Lecture 7

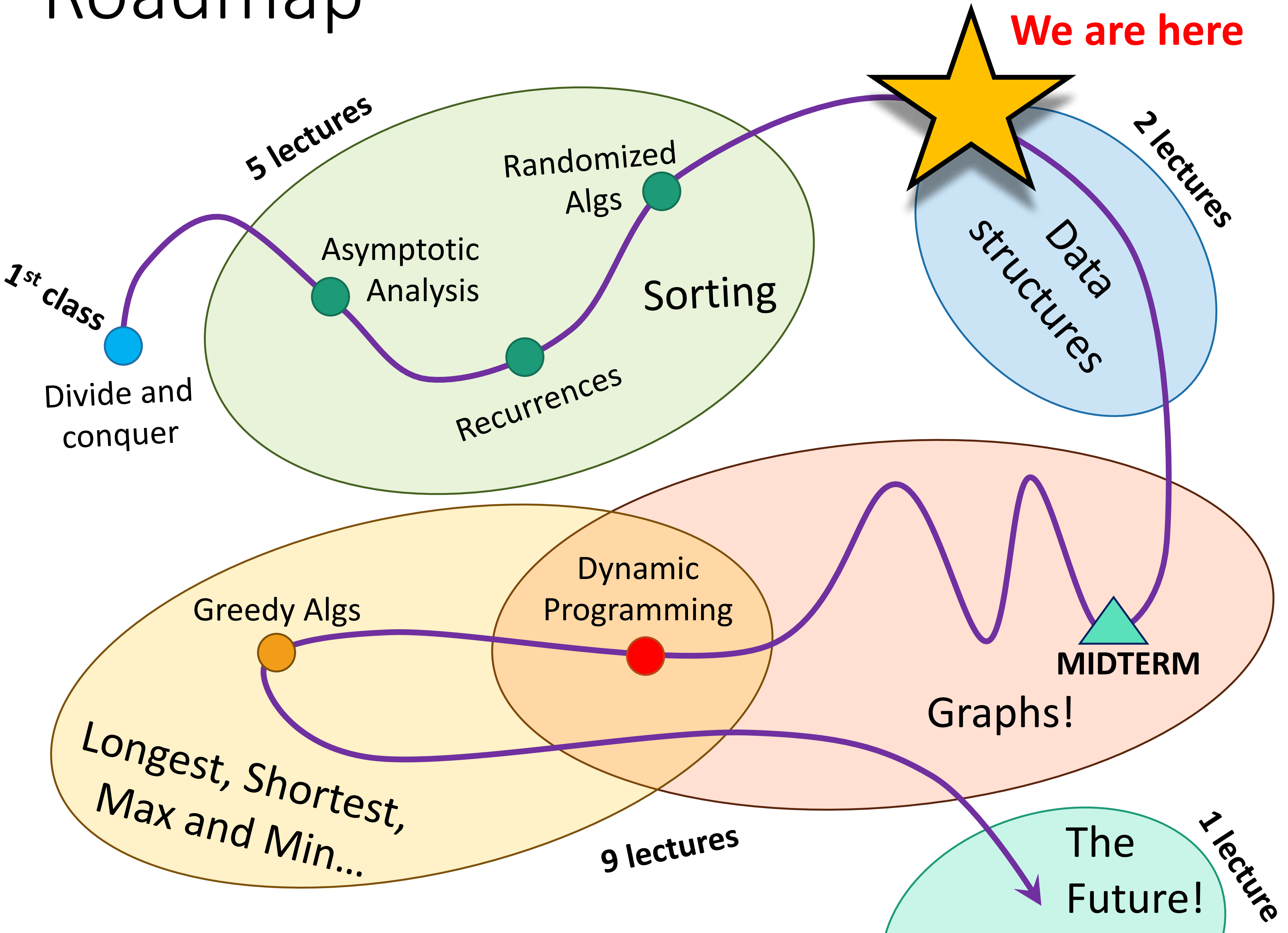
Binary Search Trees and Red-Black Trees

Announcements

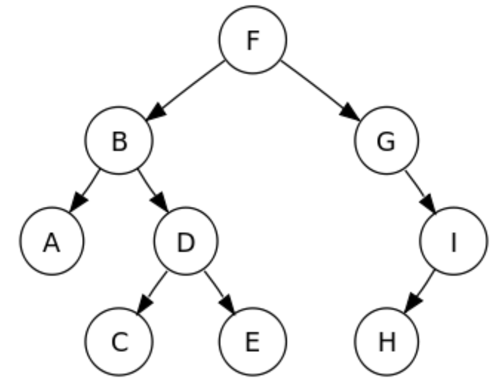
- HW 3 released! (Due Friday)
- Special guest lecturer: Sam Kim!

More detailed schedule on the website!

Roadmap

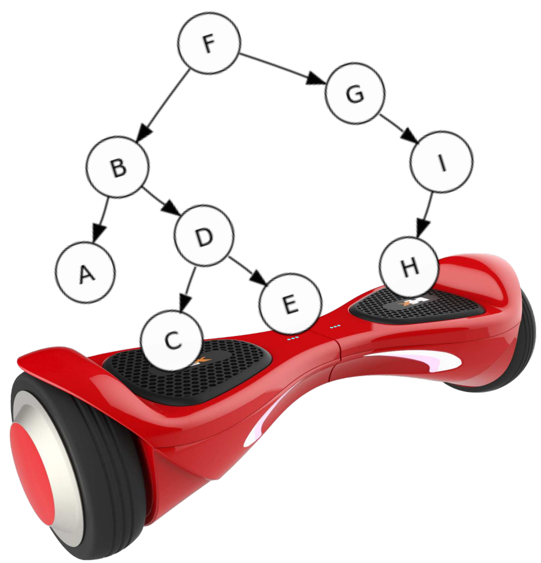


Today



- Begin a brief foray into data structures!
  - See CS 166 for more!
- Binary search trees
  - You may remember these from CS 106B
  - They are better when they're balanced.

this will lead us to…



- Self-Balancing Binary Search Trees
  - **Red-Black** trees.

Why are we studying self-balancing BSTs?

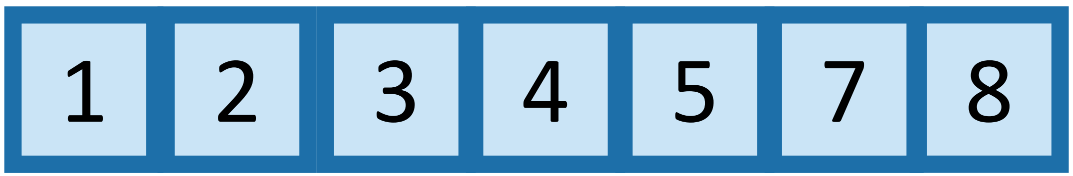
1. The punchline is **important**:
   - A data structure with O(log(n)) INSERT/DELETE/SEARCH

2. The idea behind **Red-Black Trees** is clever
   - It's good to be exposed to clever ideas.
   - Also it's just aesthetically pleasing.

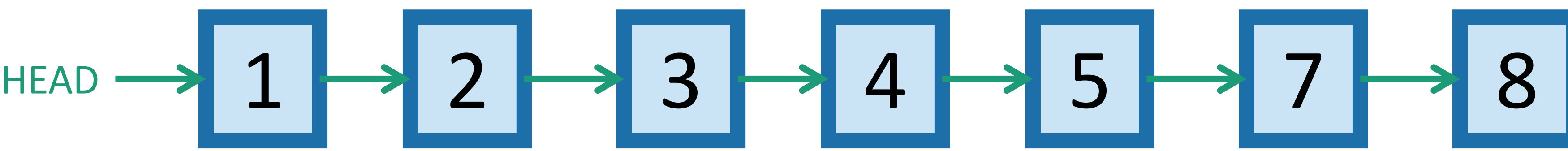
Some data structures

for storing objects like 5 (aka, nodes with keys)

- (Sorted) arrays:

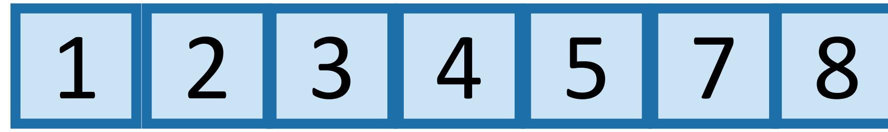


- (Sorted) linked lists:

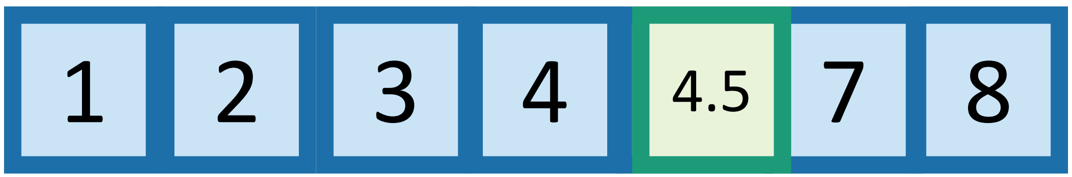


- Some basic operations:
  - INSERT, DELETE, SEARCH

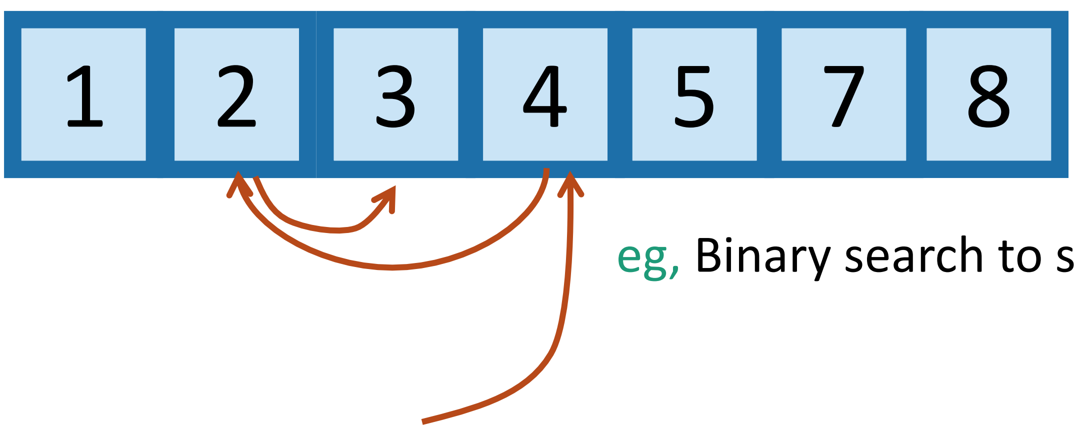
Sorted Arrays



- O(n) INSERT/DELETE:

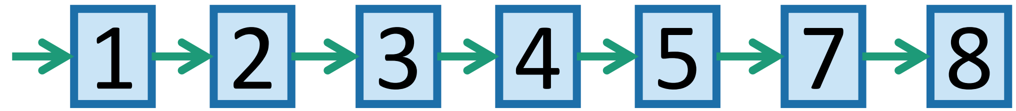


- O(log(n)) SEARCH:

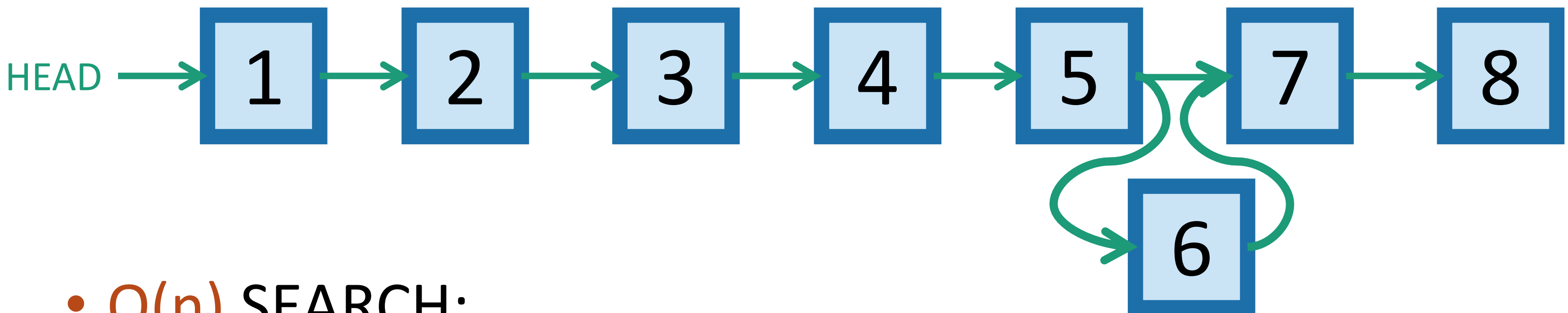


eg, Binary search to see if 3 is in A.

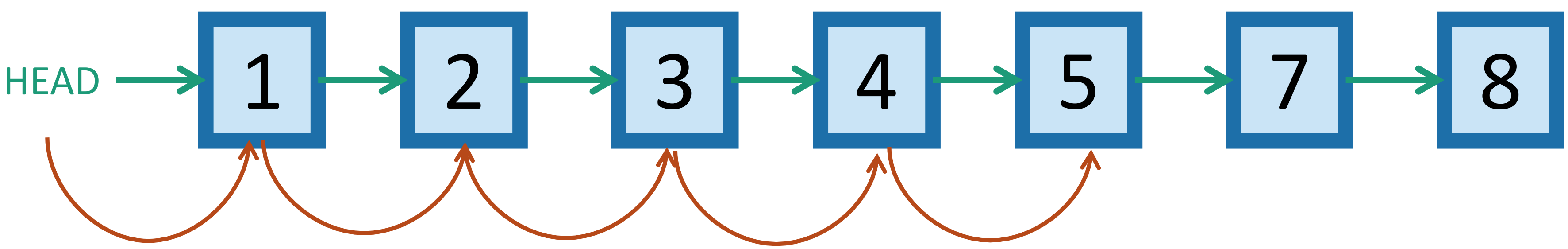
Sorted linked lists



- O(1) INSERT/DELETE:
  - (assuming we have a pointer to the location of the insert/delete)



- O(n) SEARCH:



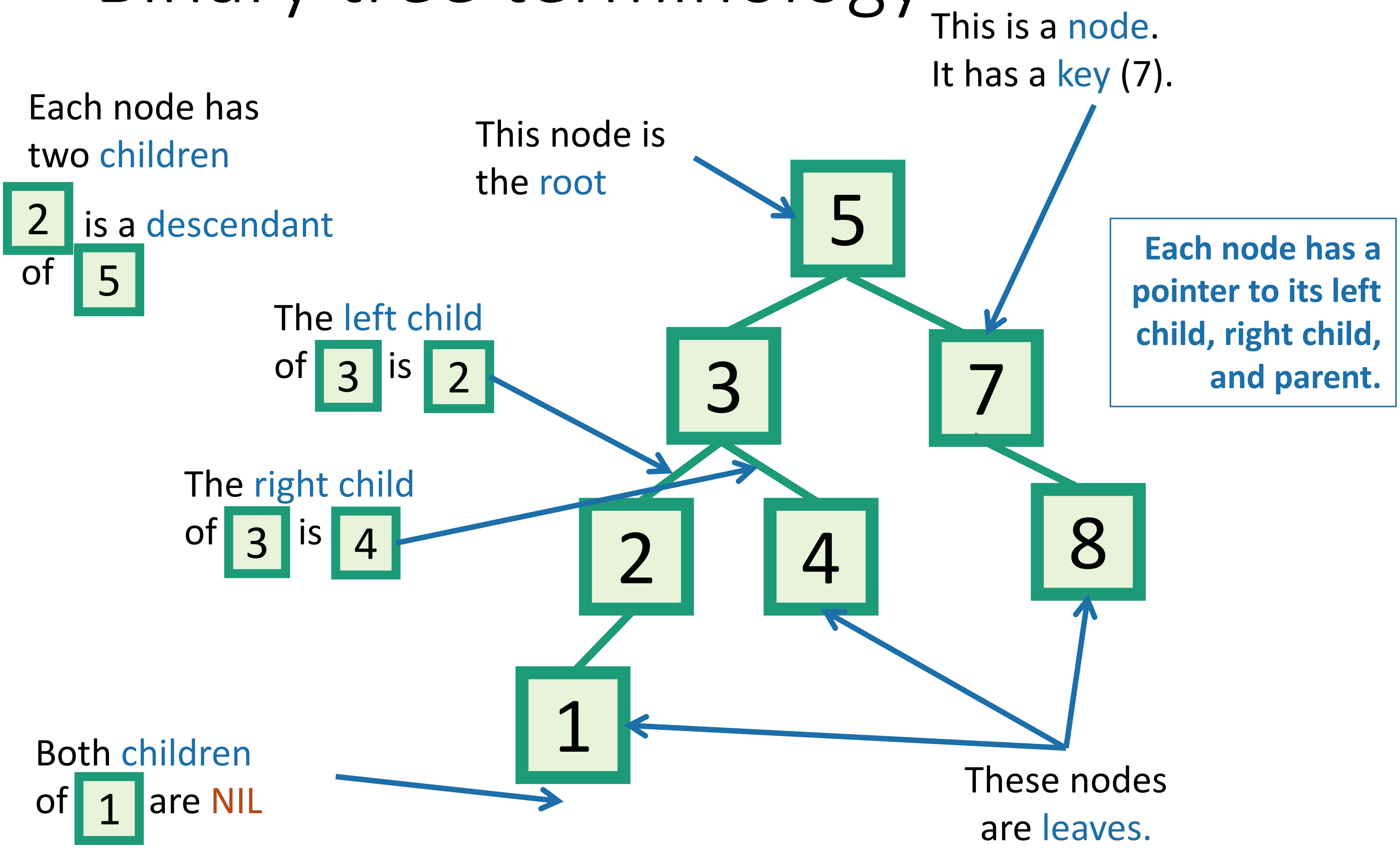
Motivation for Binary Search Trees

TODAY!

	Sorted Arrays	Linked Lists	Binary Search Trees*
Search	O(log(n)) 😃	O(n) ☹	O(log(n)) 😃
Insert/Delete	O(n) ☹	O(1) 😃	O(log(n)) 😃

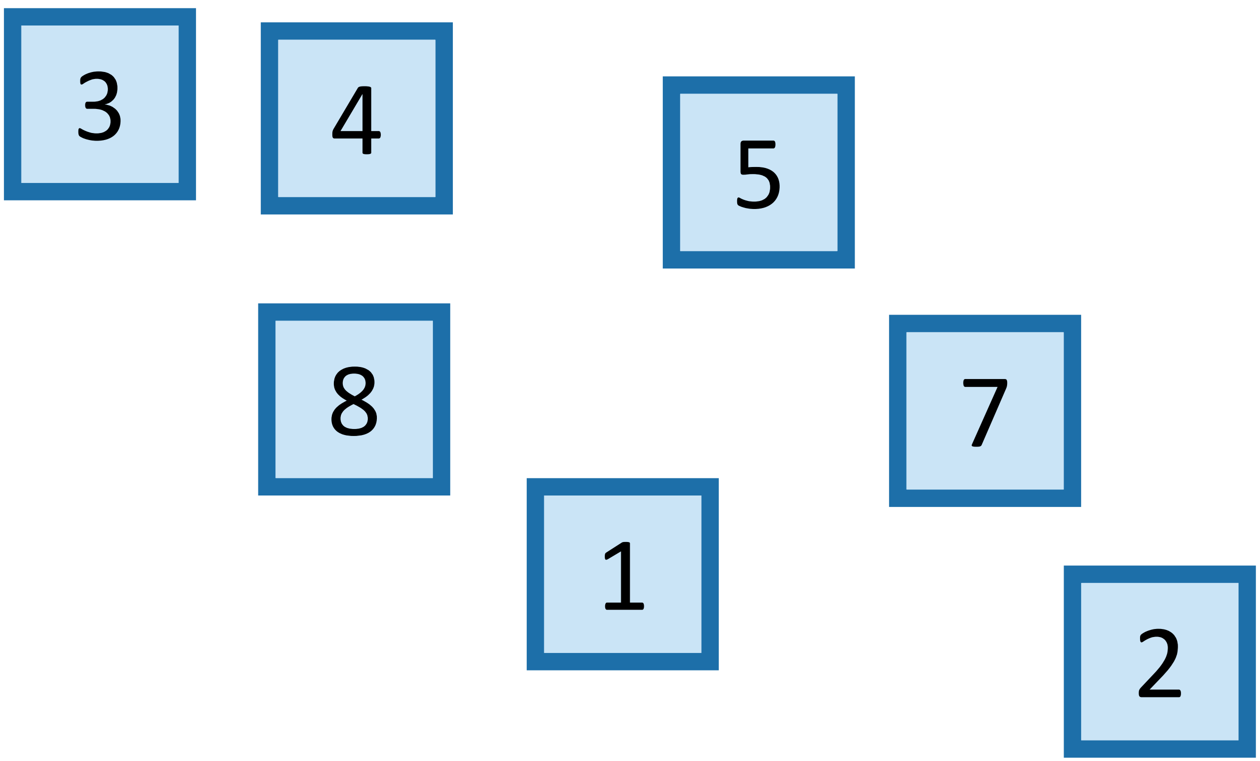
For today all keys are distinct.

Binary tree terminology



Binary Search Trees

- It's a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.
- Example of building a binary search tree:



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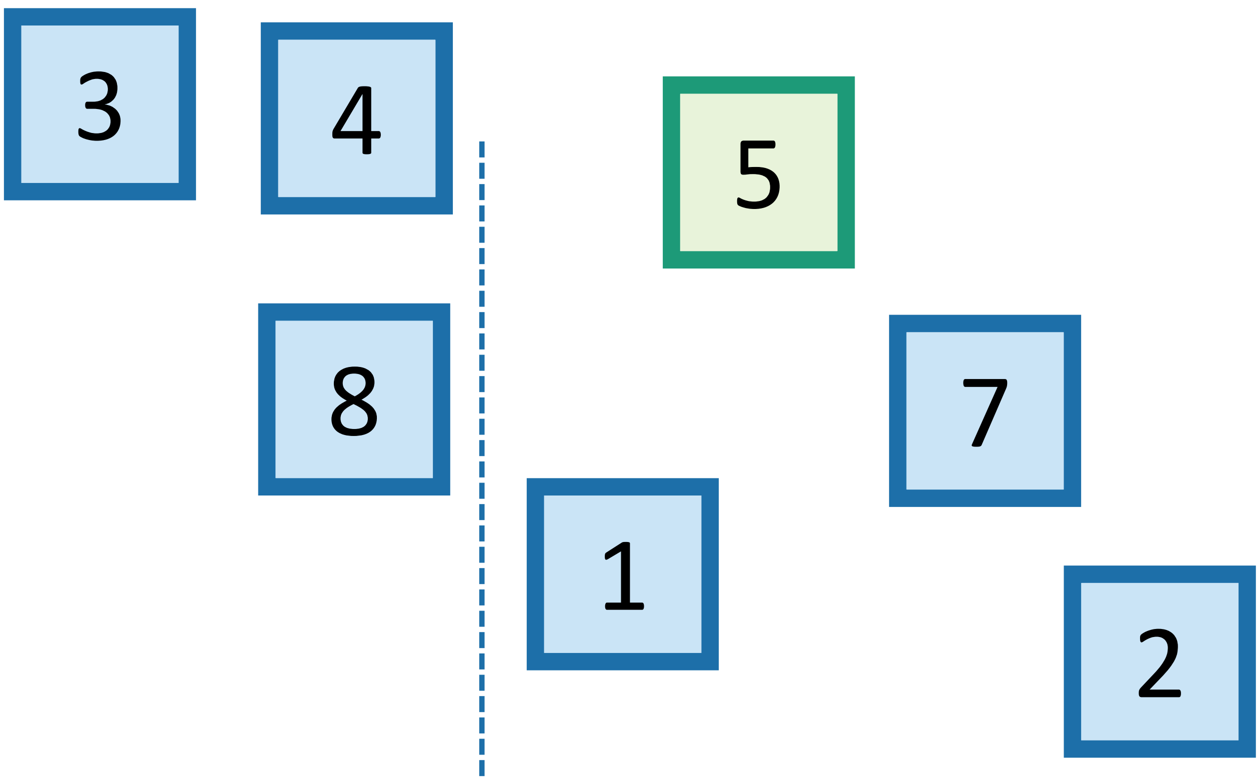

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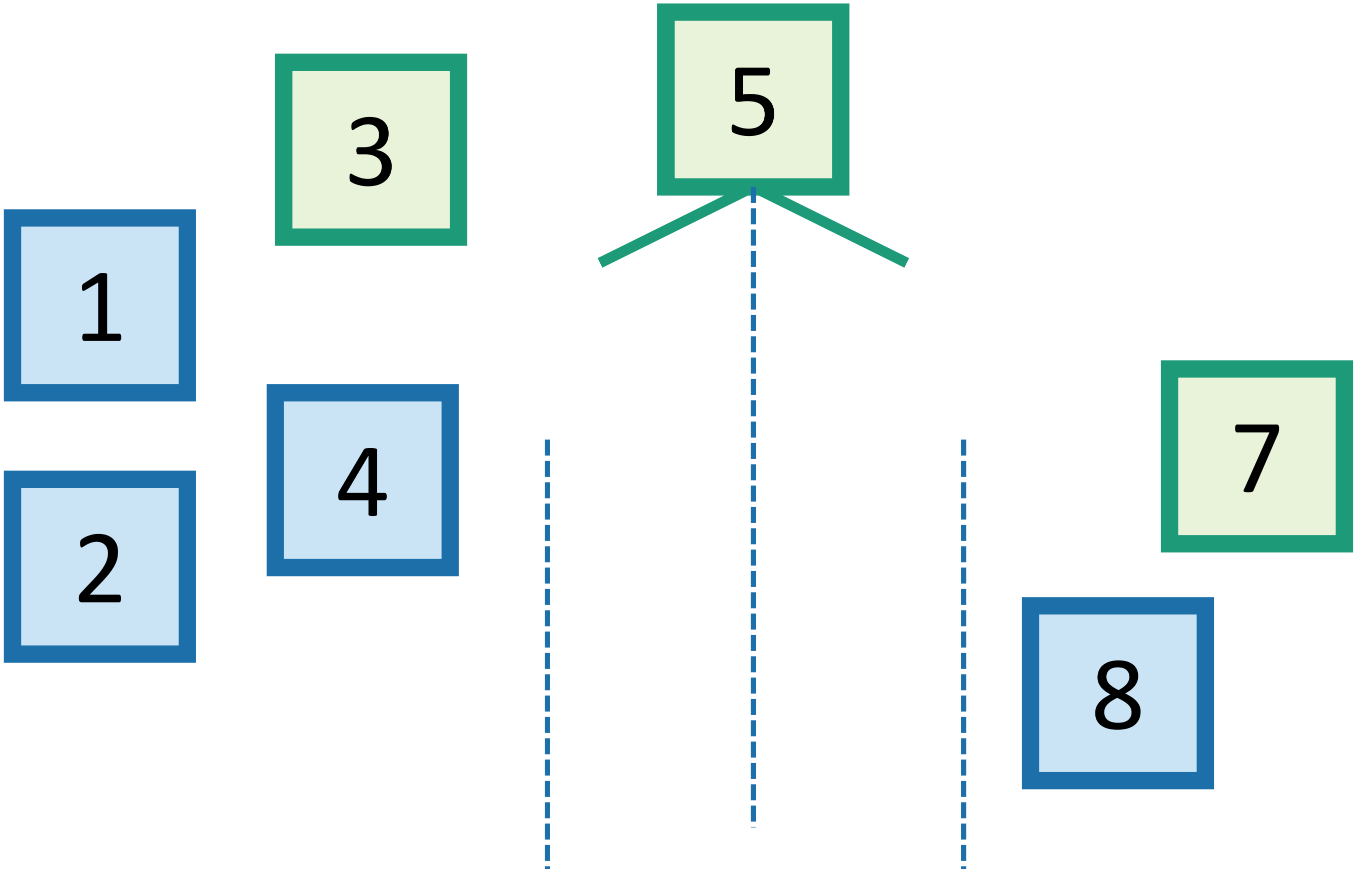

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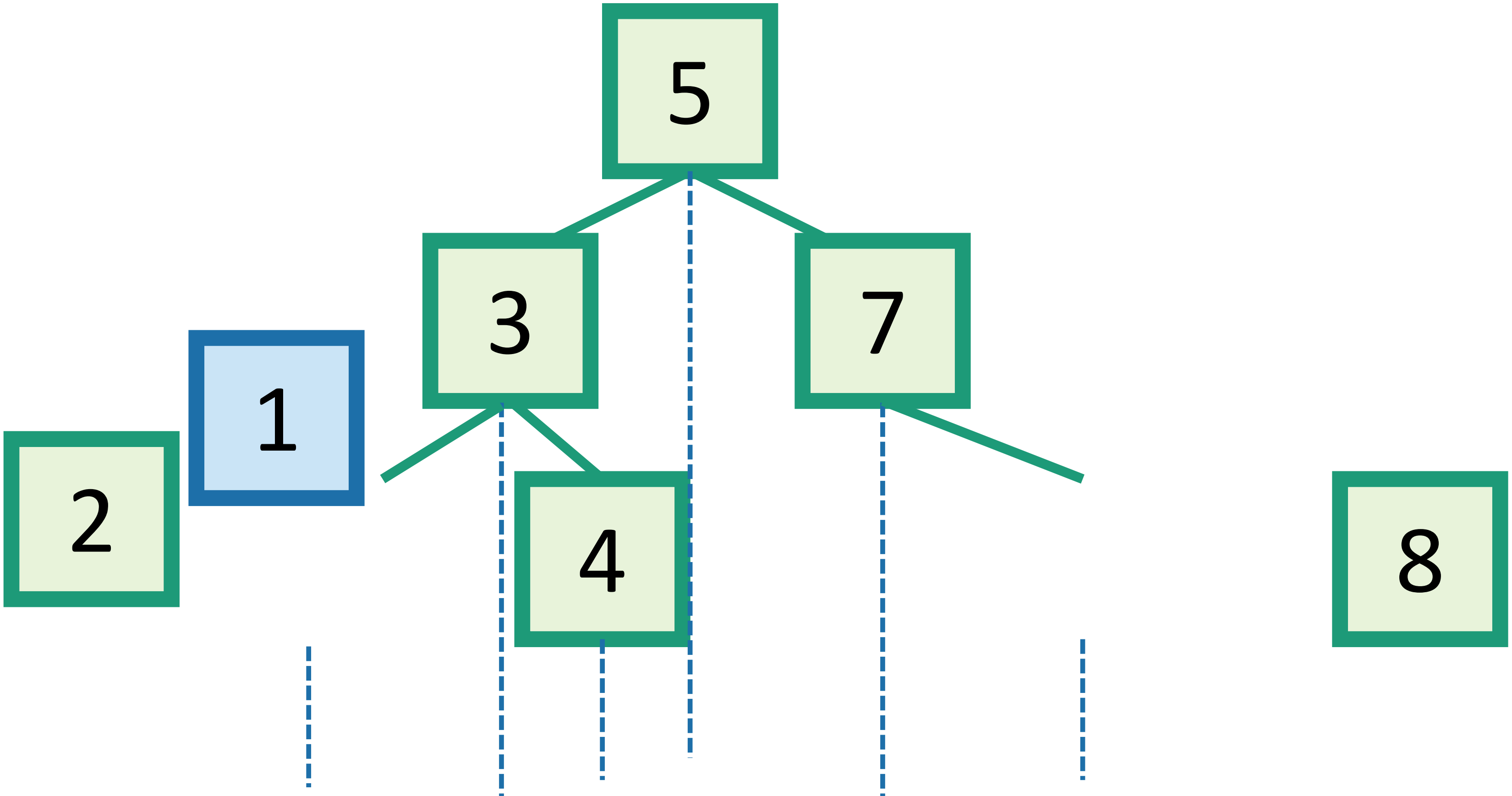

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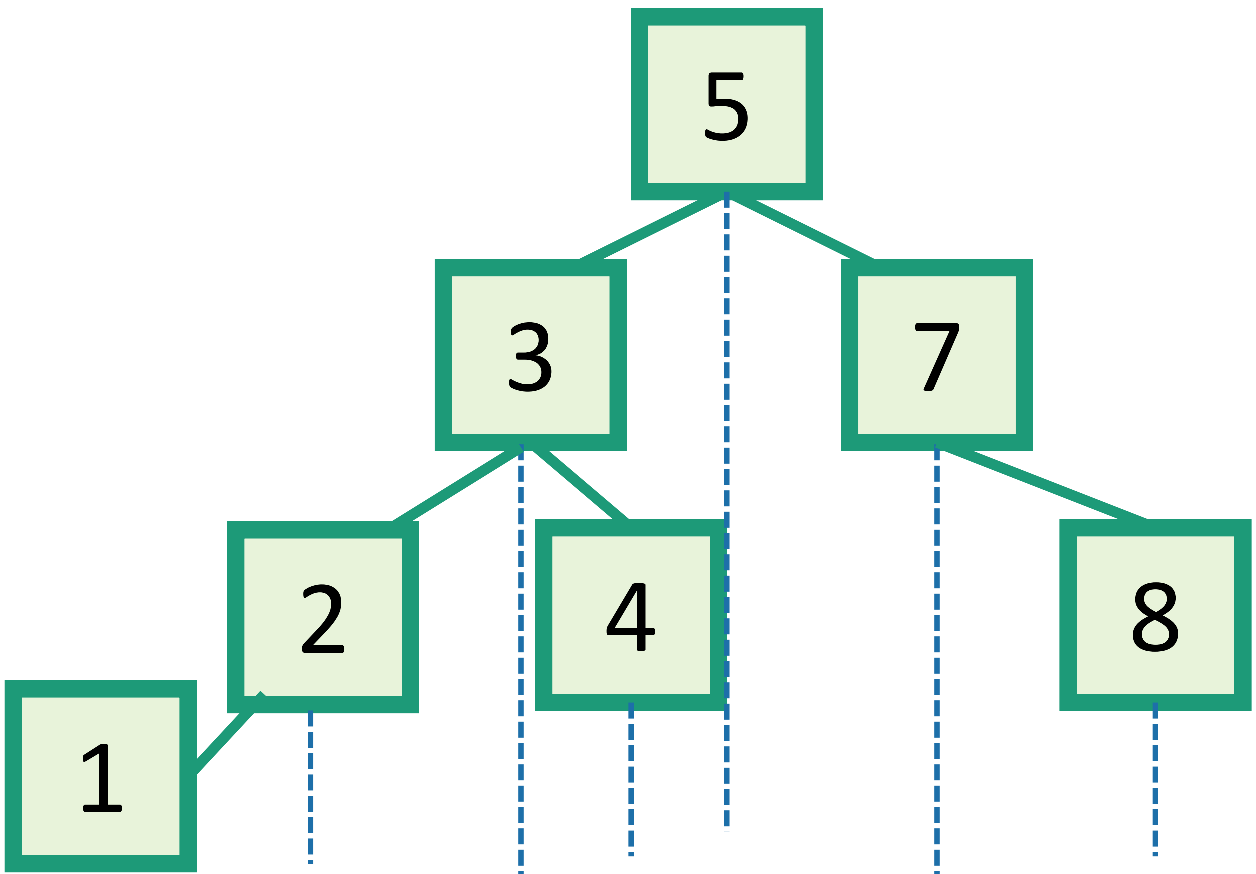
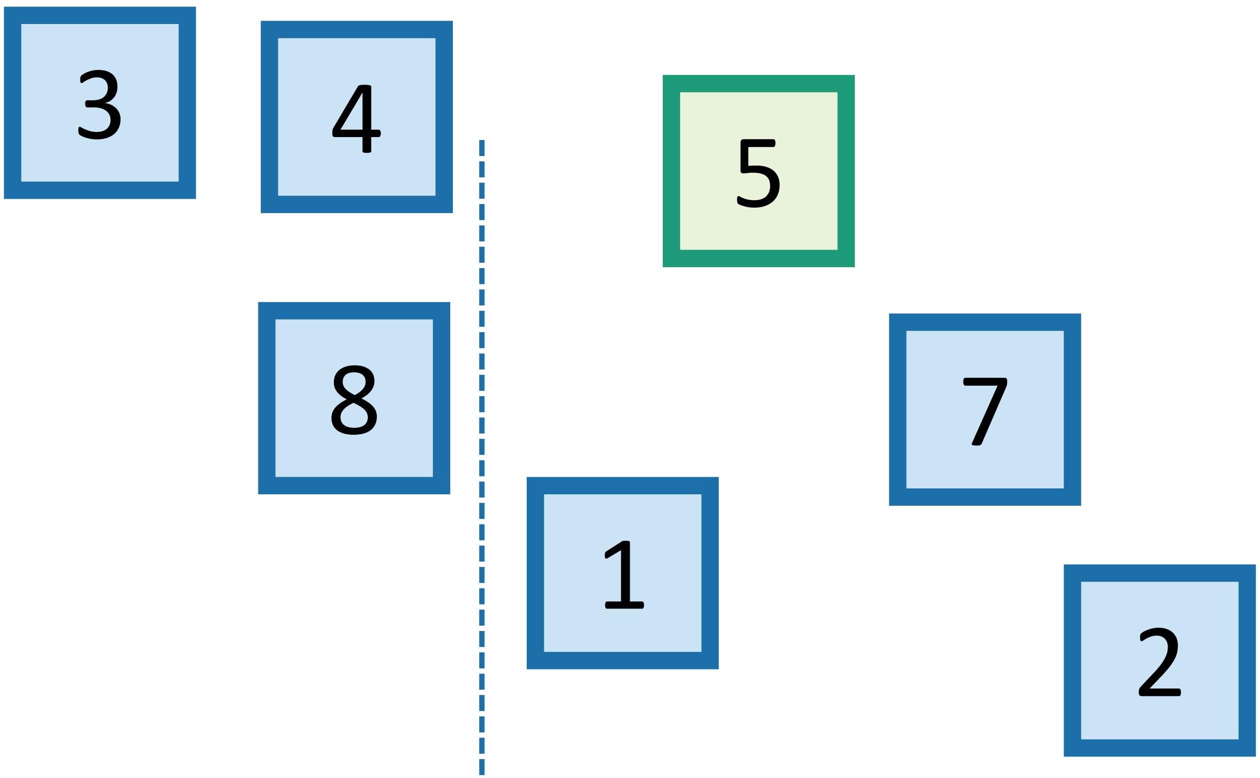


Q: Is this the only binary search tree I could possibly build with these values?

A: **No.** I made choices about which nodes to choose when. Any choices would have been fine.

Aside: this should look familiar

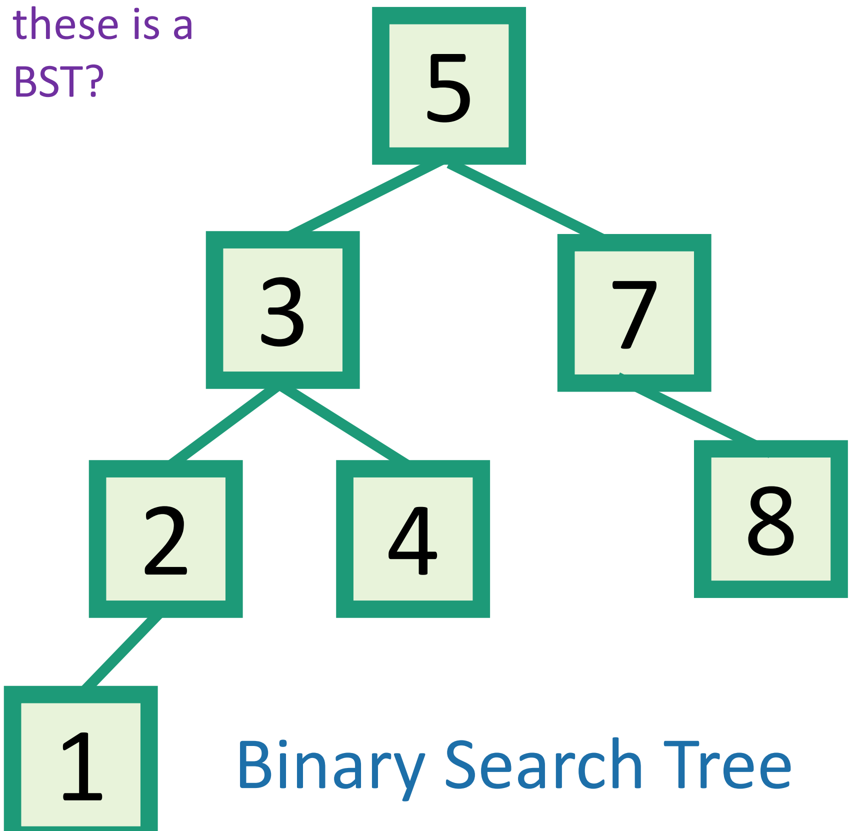
kinda like QuickSort



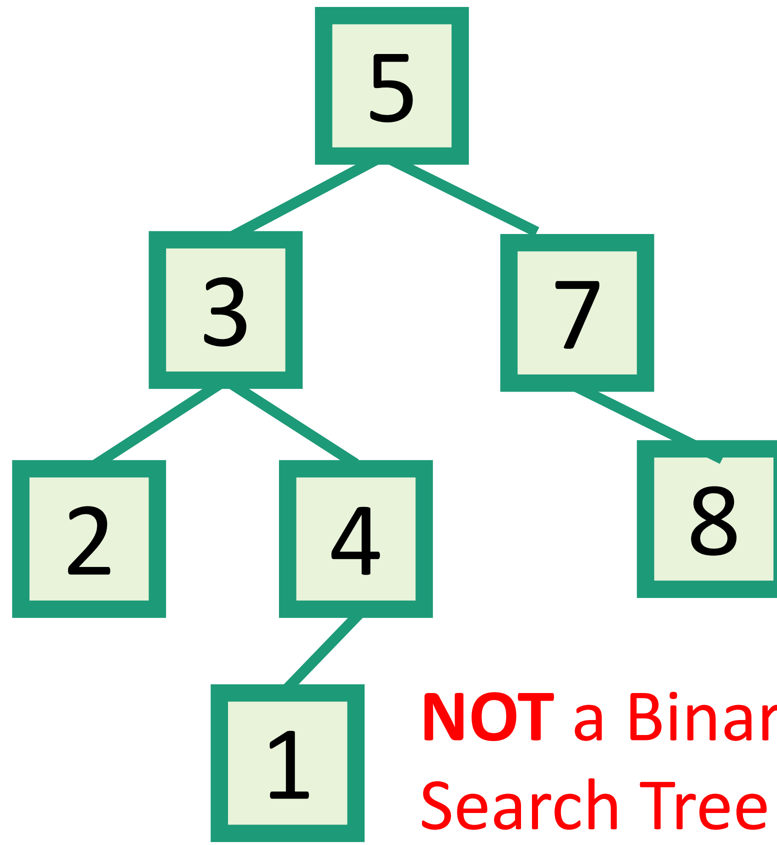
Binary Search Trees

- It's a binary tree so that:
    - **Every LEFT descendant** of a node has key less than that node.
    - **Every RIGHT descendant** of a node has key larger than that node.

Which of these is a BST?



Binary Search Tree



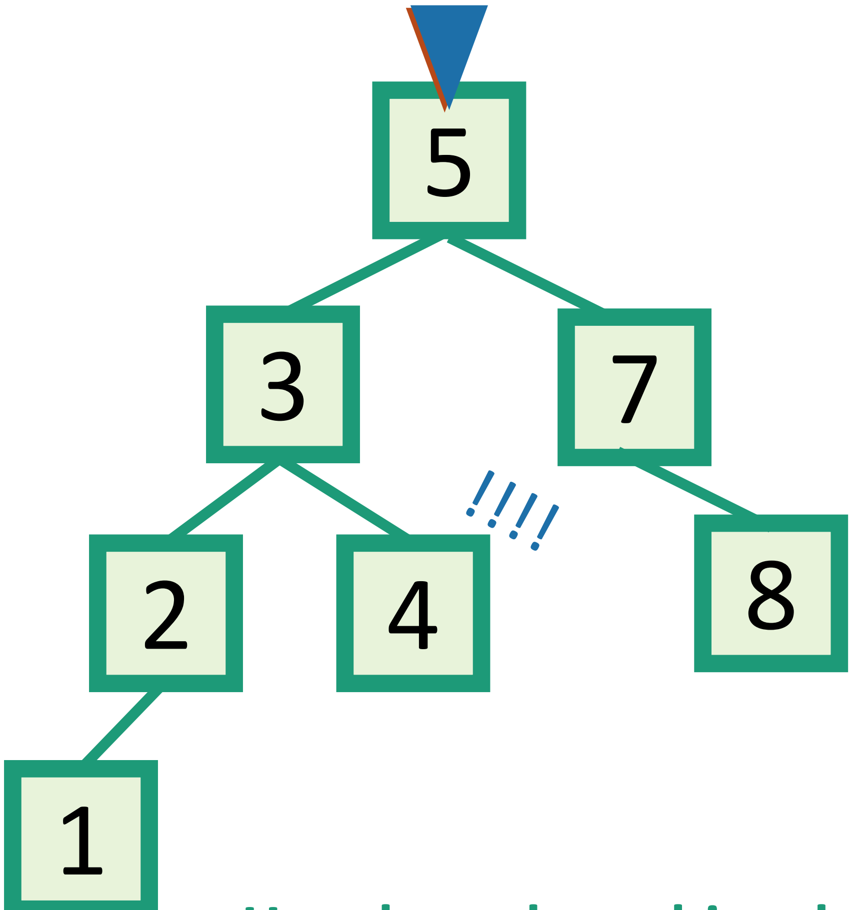
NOT a Binary Search Tree

Remember the goal

Fast SEARCH/INSERT/DELETE

Can we do these?

SEARCH in a Binary Search Tree
definition by example



EXAMPLE: Search for 4.

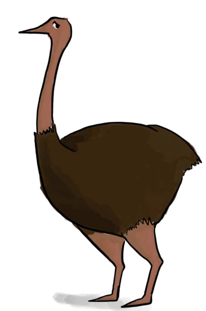
EXAMPLE: Search for 4.5

- It turns out it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

How long does this take?

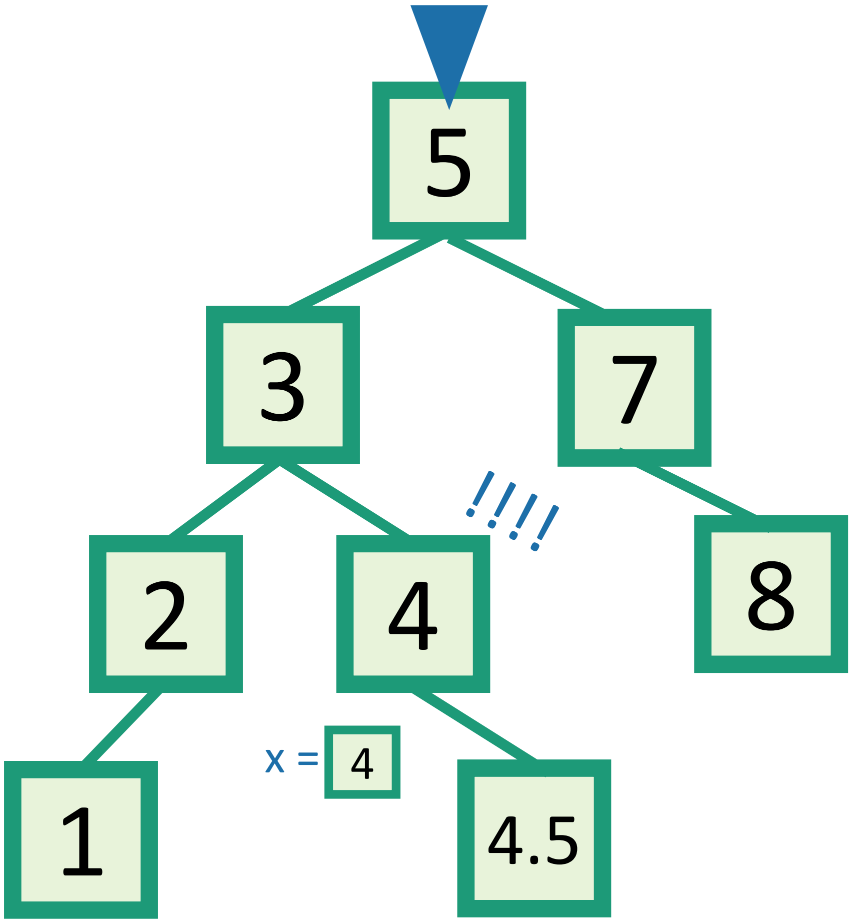
O(length of longest path) = O(height)

Write pseudocode (or actual code) to implement this!



Ollie the over-achieving ostrich

INSERT in a Binary Search Tree



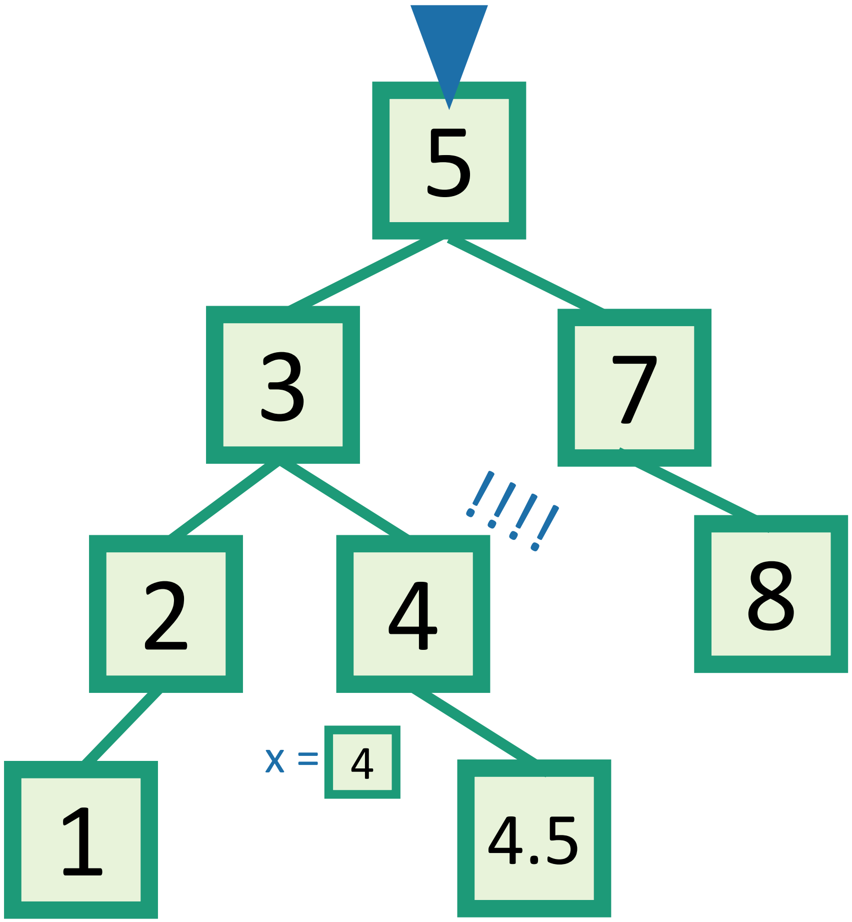
EXAMPLE: Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **Insert** a new node with desired key at x…

You thought about this on your pre-lecture exercise! (See hidden slide for pseudocode.)

INSERT in a Binary Search Tree

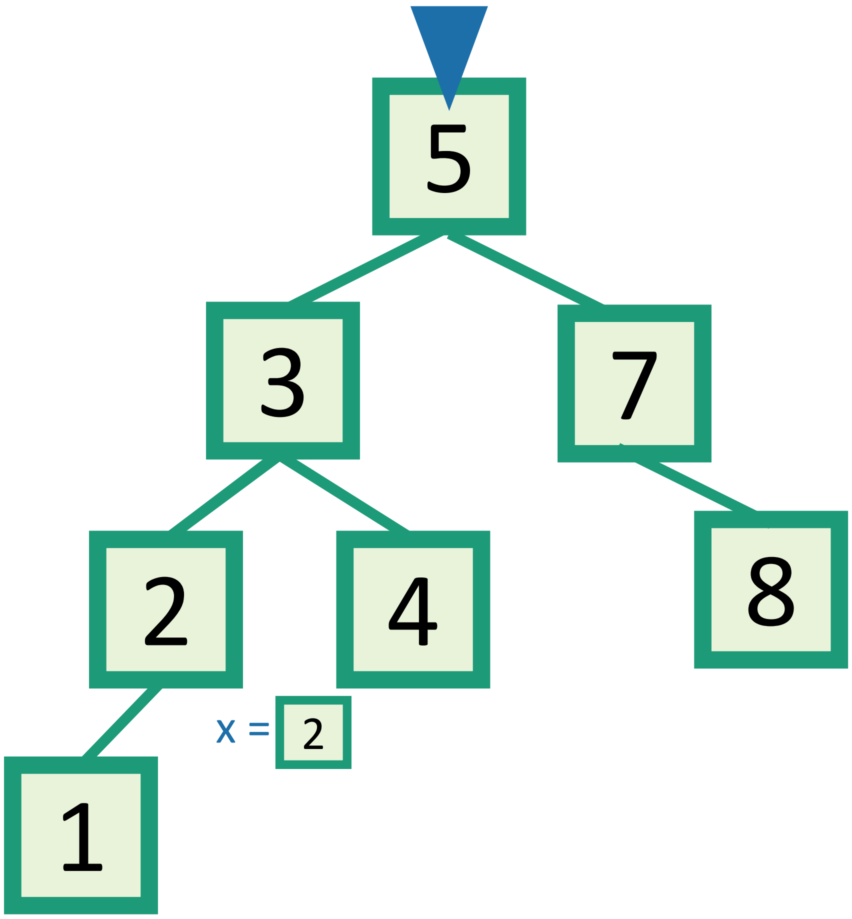
This slide skipped in class – here for reference



EXAMPLE: Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **if** key > x.key:
    - Make a new node with the correct key, and put it as the right child of x.
  - **if** key < x.key:
    - Make a new node with the correct key, and put it as the left child of x.
  - **if** x.key == key:
    - **return**

DELETE in a Binary Search Tree



EXAMPLE: Delete 2

- DELETE(key):
  - x = SEARCH(key)
  - **if** x.key == key:
    - ....delete x....

You thought about this in your pre-lecture exercise too!

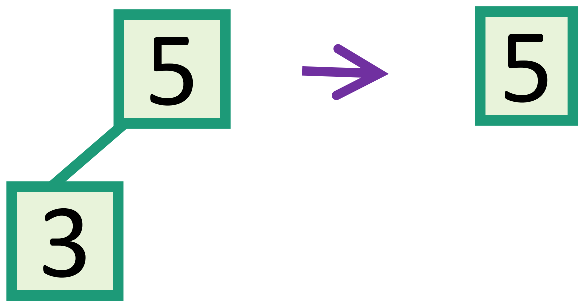
This is a bit more complicated...see the hidden slides for some pictures of the different cases.

DELETE in a Binary Search Tree

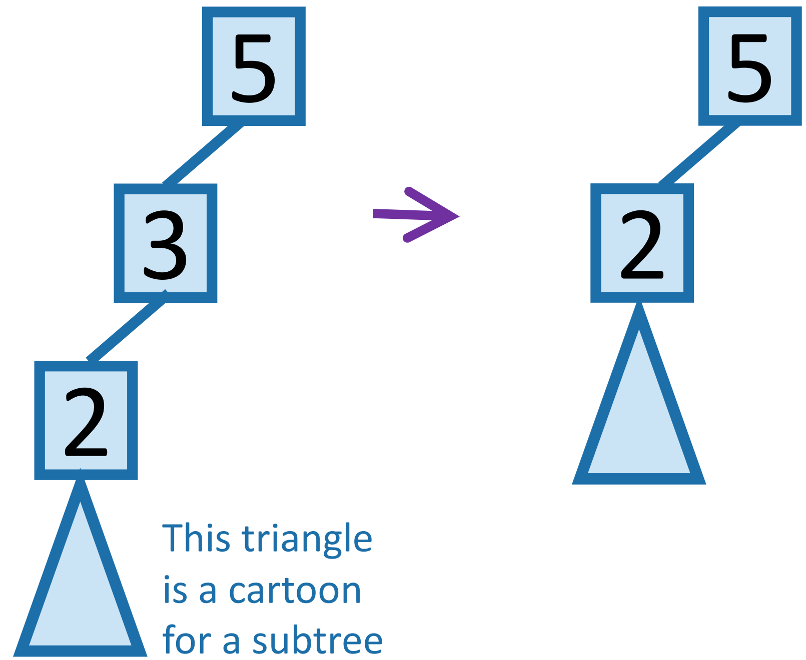
several cases (by example)

say we want to delete 3

This slide skipped in class – here for reference!

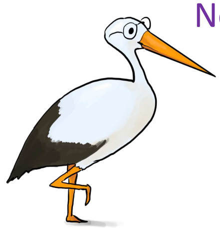


Case 1: if 3 is a leaf, just delete it.



Case 2: if 3 has just one child, move that up.

Write pseudocode for all of these! (Or see IPython Notebook for Lecture 7)



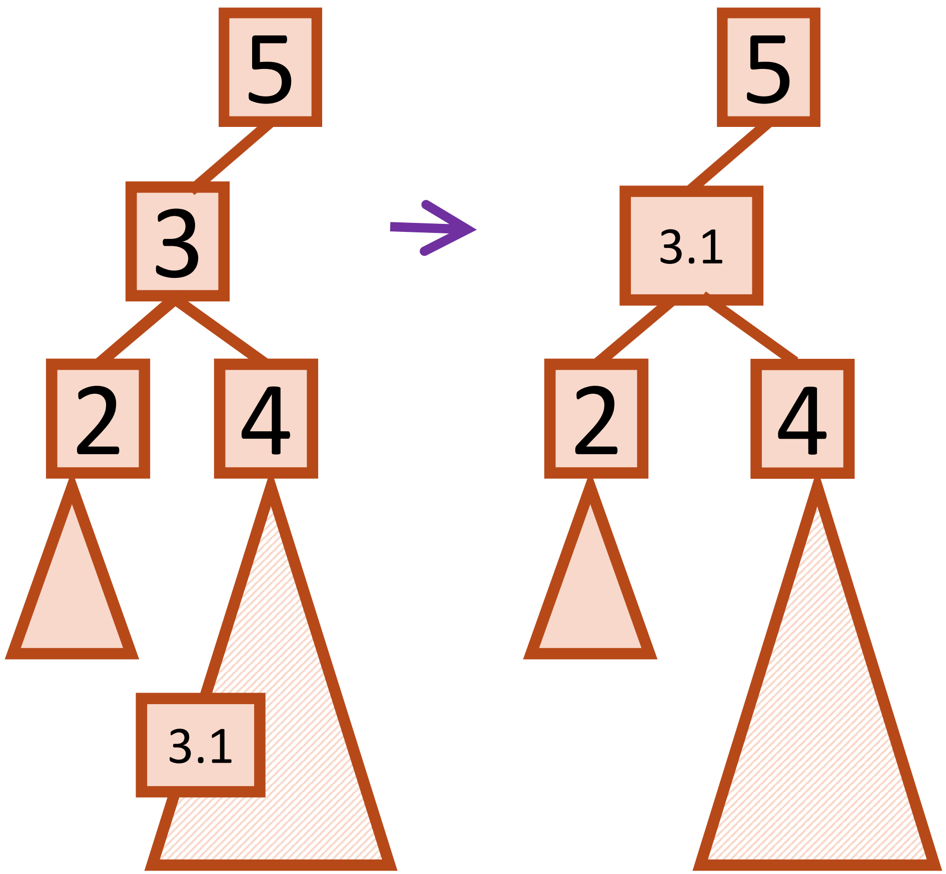
Siggi the Studious Stork

DELETE in a Binary Search Tree

ctd.

This slide skipped in class – here for reference!

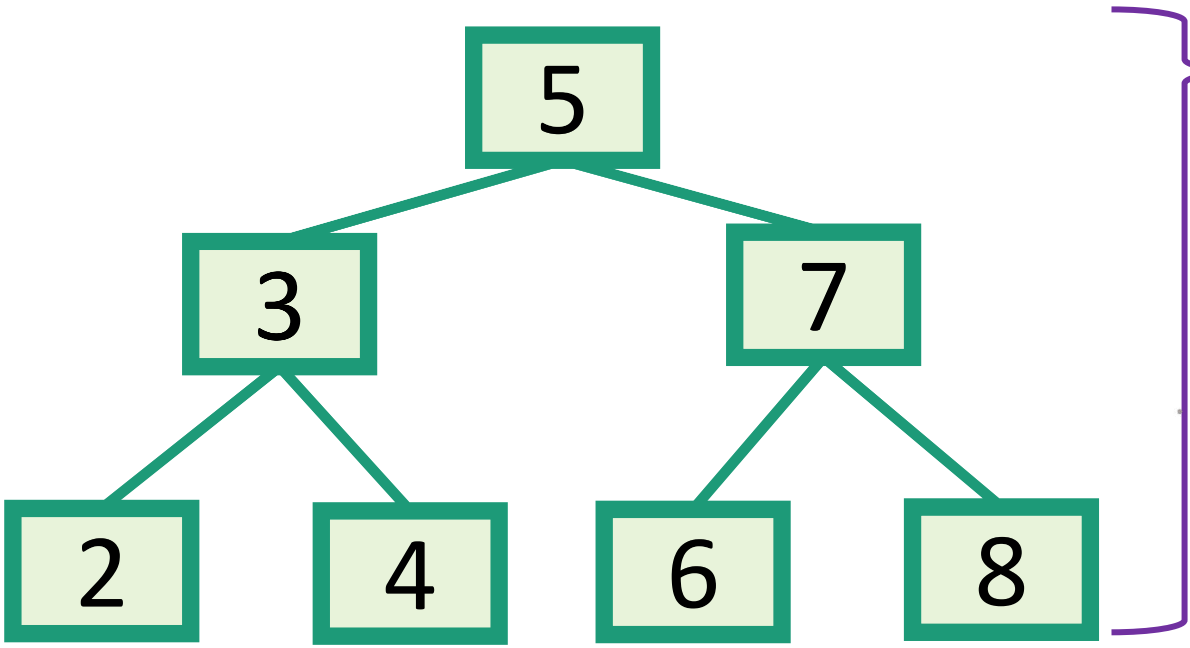
Case 3: if 3 has two children,
replace 3 with it's immediate successor.
(aka, next biggest thing after 3)



- Does this maintain the BST property?
  - Yes.
- How do we find the immediate successor?
  - SEARCH for 3 in the subtree under 3.right
- How do we remove it when we find it?
  - If [3.1] has 0 or 1 children, do one of the previous cases.
- What if [3.1] has two children?
  - It doesn't.

How long do these operations take?

- SEARCH is the big one.
  - Everything else just calls SEARCH and then does some small O(1)-time operation.



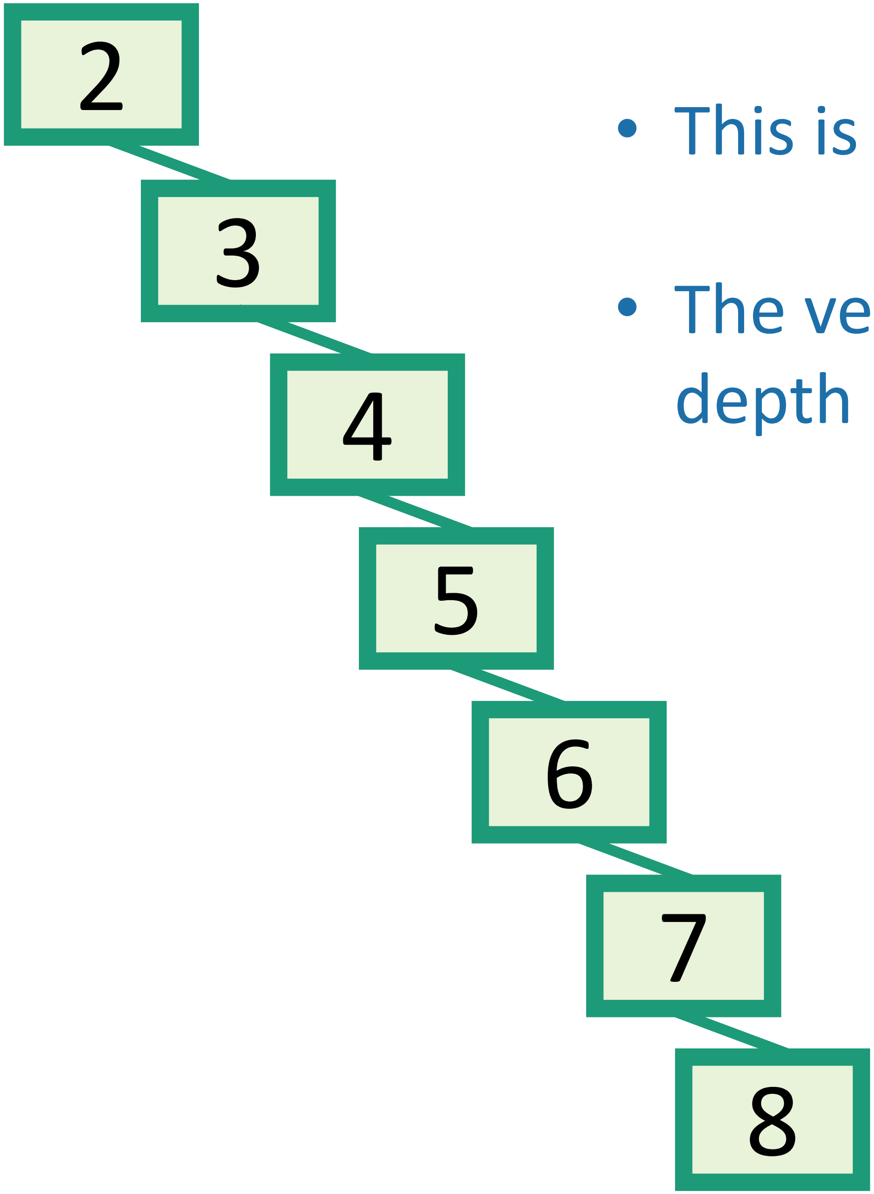
Time = O(height of tree)

How long does search take?

Trees have depth O(log(n)). **Done!**

Lucky the lackadaisical lemur.

Wait...



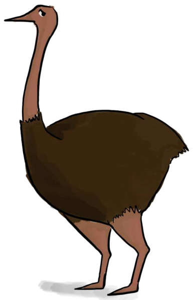
- This is a valid binary search tree.
- The version with n nodes has depth n, **not** O(log(n)).

Could such a tree show up?
In what order would I have to insert the nodes?

Inserting in the order 2,3,4,5,6,7,8 would do it.

So this ***could*** happen.

How often is “every so often” in the worst case? It’s actually pretty often!

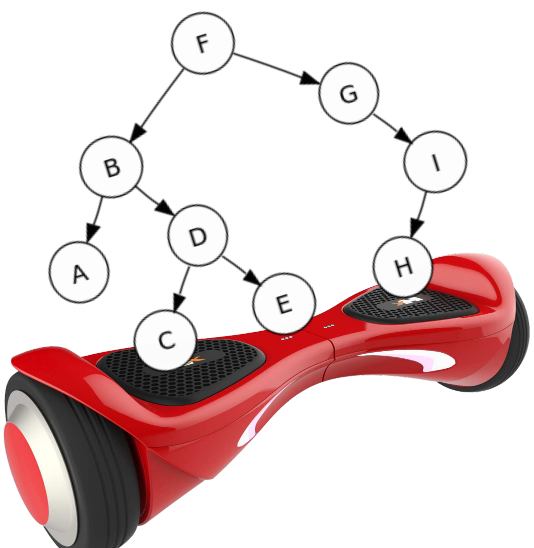


Ollie the over-achieving ostrich

What to do?

- Goal: Fast SEARCH/INSERT/DELETE
- All these things take time O(height)
- And the height might be big!!! ☹
- Idea 0:
  - Keep track of how deep the tree is getting.
  - If it gets too tall, re-do everything from scratch.
    - At least Ω(n) every so often….
- Turns out that’s not a great idea. Instead we turn to…

Self-Balancing Binary Search Trees

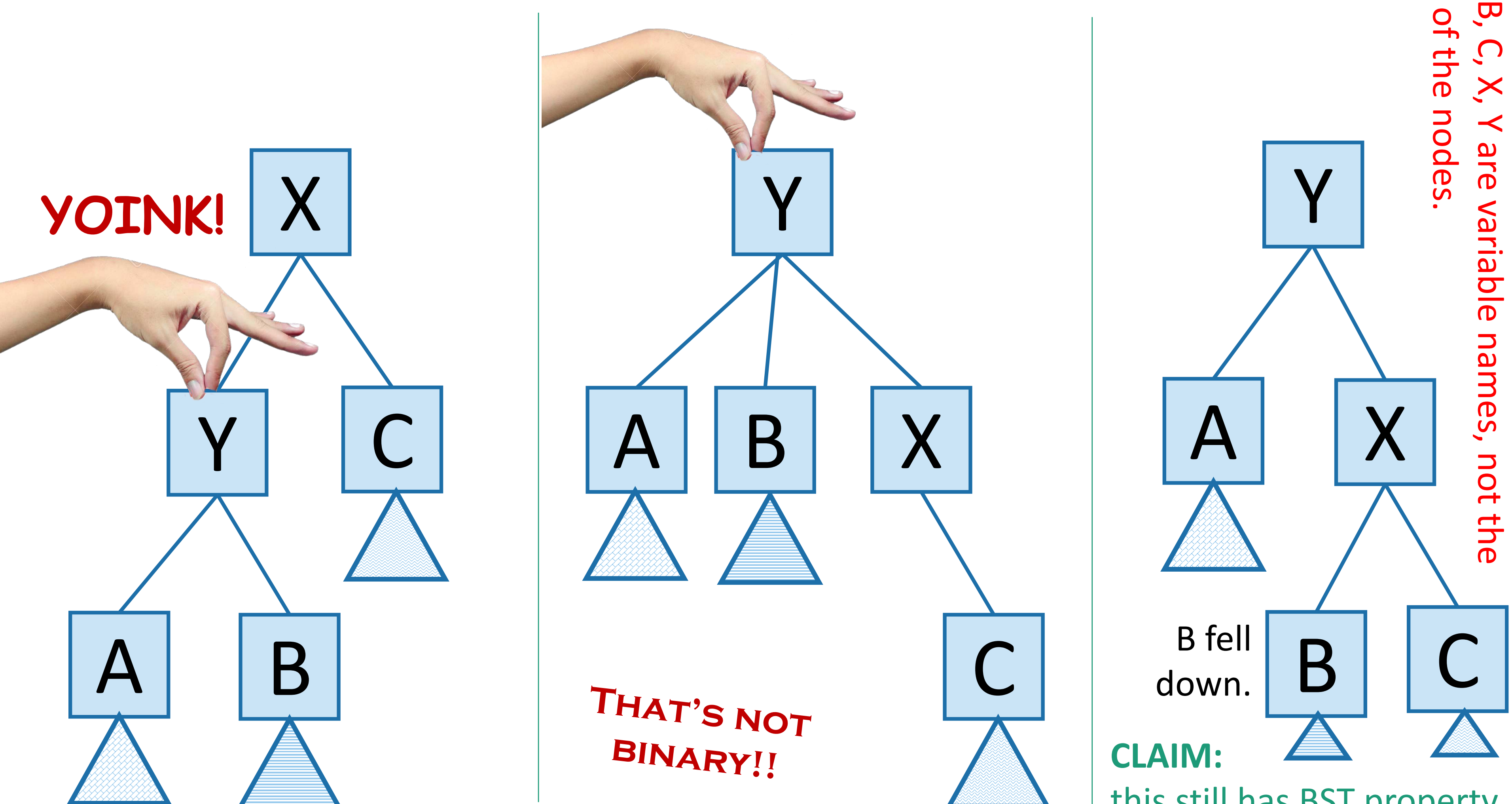


Idea 1: Rotations

No matter what lives underneath A,B,C, **this takes time O(1).** (Why?)

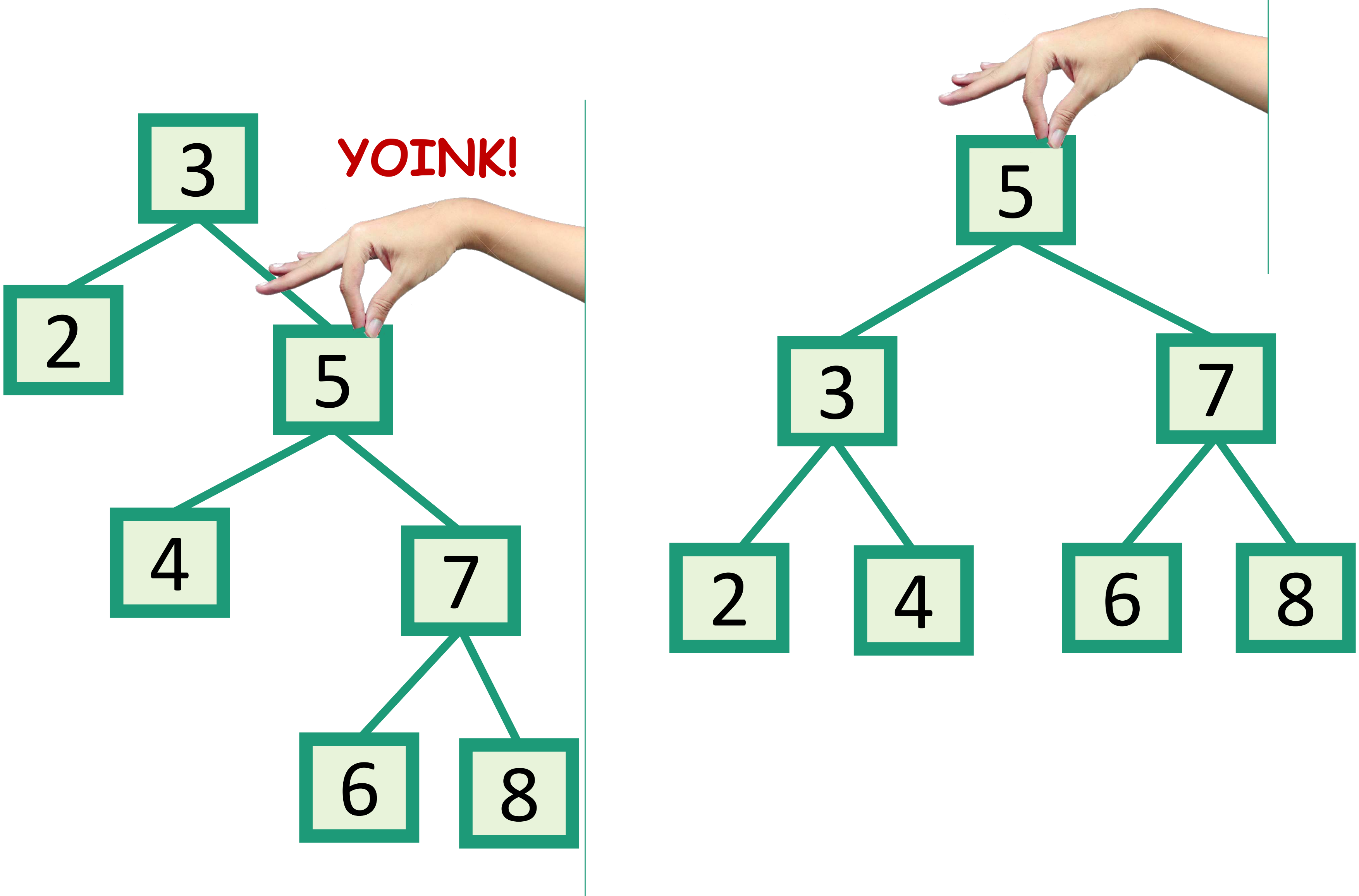
- Maintain Binary Search Tree (BST) property, while moving stuff around.

Note: A, B, C, X, Y are variable names, not the contents of the nodes.



CLAIM:
this still has BST property.

This seems helpful



Does this work?

- Whenever something seems unbalanced, do rotations until it's okay again.

Lucky the Lackadaisical Lemur

Even for me this is pretty vague. What do we mean by "seems unbalanced"? What's "okay"?

Idea 2: have some proxy for balance

- Maintaining perfect balance is too hard.
- Instead, come up with some proxy for balance:
  - If the tree satisfies [SOME PROPERTY], then it's pretty balanced.
  - We can maintain [SOME PROPERTY] using rotations.

There are actually several ways to do this, but today we'll see…

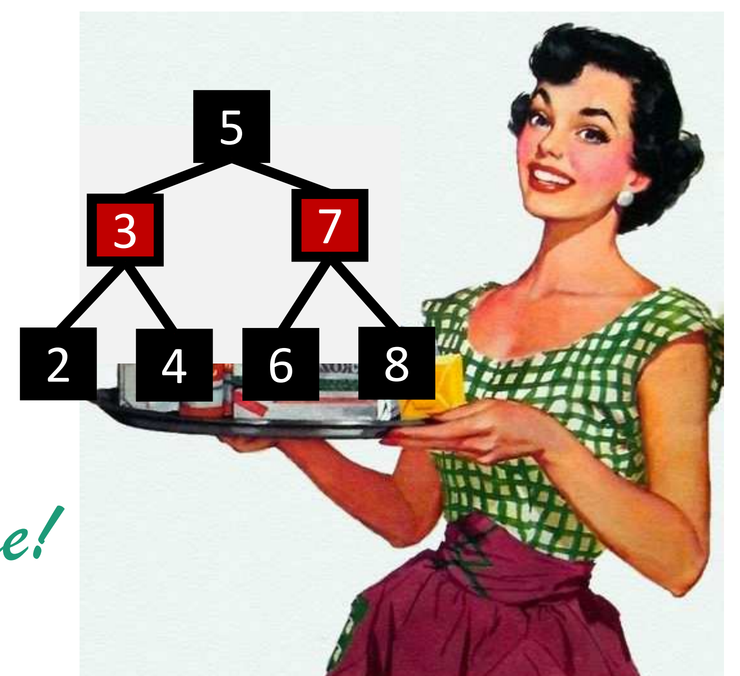
Red-Black Trees

- A Binary Search Tree that balances itself!
- No more time-consuming by-hand balancing!
- Be the envy of your friends and neighbors with the time-saving...

Red-Black tree!

Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.

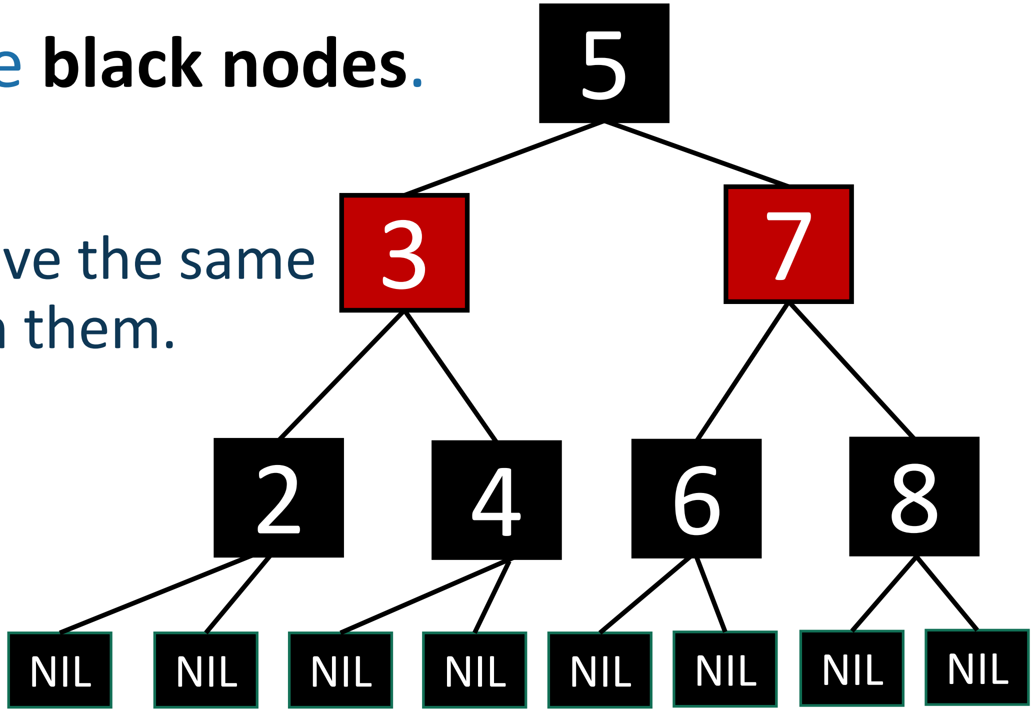
It's just good sense!



Red-Black Trees

these rules are the proxy for balance

- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NIL's have the same number of **black nodes** on them.



I'm not going to draw the NIL children in the future, but they are treated as black nodes.

Examples(?)

- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
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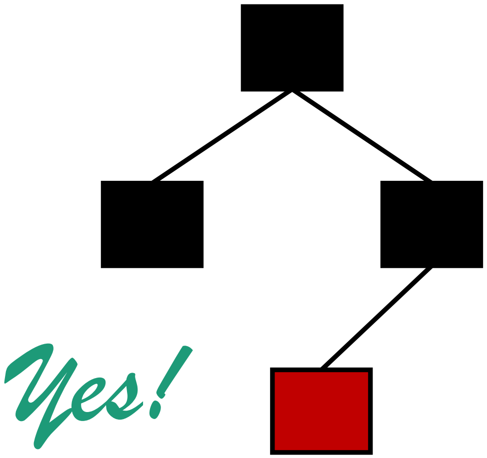
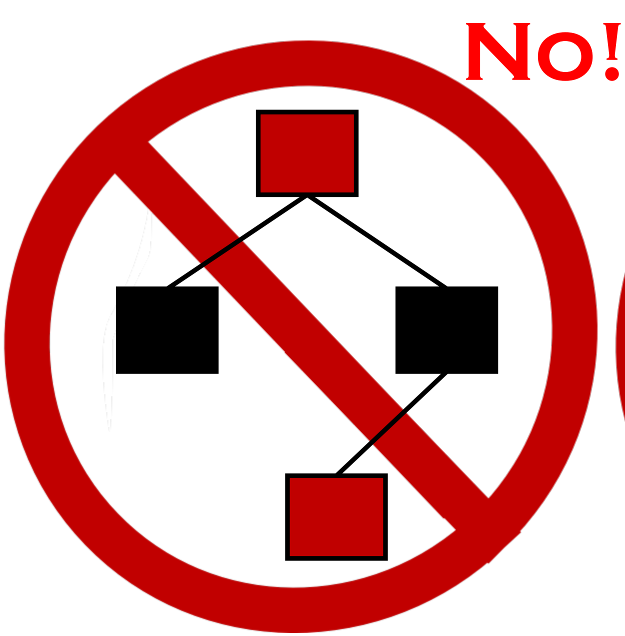
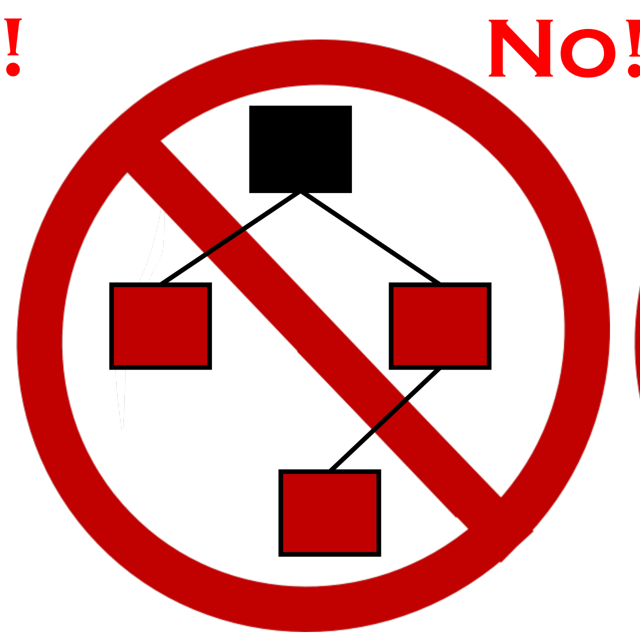
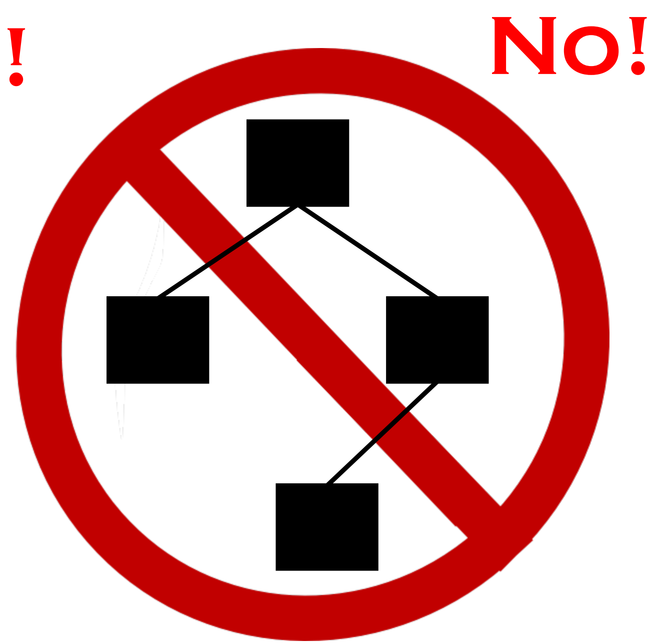


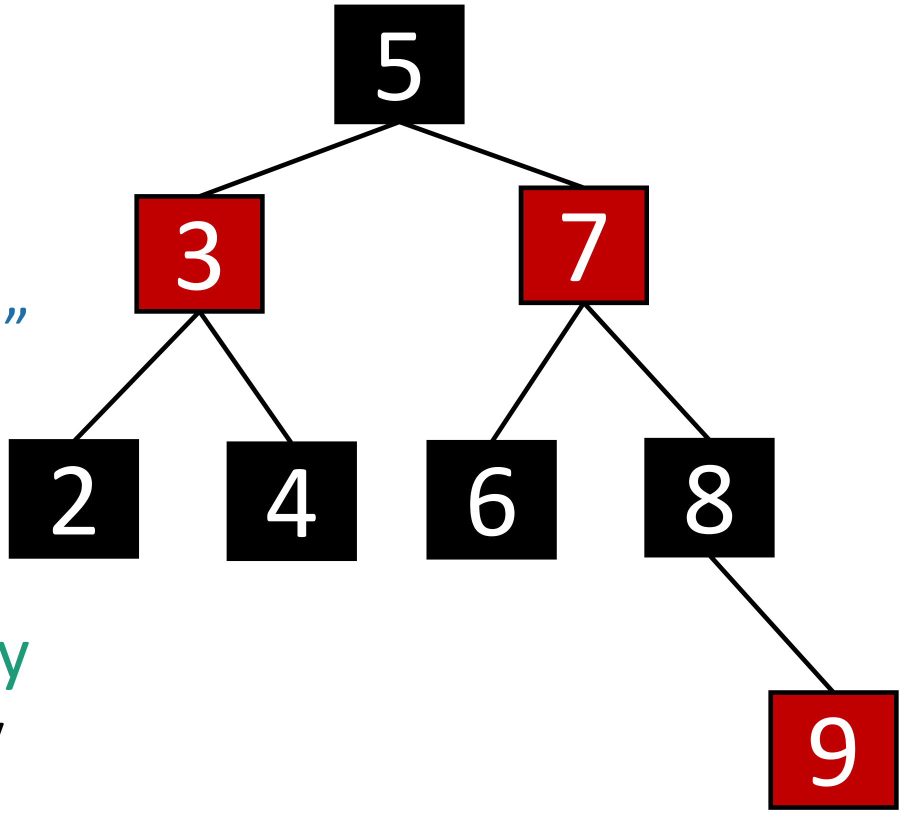






Why???????

- This is pretty balanced.
  - The **black nodes** are balanced
  - The **red nodes** are "spread out" so they don't mess things up too much.



- We can maintain this property as we insert/delete nodes, by using rotations.

This is the really clever idea!

This **Red-Black** structure is a proxy for balance.

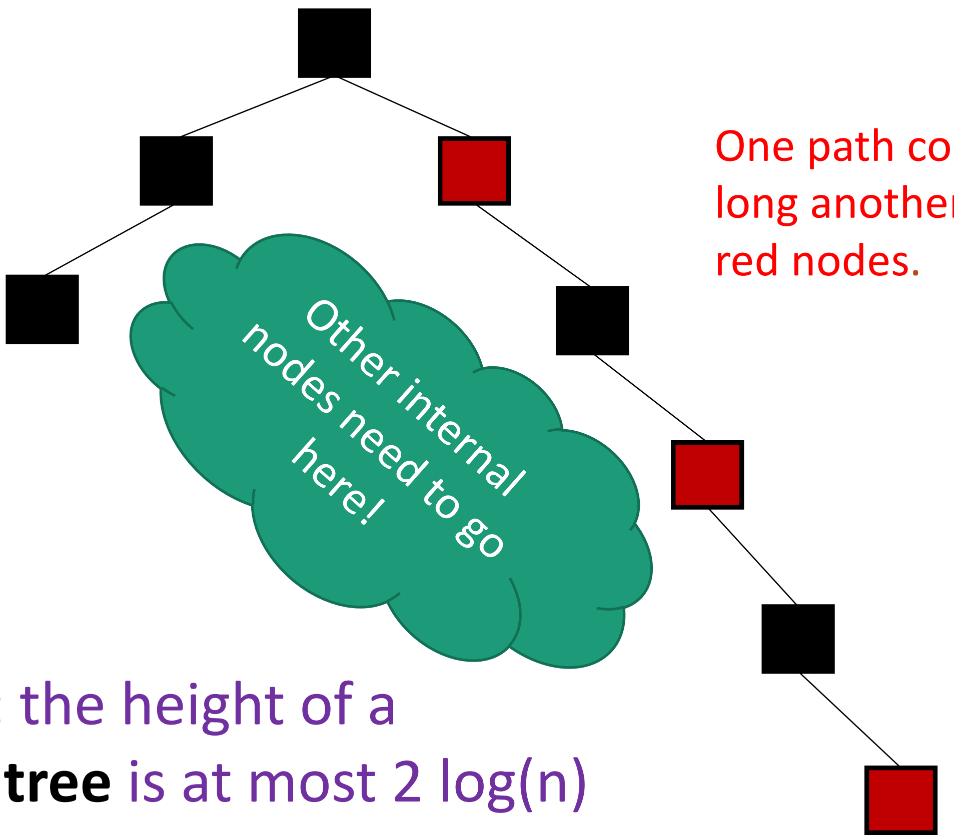
It's just a smidge weaker than perfect balance, but we can actually maintain it!

Let’s build some intuition!

Lucky the lackadaisical lemur

This is “pretty balanced”

- To see why, intuitively, let’s try to build a Red-Black Tree that’s unbalanced.



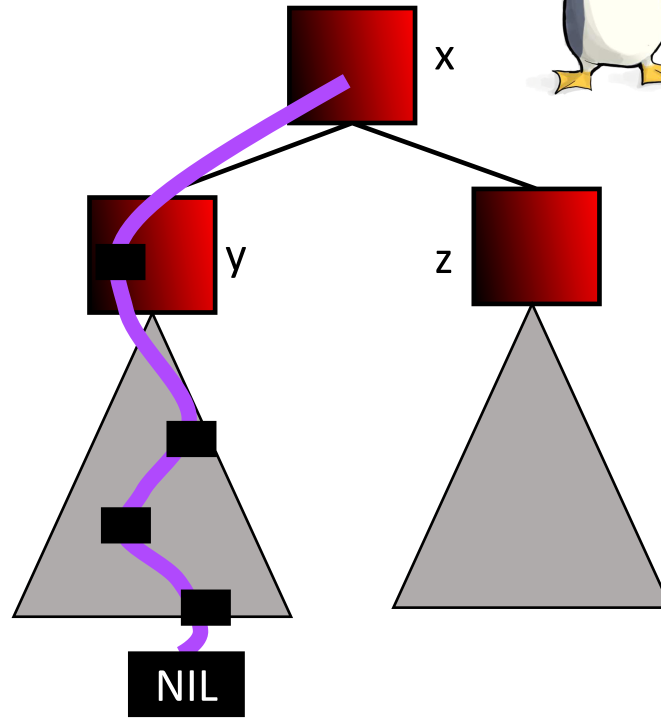
One path could be twice as long another if we pad it with red nodes.

Conjecture: the height of a **red-black tree** is at most 2 log(n)

That turns out to be basically right.

[proof sketch]

- Say there are b(x) black nodes in any path from x to NIL.
  - (excluding x, including NIL).
- **Claim:**
  - Then there are at least $2^{b(x)} - 1$ non-NIL nodes in the subtree underneath x. (Including x).
- [Proof by induction – on board if time]



Then:

$$n \geq 2^{b(root)} - 1$$ using the **Claim**

$$\geq 2^{height/2} - 1$$ b(root) >= height/2 because of RBTree rules.

Rearranging:

$$n + 1 \geq 2^{height/2} \Rightarrow height \leq 2\log(n+1)$$

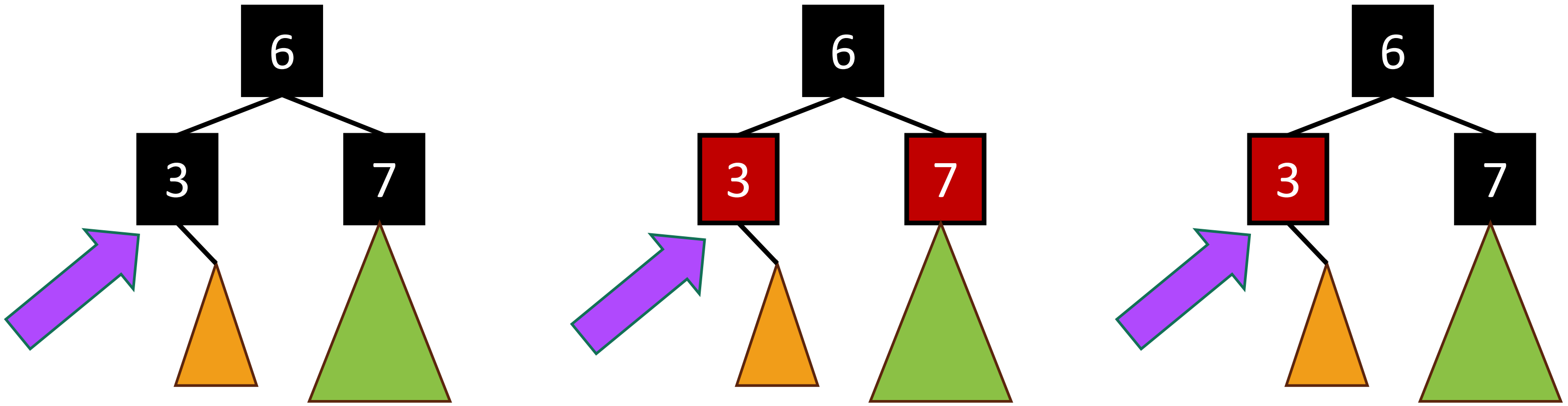
Okay, so it's balanced…
…but can we maintain it?

- Yes!

- For the rest of lecture:
  - sketch of how we'd do this.

- See CLRS for more details.
- (You are not responsible for the details for this class – but you should understand the main ideas).

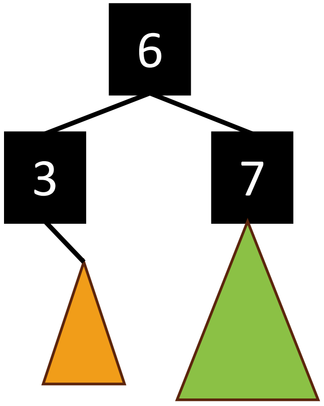
Many cases



- Suppose we want to insert **here**.
  - eg, want to insert 0.

- And then there are 9 more cases for all of the various symmetries of these 3 cases…

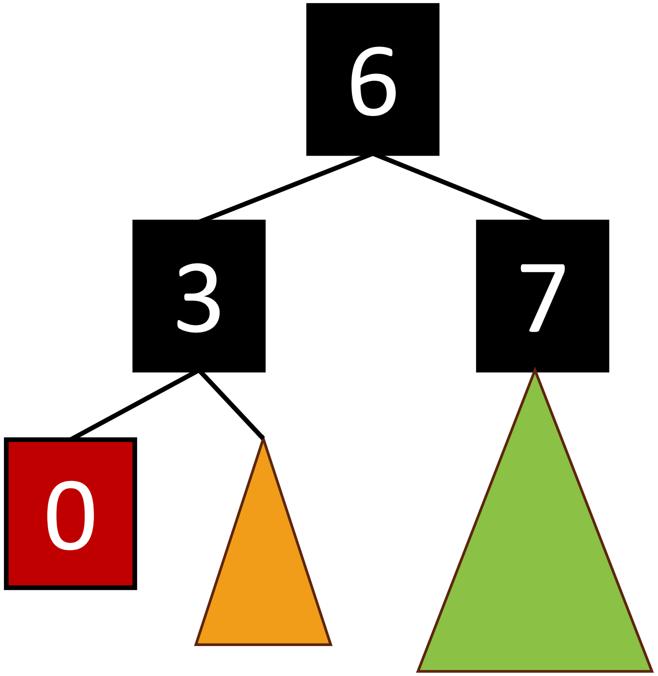
Inserting into a Red-Black Tree



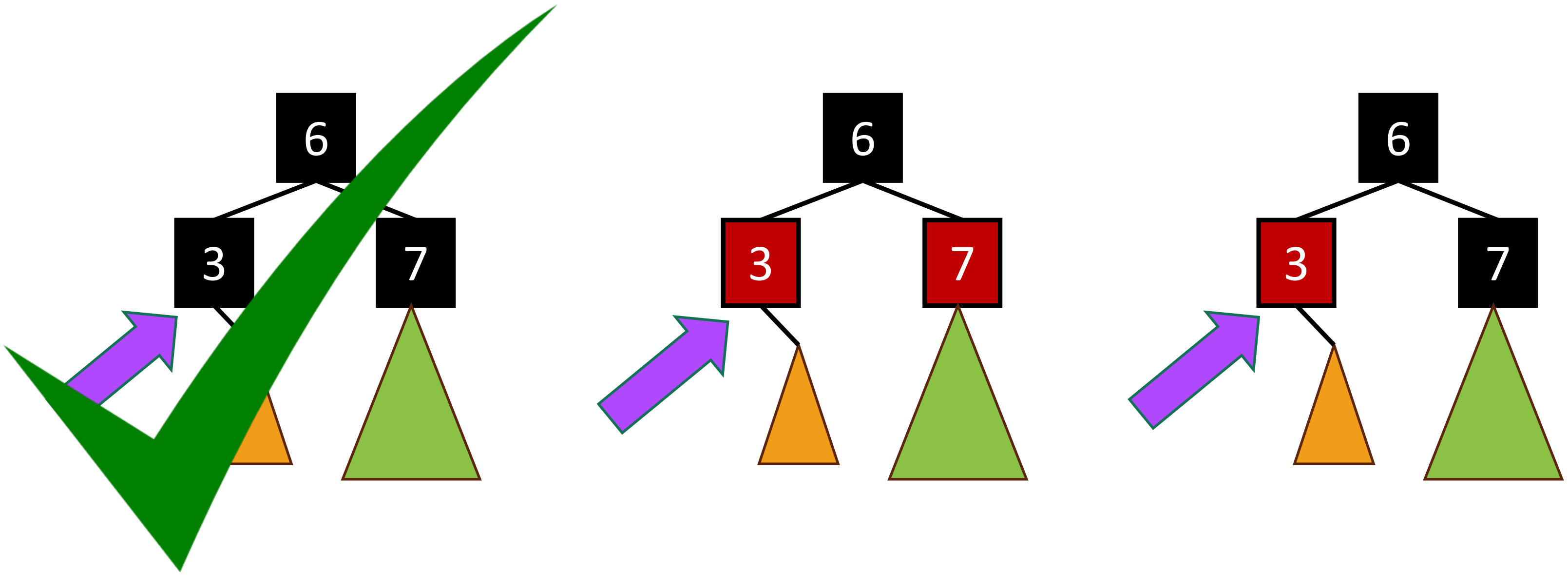
What if it looks like this?

- Make a new **red node**.
- Insert it as you would normally.

Example: insert 0



Many cases



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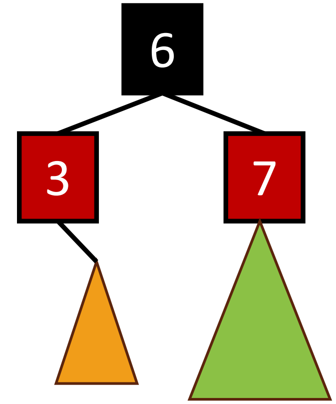
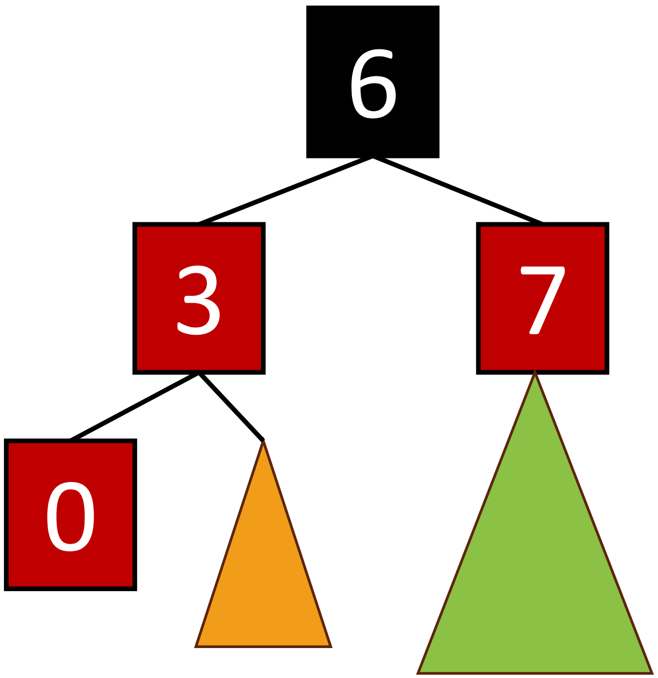
Inserting into a Red-Black Tree



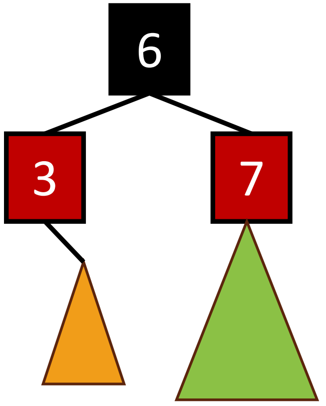
What if it looks like this?

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.

Example: insert 0



Inserting into a Red-Black Tree



What if it looks like this?

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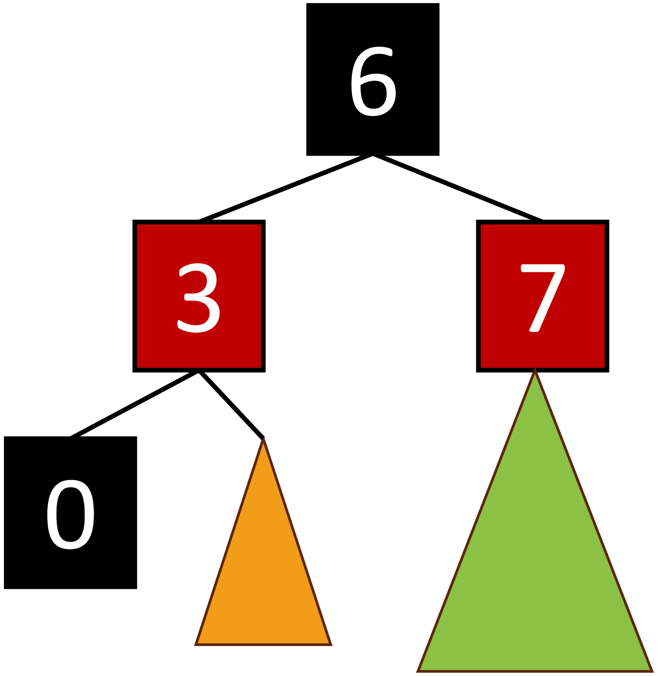
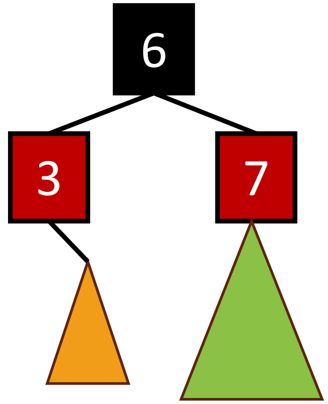


Example: insert 0

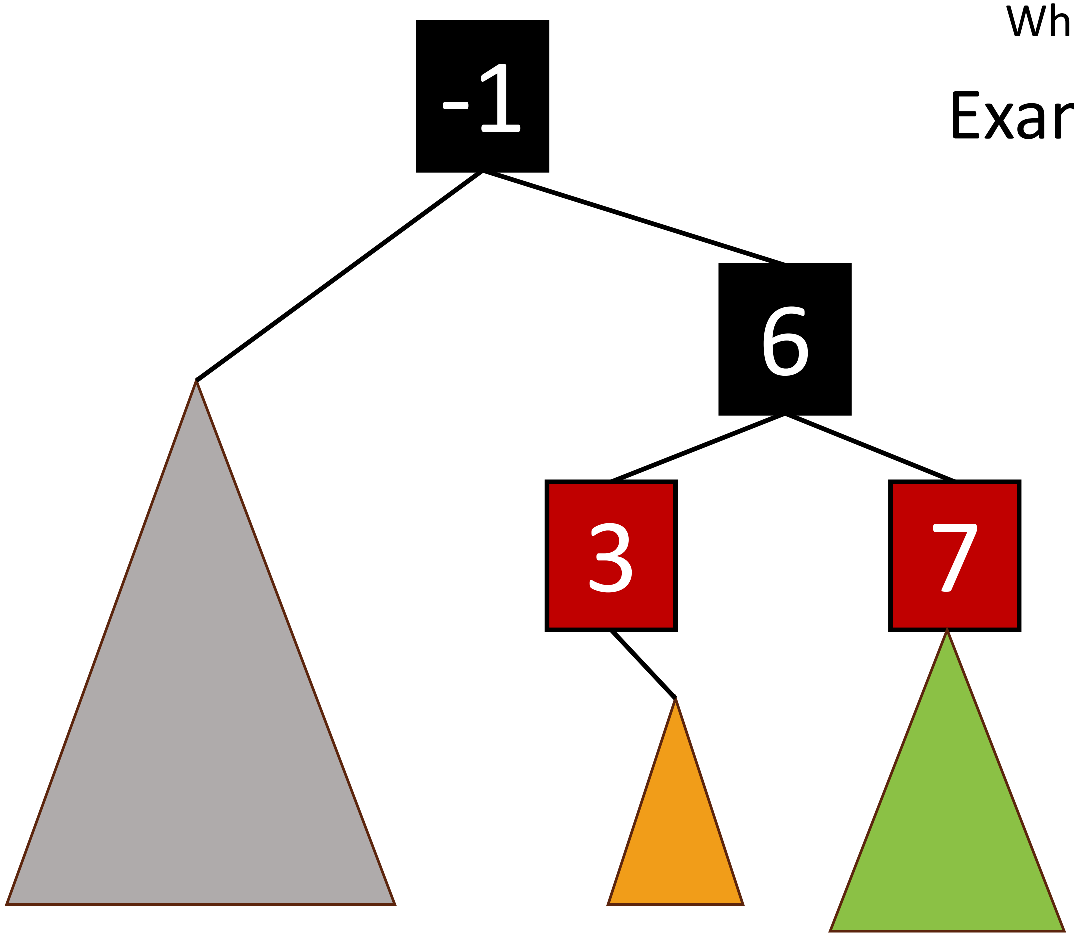
Can’t we just insert 0 as a **black node?**

We need a bit more context



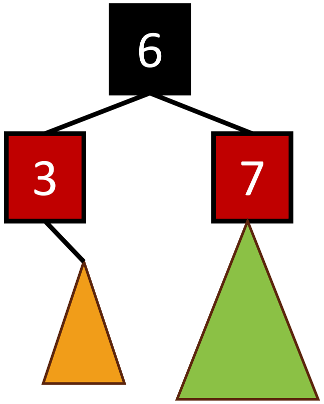
What if it looks like this?

Example: insert 0



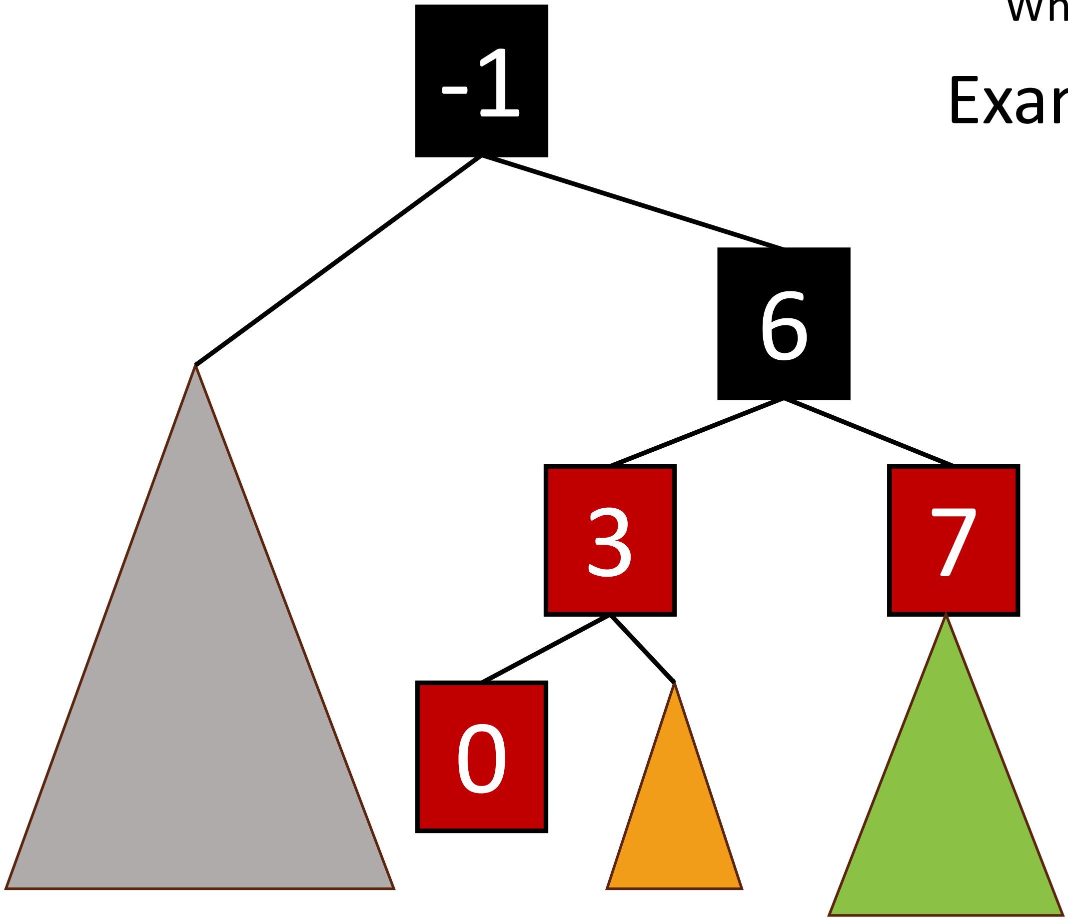
We need a bit more context

- Add 0 as a red node.



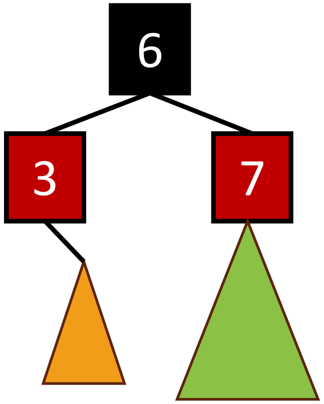
What if it looks like this?

Example: insert 0

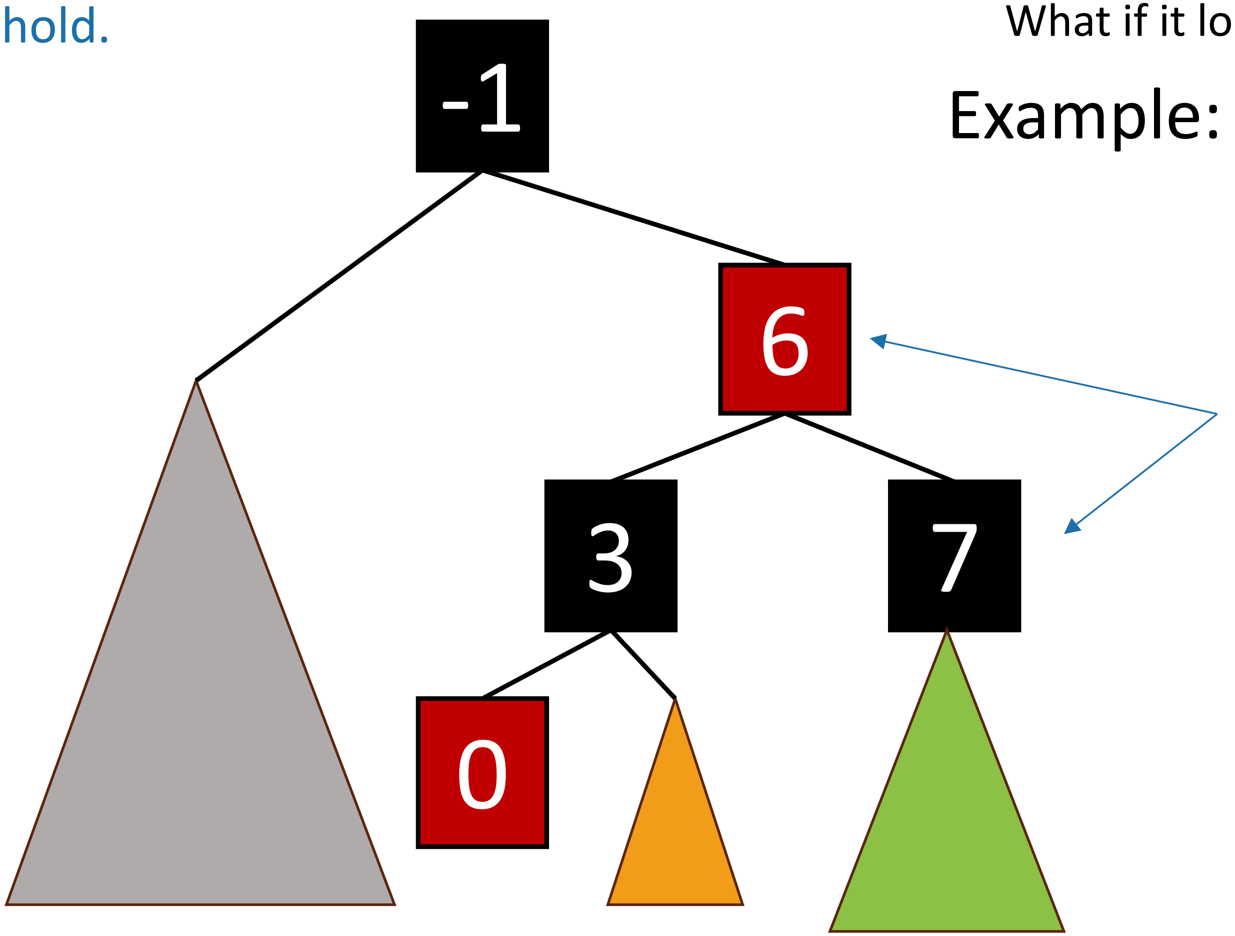


We need a bit more context

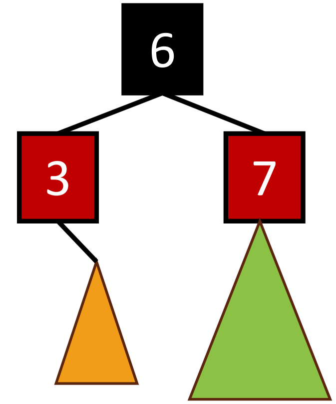
- Add 0 as a red node.
- **Claim:** RB-Tree properties still hold.



Example: insert 0

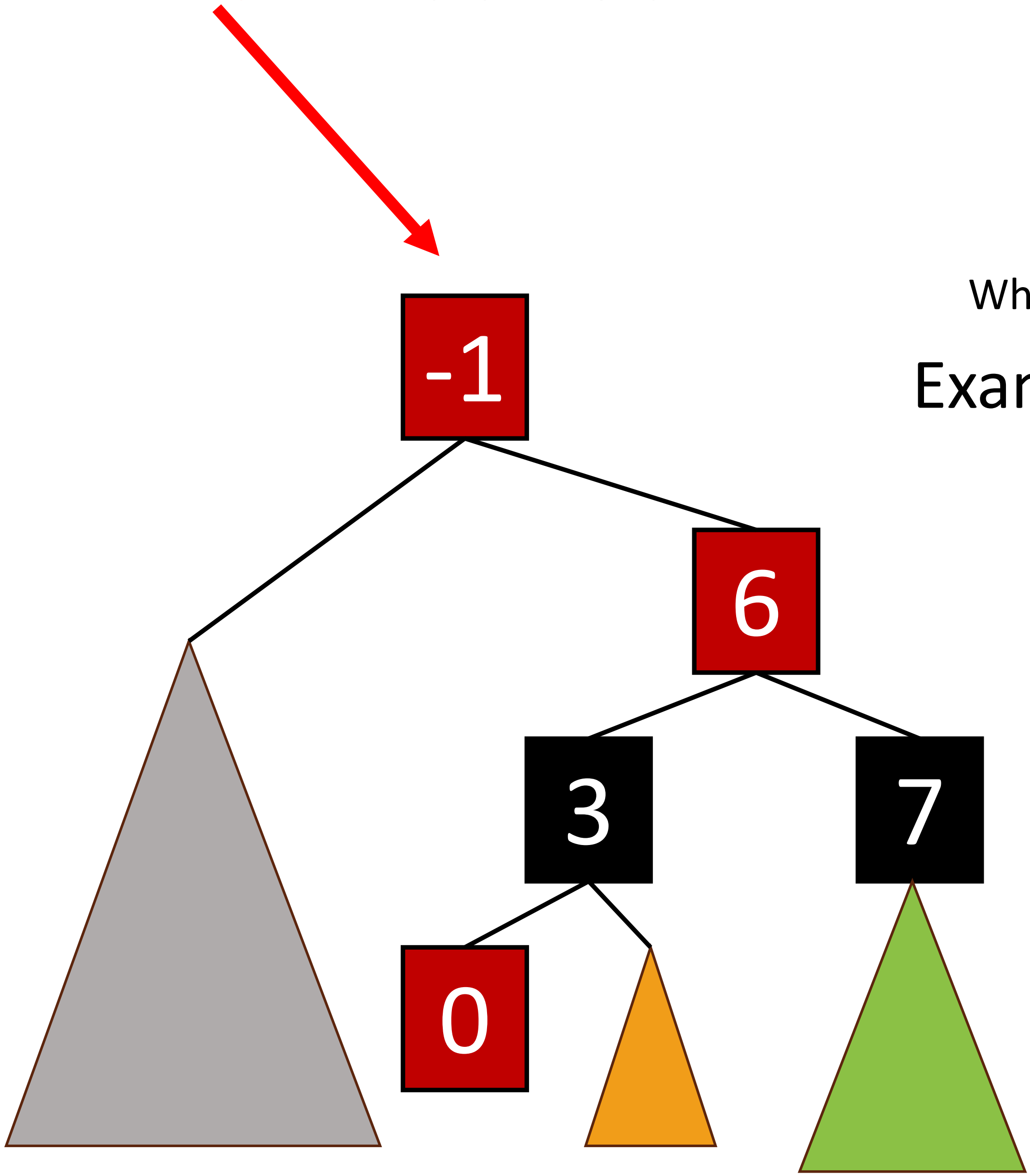


But what if that was red?

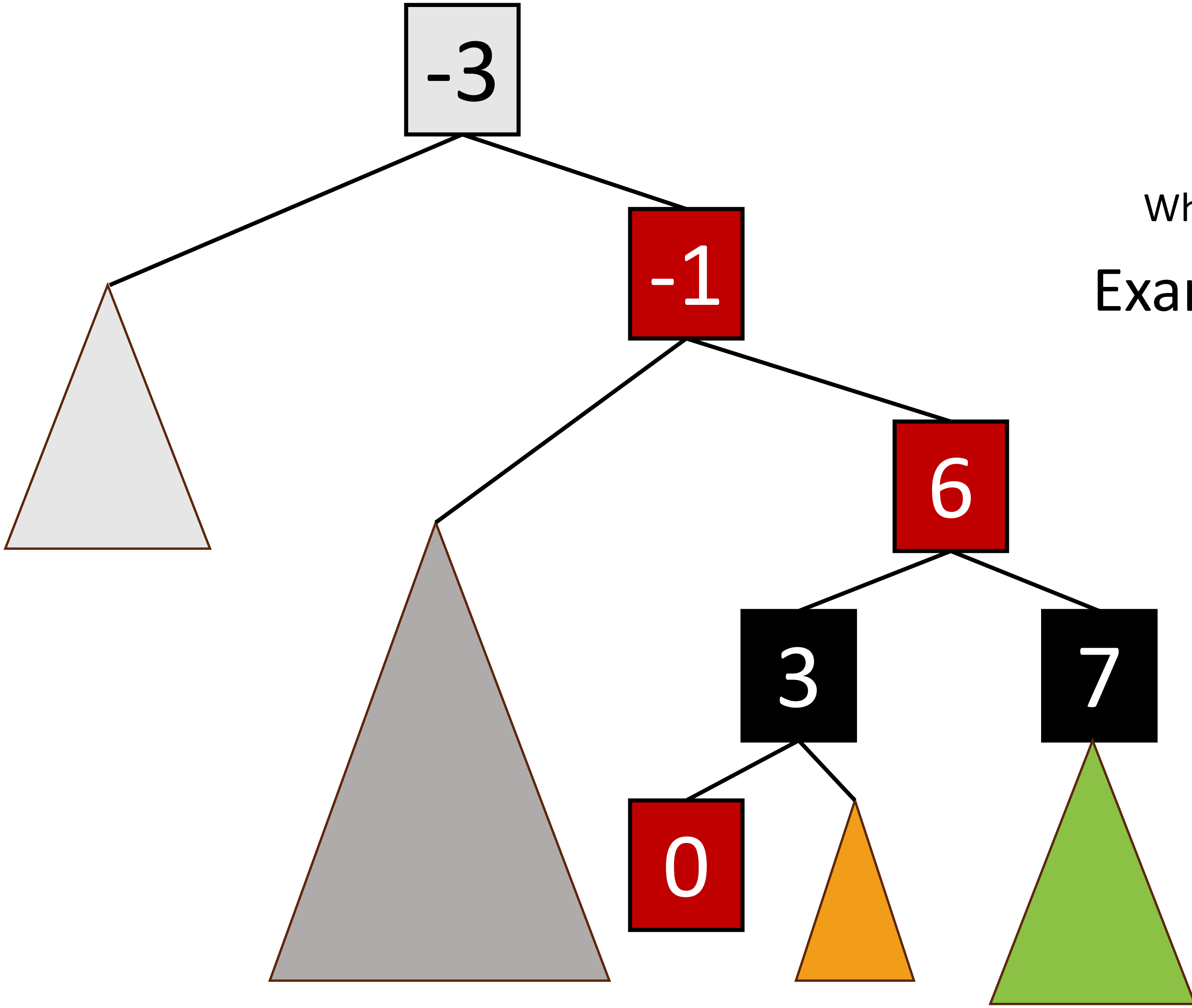


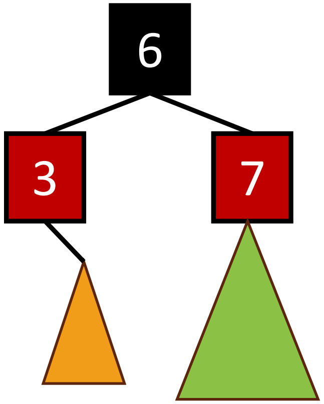
What if it looks like this?

Example: insert 0



More context…

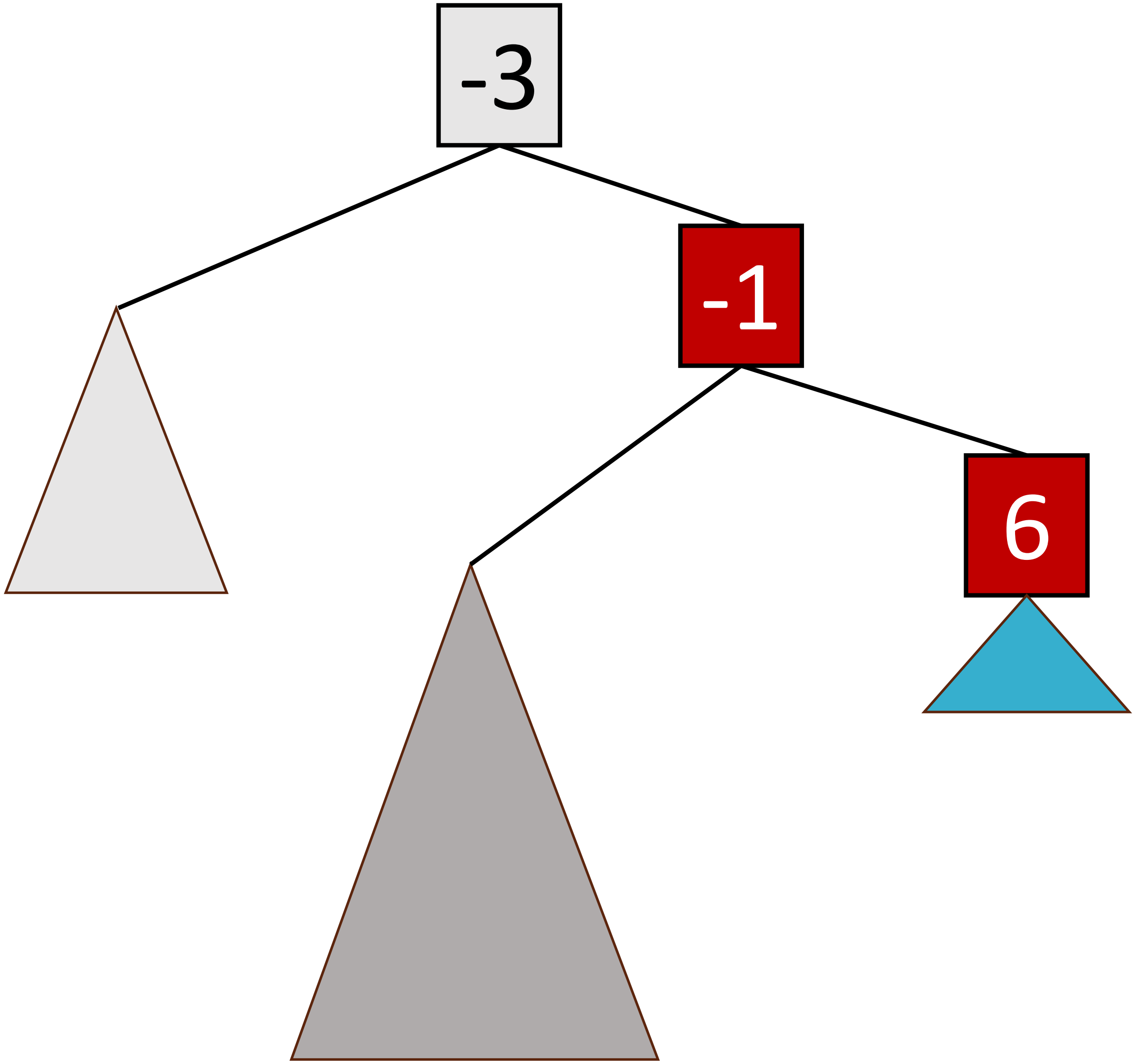


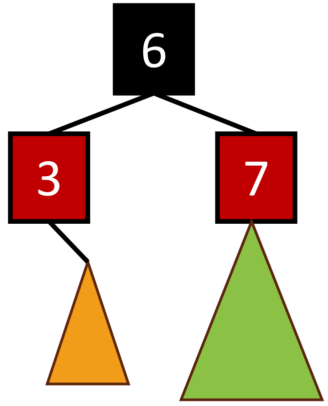


What if it looks like this?

Example: insert 0

More context…



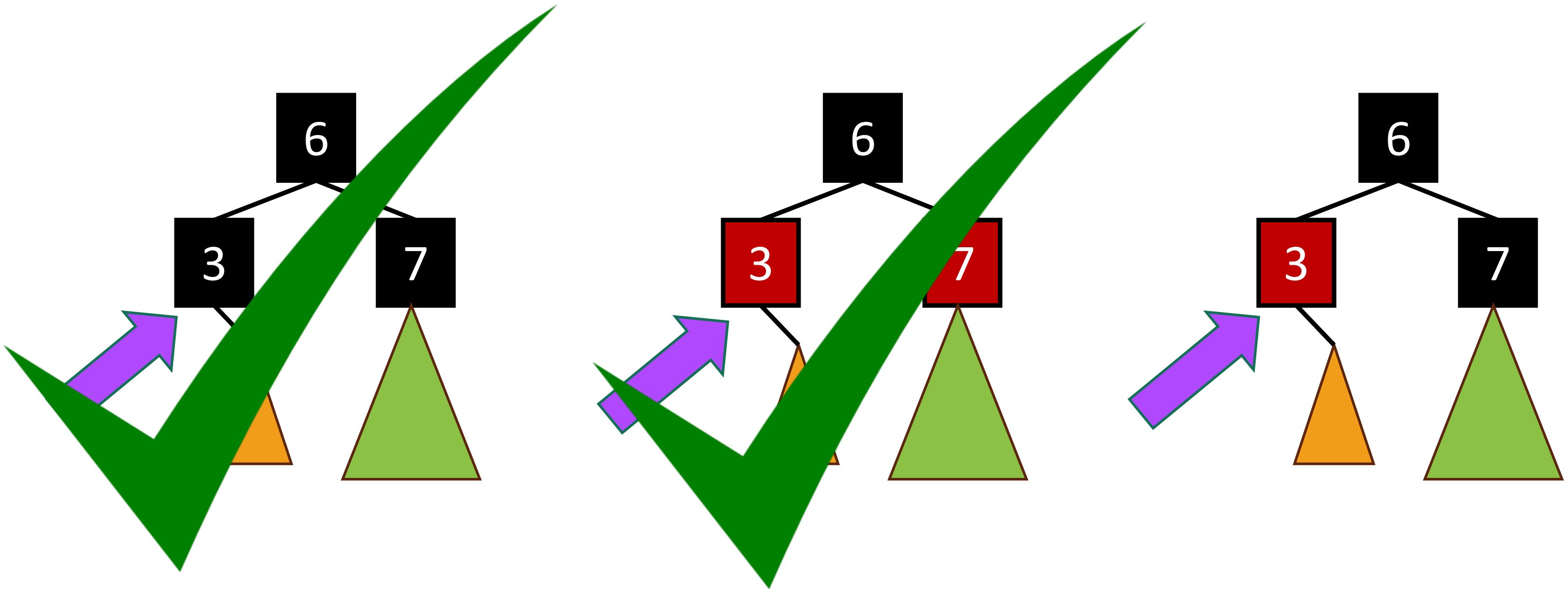


What if it looks like this?

Example: insert 0

Now we're basically inserting 6 into some smaller tree. Recurse!

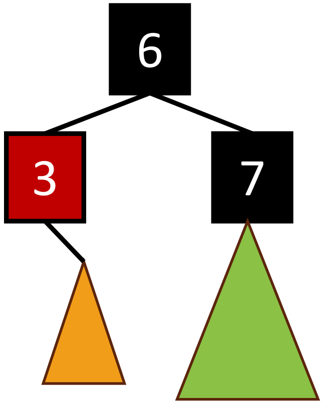
Many cases



- Suppose we want to insert **here**.
    - eg, want to insert 0.

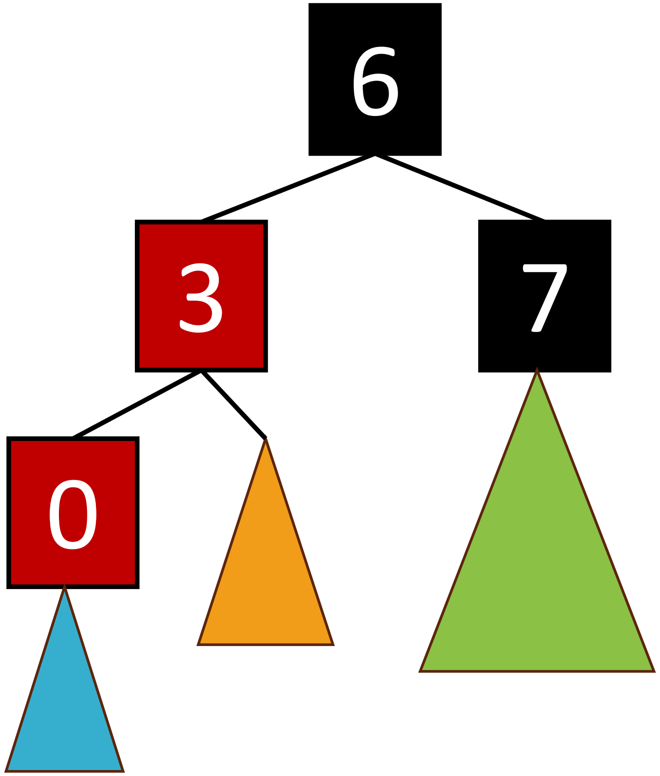
- And then there are 9 more cases for all of the various symmetries of these 3 cases…

Inserting into a Red-Black Tree



What if it looks like this?

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.

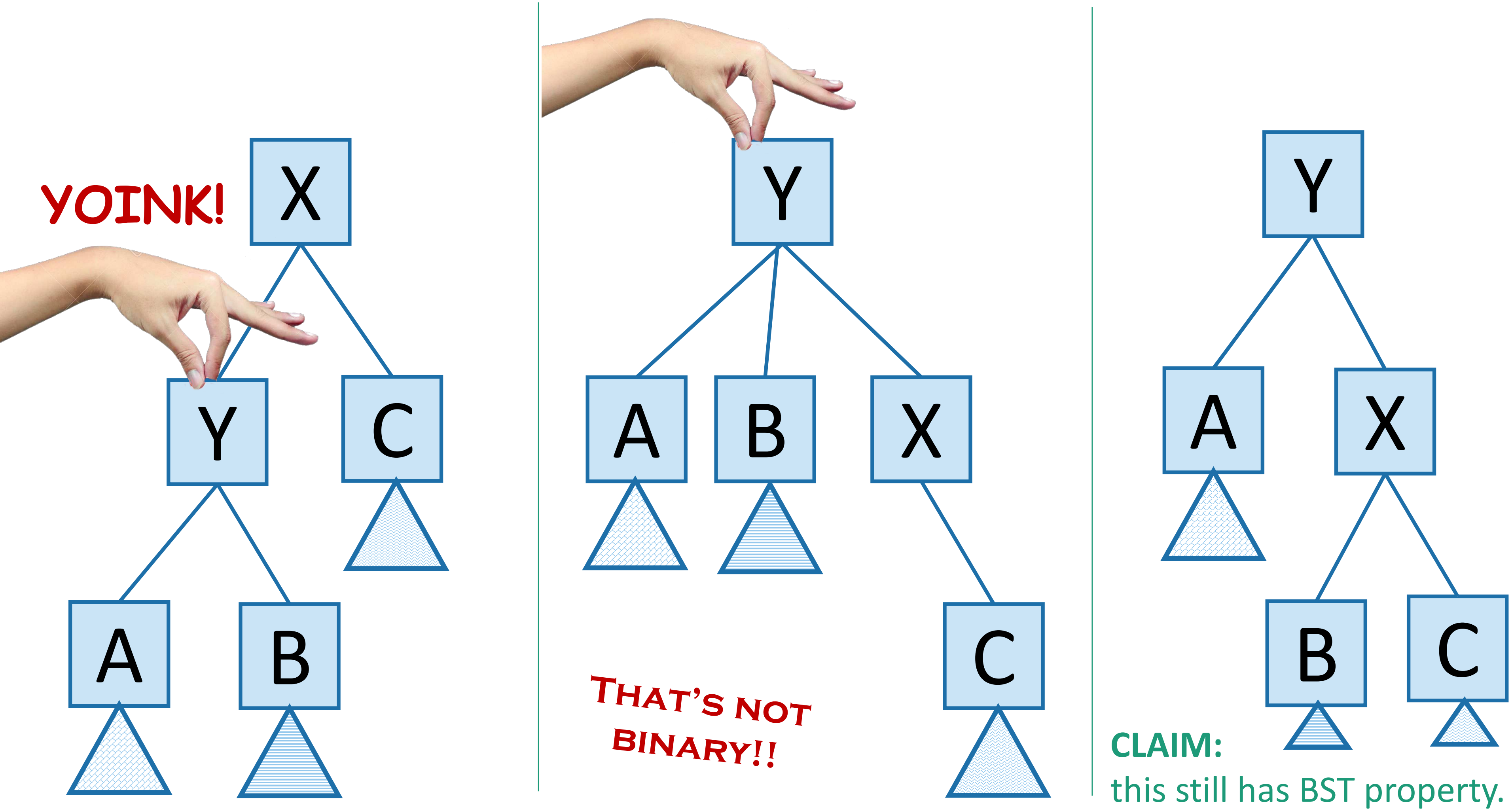


Example: Insert 0.

- Actually, **this can't happen?**
  - **6**-**3** path has one black node
  - **6**-**7**-… has at least two
- It might happen that we just turned 0 red from the previous step.
- Or it could happen if 7 is actually **NIL**.

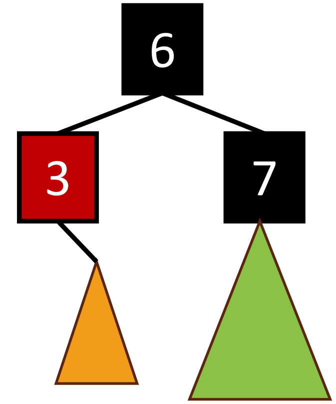
Recall Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.

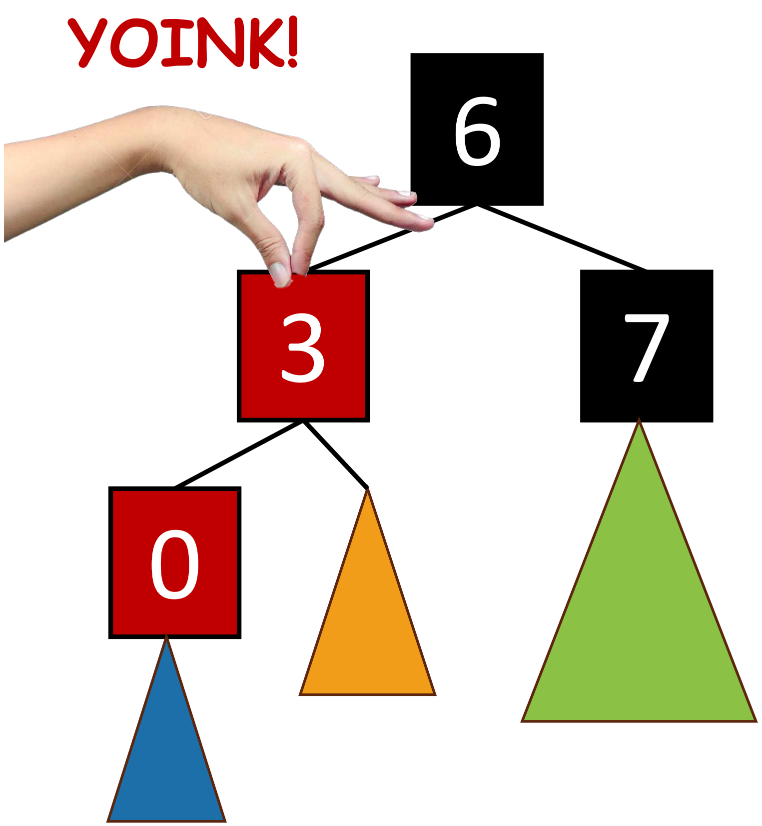


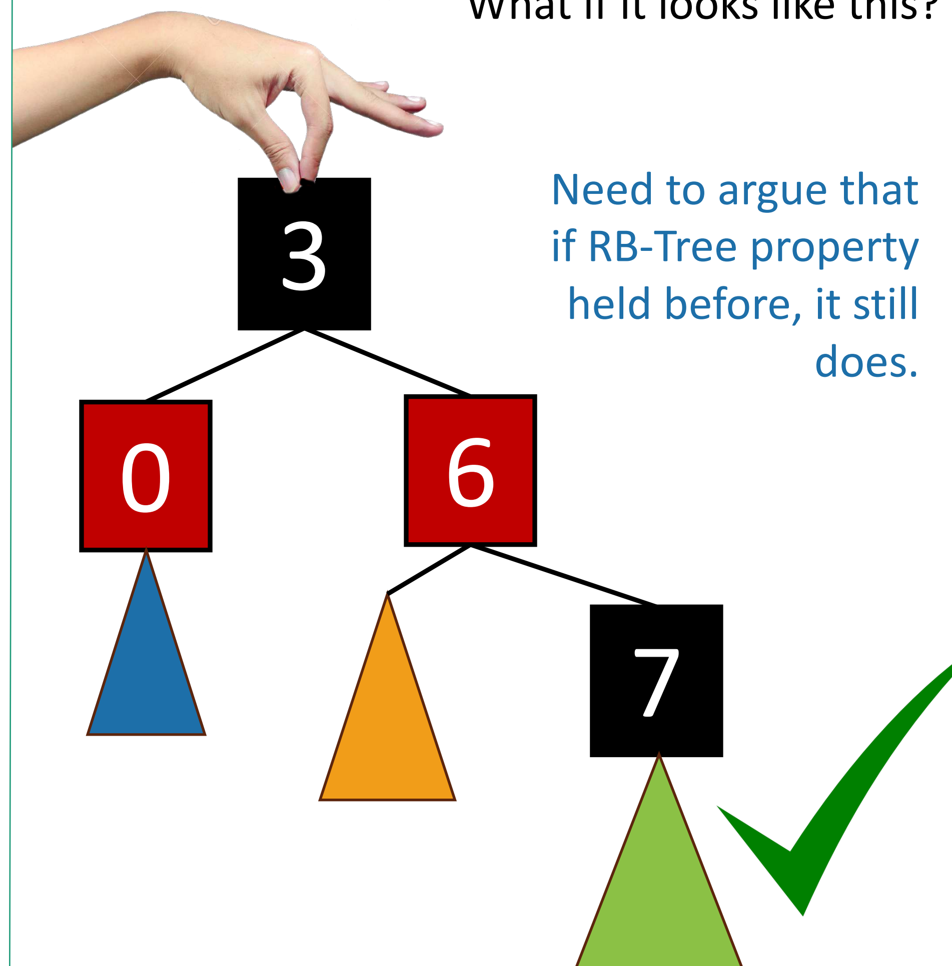
Inserting into a Red-Black Tree

- Make a new **red node**.
- Insert it as you would normally.
- Fix things up if needed.



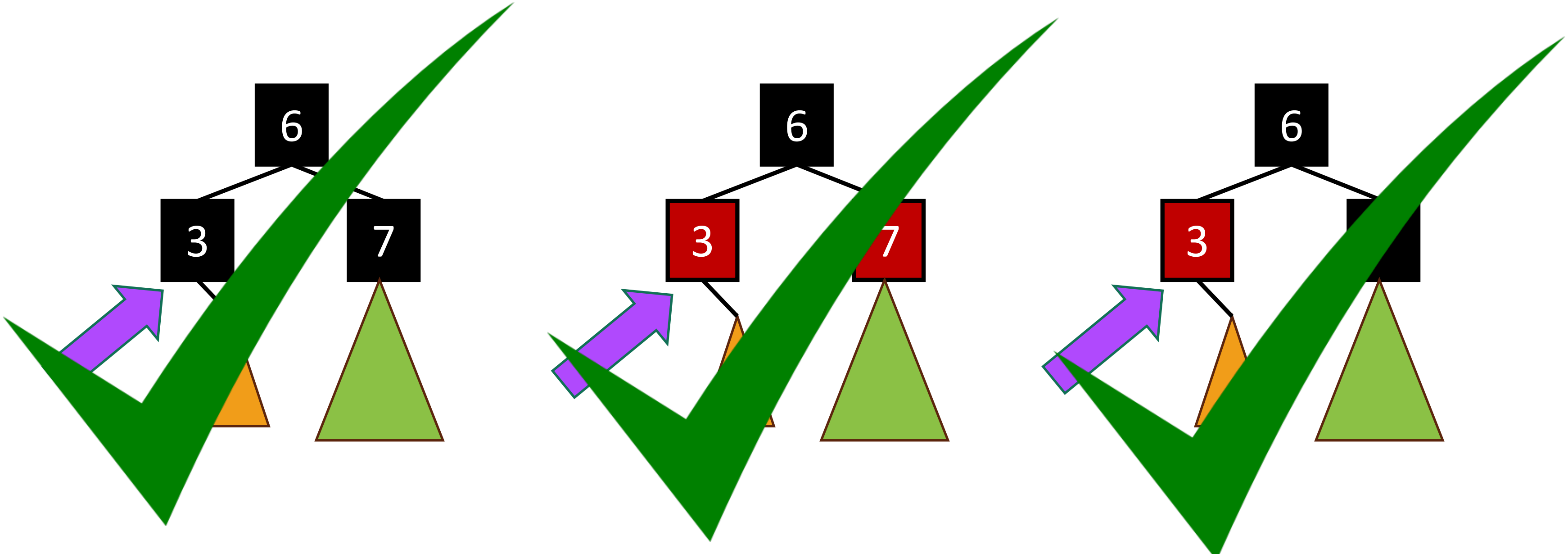
What if it looks like this?





Need to argue that if RB-Tree property held before, it still does.

Many cases

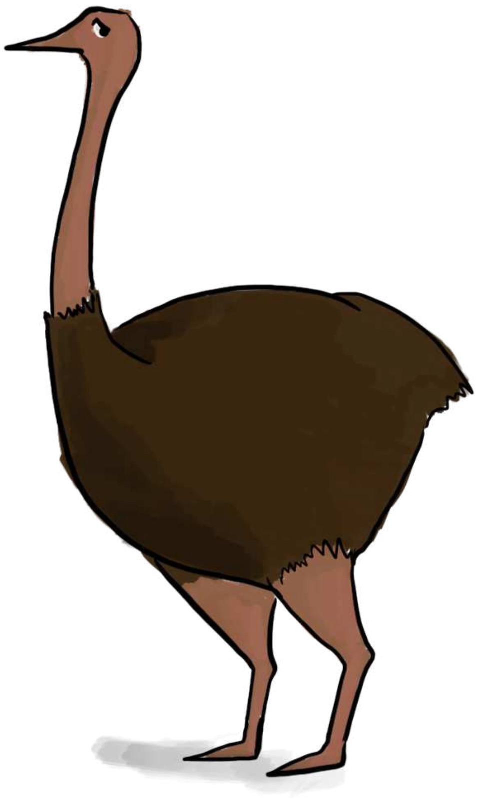


- Suppose we want to insert **here**.
  - eg, want to insert 0.

- And then there are 9 more cases for all of the various symmetries of these 3 cases…

Deleting from a Red-Black tree

Fun exercise!



Ollie the over-achieving ostrich

That's a lot of cases

- You are **not responsible** for the nitty-gritty details of Red-Black Trees. (For this class)
  - Though implementing them is a great exercise!
- You should know:
  - What are the properties of an RB tree?
  - And (more important) why does that guarantee that they are balanced?

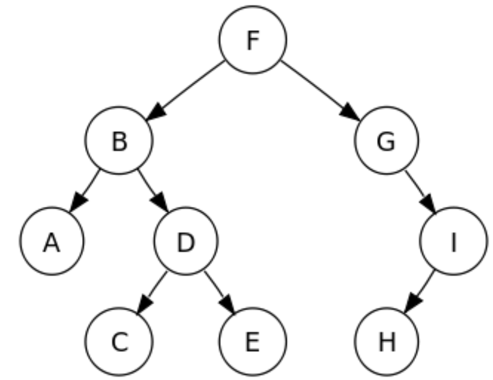
What was the point again?

- Red-Black Trees always have height at most 2log(n+1).
- As with general Binary Search Trees, all operations are O(height)
- So all operations are O(log(n)).

Conclusion: The best of both worlds

	Sorted Arrays	Linked Lists	Balanced Binary Search Trees
Search	O(log(n)) 😃	O(n) ☹️	O(log(n)) 😃
Insert/Delete	O(n) ☹️	O(1) 😃	O(log(n)) 😃

Today

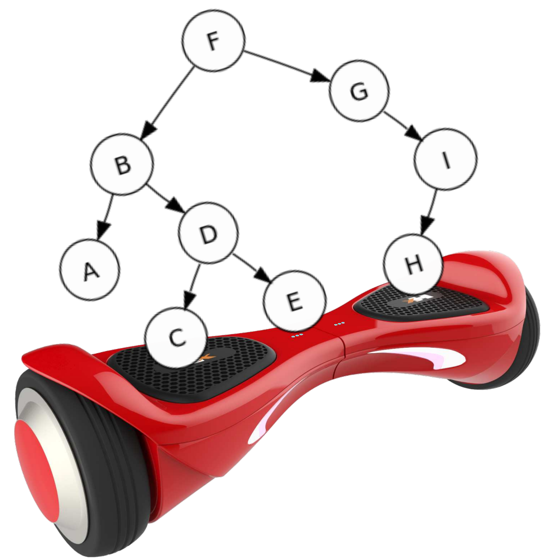


- Begin a brief foray into data structures!
  - See CS 166 for more!
- Binary search trees
  - You may remember these from CS 106B
  - They are better when they're balanced.

this will lead us to...

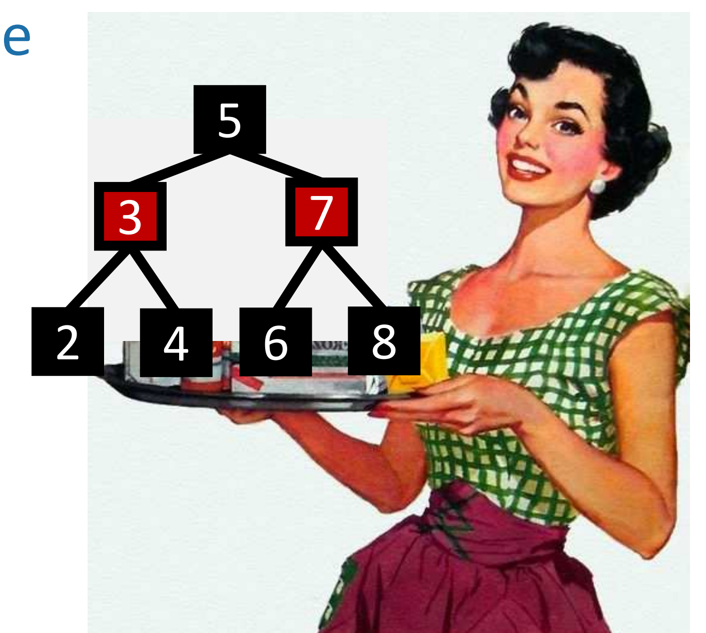
- Self-Balancing Binary Search Trees
  - **Red-Black** trees.

Recap



Recap

- Balanced binary trees are the best of both worlds!
- But we need to keep them balanced.
- **Red-Black Trees** do that for us.
  - We get O(log(n))-time INSERT/DELETE/SEARCH
  - Clever idea: have a proxy for balance



Next time

- **Hashing!**

Before next time

- Pre-lecture exercise for Lecture 8
  - (More) fun with probability!