Pre-Lecture Exercise for Lecture 3
Date: October 2, 2017

Pre-lecture exercises will not be collected for credit. However, you will get more out of each lecture if you do them, and they will be referenced during lecture. We recommend **writing out** your answers to pre-lecture exercises before class. Pre-lecture exercises usually should not take you more than 20 minutes.

In this pre-lecture exercise, you'll explore three *recurrence relations.* For each of the following expressions, try to figure out a closed-form expression for $T(n)$, **when** n **is a power of** 2**.** (Don't worry about when n isn't a power of 2 for now).

If you are feeling stuck, **we've done the first one for you in two different ways on the next page** to give you some inspiration for how to attack the second and the third.

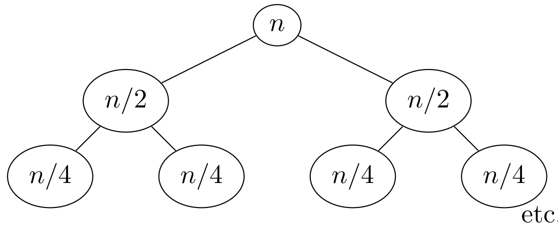
1. $T(n) = \begin{cases} 2 \cdot T(n/2) + n & n = 2^i, i > 0 \\ T(n) = 1 & n = 1 \end{cases}.$

2. $T(n) = \begin{cases} T(n/2) + n & n = 2^i, i > 0 \\ T(n) = 1 & n = 1 \end{cases}.$

3. $T(n) = \begin{cases} 4 \cdot T(n/2) + n & n = 2^i, i > 0 \\ T(n) = 1 & n = 1 \end{cases}.$

SPOILER ALERT: Here are two solutions to Exercise 1, which you can look at to help you figure out how to do Exercises 2 and 3.

SOLUTION 1. We do as we did with MergeSort, and imagine a tree with $\log(n) + 1$ levels. The top node is labeled "n", its two children are labeled "$n/2$", and so on.



Consider $T(n) = T(n/2) + T(n/2) + n$. In the context of the tree above, that means that $T(n) = n +$ (stuff contributed by things in the tree lower than the root). That is,

$$T(n) = (\text{label on the root}) + (\text{stuff contributed by things lower than the root}).$$

We can repeat this logic recursively to figure out what that second term is, all the way down to the bottom of the tree, where we have $T(1) = 1$. We conclude that $T(n)$ is equal to the sum, over all the nodes, of the labels on the nodes[1].

If the root is level 0, then at level $j \leq \log(n)$, there are 2^j nodes, each which have label $n/2^j$. So

$$\sum_{j=0}^{\log(n)} 2^j \cdot \frac{n}{2^j} = n(\log(n) + 1).$$

is our answer.

SOLUTION 2. We can do the exact same calculation without the tree, by repeatedly applying our formula.

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2\left(2T(n/4) + n/2\right) + n \\ &= 4T(n/4) + 2n \\ &= 4(2T(n/8) + n/4) + 2n \\ &= 8T(n/8) + 3n \end{aligned}$$

and at this point we can spot the pattern: for all $j \leq \log(n)$,

$$T(n) = 2^j T(n/2^j) + jn.$$

In order to formally prove that this is true, we should use a proof by induction; that's called the *substitution method* and we'll talk about it on Wednesday. But for now you can convince yourself that this is true.

Once we have this, we can just plug in $j = \log(n)$, and get

$$T(n) = 2^{\log(n)} T(n/2^{\log(n)}) + n\log(n) = n \cdot T(1) + n\log(n) = n(\log(n) + 1),$$

just as before.

[1]Notice that this is a special consequence of the fact that the term we are adding on is exactly n; if it were, say $11 \cdot n$, we'd have to multiply all the labels by 11 before counting their contribution. Or if it were $\sqrt{n}$, we'd have to take the square root, and so on.