Lecture 3

Recurrence Relations and the Master Theorem!

Announcements!

- **HW1** is posted!
  - Due Friday.
- **Sections** will be Tuesdays 4:30-5:20 , room 380-381U!
  - They are optional.
  - But valuable!
- Sign up for **Piazza!**
  - There's a link on the course website.
  - Course announcements will be posted on Piazza.

More announcements

Last time….

- Sorting: InsertionSort and MergeSort
- Analyzing correctness of iterative + recursive algs
  - Via “loop invariant” and induction
- Analyzing running time of recursive algorithms
  - By writing out a tree and adding up all the work done.

- How do we measure the runtime of an algorithm?
  - Worst-Case Analysis
  - Big-Oh Notation

Today

- Recurrence Relations!
  - How do we measure the runtime a recursive algorithm?
  - Like **Integer Multiplication** and **MergeSort**?

- The ***Master Method***
  - A useful theorem so we don’t have to answer this question from scratch each time.

Running time of MergeSort

- Let's call this running time T(n).
  - when the input has length n.
- We know that T(n) = O(nlog(n)).
- But if we didn't know that…

$$T(n) \leq 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$$

From last time

```
MERGESORT(A):
    n = length(A)
    if n ≤ 1:
        return A
    L = MERGESORT(A[:n/2])
    R = MERGESORT(A[n/2:])
    return MERGE(L,R)
```

Recurrence Relations

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$ is a **recurrence relation.**
- It gives us a formula for T(n) in terms of T(less than n)

- The challenge:

Given a recurrence relation for T(n), find a closed form expression for T(n).

- For example, T(n) = O(nlog(n))

Technicalities I

Base Cases

Plucky the Pedantic Penguin

- Formally, we should always have base cases with recurrence relations.
- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$ with $T(1) = 1$

is not the same as

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$ with $T(1) = 1000000000$
- However, T(1) = O(1), so sometimes we'll just omit it.

Why does T(1) = O(1)?

Siggi the Studious Stork

On your pre-lecture exercise

- You played around with these examples (when n is a power of 2):

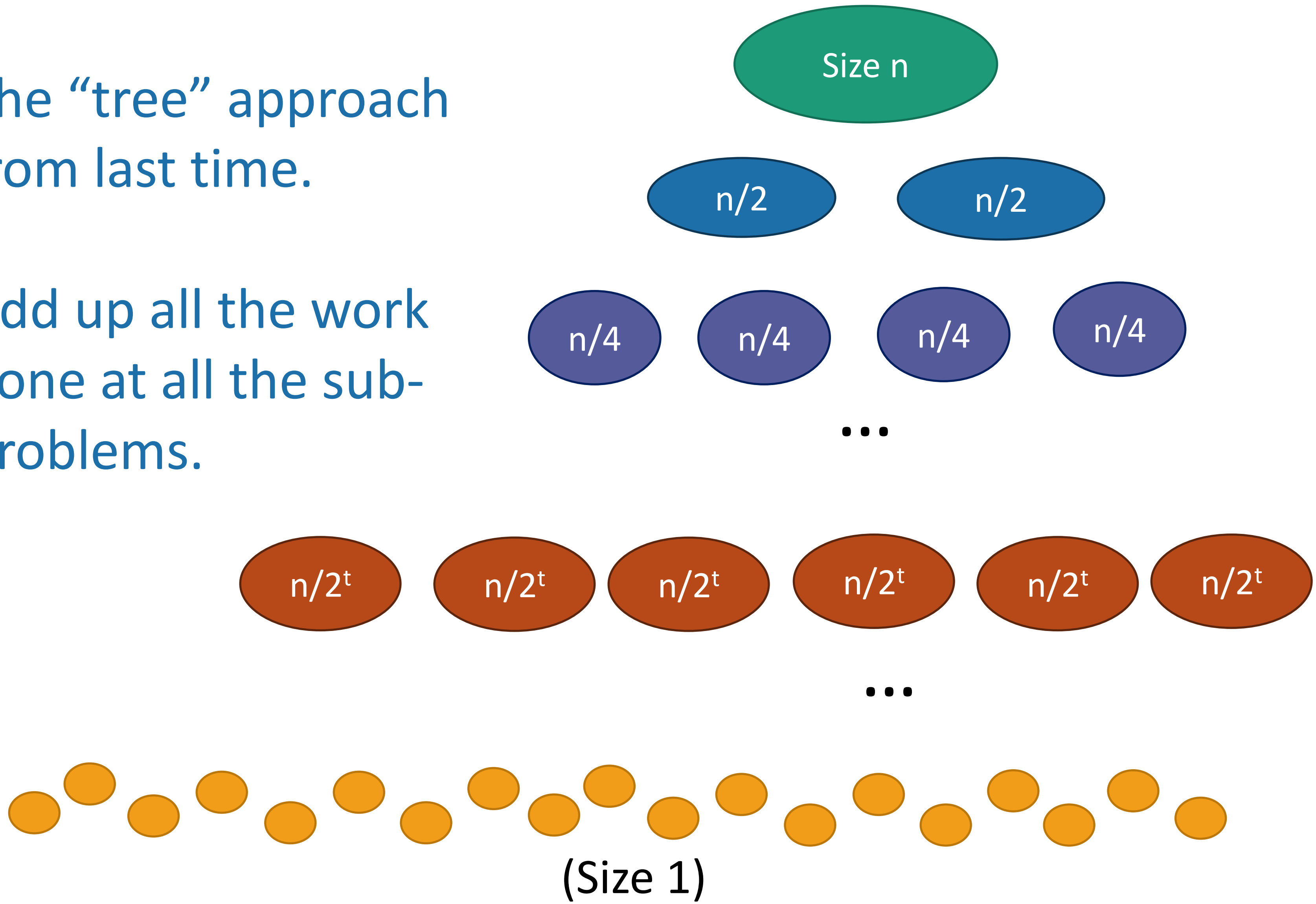
  1. $T(n) = T\left(\frac{n}{2}\right) + n, \qquad T(1) = 1$
  2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n, \qquad T(1) = 1$
  3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n, \qquad T(1) = 1$

- What are the closed forms?

  [on board]

One approach for all of these

- The "tree" approach from last time.
- Add up all the work done at all the sub-problems.



Another approach:

Recursively apply the relationship a bunch until you see a pattern.

- $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 11 \cdot n$
- $T(n) = 2 \cdot \left(2 \cdot T\left(\frac{n}{4}\right) + \frac{11 \cdot n}{2}\right) + 11 \cdot n$
- $T(n) = 4 \cdot T\left(\frac{n}{4}\right) + 22 \cdot n$
- $T(n) = 4 \cdot \left(2 \cdot T\left(\frac{n}{8}\right) + \frac{11 \cdot n}{4}\right) + 22 \cdot n$
- $T(n) = 8 \cdot T\left(\frac{n}{8}\right) + 33 \cdot n$

Formally, this should be accompanied with a proof that the pattern holds!

More next time.

- Following the pattern…
- $\boldsymbol{T(n) = n \cdot T(1) + 11 \cdot log(n) \cdot n = O\left(n \cdot \log(n)\right)}$

This slide skipped in class, provided here in case this way makes more sense to you.

More examples

T(n) = time to solve a problem of size n.

- Needlessly recursive integer multiplication
- $T(n) = 4\,T(n/2) + O(n)$
- $T(n) = O(n^2)$

- Karatsuba integer multiplication
- $T(n) = 3\,T(n/2) + O(n)$
- $T(n) = O(n^{\log_2(3)} \approx n^{1.6})$

- MergeSort
- $T(n) = 2T(n/2) + O(n)$
- $T(n) = O(n\log(n))$

These two are the same as the ones on your pre-lecture exercise.

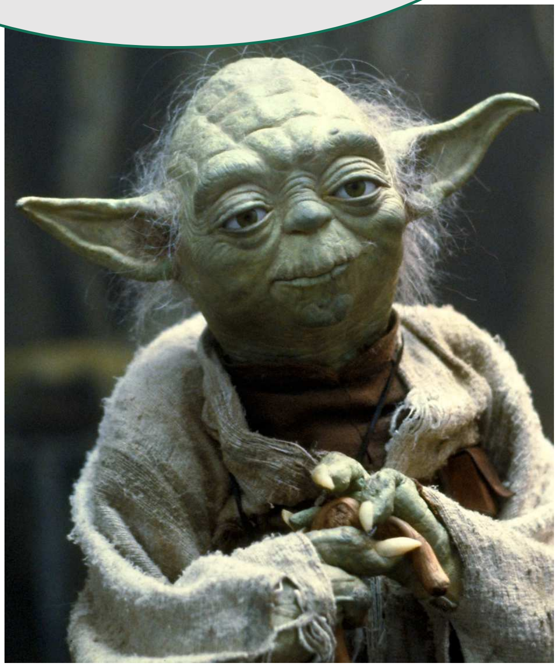
What's the pattern?!?!?!?!

The master theorem

- A **formula** that solves recurrences when all of the sub-problems are the same size.
  - We'll see an example Wednesday when not all problems are the same size.

- "Generalized" tree method.

A useful formula it is.
Know why it works you should.



Jedi master Yoda

The master theorem

We can also take n/b to mean either $\left\lfloor \frac{n}{b} \right\rfloor$ or $\left\lceil \frac{n}{b} \right\rceil$ and the theorem is still true.

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Three parameters:
a : number of subproblems
b : factor by which input size shrinks
d : need to do n^d work to create all the subproblems and combine their solutions.

Many symbols those are….

Technicalities II

Integer division

Plucky the
Pedantic Penguin

- If n is odd, I can't break it up into two problems of size n/2.

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lceil \frac{n}{2} \right\rceil\right) + O(n)$$

- However (see CLRS, Section 4.6.2), one can show that the Master theorem works fine if you pretend that what you have is:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + \ O(n)$$

- From now on we'll mostly ignore floors and ceilings in recurrence relations.

Examples

(details on board)

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- Needlessly recursive integer mult.
  - $T(n) = 4\, T(n/2) + O(n)$
  - $T(n) = O(n^2)$

$a = 4$
$b = 2$
$d = 1$

$a > b^d$ ✓

- Karatsuba integer multiplication
  - $T(n) = 3\, T(n/2) + O(n)$
  - $T(n) = O(n^{\log\_2(3)} \approx n^{1.6})$

$a = 3$
$b = 2$
$d = 1$

$a > b^d$ ✓

- MergeSort
  - $T(n) = 2T(n/2) + O(n)$
  - $T(n) = O(n\log(n))$

$a = 2$
$b = 2$
$d = 1$

$a = b^d$ ✓

- That other one
  - $T(n) = T(n/2) + O(n)$
  - $T(n) = O(n)$

$a = 1$
$b = 2$
$d = 1$

$a < b^d$ ✓

Proof of the master theorem

- We'll do the same recursion tree thing we did for MergeSort, but be more careful.
- Suppose that $T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$.

Hang on! The hypothesis of the Master Theorem was the the extra work at each level was $O(n^d)$. That's NOT the same as work <= cn^d for some constant c.

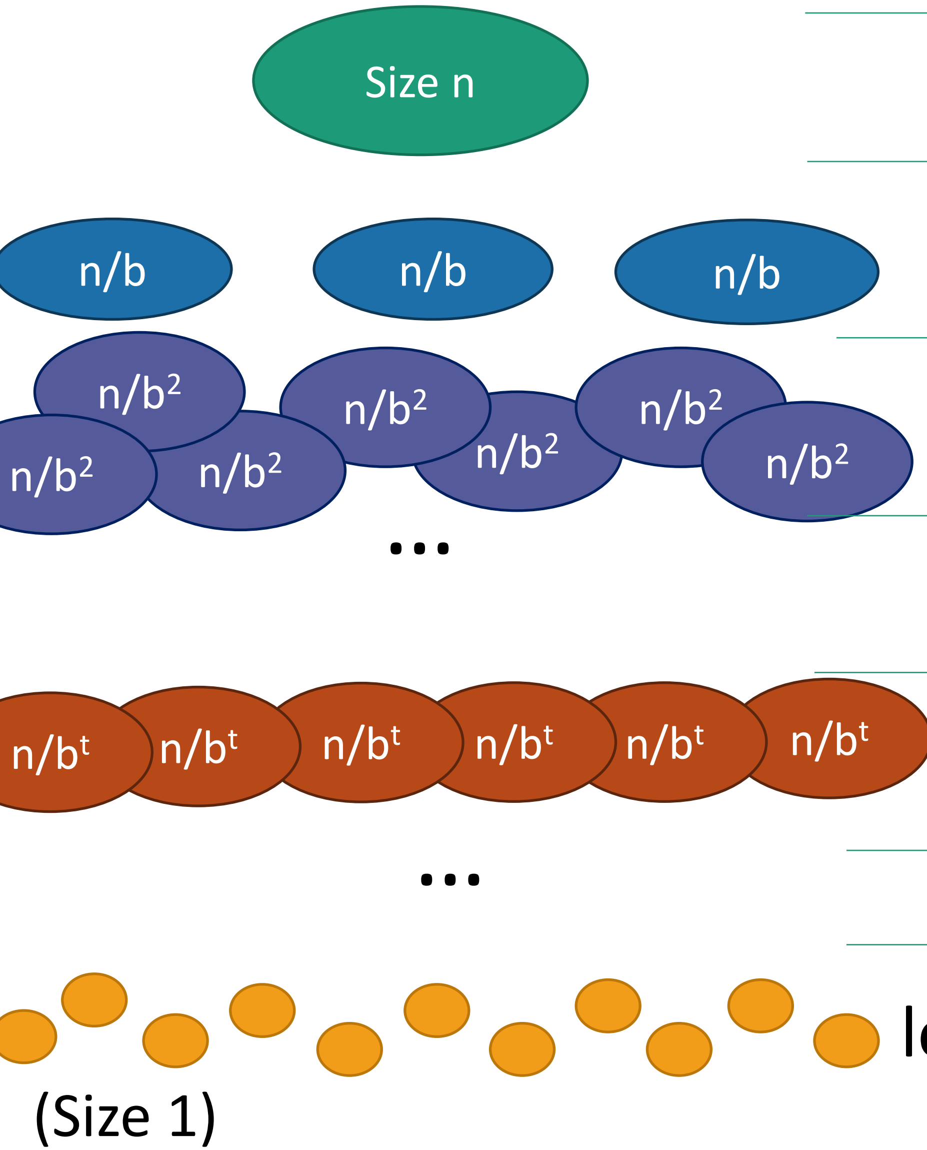
Plucky the Pedantic Penguin

That's true … we'll actually prove a weaker statement that uses this hypothesis instead of the hypothesis that $T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d)$. It's a good exercise to make this proof work rigorously with the O() notation.

Siggi the Studious Stork

Recursion tree

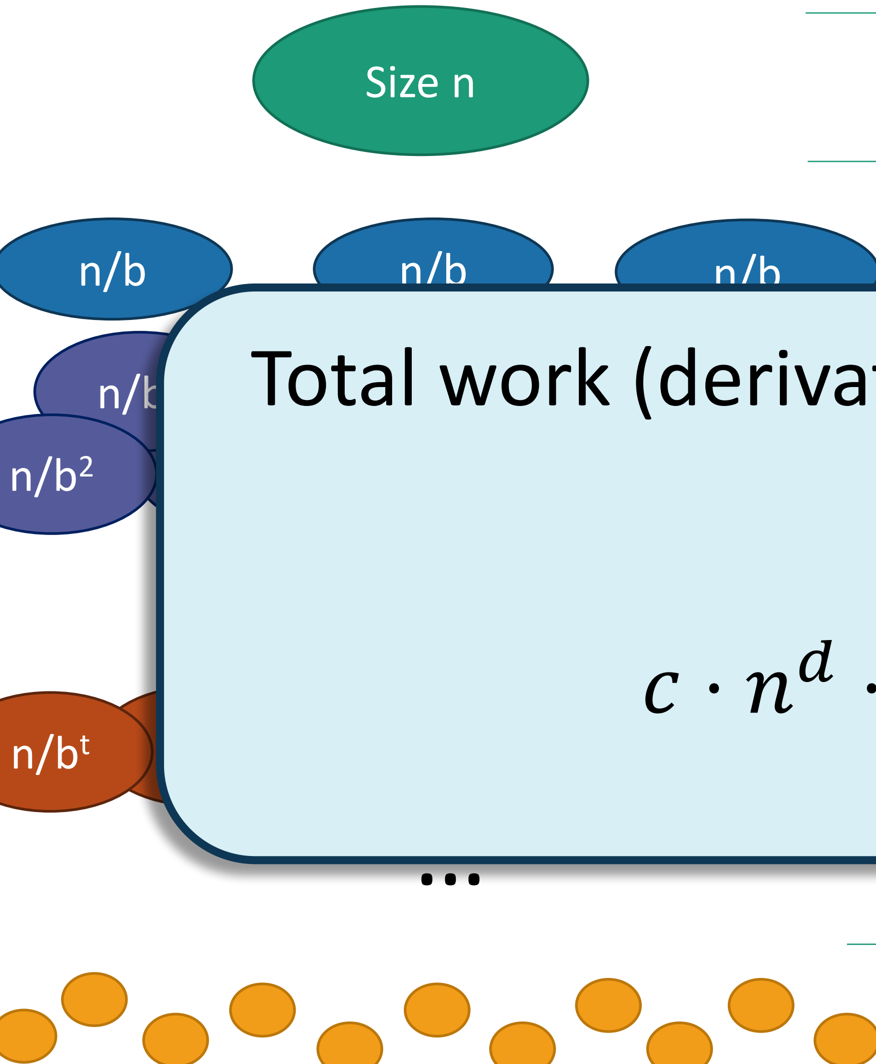
$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
2	a^2	n/b^2	$a^2 c\left(\frac{n}{b^2}\right)^d$
t	a^t	n/b^t	$a^t c\left(\frac{n}{b^t}\right)^d$
…			
$\log_b(n)$	$a^{\log_b(n)}$	1	$a^{\log_b(n)} c$

Recursion tree

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d$$



Level	# problems	Size of each problem	Amount of work at this level
0	1	n	$c \cdot n^d$
1	a	n/b	$ac\left(\frac{n}{b}\right)^d$
...			
$\log_b(n)$	$a^{\log_b(n)}$	1	$a^{\log_b(n)} c$

Total work (derivation on board) is at most:

$$c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t$$

Now let's check all the cases
(on board)

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

Even more generally, for T(n) = aT(n/b) + f(n)...

Theorem 3.2 (Master Theorem). *Let $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$ be a recurrence where $a \geq 1$, $b > 1$. Then,*

- *If $f(n) = O\left(n^{\log_b a - \epsilon}\right)$ for some constant $\epsilon > 0$, $T(n) = \Theta\left(n^{\log_b a}\right)$.*
- *If $f(n) = \Theta\left(n^{\log_b a}\right)$, $T(n) = \Theta\left(n^{\log_b a} \log n\right)$.*
- *If $f(n) = \Omega\left(n^{\log_b a + \epsilon}\right)$ for some constant $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.*

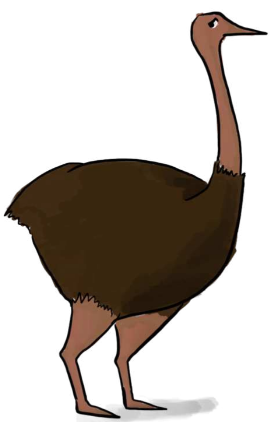


Figure out how to adapt the proof we gave to prove this more general version!

[From CLRS]

Ollie the Over-Achieving Ostrich


LIFE IS SO GOOD
BUT I WILL NOT BECOME COMPLACENT AND I CAN TOTALLY SOLVE RECURRENCES FROM SCRATCH IF I WANT TO
imgflip.com

Understanding the Master Theorem

- Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + O\left(n^d\right)$. Then

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- What do these three cases mean?

The eternal struggle

Branching causes the number of problems to explode!
The most work is at the bottom of the tree!

The problems lower in the tree are smaller!
The most work is at the top of the tree!

Consider our three warm-ups

1. $T(n) = T\left(\frac{n}{2}\right) + n$
2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$
3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n$

First example: tall and skinny tree

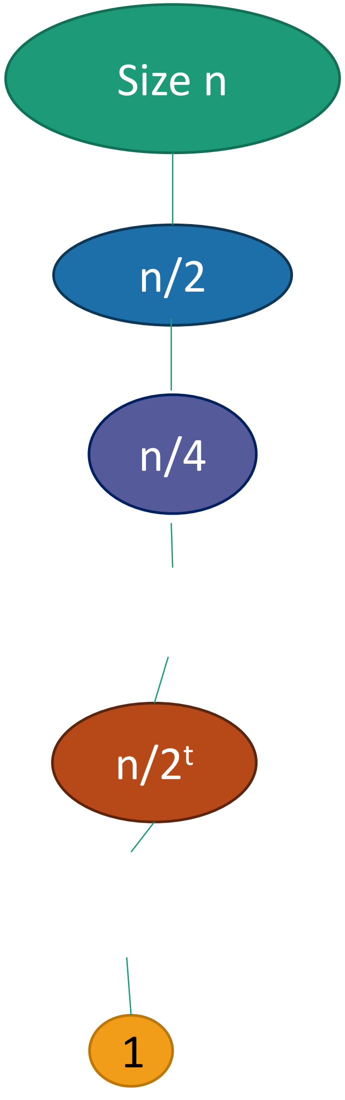
1. $T(n) = T\left(\frac{n}{2}\right) + n, \qquad \left(a < b^d\right)$

- The amount of work done at the top (the biggest problem) swamps the amount of work done anywhere else.

- T(n) = O(work at top) = O(n)

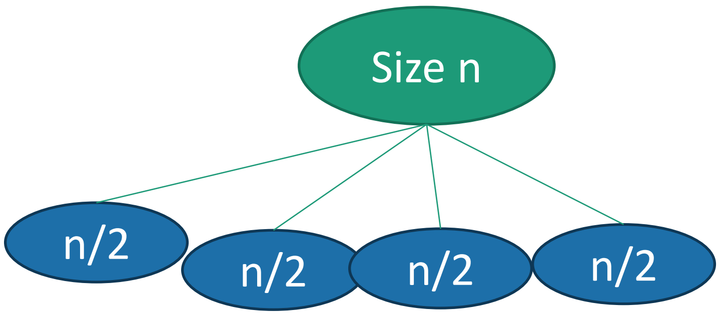
WINNER

Most work at the top of the tree!



Third example: bushy tree

3. $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + n, \qquad (a > b^d)$

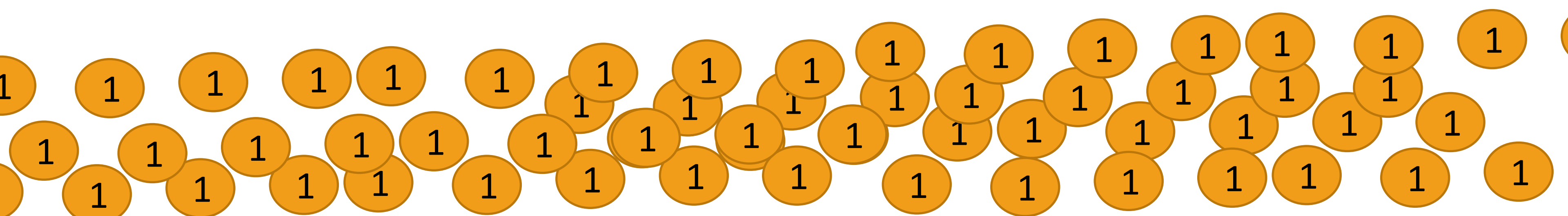


WINNER

Most work at the bottom of the tree!

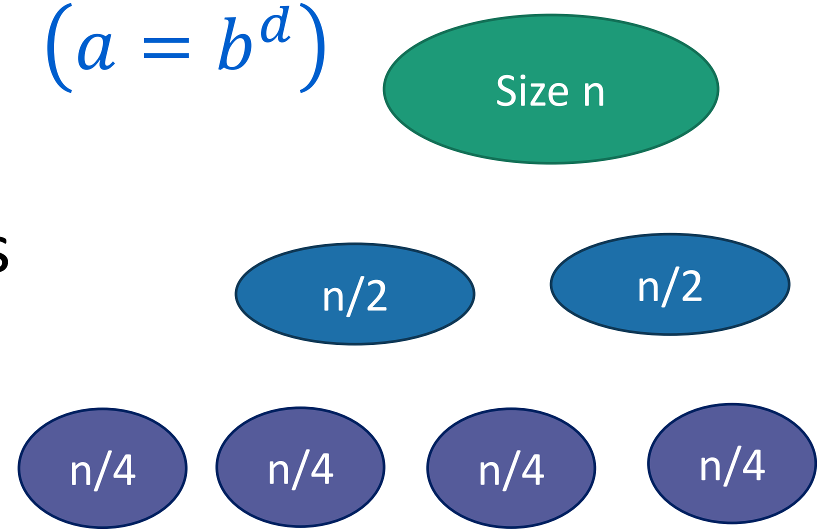
- There are a HUGE number of leaves, and the total work is dominated by the time to do work at these leaves.

- T(n) = O(work at bottom) = O($4^{\text{depth of tree}}$) = O(n^2)



Second example: just right

2. $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n, \qquad \left(a = b^d\right)$



- The branching **just** balances out the amount of work.
- The same amount of work is done at every level.
- T(n) = (number of levels) * (work per level)
- = log(n) * O(n) = O(nlog(n))

Recap

- The ”Master Method” makes our lives easier.
- But it’s basically just codifying a calculation we could do from scratch if we wanted to.

Next Time

- What if the sub-problems are different sizes?
- And when might that happen?

BEFORE Next Time

- Pre-Lecture Exercise 4!
  - Which should be easier if you did Pre-Lecture Exercise 3…