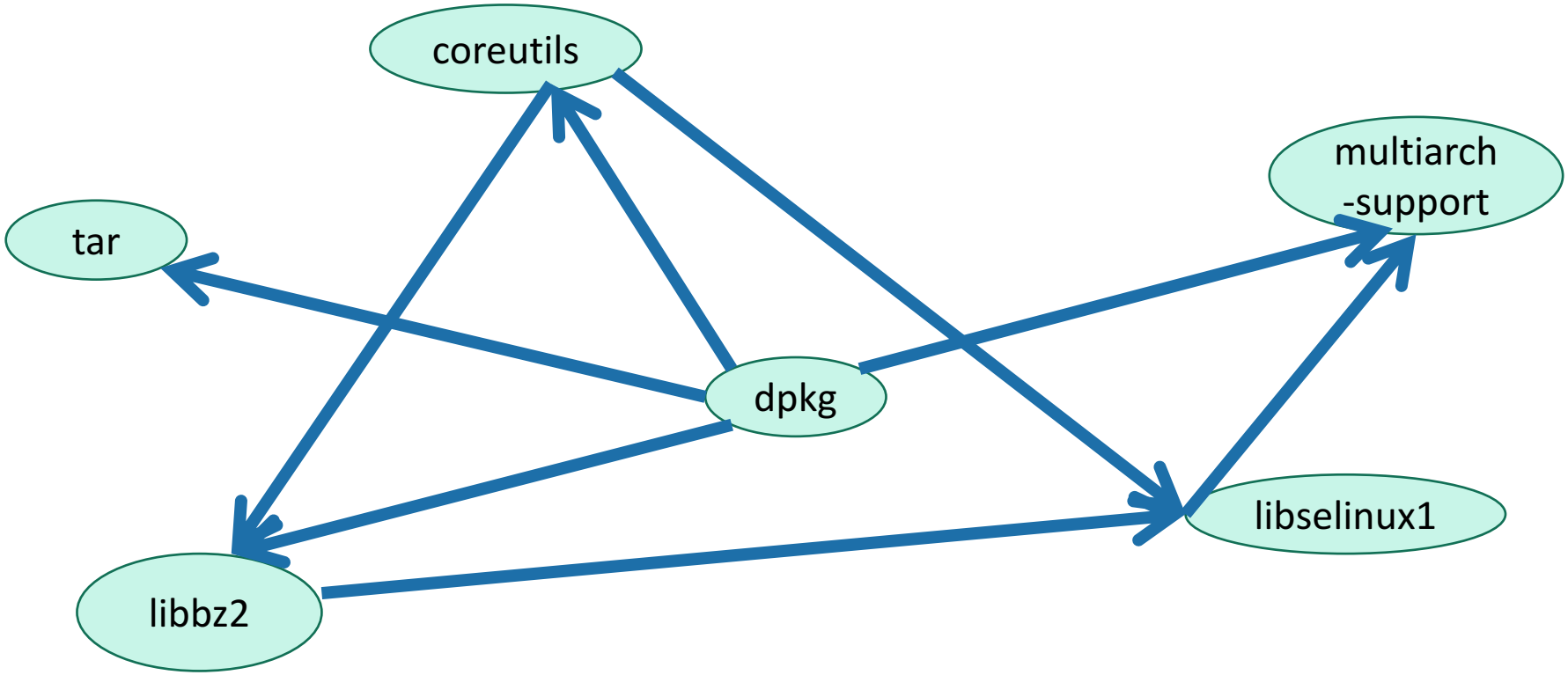
- Since the graph has no cycles, B must be a descendent of A in that tree.
  - All edges go down the tree.

- Then

- aka, B.finishTime < A.finishTime.

Back to this problem

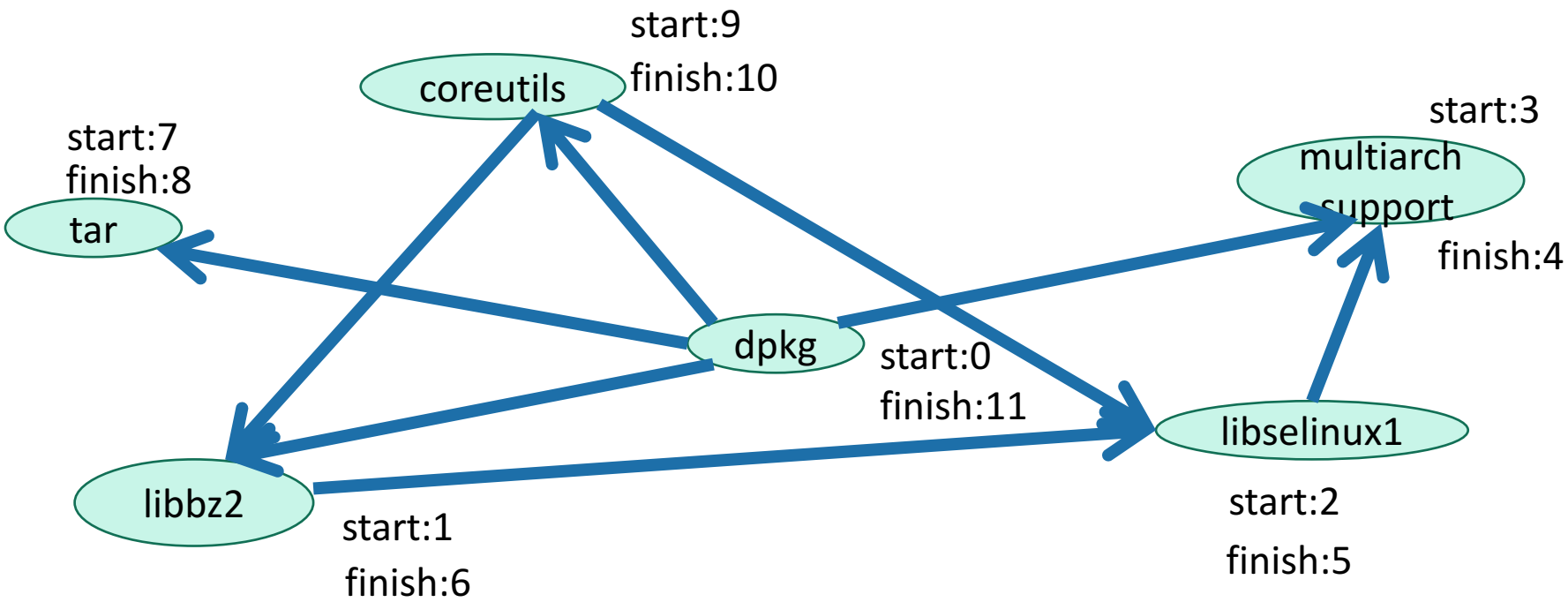
- Example: package dependency graph
- Question: in what order should I install packages?



In reverse order of finishing time

- Do DFS
- Maintain a list of packages, in the order you want to install them.
- When you mark a vertex as **all done**, put it at the **beginning** of the list.

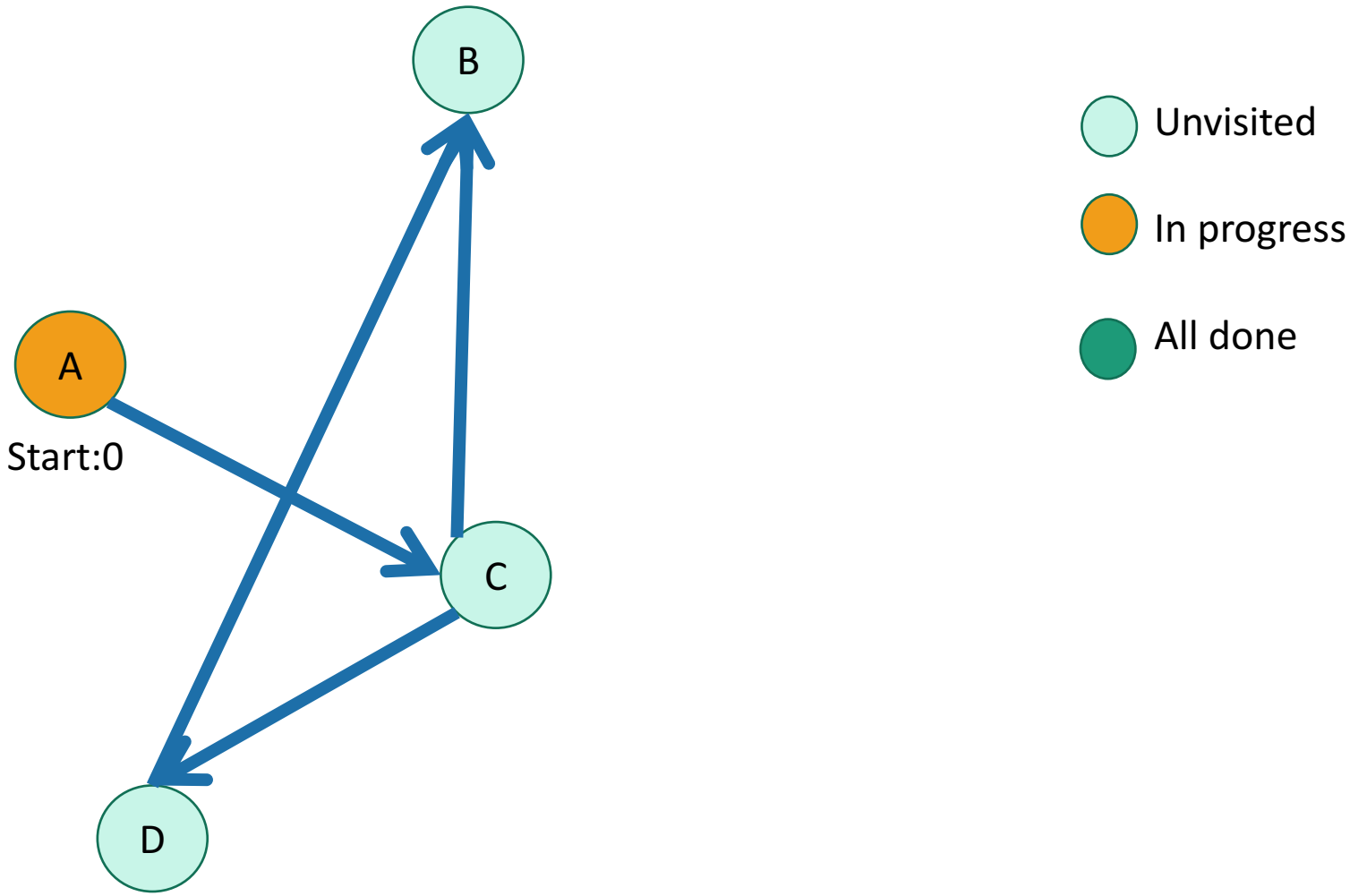
- `dpkg`
- `coreutils`
- `tar`
- `libbz2`
- `libselinux1`
- `multiarch_support`



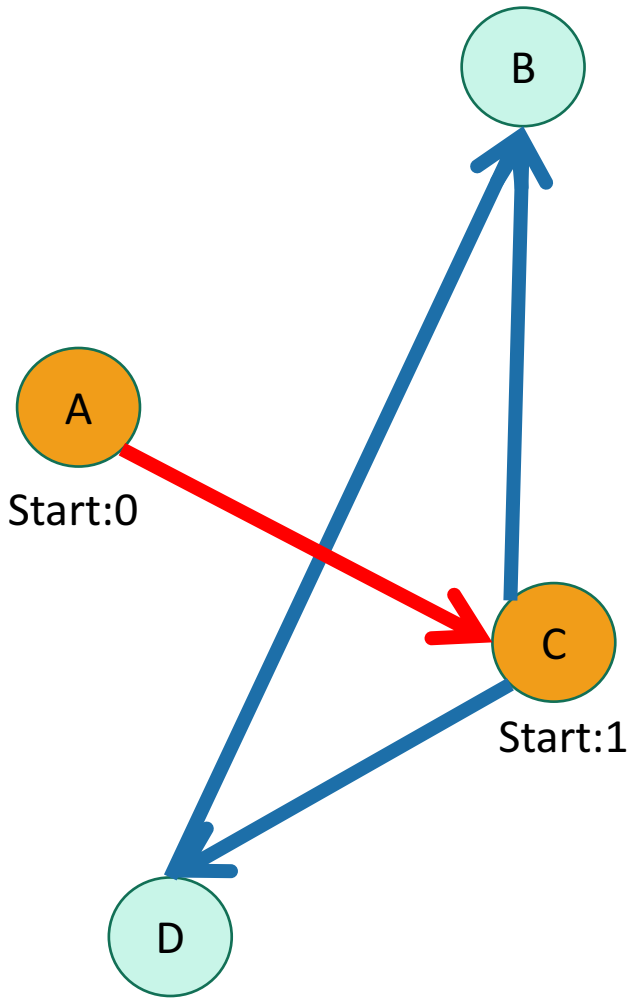
What did we just learn?

- DFS can help you solve the **TOPOLOGICAL SORTING PROBLEM**
  - That's the fancy name for the problem of finding an ordering that respects all the dependencies
- Thinking about the DFS tree is helpful.

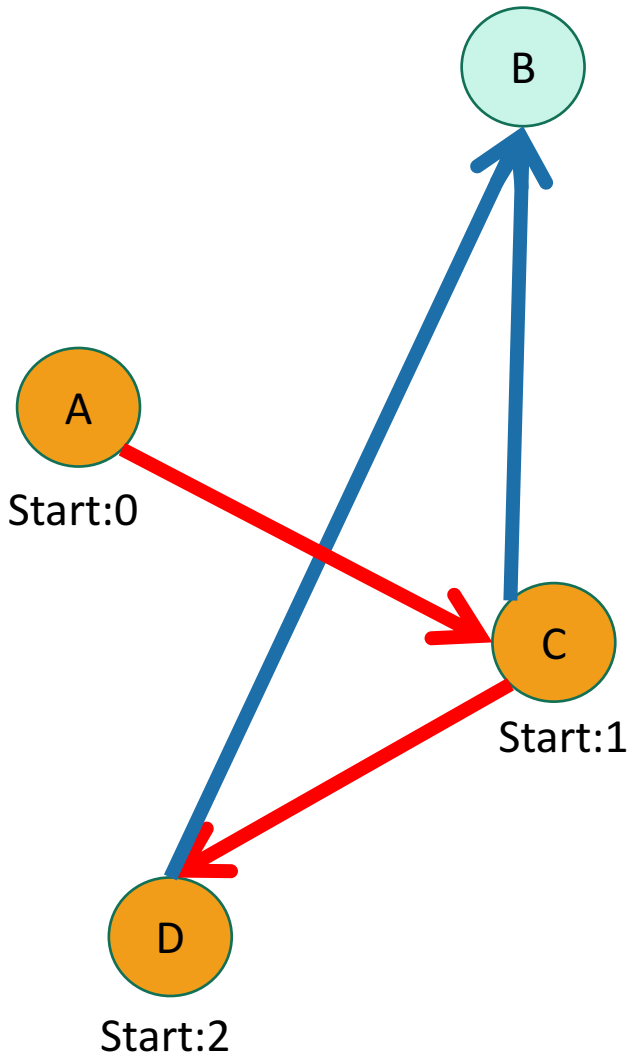
Example:



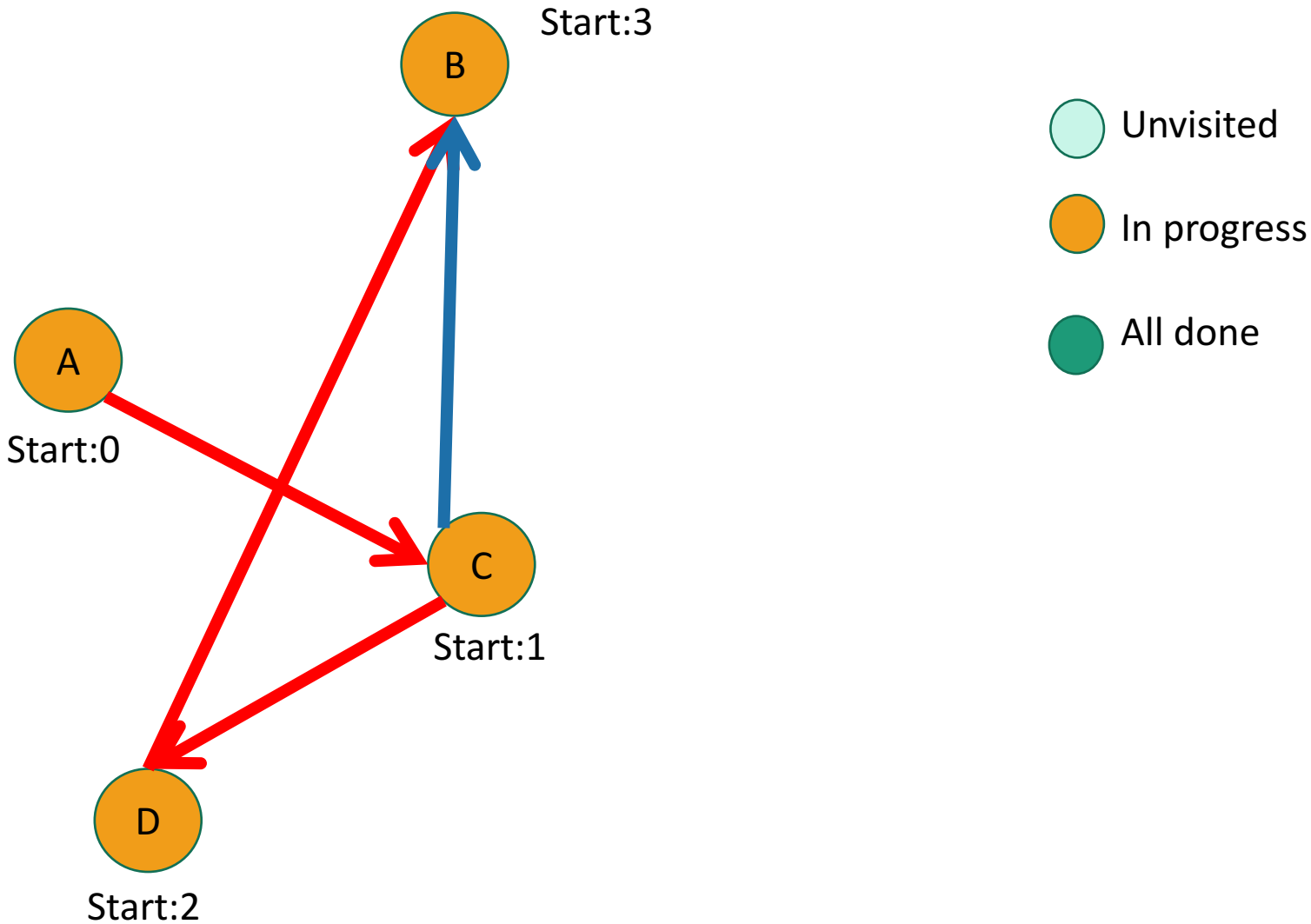
Example



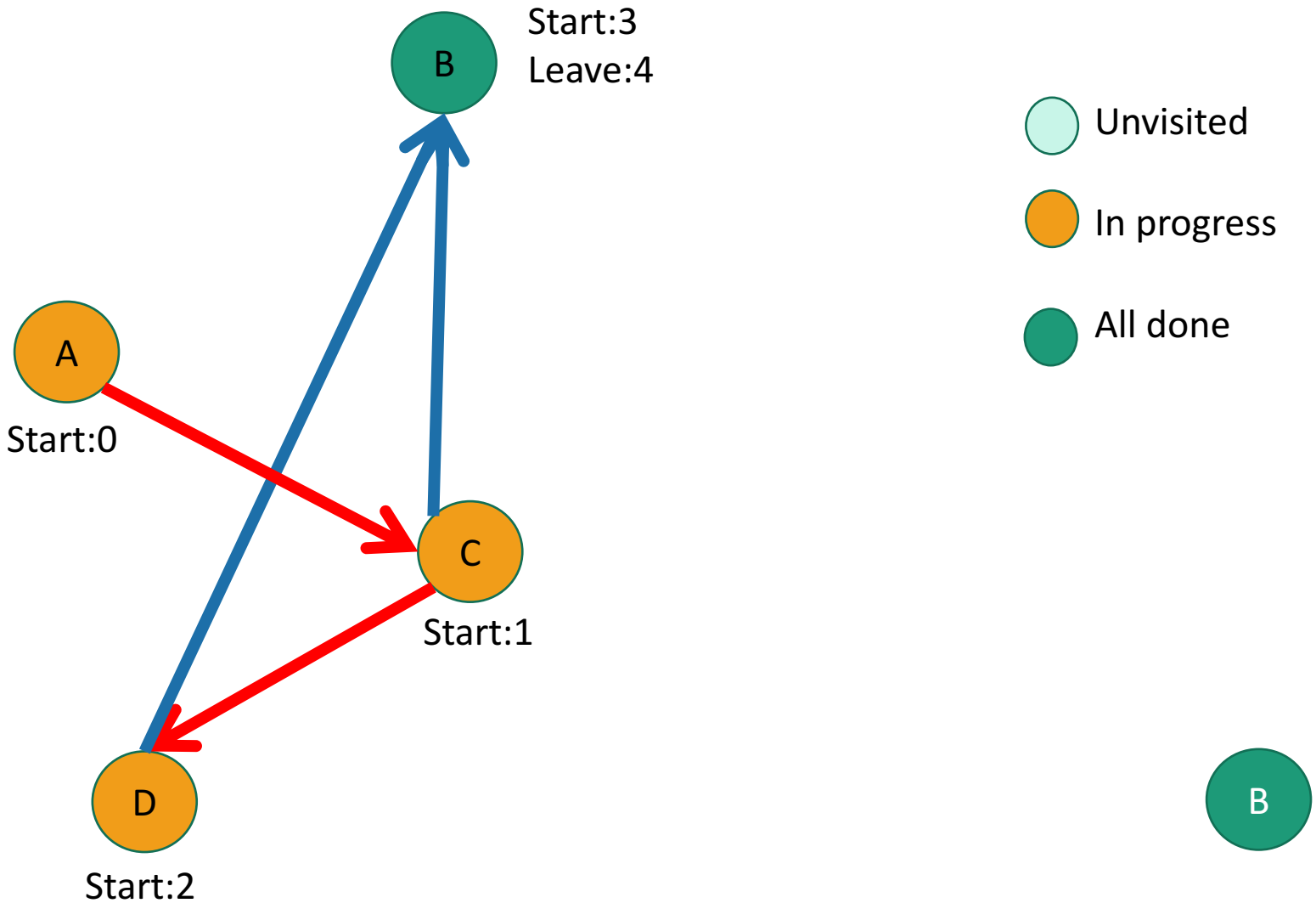
Example



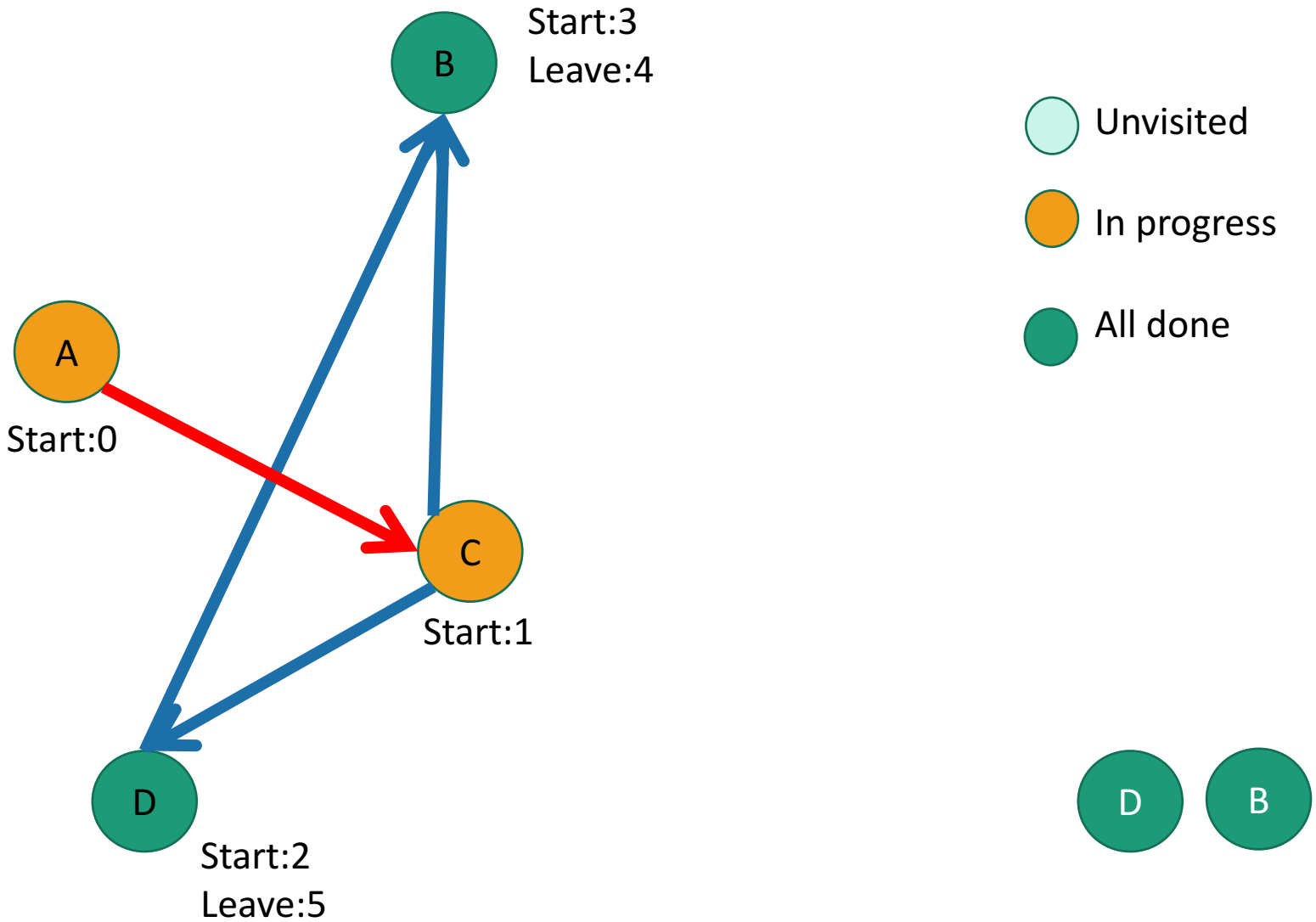
Example


Start:3
B
A
Start:0
C
Start:1
D
Start:2
Unvisited
In progress
All done

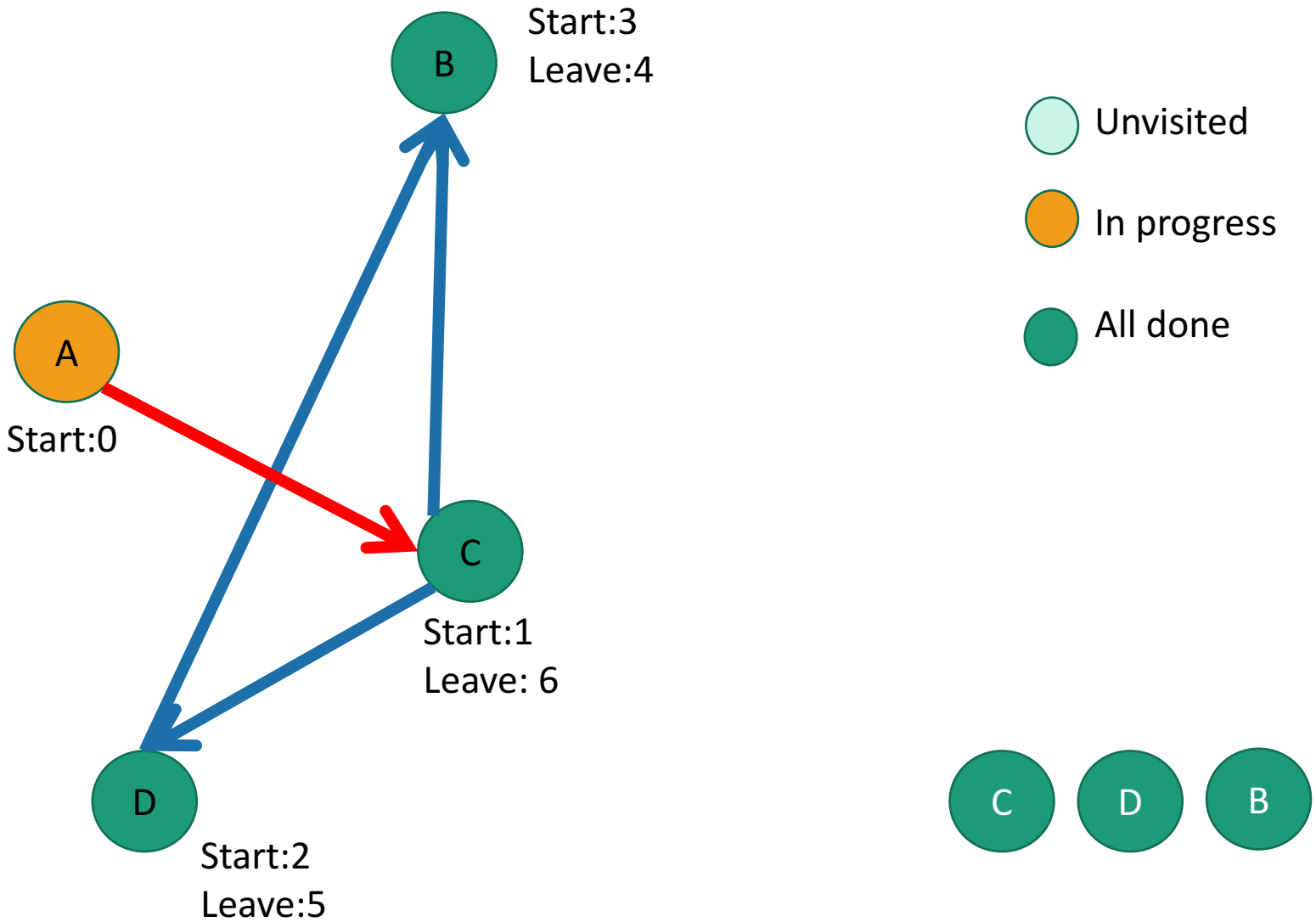
Example


B
Start:3
Leave:4
A
Start:0
C
Start:1
D
Start:2
Unvisited
In progress
All done
B

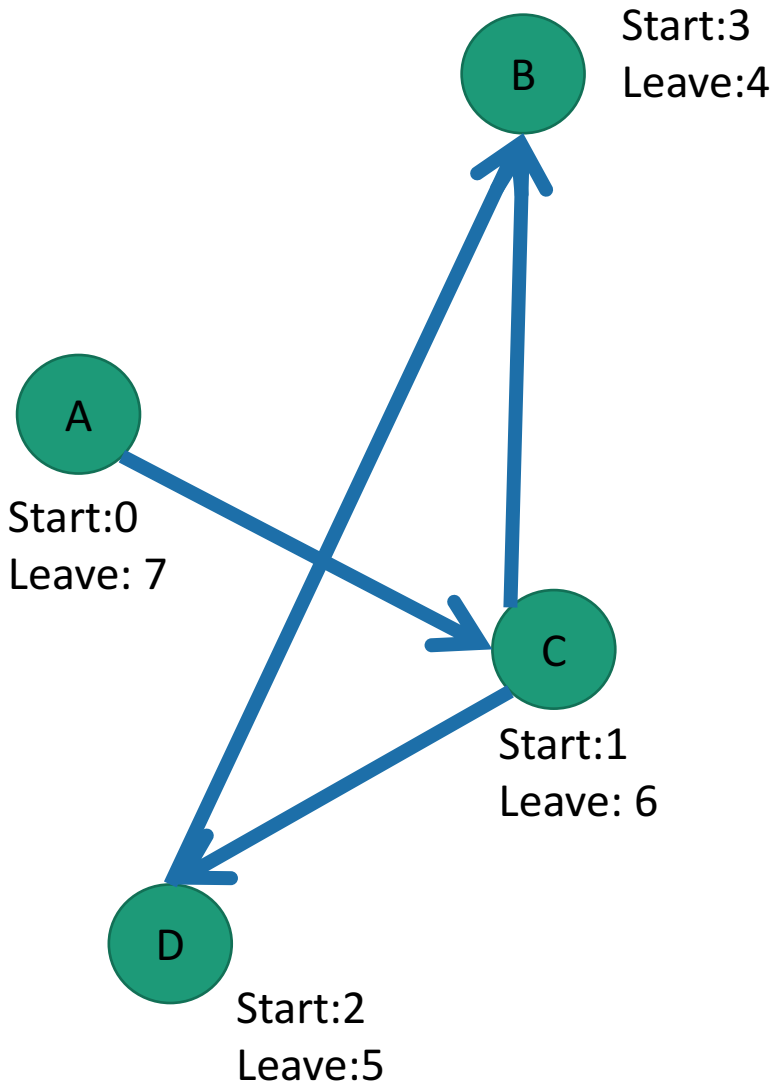
Example


B
Start:3
Leave:4
A
Start:0
C
Start:1
D
Start:2
Leave:5
Unvisited
In progress
All done
D
B

Example


B
Start:3
Leave:4
A
Start:0
C
Start:1
Leave: 6
D
Start:2
Leave:5
Unvisited
In progress
All done
C
D
B

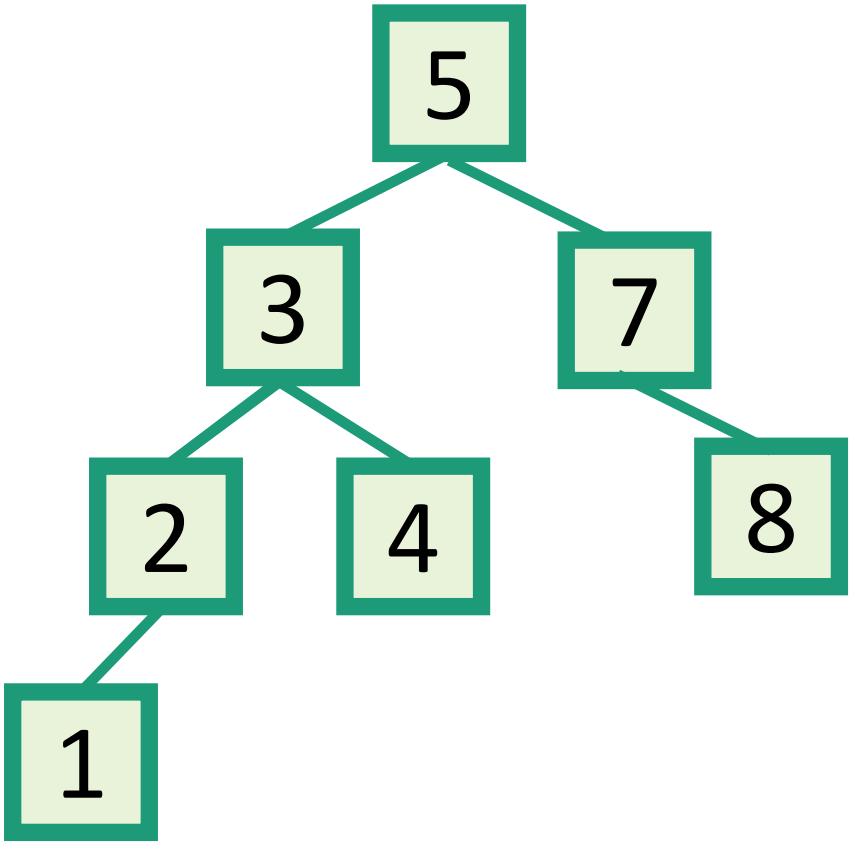
Example



Do them in this order:

Another use of DFS

- In-order enumeration of binary search trees



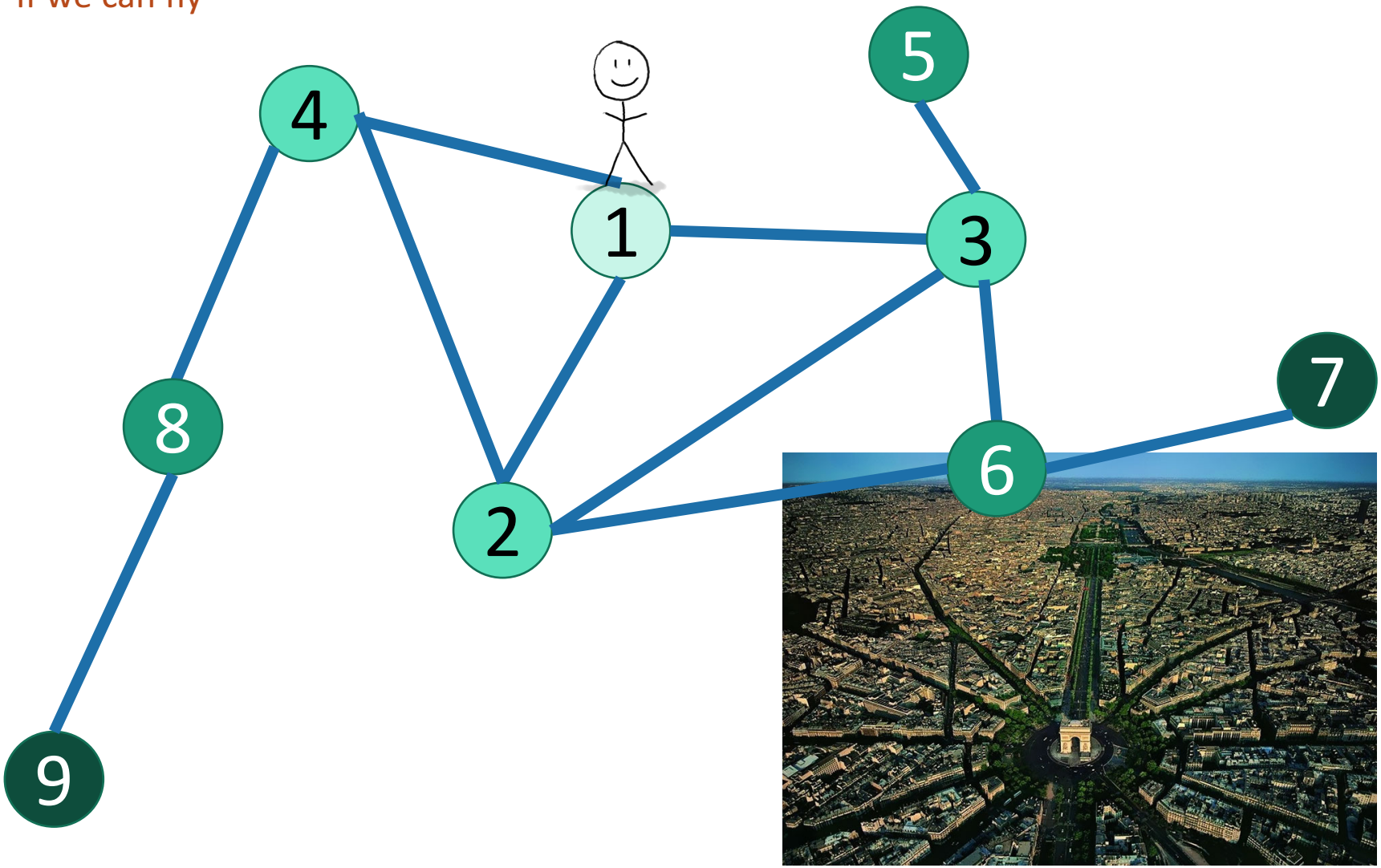
Given a binary search tree, output all the nodes **in order.**

Instead of outputting a node when you are done with it, output it when you are done with the left child and before you begin the right child.

Part 2: breadth-first search

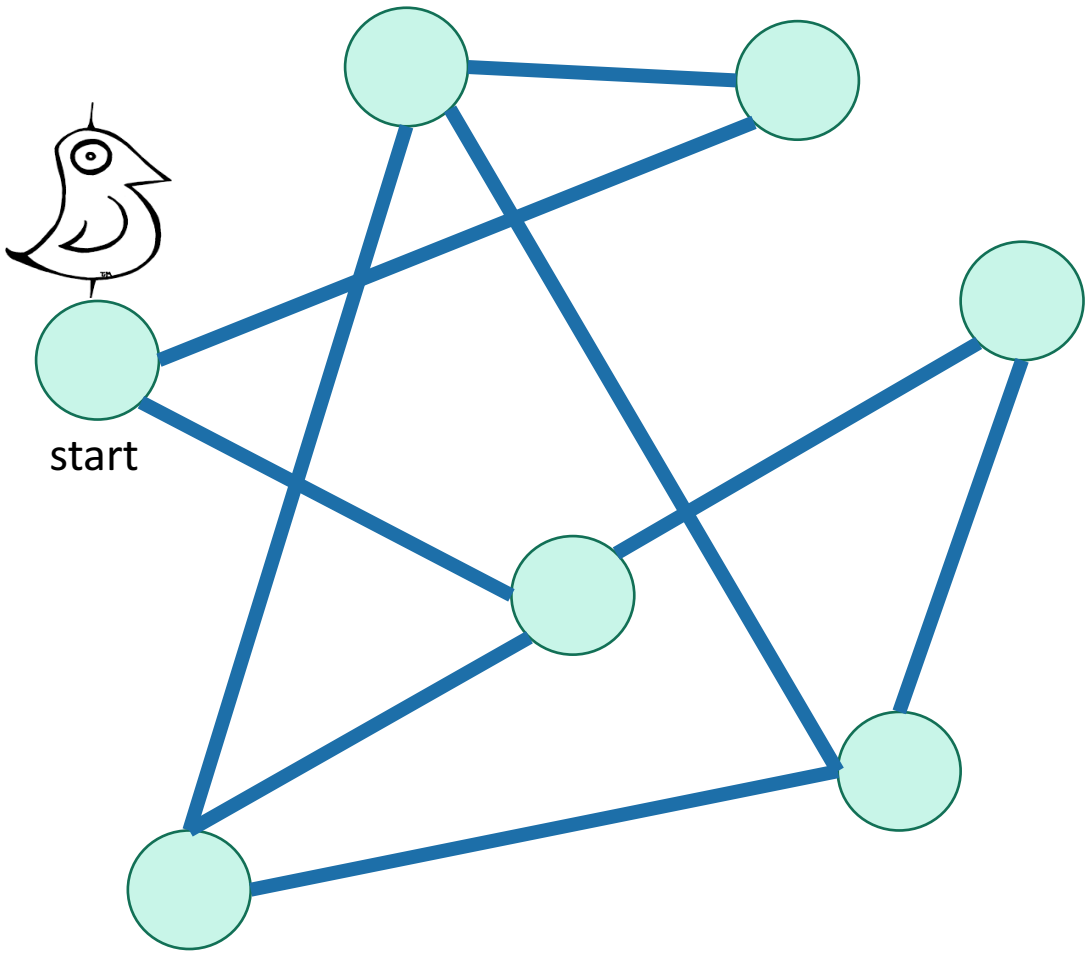
How do we explore a graph?

If we can fly



Breadth-First Search

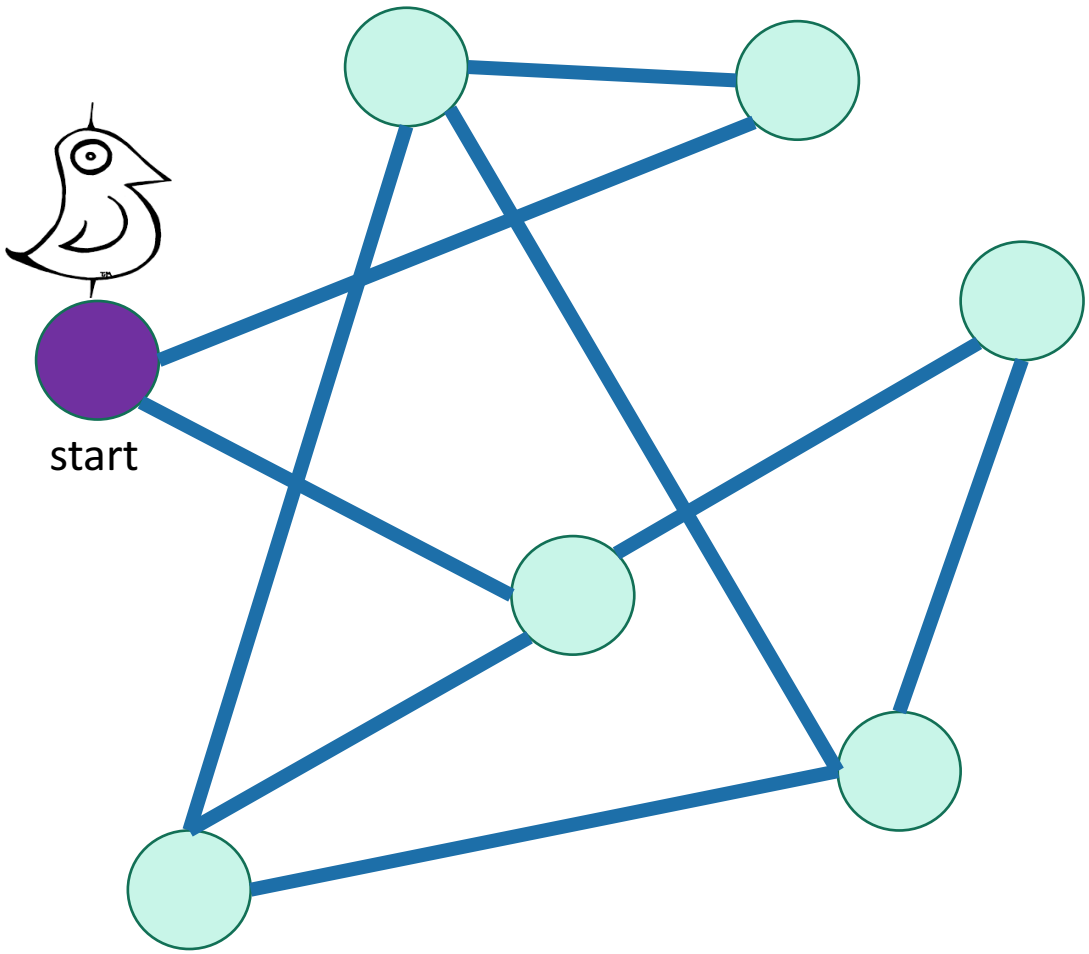
Exploring the world with a bird's-eye view



- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
- Can reach there in two steps
- Can reach there in three steps

Breadth-First Search

Exploring the world with a bird's-eye view



- Not been there yet
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- Can reach there in one step
- Can reach there in two steps
- Can reach there in three steps

Breadth-First Search

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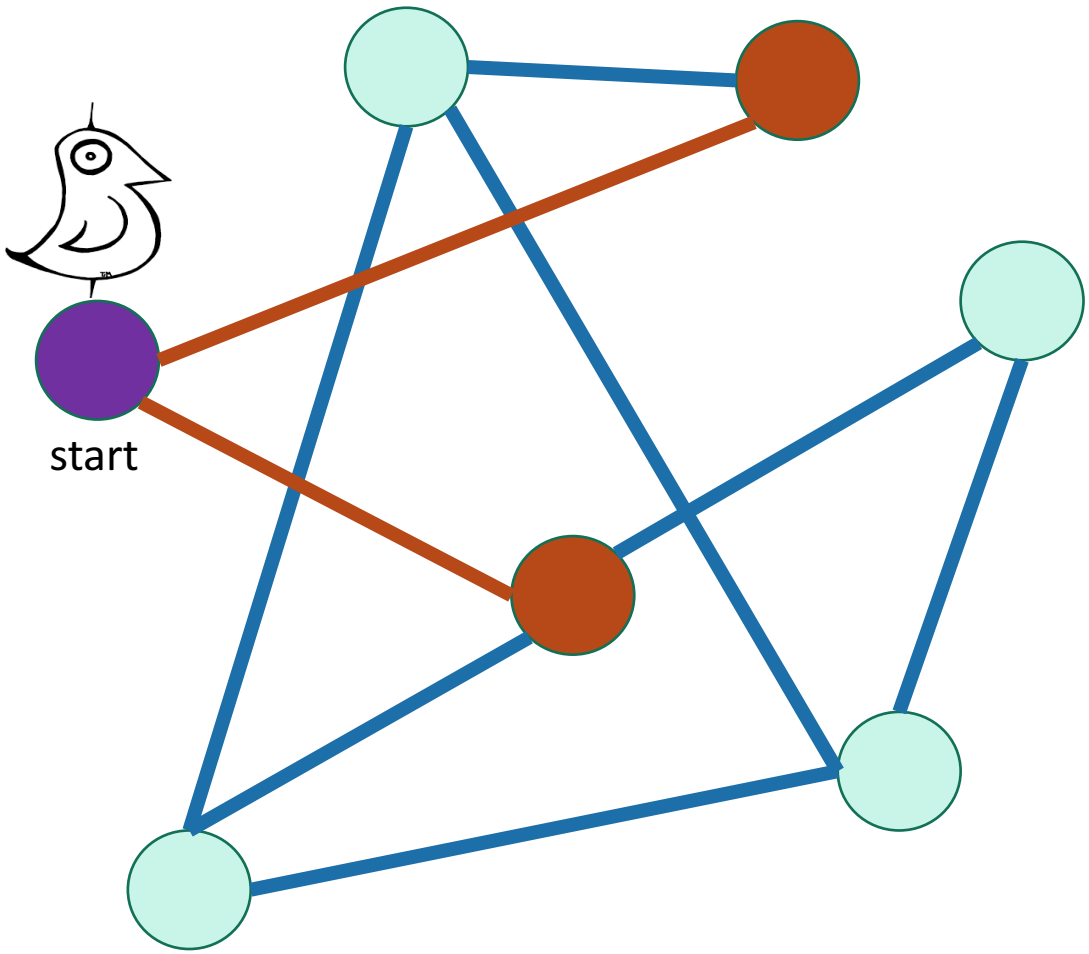

- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
- Can reach there in two steps
- Can reach there in three steps

Breadth-First Search

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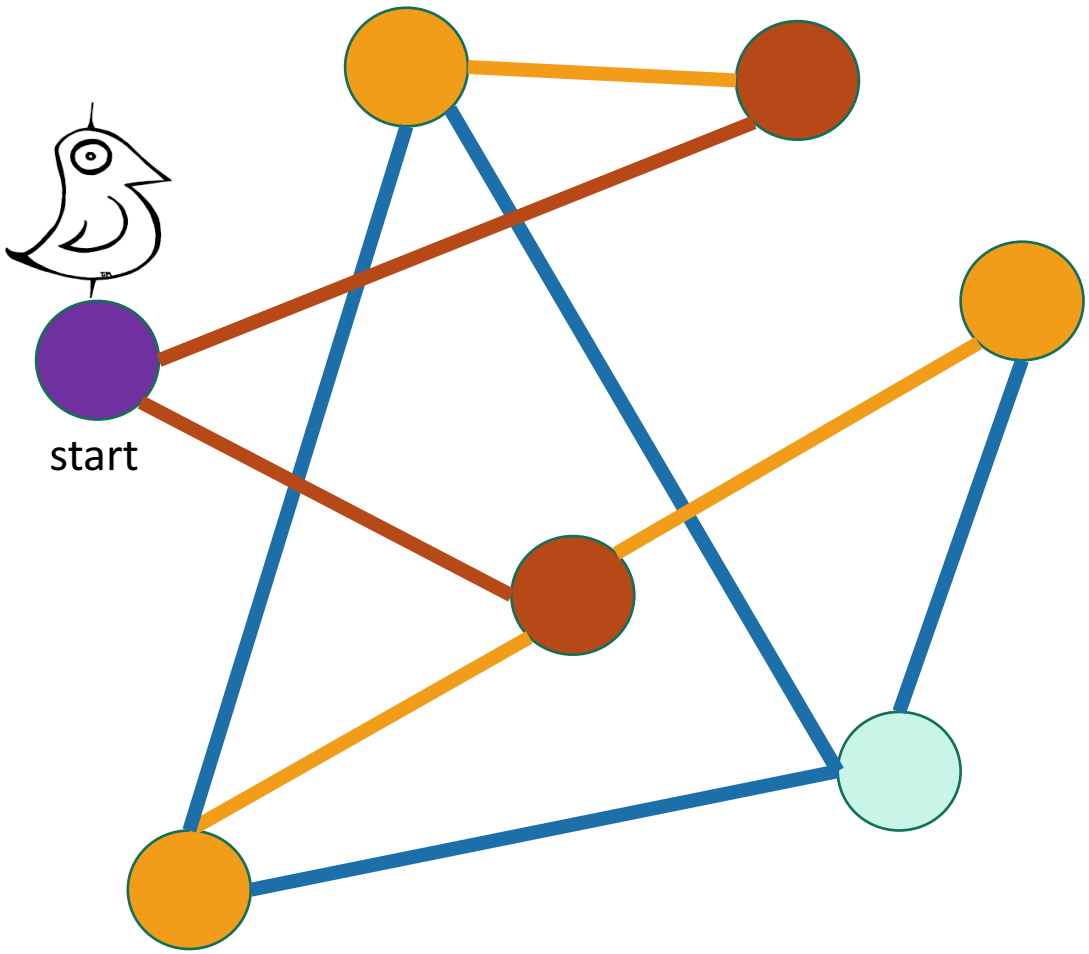

- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
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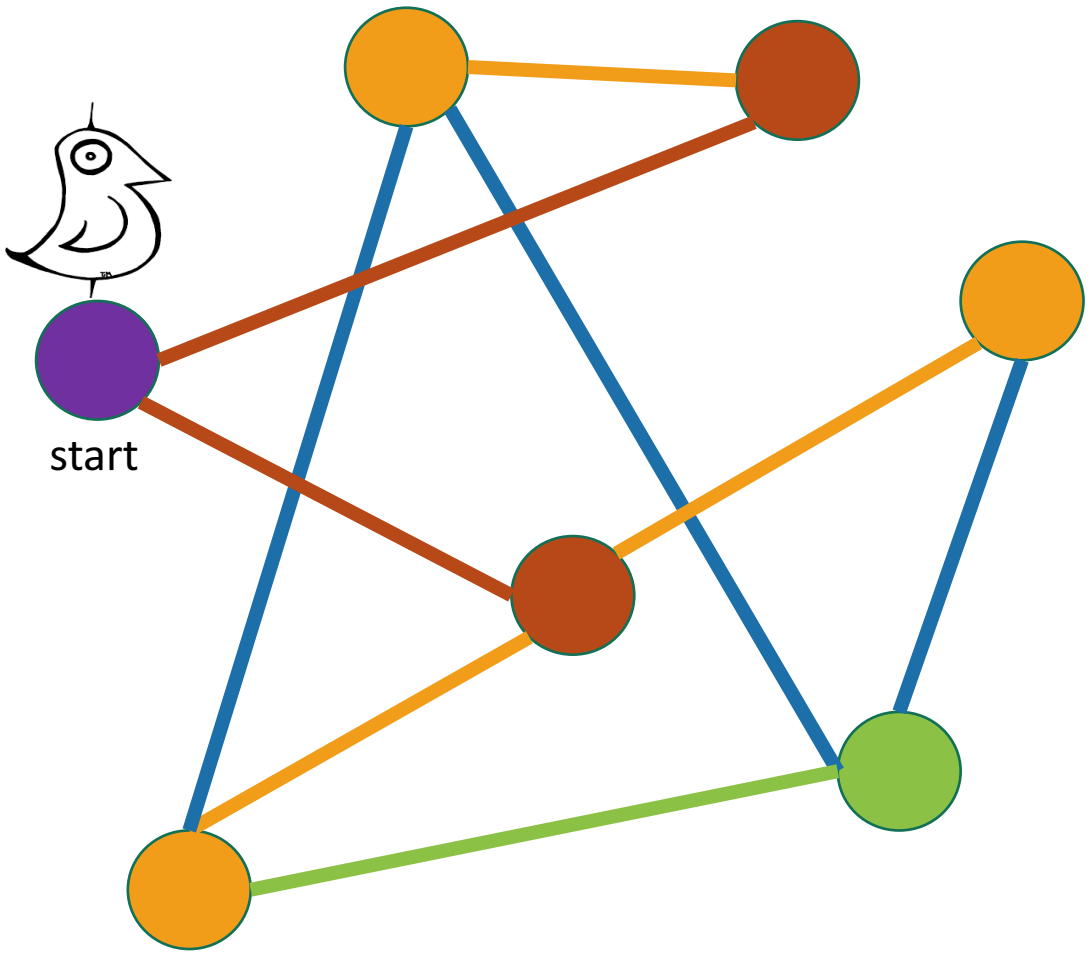
- Can reach there in three steps

Breadth-First Search

Exploring the world with a bird's-eye view



- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
- Can reach there in two steps
- Can reach there in three steps

World:

EXPLORED!

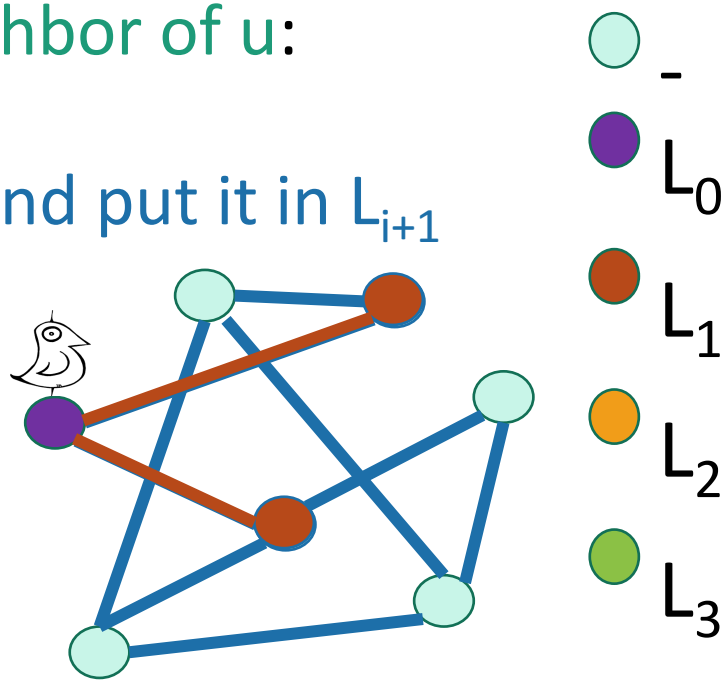
Same disclaimer as for DFS: you may have seen other ways to implement this, this will be convenient for us.

Breadth-First Search

Exploring the world with pseudocode

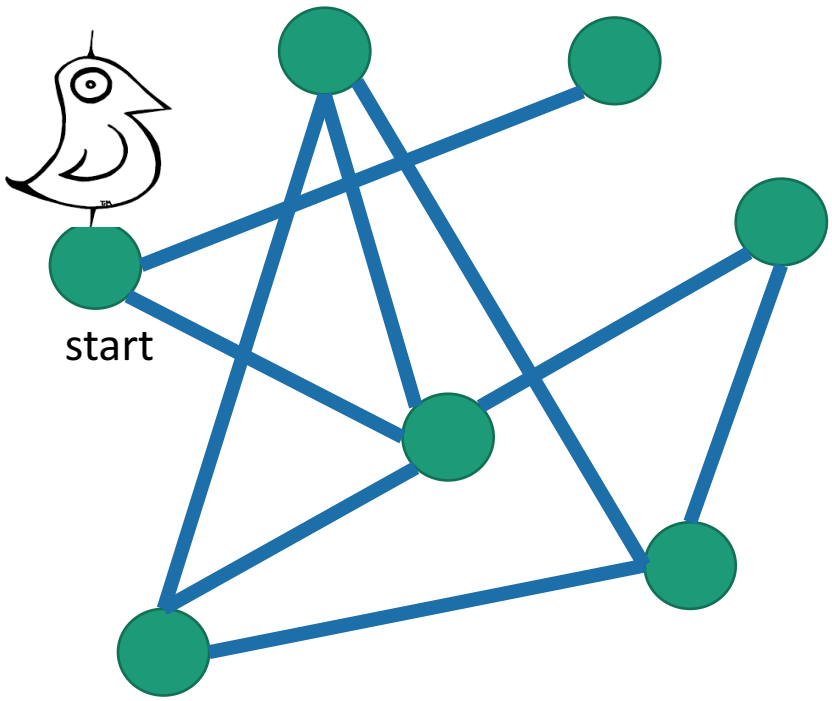
- Set $L_i = \{\}$ for $i=1,\ldots,n$
- $L_0 = \{w\}$, where w is the start node
- **For** $i = 0, \ldots, n-1$:
  - **For** u in L_i:
    - **For** each v which is a neighbor of u:
      - **If** u isn't yet visited:
        - mark u as visited, and put it in L_{i+1}

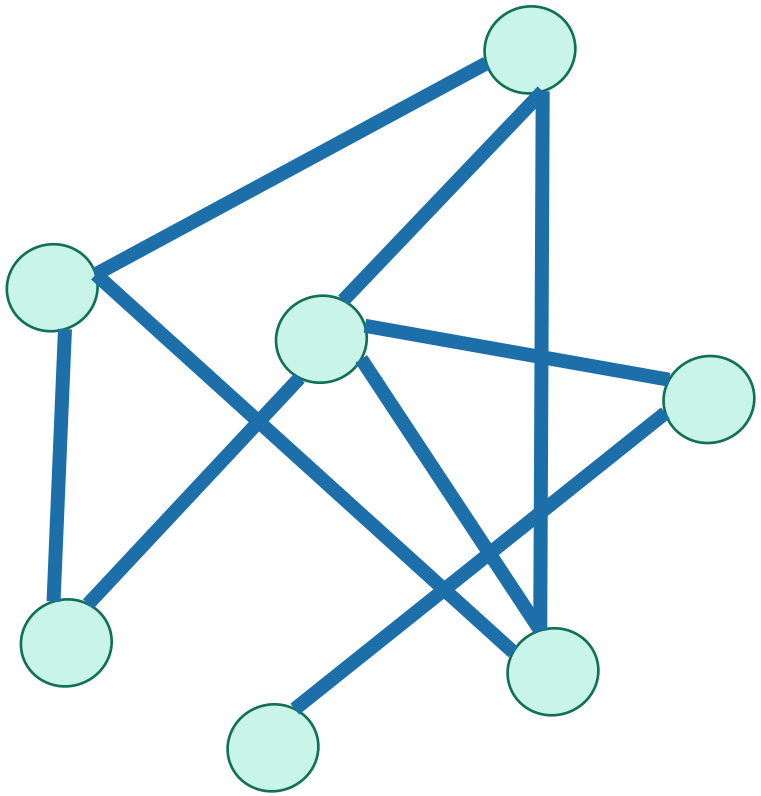
L_i is the set of nodes we can reach in i steps from w



Go through all the nodes in L_i and add their unvisited neighbors to L_{i+1}

BFS also finds all the nodes reachable from the starting point





It is also a good way to find all the **connected components.**

Running time

To explore the whole thing

- Explore the connected components one-by-one.
- Same argument as DFS: running time is

$$O(n + m)$$

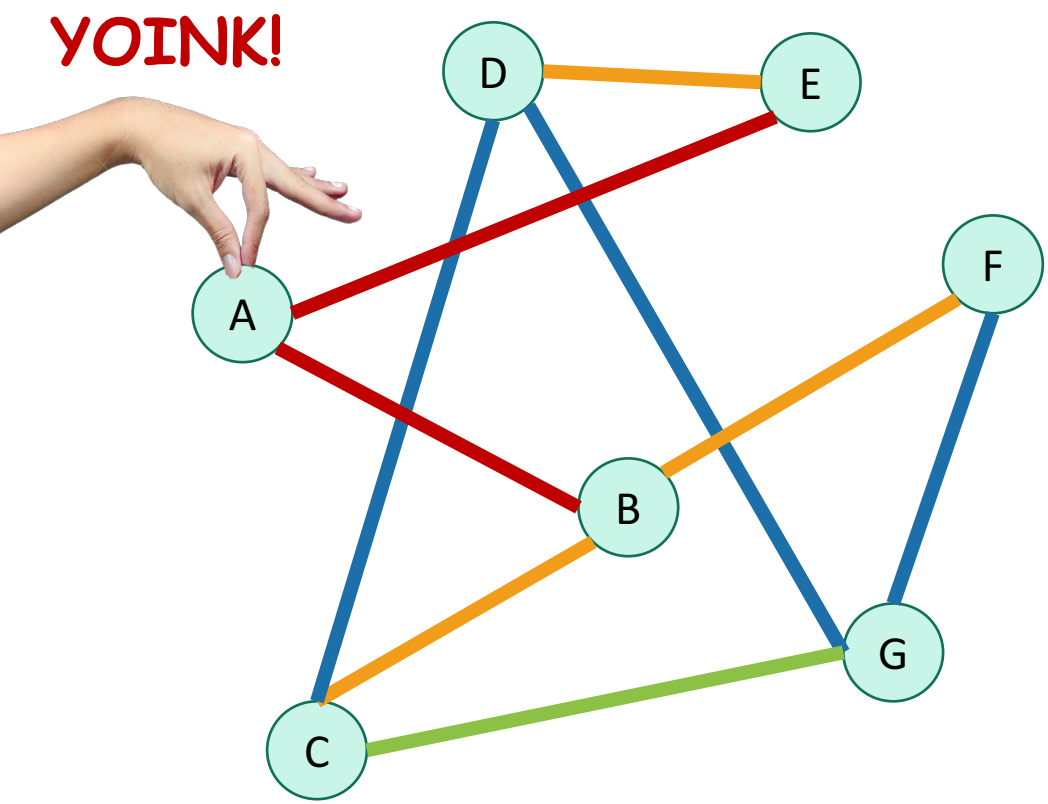
Verify these!

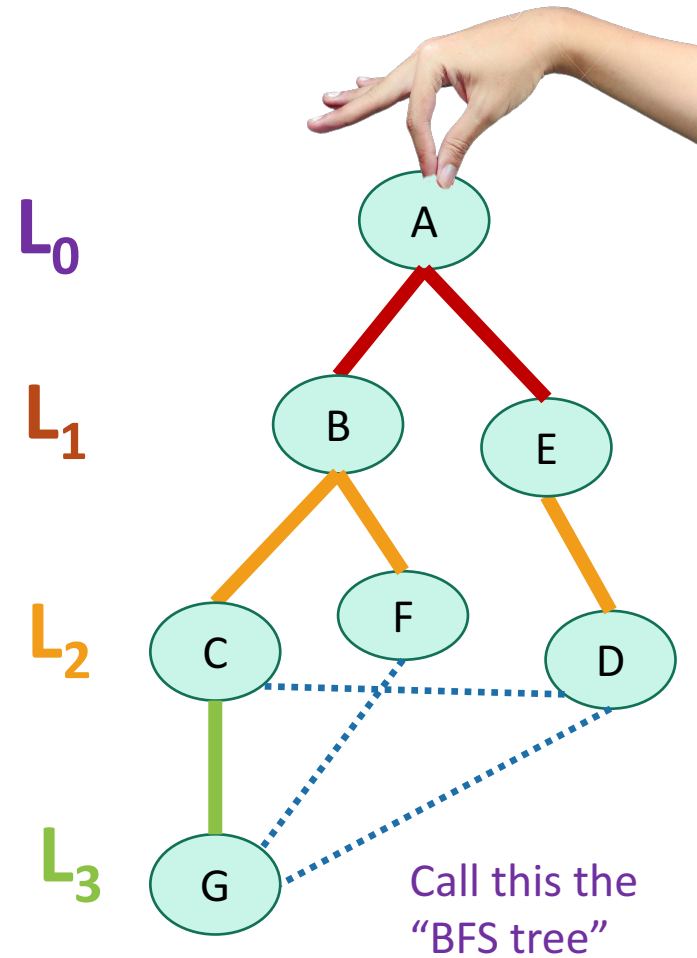
Ollie the over-achieving ostrich

- Like DFS, BFS also works fine on directed graphs.

Why is it called breadth-first?

- We are implicitly building a tree:

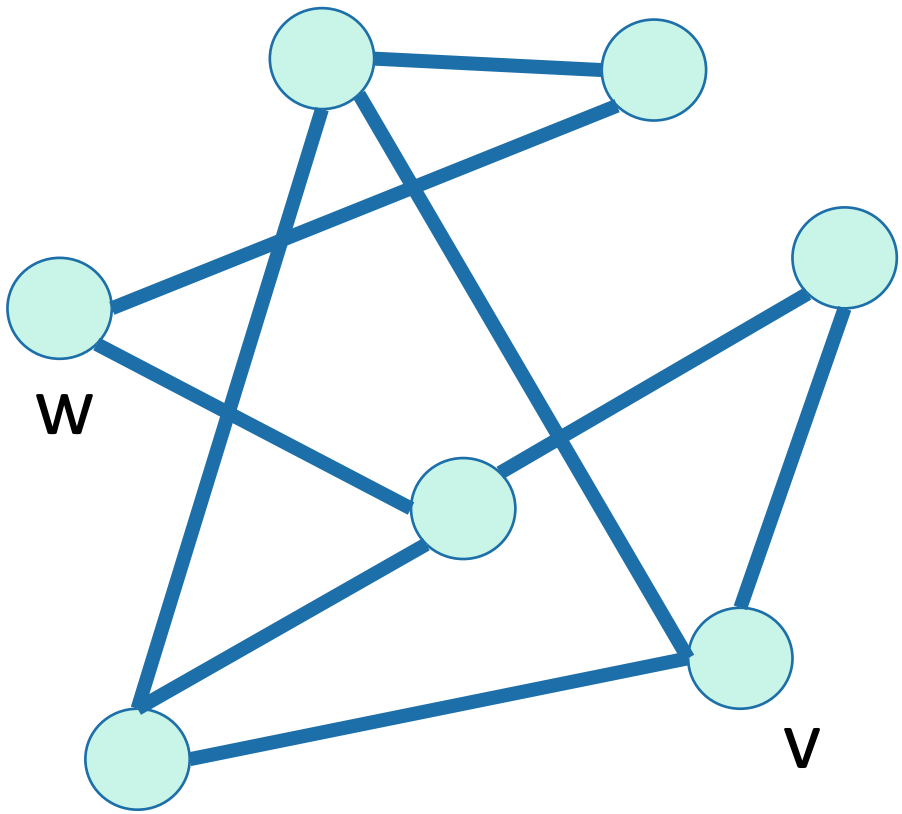




- And first we go as broadly as we can.

Application: shortest path

- How long is the shortest path between w and v?



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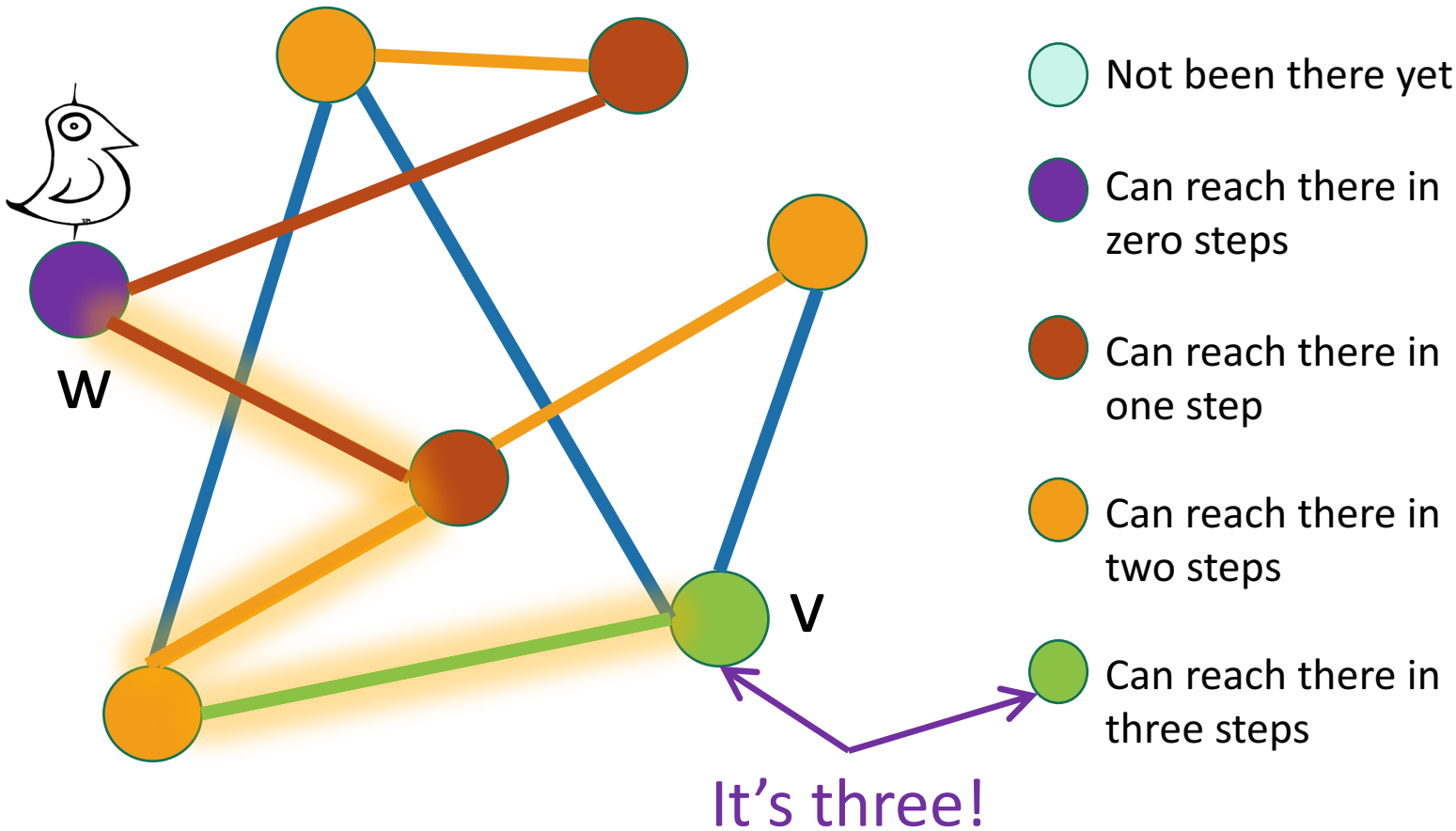


To find the distance between w and all other vertices v

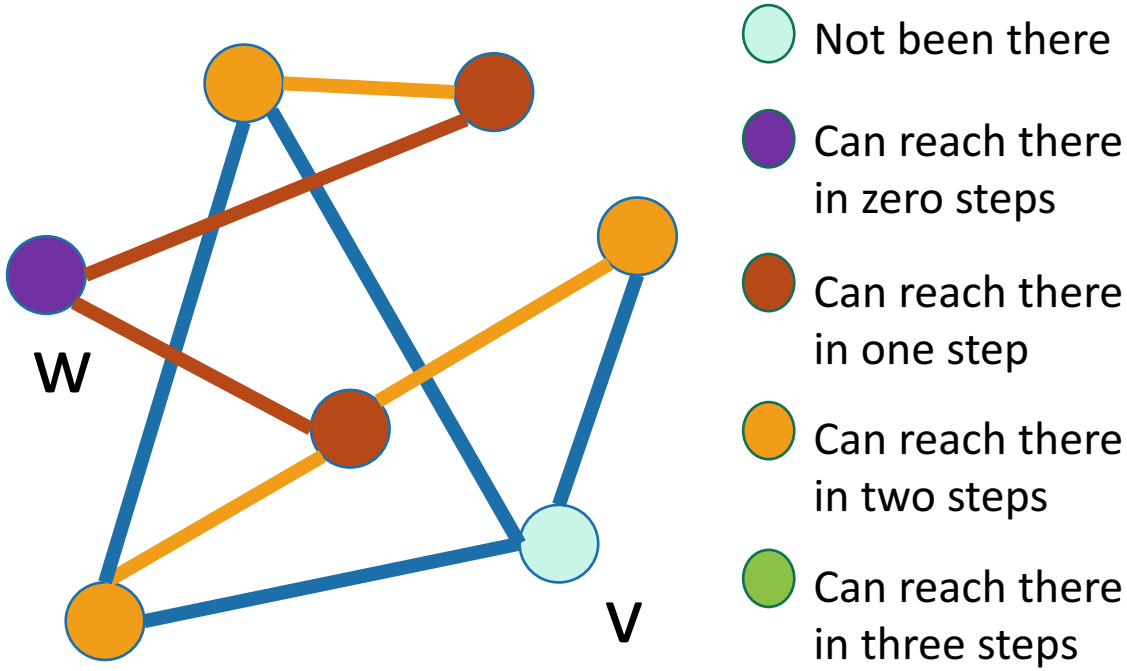
The distance between two vertices is the length of the shortest path between them.

- Do a DFS starting at w
- For all v in L_i (the i'th level of the BFS tree)
  - The shortest path between w and v has length i
  - A shortest path between w and v is given by the path in the BFS tree.
- If we never found v, the distance is infinite.

Proof idea

Just the idea…see CLRS for details!

- Suppose by **induction** it's true for vertices in L_0, L_1, L_2
  - For all $i < 3$, the vertices in L_i have distance i from v.
- **Want to show**: it's true for vertices of distance 3 also.
  - aka, the shortest path between w and v has length 3.
- **Well, it has distance at most 3**
  - Since we just found a path of length 3
- **And it has distance at least 3**
  - Since if it had distance $i < 3$, it would have been in L_i.



What did we just learn?

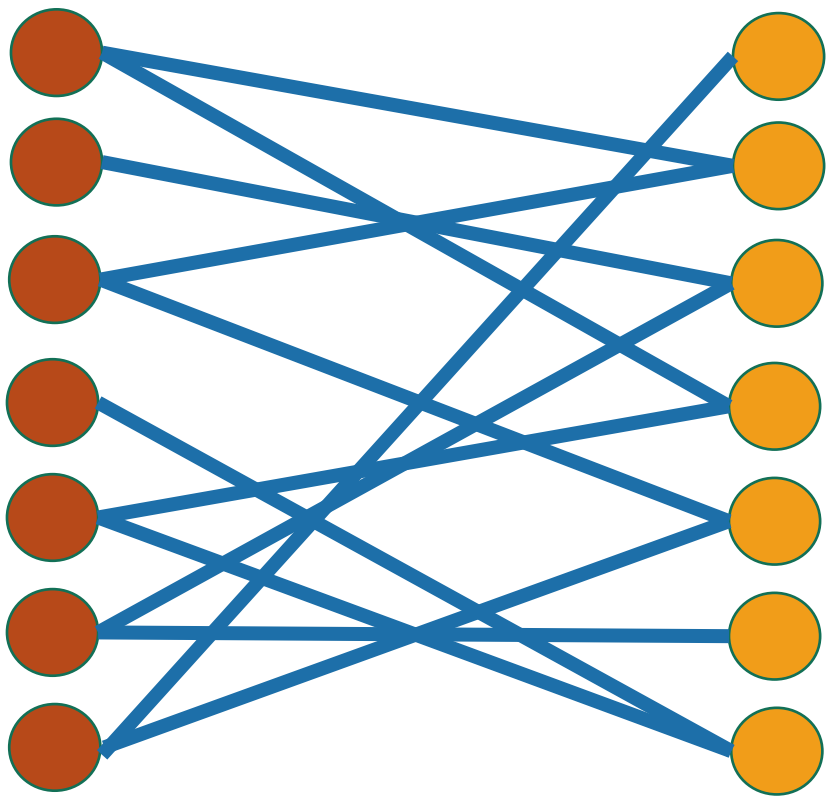
- The BFS tree is useful for computing distances between pairs of vertices.
- We can find the shortest path between u and v in time O(m).

The BSF tree is also helpful for:

- Testing if a graph is bipartite or not.

Application: testing if a graph is bipartite

- Bipartite means it looks like this:



Can color the vertices red and orange so that there are no edges between any same-colored vertices

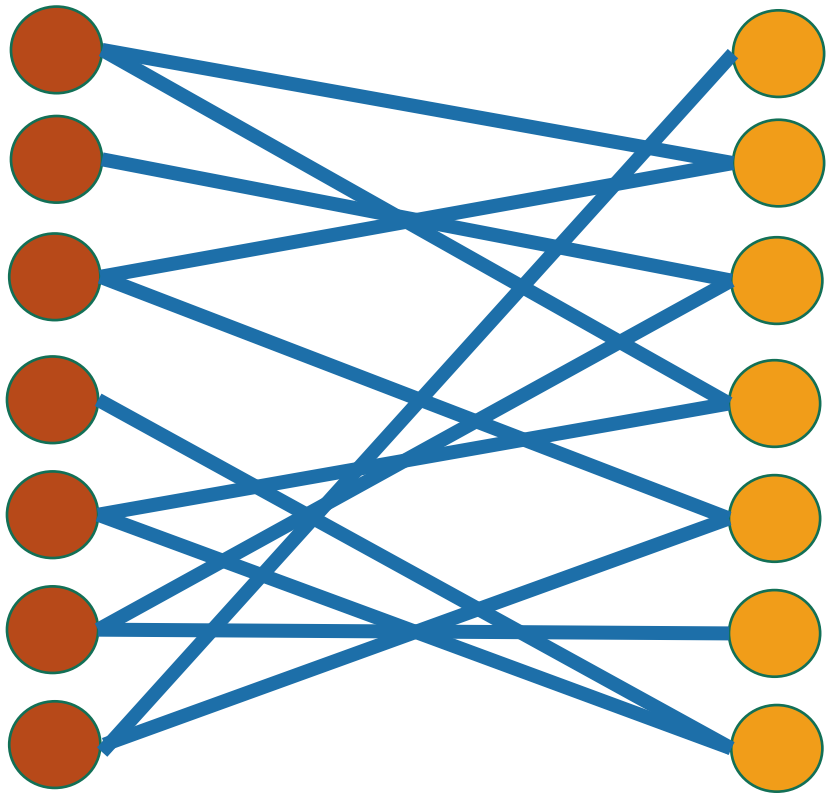
Example:

are students

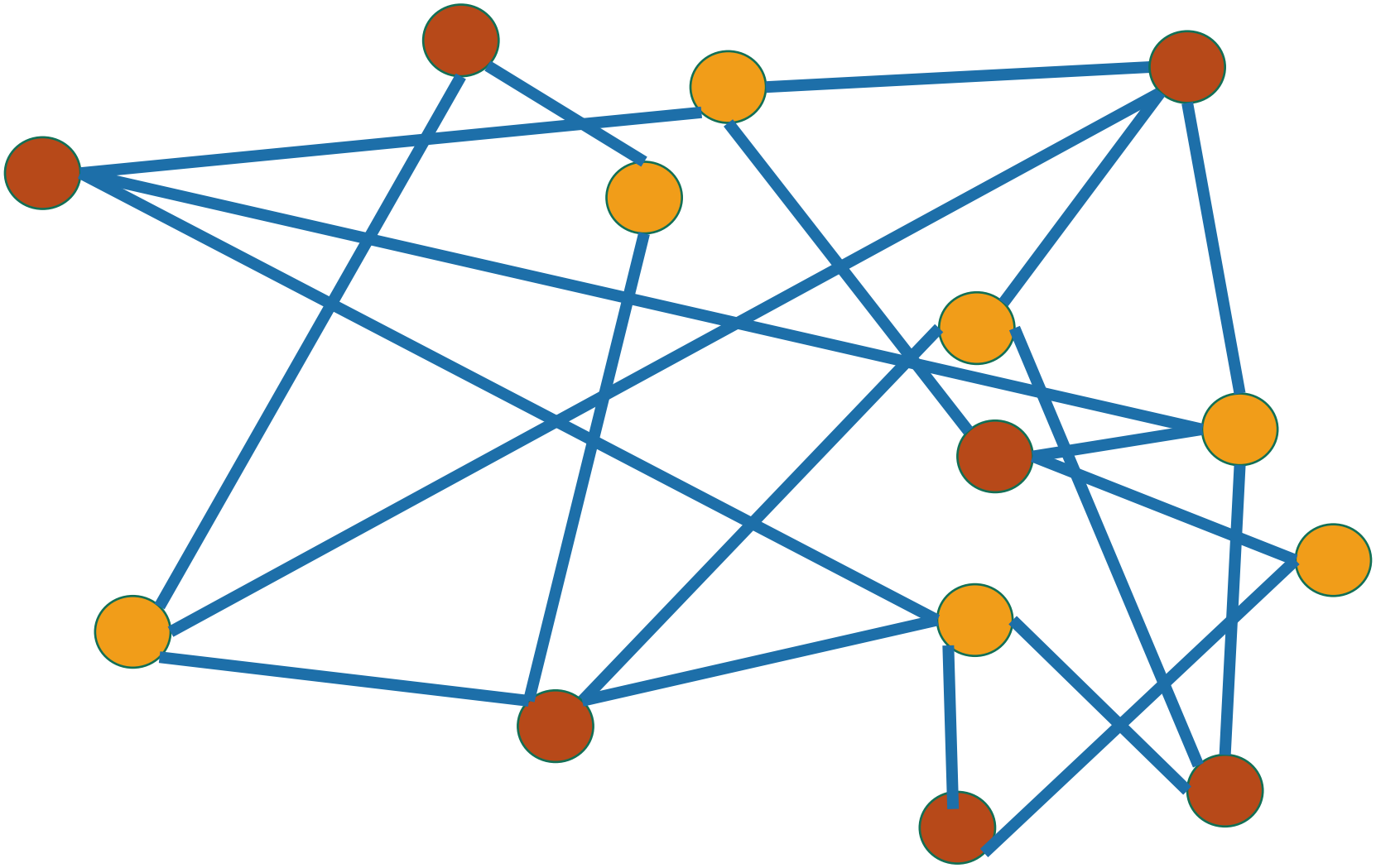
are classes

if the student is enrolled in the class

Is this graph bipartite?



How about this one?



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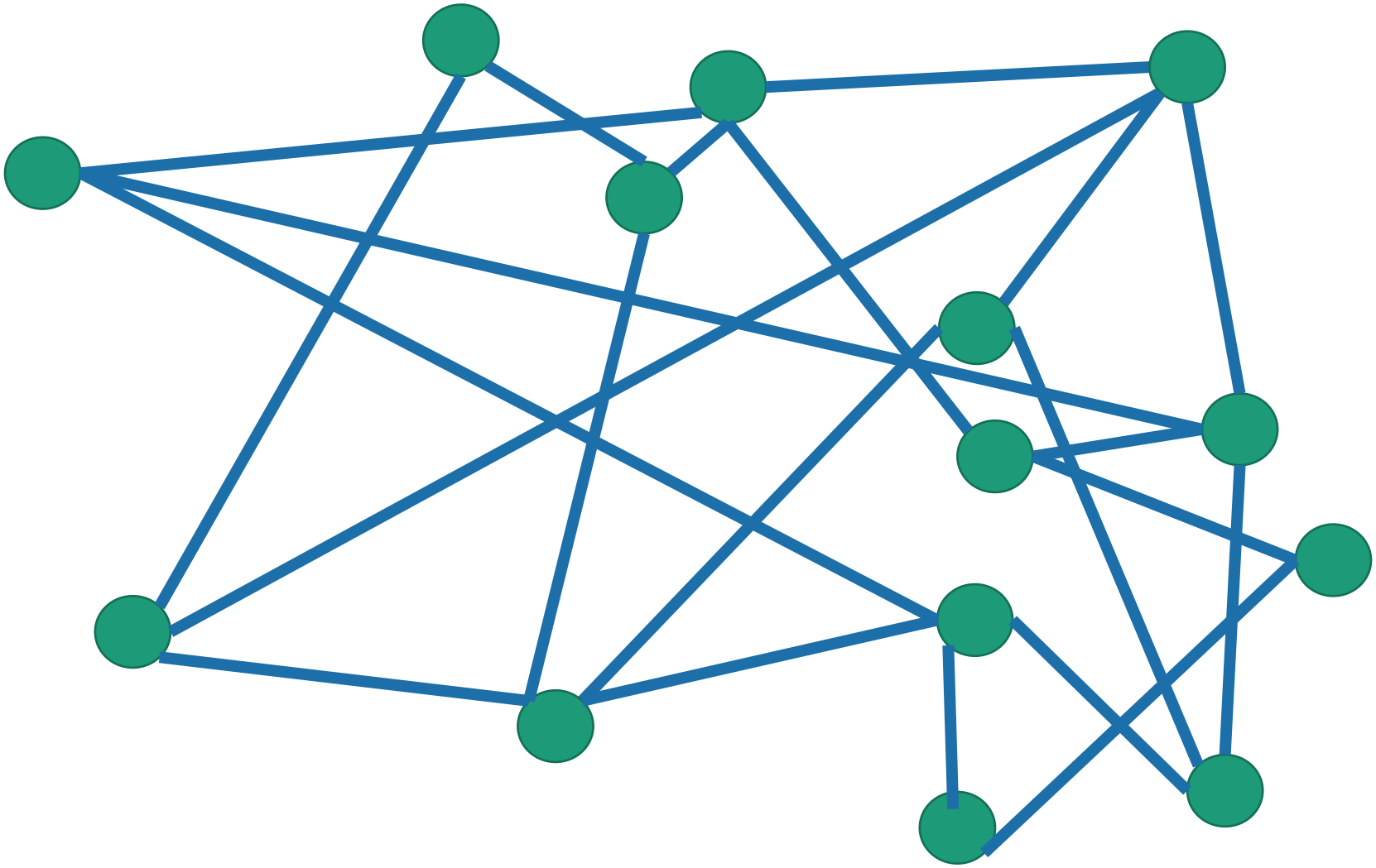
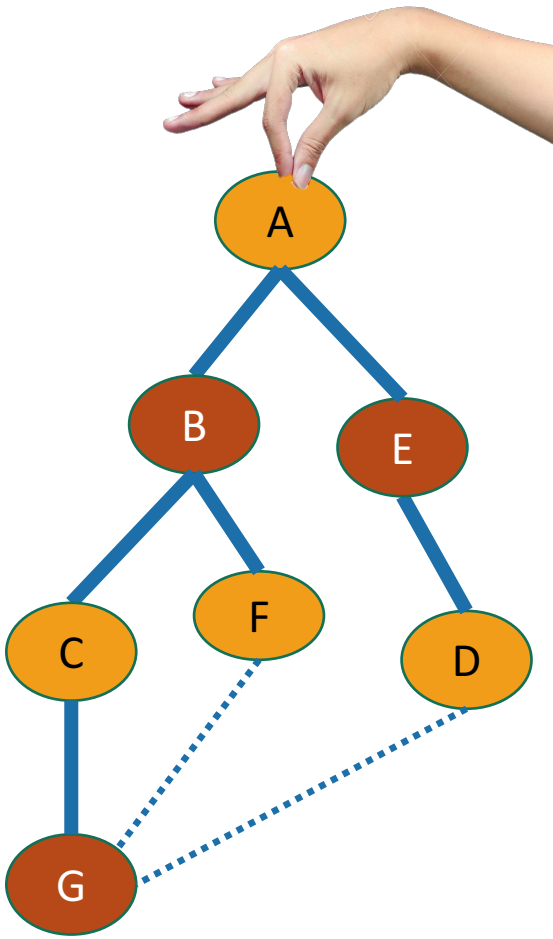


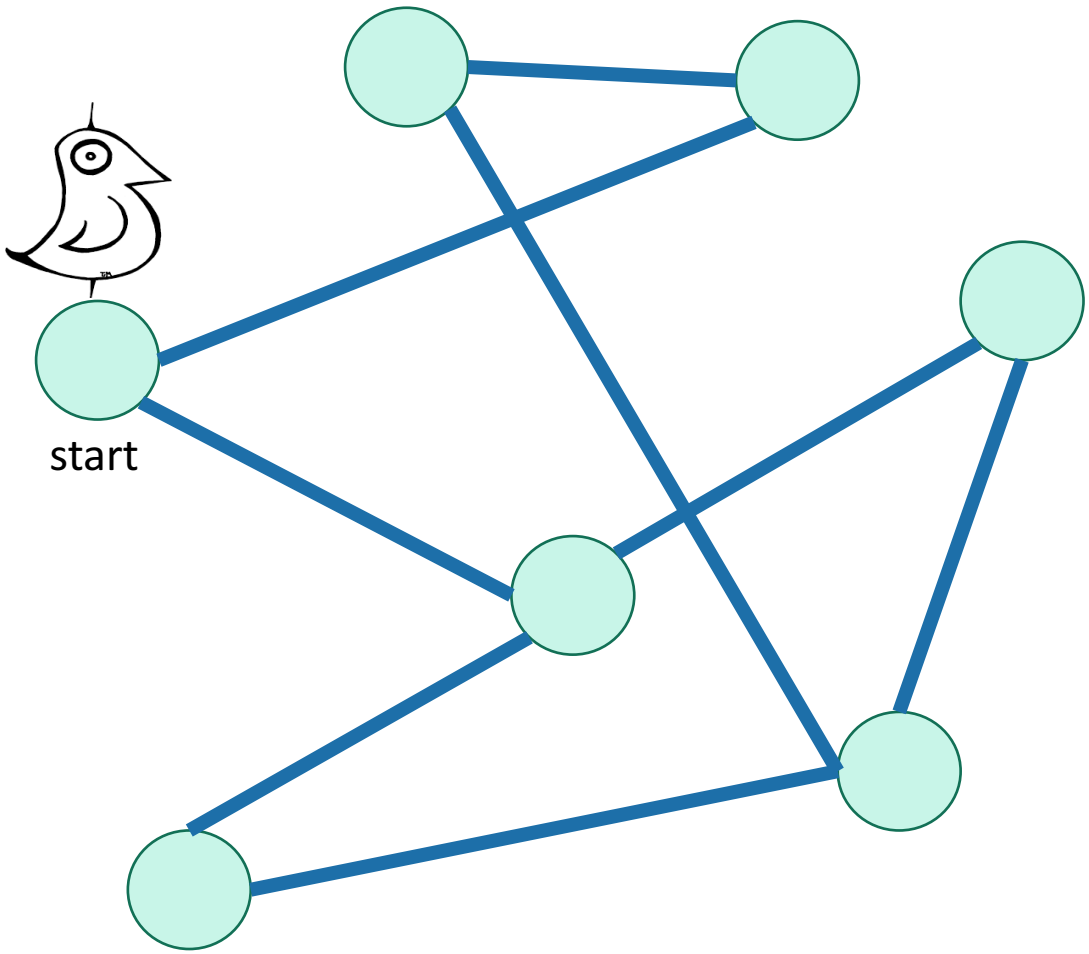
Solution using BFS

- Color the levels of the BFS tree in alternating colors.
- If you ever color a node so that you never color two connected nodes the same, then it is bipartite.
- Otherwise, it's not.



Breadth-First Search

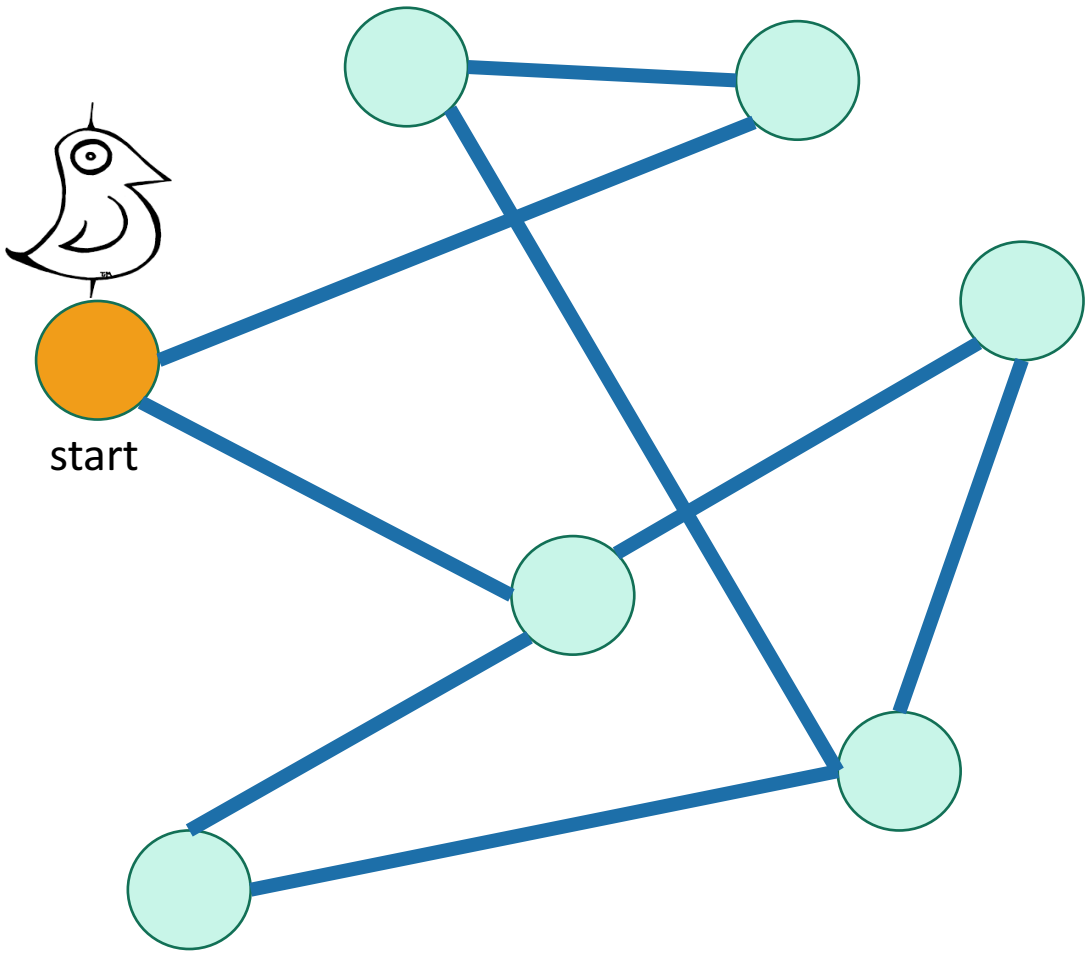
For testing bipartite-ness



- Not been there yet
- Can reach there in zero steps
- Can reach there in one step
- Can reach there in two steps
- Can reach there in three steps

Breadth-First Search

For testing bipartite-ness



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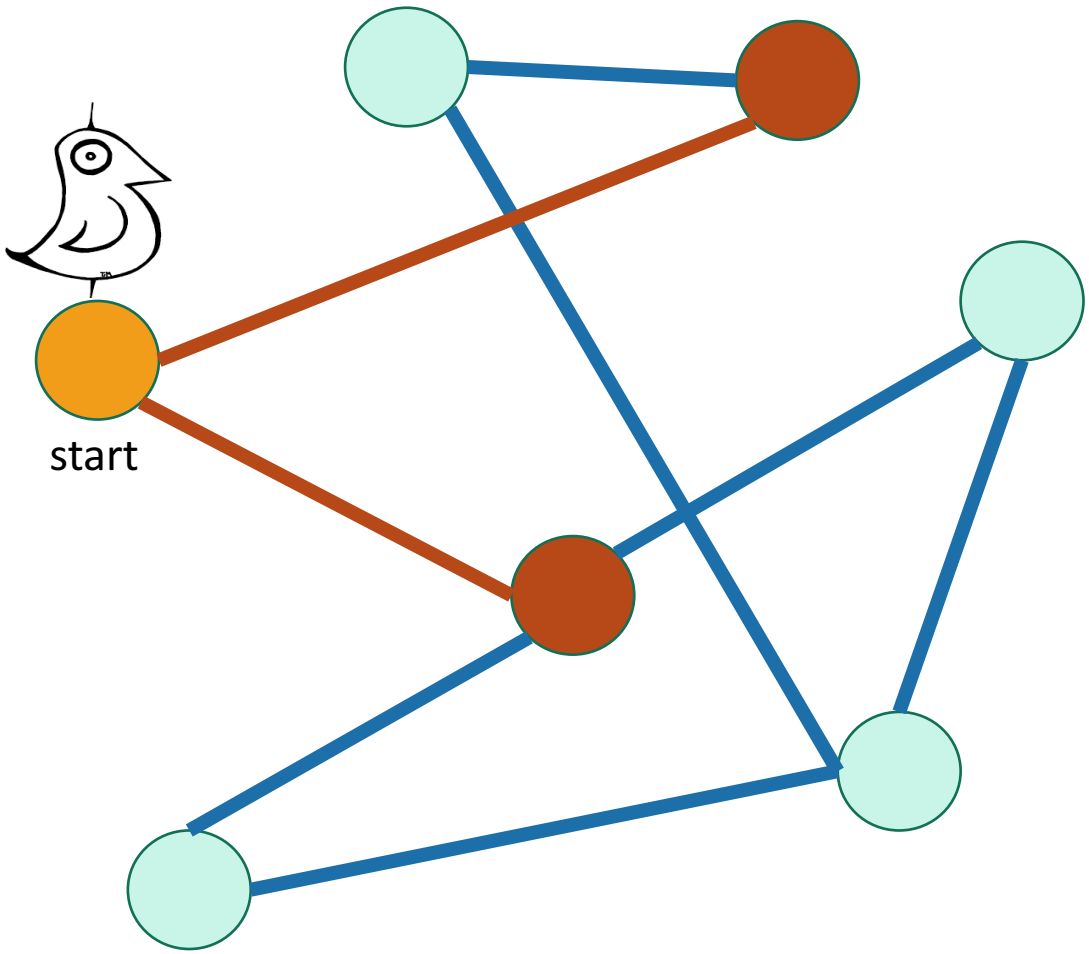

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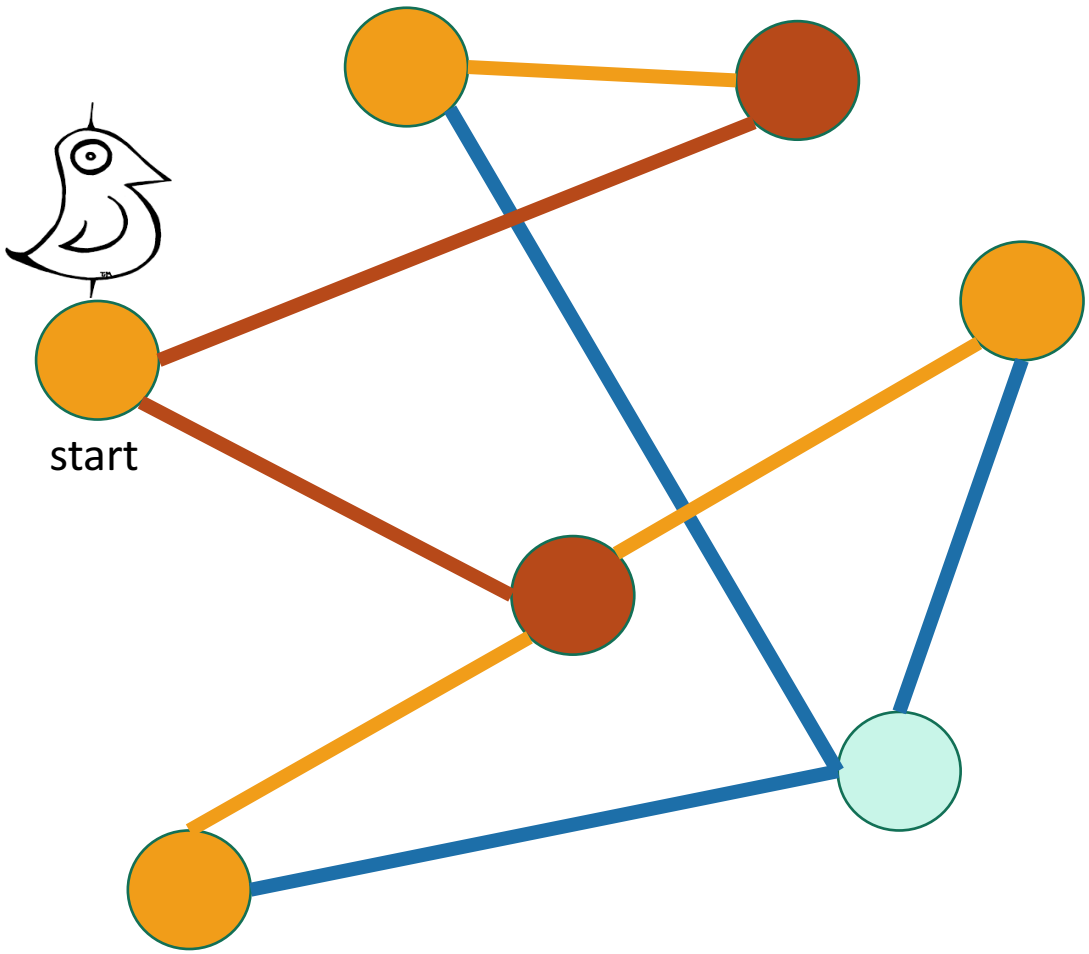

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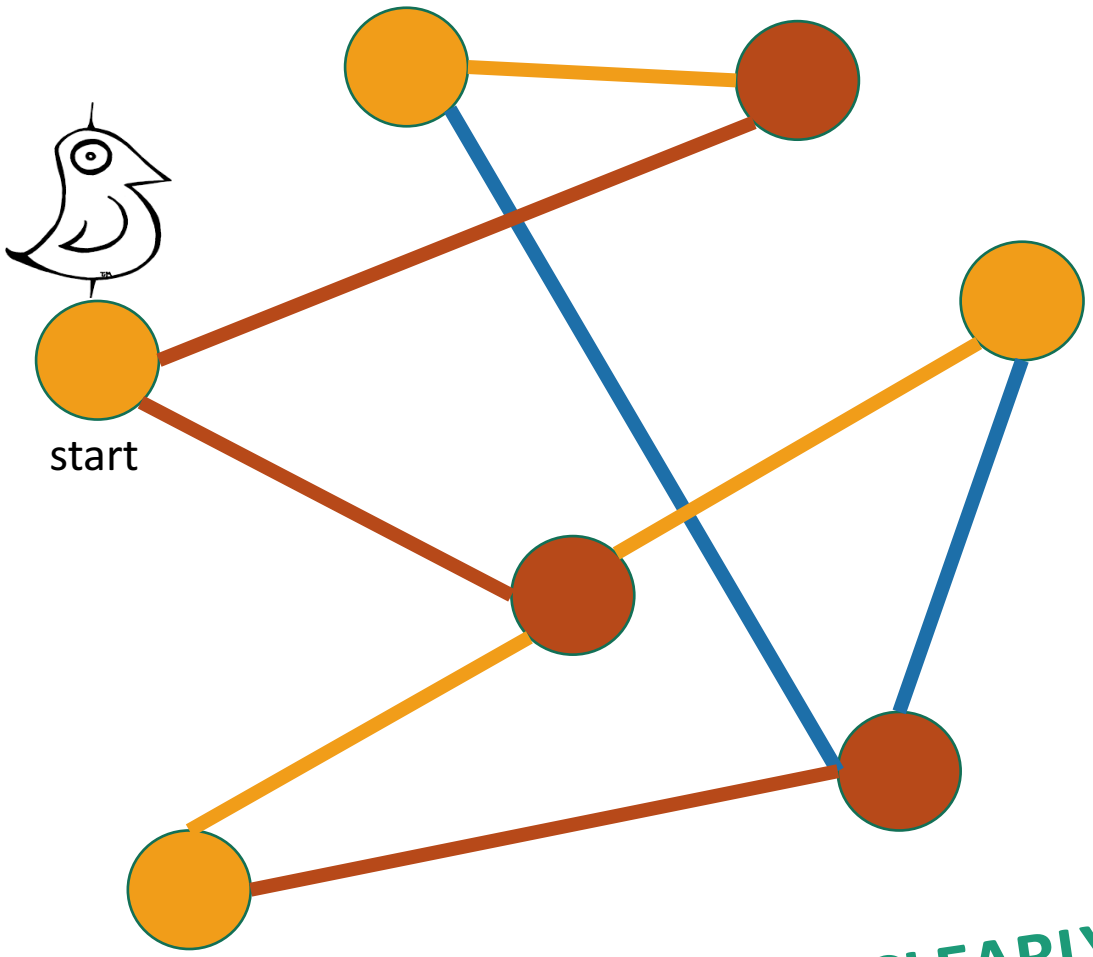

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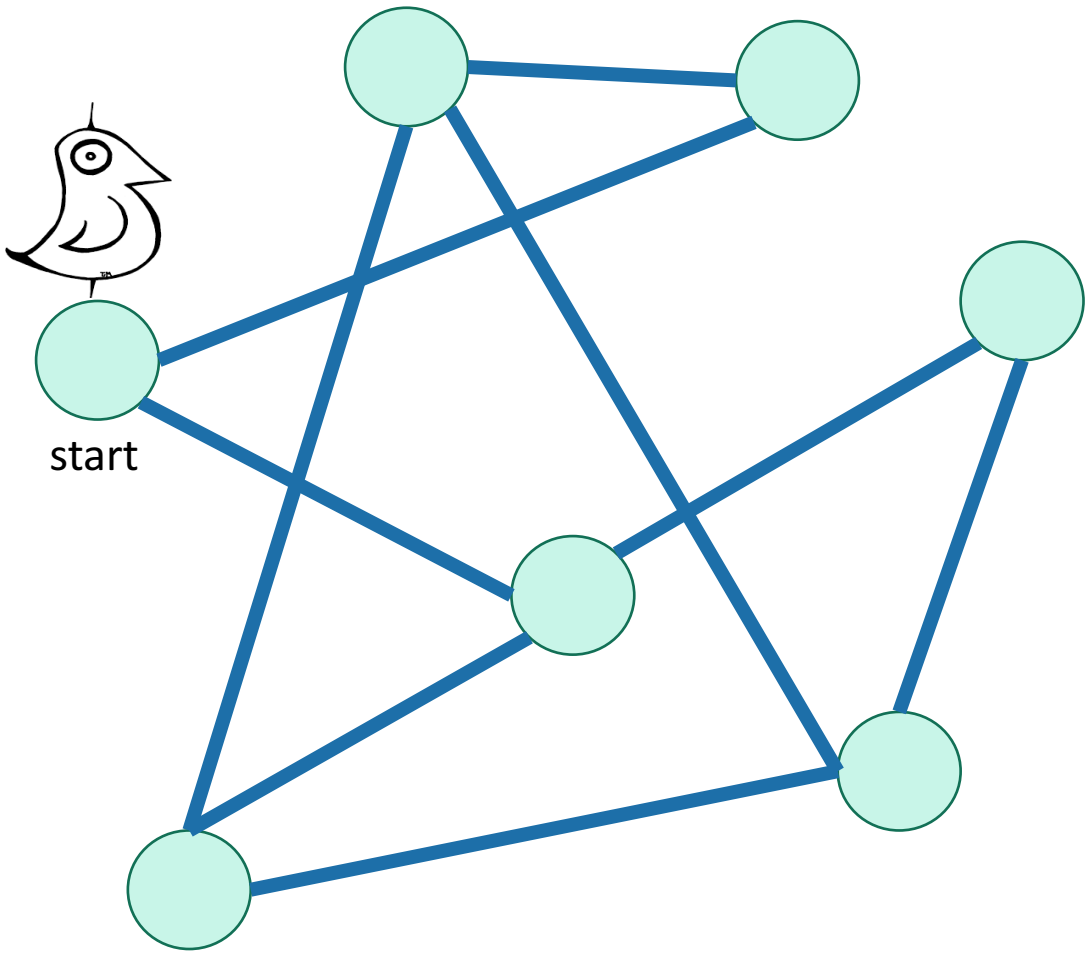
CLEARLY BIPARTITE!

Breadth-First Search

For testing bipartite-ness



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Breadth-First Search

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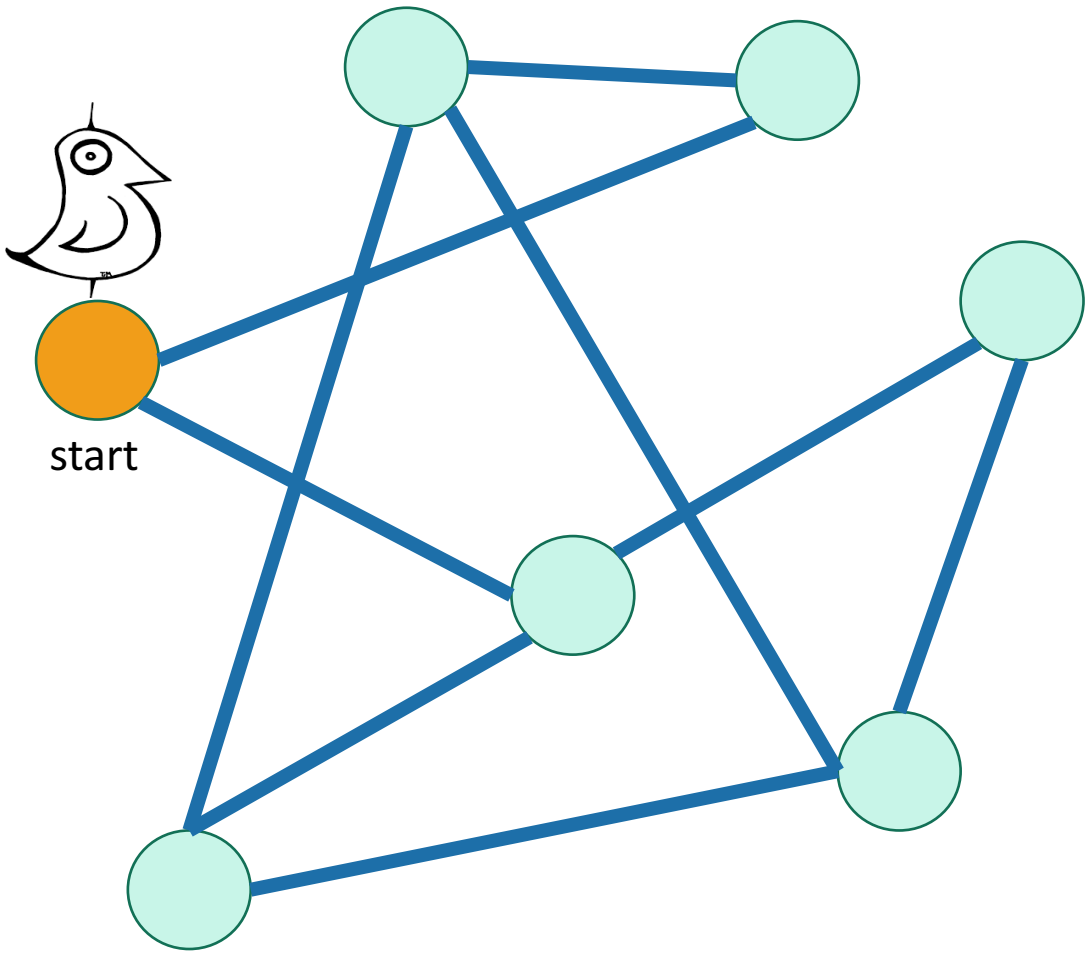

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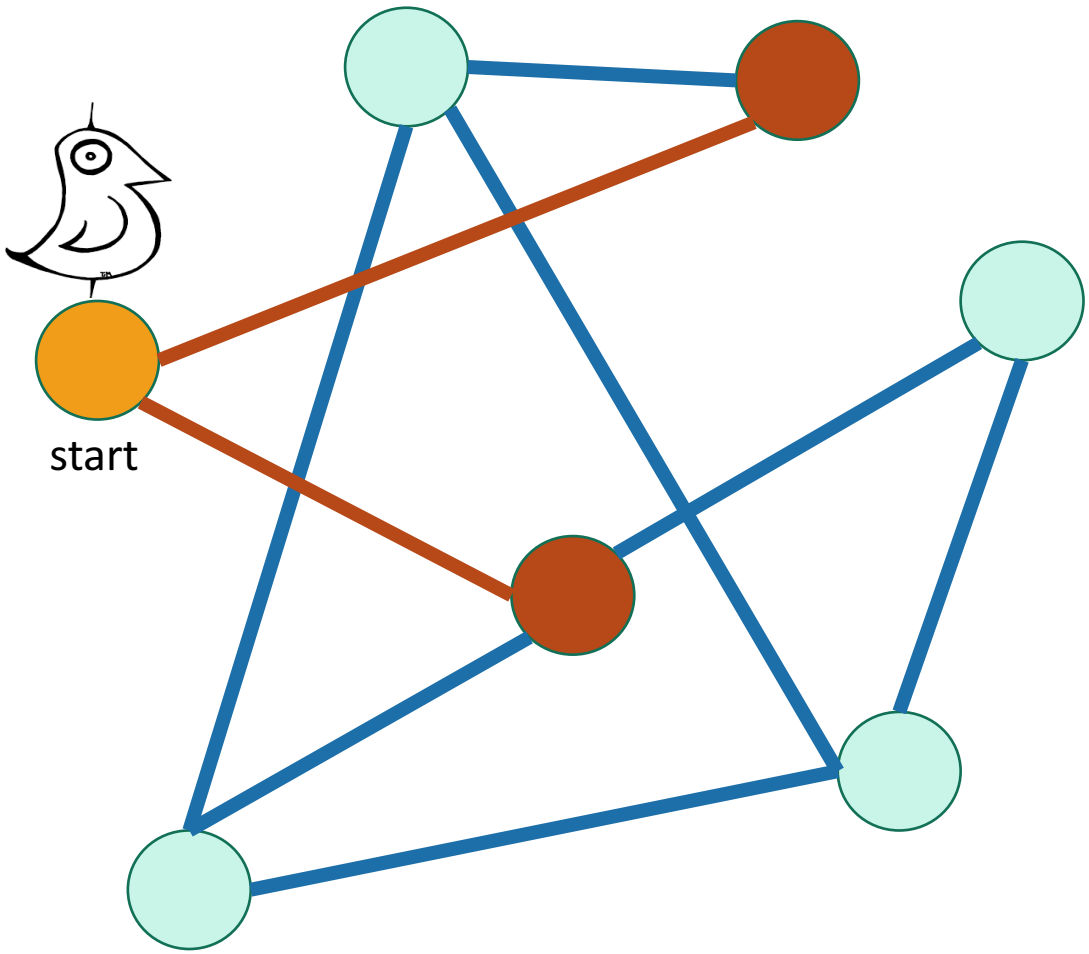

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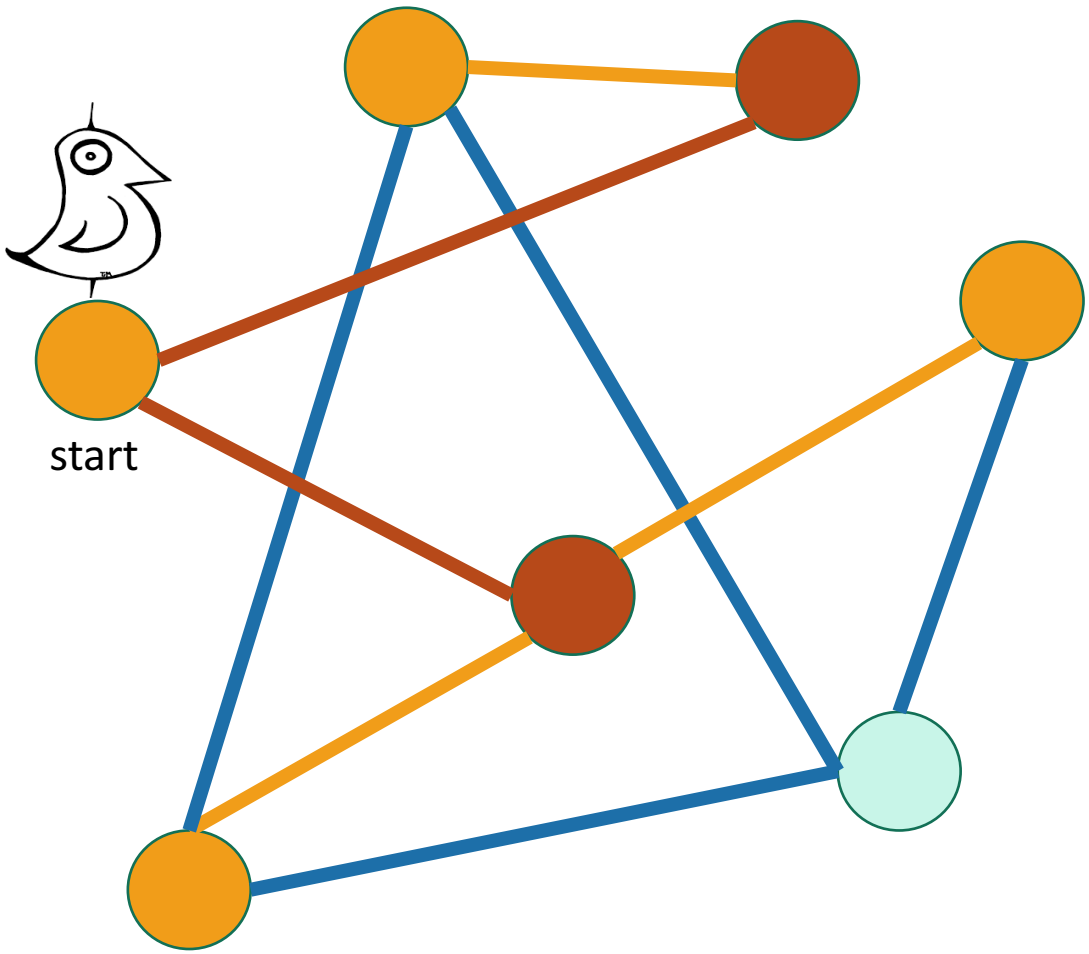

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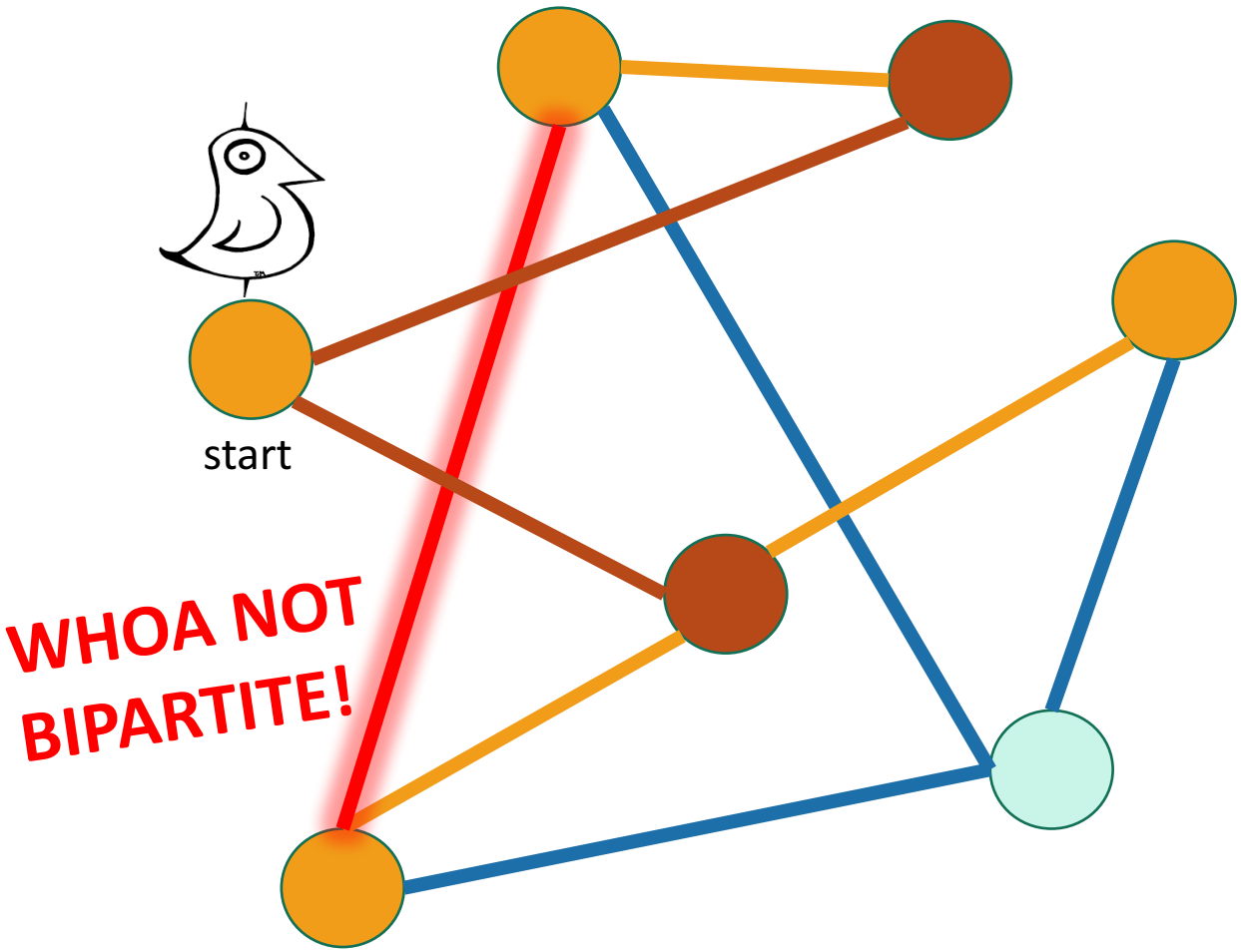

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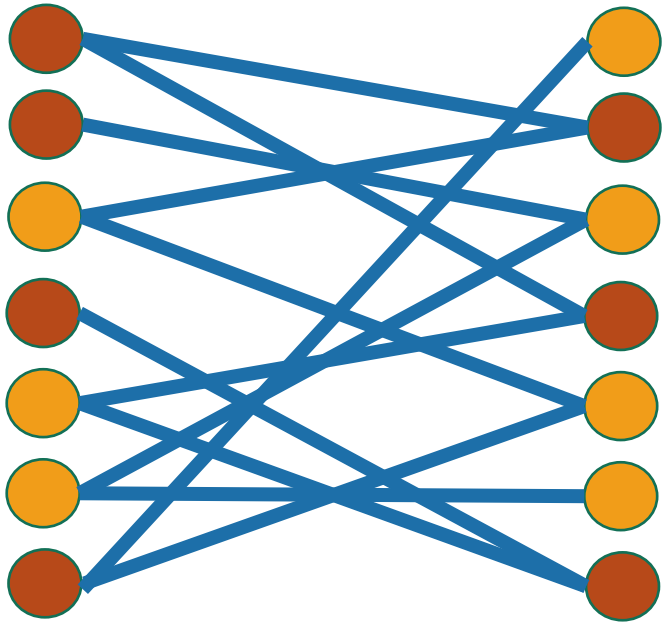
Hang on now.

- Just because **this** coloring doesn't work, why does that mean that there is **no** coloring that works?



I can come up with plenty of bad colorings on this legitimately bipartite graph…

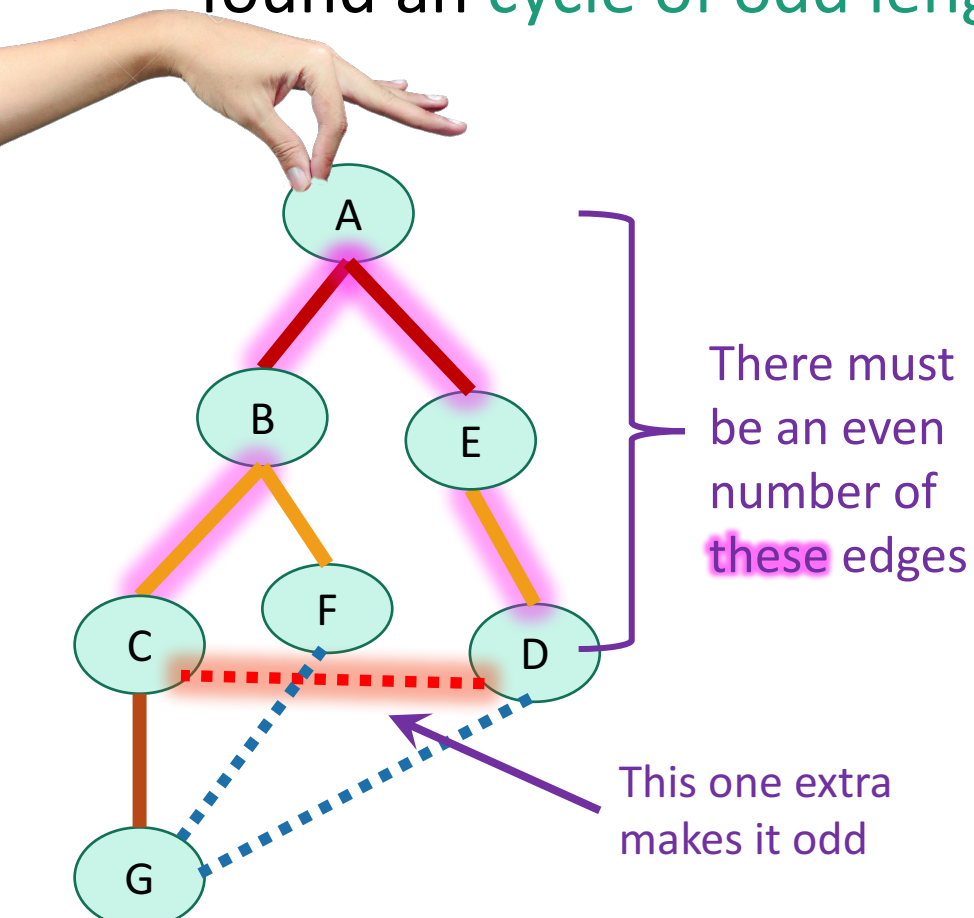
Plucky the pedantic penguin

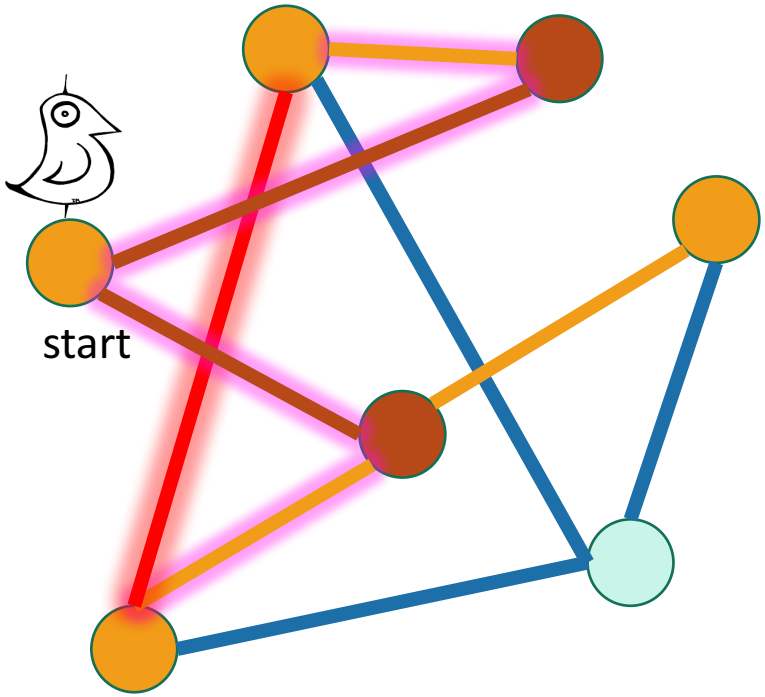
Some proof required

Make this proof sketch formal!

Ollie the over-achieving ostrich

- If BFS colors two neighbors the same color, then it's found an cycle of odd length in the graph.





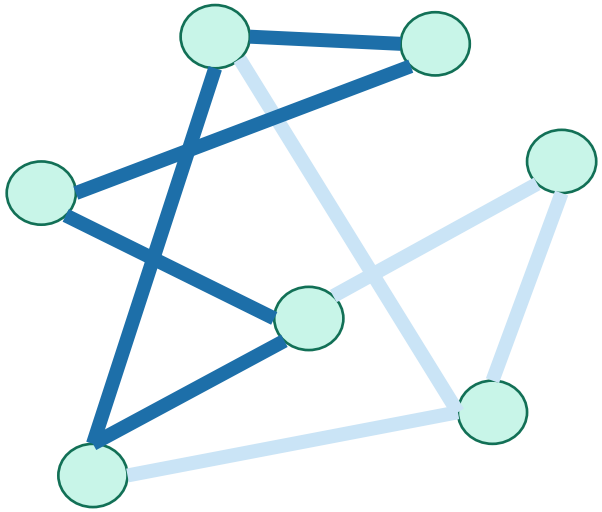
Some proof required

Make this proof sketch formal!

Ollie the over-achieving ostrich

- If BFS colors two neighbors the same color, then it's found an cycle of odd length in the graph.
- So the graph has an odd cycle as a subgraph.
- But you can never color an odd cycle with two colors so that no two neighbors have the same color.
  - [Fun exercise!]

- So you can't legitimately color the whole graph either.
- **Thus it's not bipartite.**



What did we just learn?

BFS can be used to detect bipartite-ness in time $O(n + m)$.

Recap

- Depth-first search
  - Useful for topological sorting
  - Also in-order traversals of BSTs
- Breadth-first search
  - Useful for finding shortest paths
  - Also for testing bipartiteness
- Both DFS, BFS:
  - Useful for exploring graphs, finding connected components, etc

Still open (next few classes)

- We can now find components in undirected graphs…
  - What if we want to find strongly connected components in directed graphs?

- How can we find shortest paths in weighted graphs?

To be continued…