Lecture 8

HASHING!!!!!

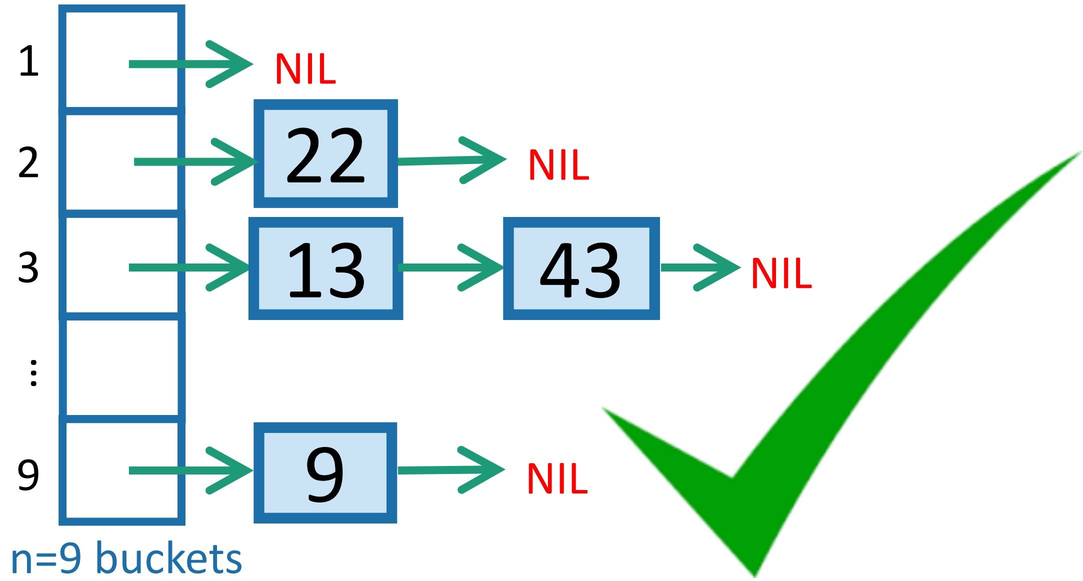
Announcements

- HW3 due Friday!
- HW4 posted Friday!

- Q: Where can I see examples of proofs?
  - Lecture Notes
  - CLRS
  - HW Solutions

- Office hours: lines are long ☹
- Solutions:
  - We will be (more) mindful of throughput.
  - ~~Get more TAs~~
  - ~~Stop assigning homework~~
  - Use Piazza!
  - Start early. (There are no lines on Monday!)

Today: hashing



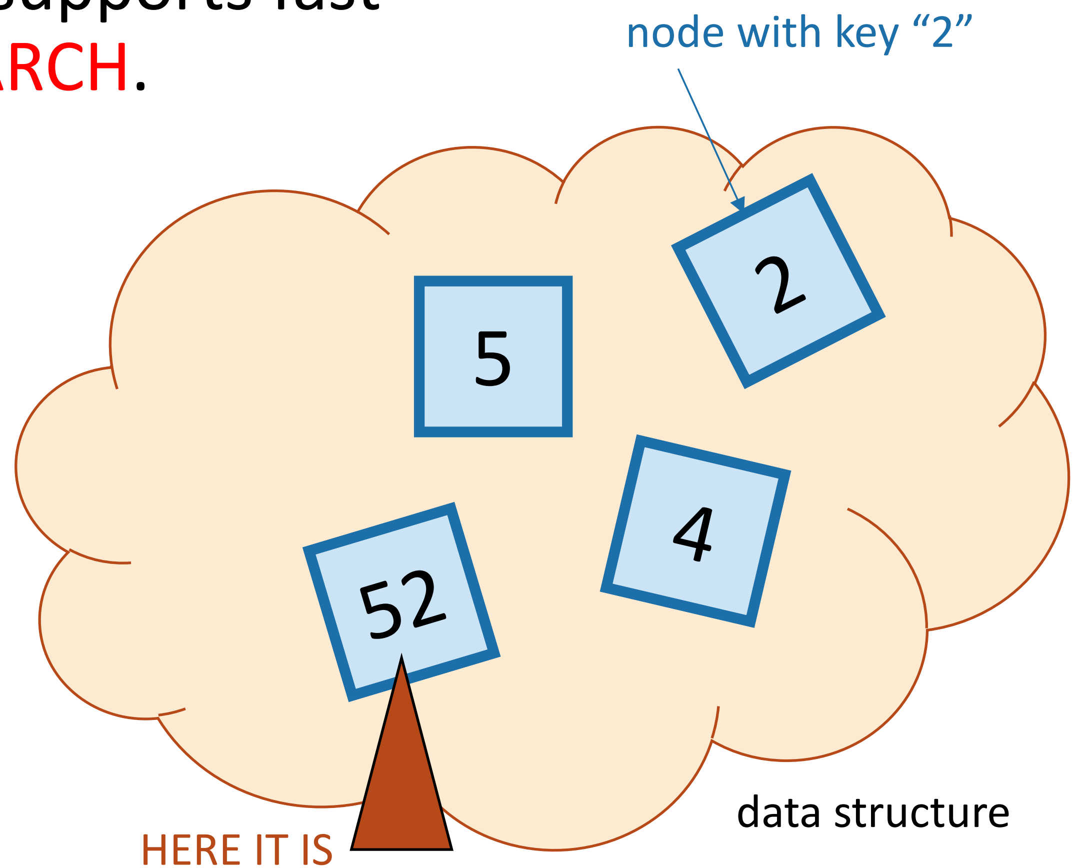
Outline

- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.
- **Hash families** are the magic behind hash tables.
- **Universal hash families** are even more magic.

Goal:
Just like on Monday

- We are interesting in putting nodes with keys into a data structure that supports fast INSERT/DELETE/SEARCH.

- INSERT 5
- DELETE 4
- SEARCH 52

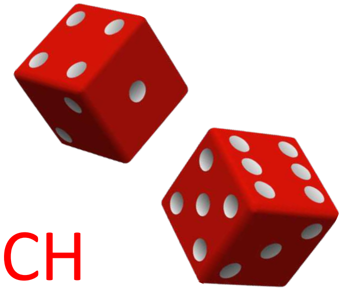


On Monday:

- Self balancing trees:
  - O(log(n)) deterministic INSERT/DELETE/SEARCH

#prettysweet

Today:



- Hash tables:
  - O(1) expected time INSERT/DELETE/SEARCH
- Worse worst-case performance, but often great in practice.

#evensweeterinpractice

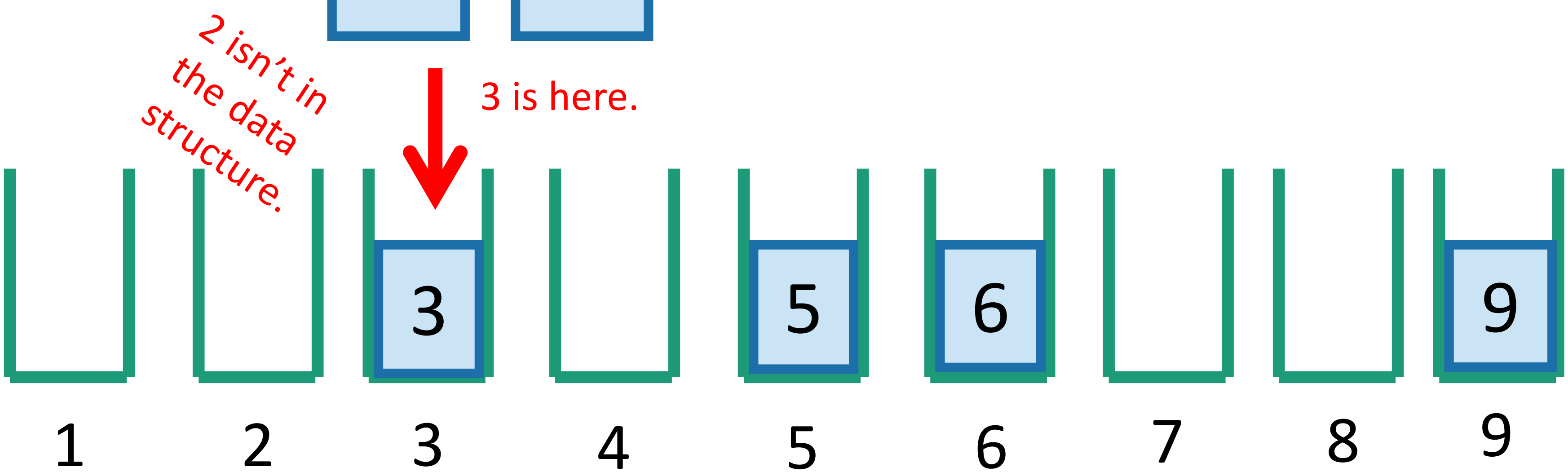
eg, Python's `dict`, Java's `HashSet`/`HashMap`, C++'s `unordered_map`

Hash tables are used for databases, caching, object representation, …

One way to get O(1) time

This is called “direct addressing”

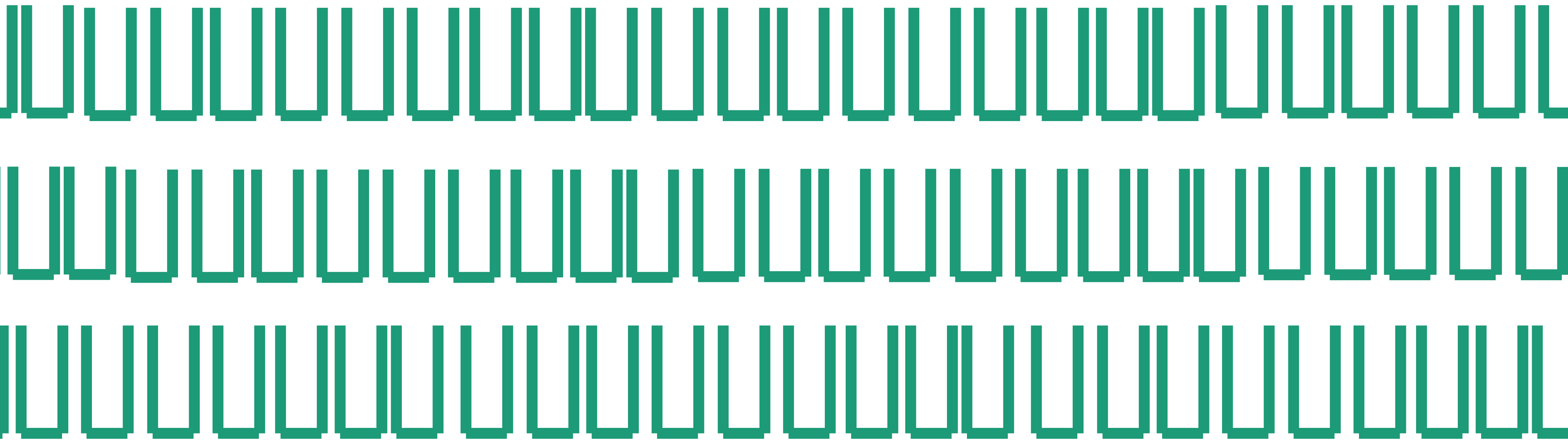
- Say all keys are in the set {1,2,3,4,5,6,7,8,9}.
- INSERT: 9 6 3 5
- DELETE: 6
- SEARCH: 3 2



That should look familiar

The universe is really big!

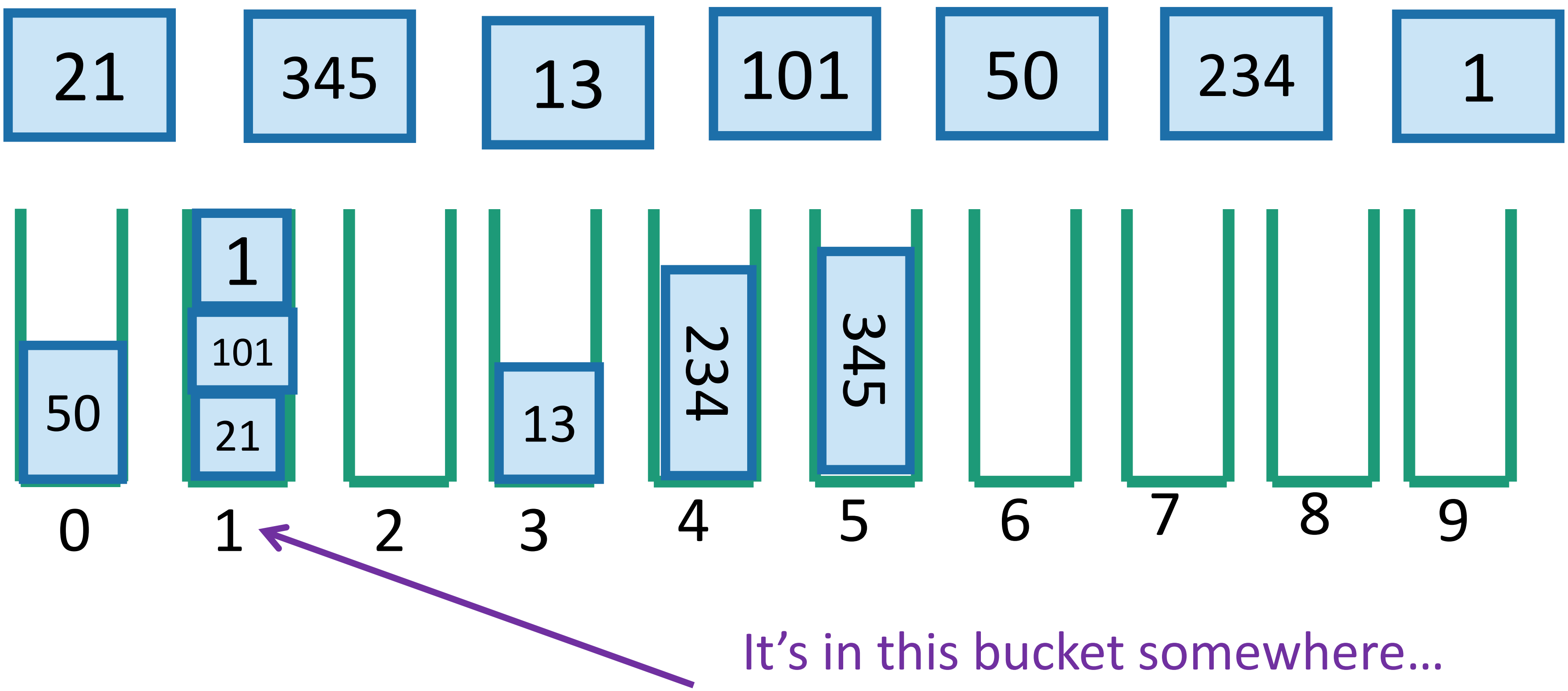
- Kind of like BUCKETSORT from Lecture 6.
- Same problem: if the keys may come from a universe U = {1,2, …., 10000000000}….



The solution then was…

- Put things in buckets based on one digit.

INSERT:

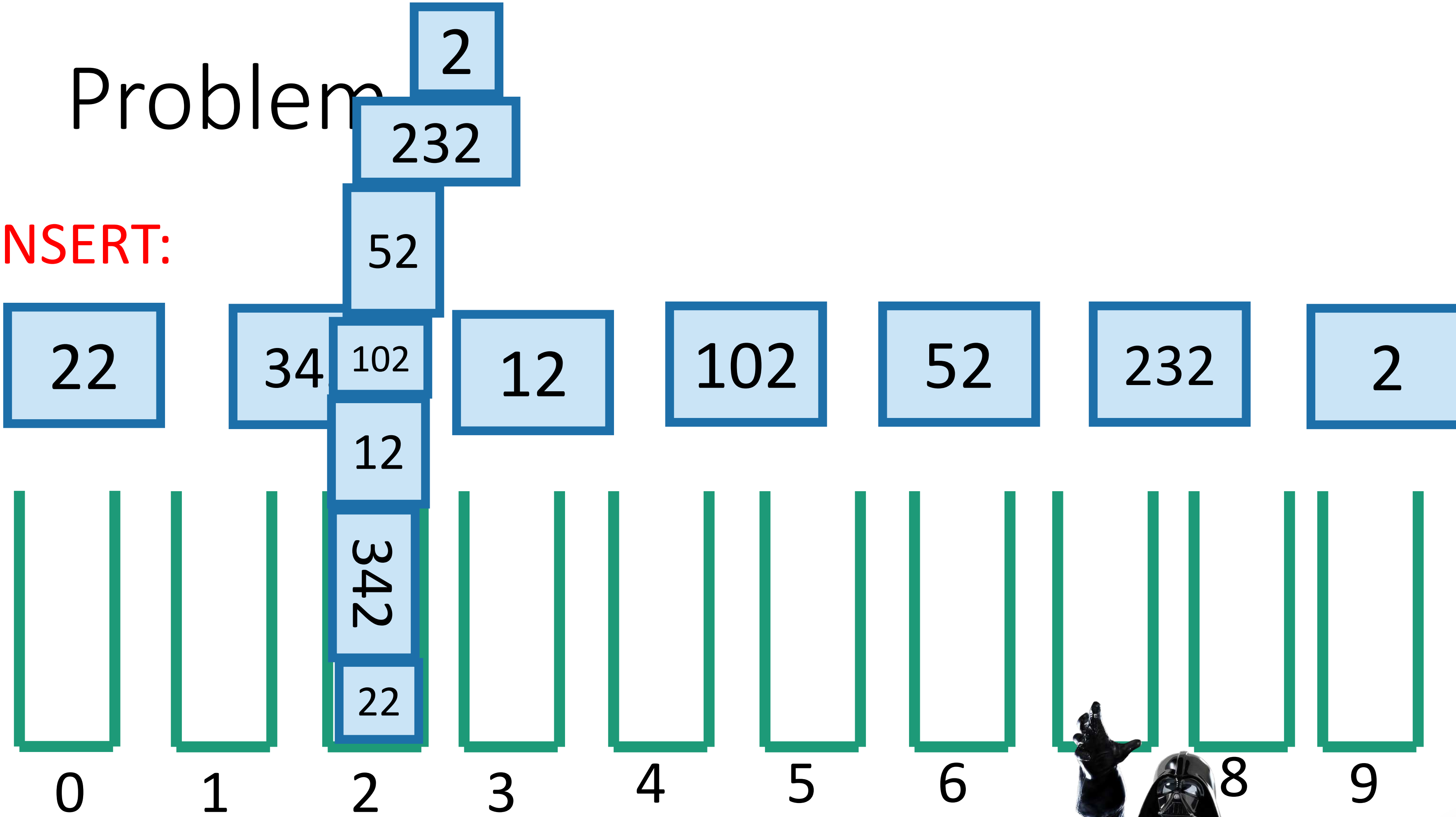


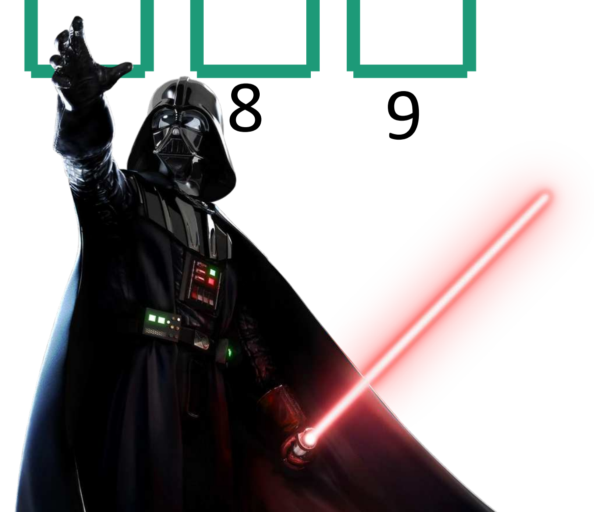
It’s in this bucket somewhere…
go through until we find it.

Now SEARCH 21

Problem

INSERT:





Now SEARCH 22

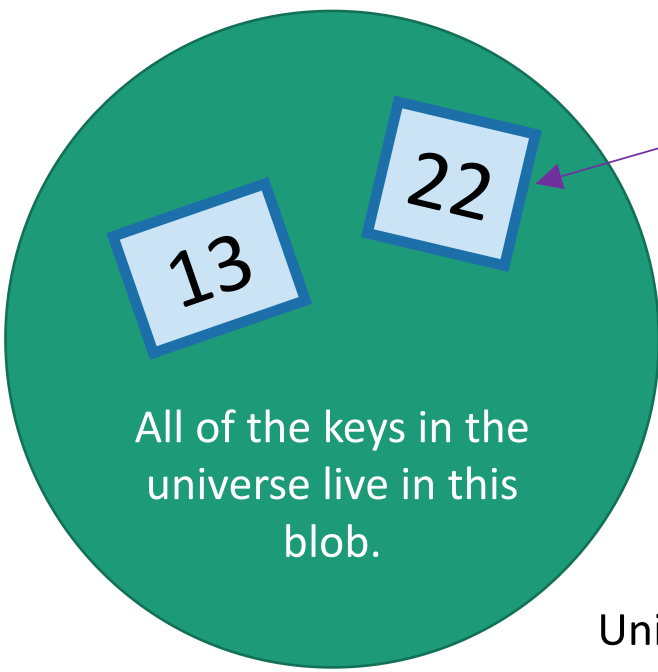
....this hasn't made our lives easier...

Hash tables

- That was an example of a hash table.
  - not a very good one, though.
- We will be **more clever** (and less deterministic) about our bucketing.
- This will result in fast (expected time) INSERT/DELETE/SEARCH.

But first! Terminology.

- We have a universe U, of size M.
  - M is really big.
- But only a few (say at most n for today’s lecture) elements of M are ever going to show up.
  - M is waaaayyyyyyy bigger than n.
- But we don’t know which ones will show up in advance.



Universe U

A few elements are special and will actually show up.

Example: U is the set of all strings of at most 140 ascii characters. (128^{140} of them).

The only ones which I care about are those which appear as trending hashtags on twitter. #hashinghashtags

There are way fewer than 128^{140} of these.

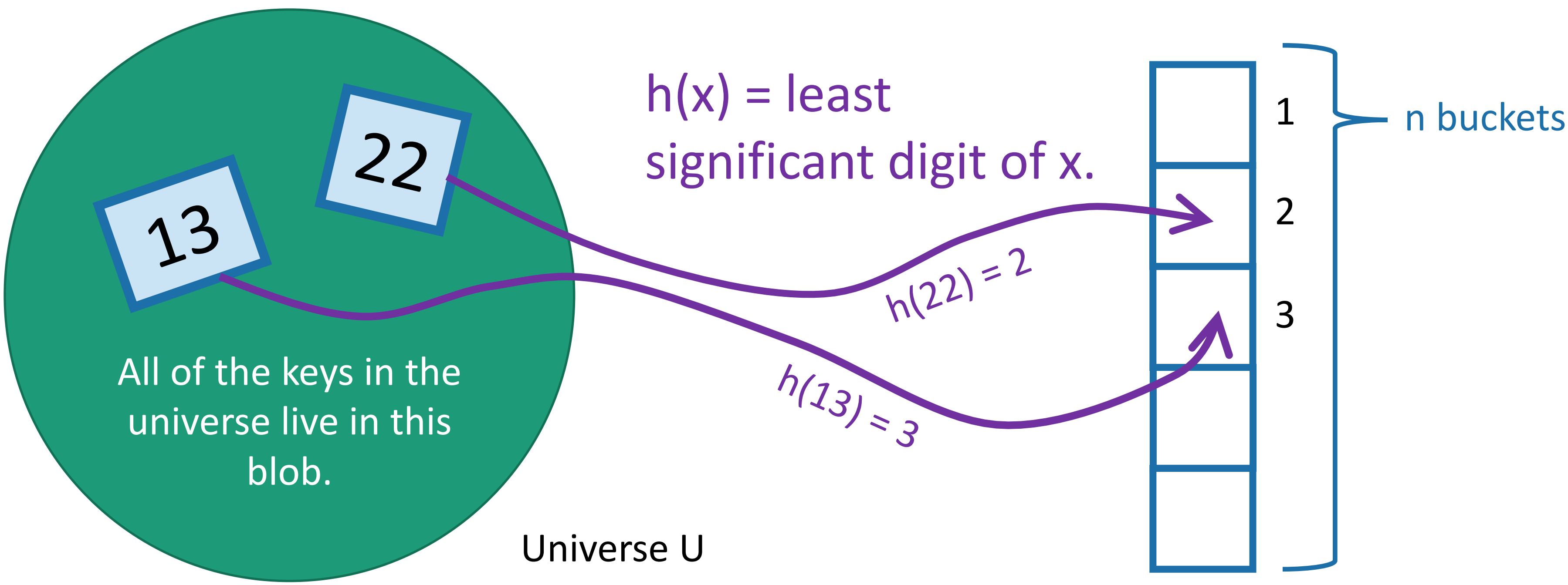
Examples aside, I’m going to draw elements like I always do, as blue boxes with integers in them…

The previous example

with this terminology

For this lecture, I'm assuming that the number of things is the same as the number of buckets, both are n.
This doesn't have to be the case, although we do want:
#buckets = O(#things which show up)

- We have a universe U, of size M.
  - at most n of which will show up.
- M is waaaayyyyyy bigger than n.
- We will put items of U into n buckets.
- There is a <u>hash function</u> h:U → {1,…,n} which says what element goes in what bucket.

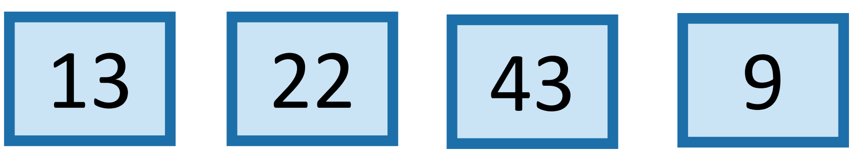


This is a **hash table** (with chaining)

For demonstration purposes only!
This is a terrible hash function! Don't use this!

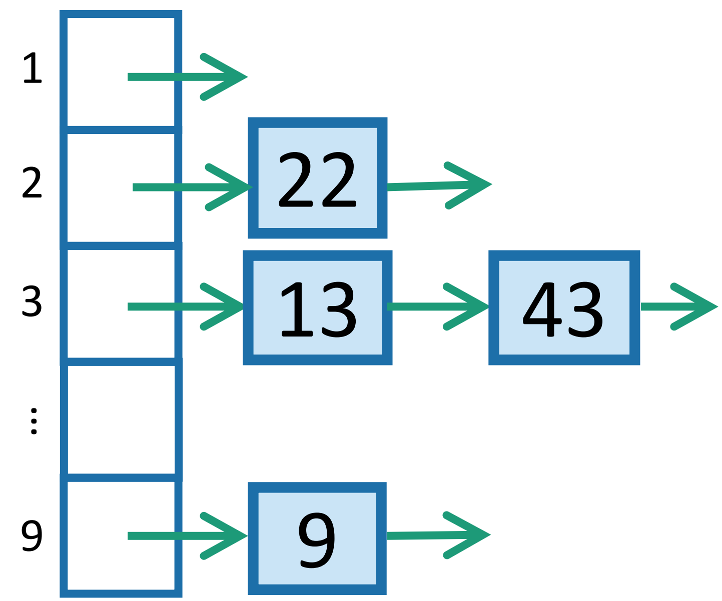
- Array of n buckets.
- Each bucket stores a linked list.
  - We can insert into a linked list in time O(1)
  - To find something in the linked list takes time O(length(list)).
- h:U → {1,…,n} can be any function:
  - but for concreteness let's stick with h(x) = least significant digit of x.

INSERT:



SEARCH 43:

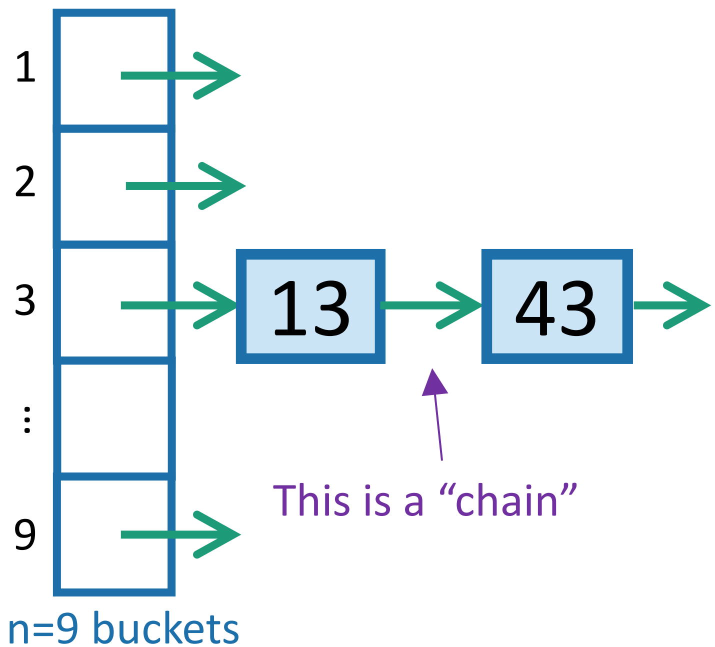
Scan through all the elements in bucket h(43) = 3.

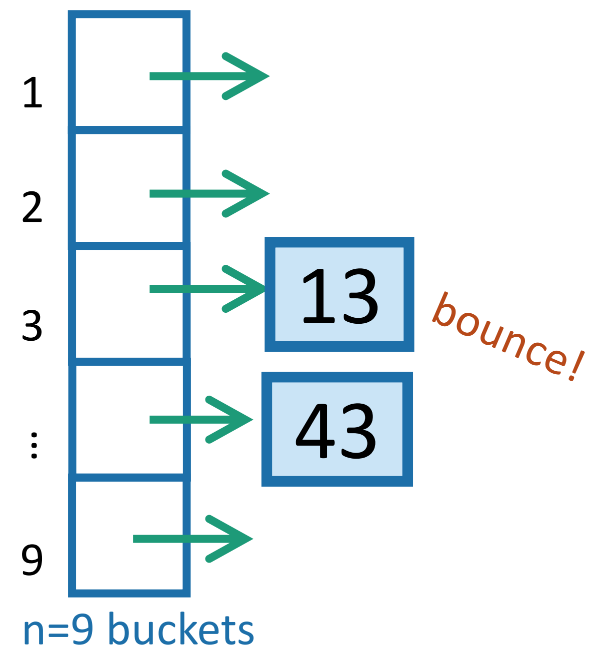


n buckets (say n=9)

Aside: Hash tables with open addressing

- The previous slide is about hash tables with chaining.
- There's also something called "open addressing"
- Read in CLRS if you are interested!





\end{Aside}

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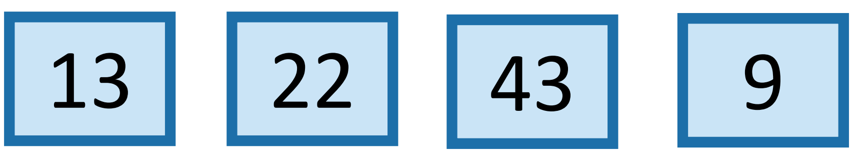
INSERT:



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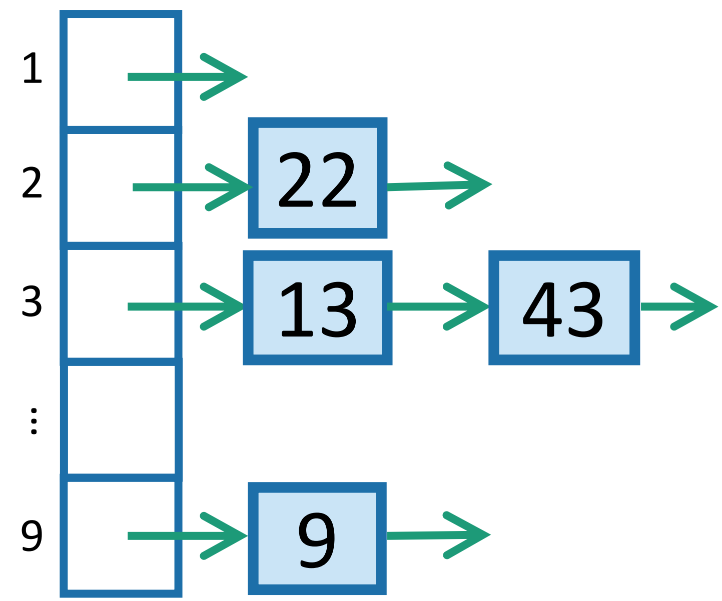


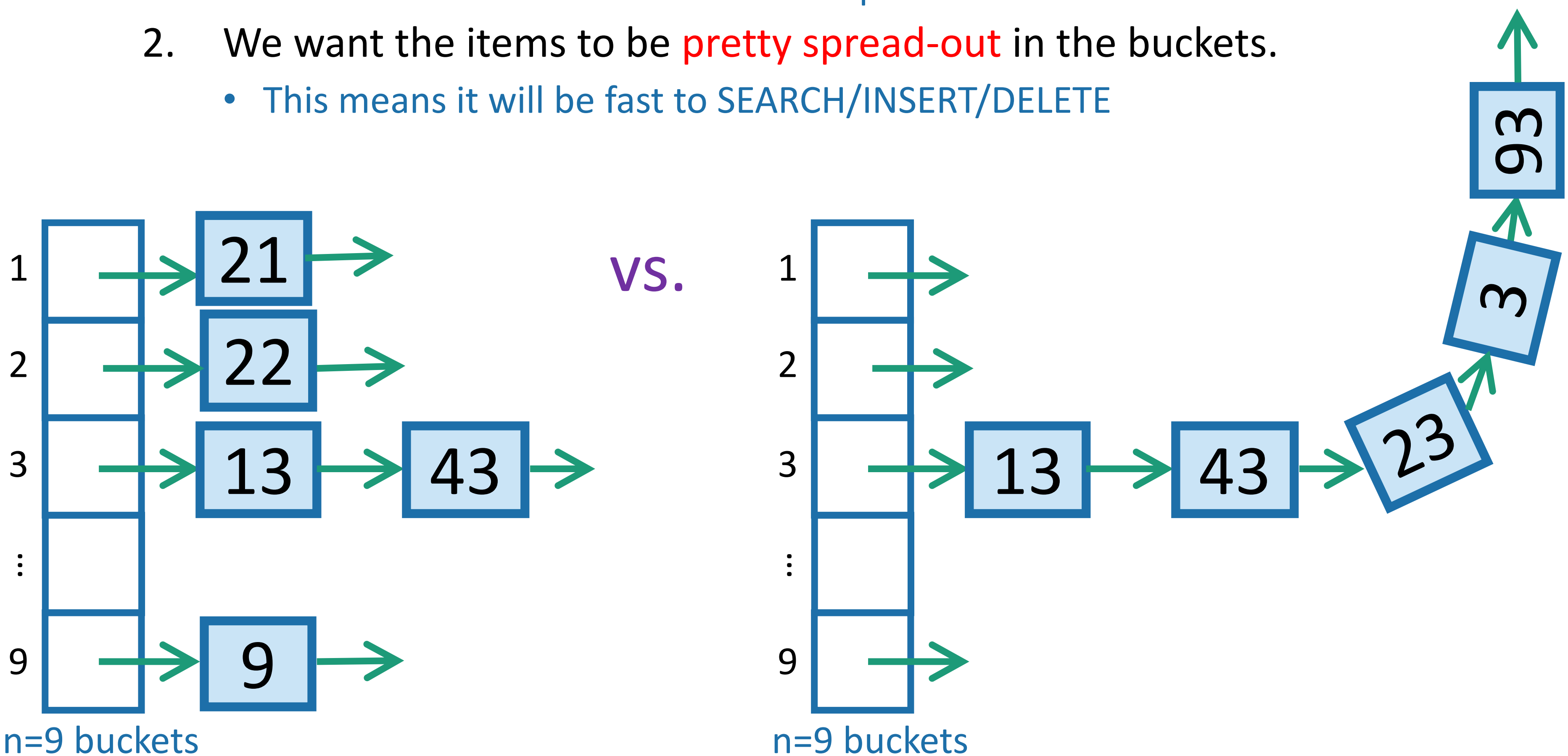
n buckets (say n=9)

IPython notebook time

- (Seems to work!)
- (Will this example be a good idea?)

Sometimes this a good idea
Sometimes this is a bad idea

- How do we pick that function so that this is a good idea?
  1. We want there to be not many buckets (say, n).
     - This means we don't use too much space
  2. We want the items to be pretty spread-out in the buckets.
     - This means it will be fast to SEARCH/INSERT/DELETE



Worst-case analysis

- Design a function h: U -> {1,…,n} so that:
  - No matter what input (fewer than n items of U) **a bad guy** chooses, the buckets will be balanced.
  - Here, balanced means O(1) entries per bucket.

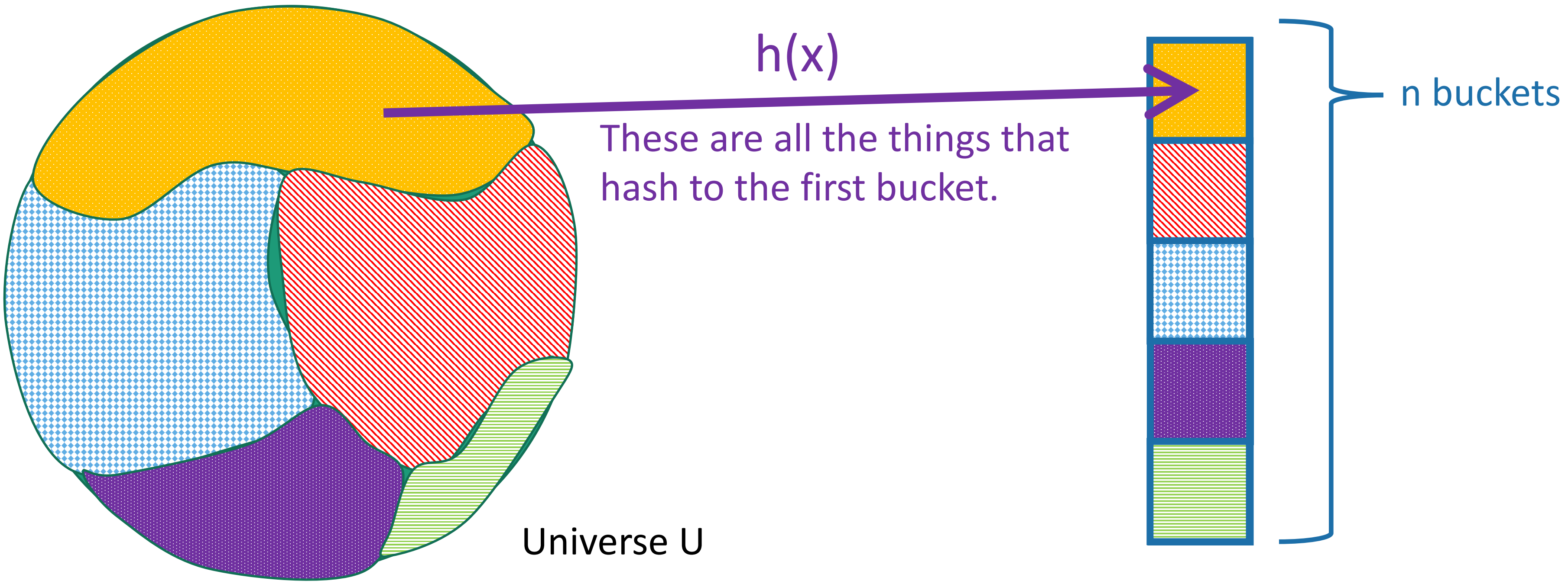
- If we had this, then we'd achieve our dream of O(1) INSERT/DELETE/SEARCH

Can you come up with such a function?


YOU CANNOT ESCAPE THE DARK SIDE
WITH DETERMINISTIC
HASH FUNCTIONS
imgflip.com

We really can’t beat the bad guy here.

- The universe U has M items
- They get hashed into n buckets
- At least one bucket has at least M/n items hashed to it.
- M is WAAYYYYY bigger then n, so M/n is bigger than n.
- **Bad guy chooses n of the items that landed in this very full bucket.**



Solution:

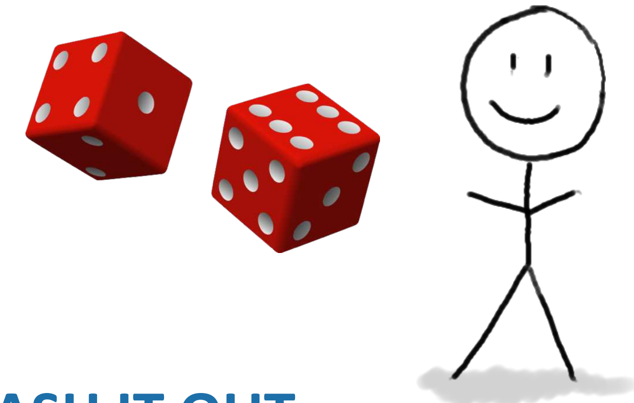
Randomness

The game

What does **random** mean here? Uniformly random?

Plucky the pedantic penguin

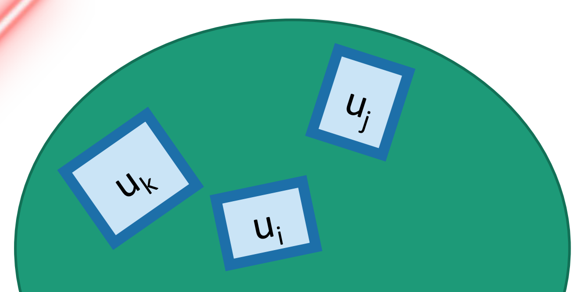
2. You, the algorithm, chooses a **random** hash function $h: U \to \{1, \dots, n\}$.



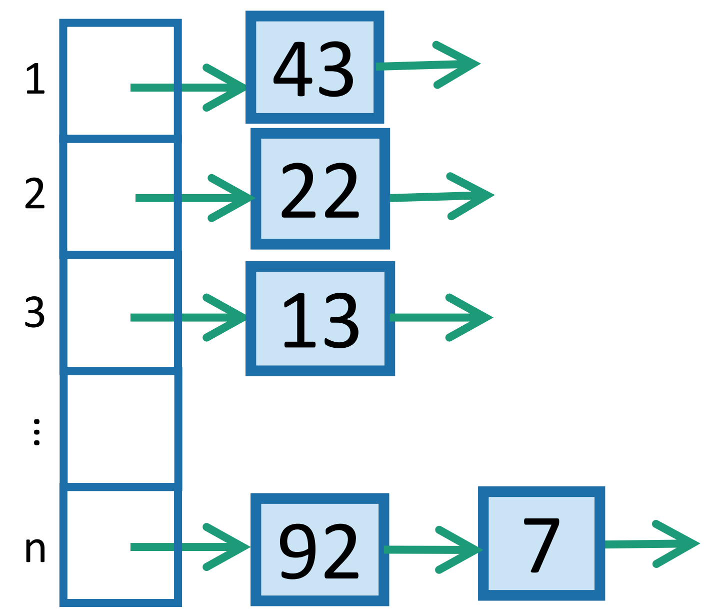
1. An adversary chooses any n items $u_1, u_2, \dots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

13	22	43	92	7

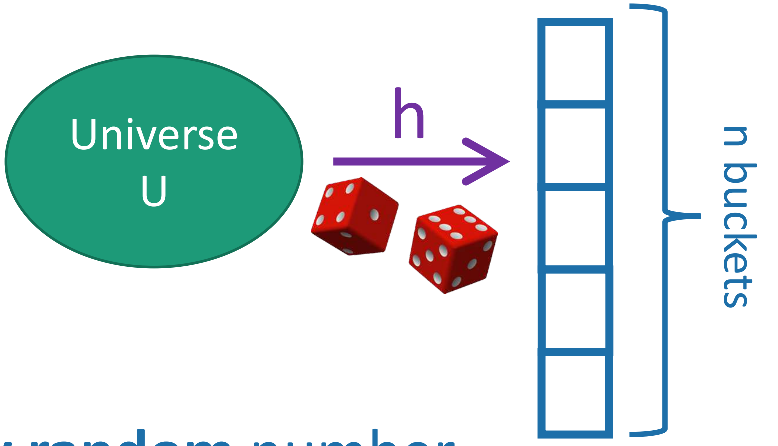
INSERT 13, INSERT 22, INSERT 43, INSERT 92, INSERT 7, SEARCH 43, DELETE 92, SEARCH 7, INSERT 92



3. **HASH IT OUT** #hashpuns



Example



- Say that h *is* uniformly random.
  - That means that **h(1)** is a **uniformly random** number between 1 and n.
  - **h(2)** is also a **uniformly random** number between 1 and n, independent of h(1).
  - **h(3)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2).
  - …
  - **h(n)** is also a **uniformly random** number between 1 and n, independent of h(1), h(2), …, h(n-1).

Why should that help?

Intuitively: The bad guy can't foil a hash function that he doesn't yet know.

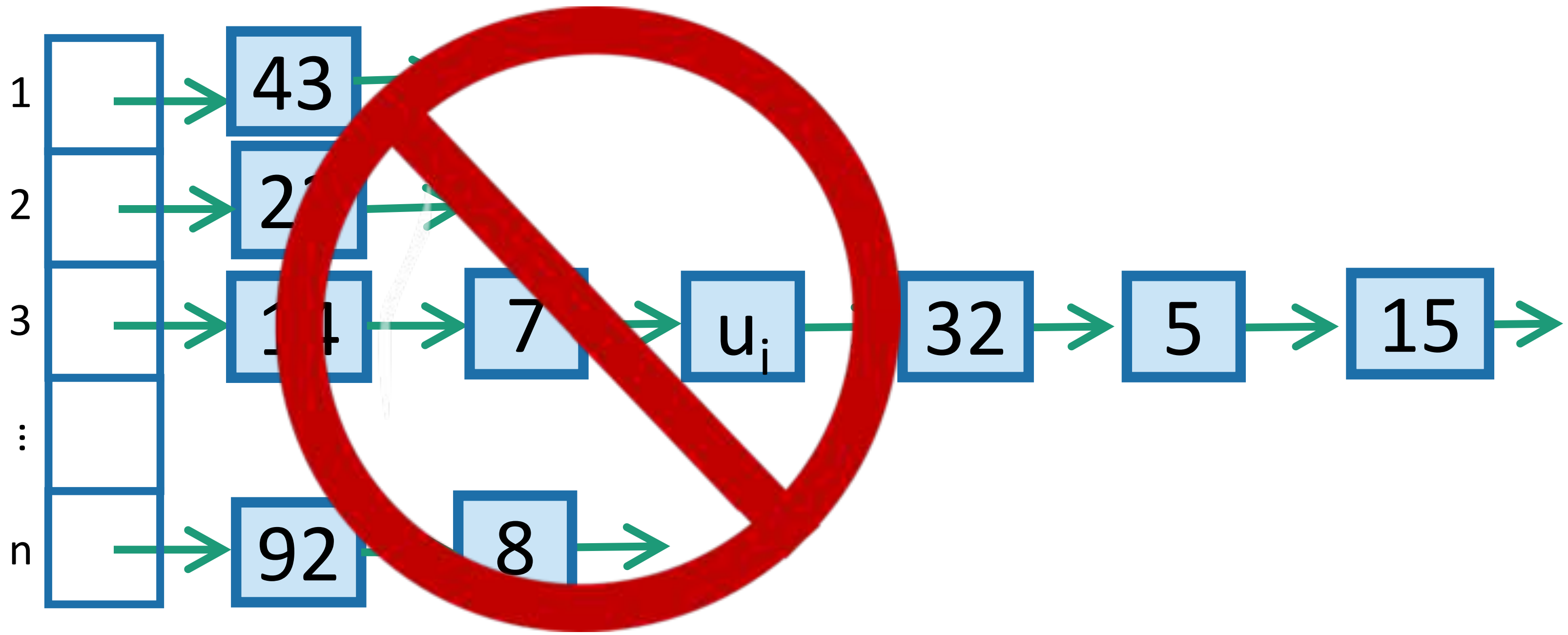
Why not? What if there's some strategy that foils a random function with high probability?

We'll need to do some analysis…

What do we want?

It's **bad** if lots of items land in u_i's bucket.

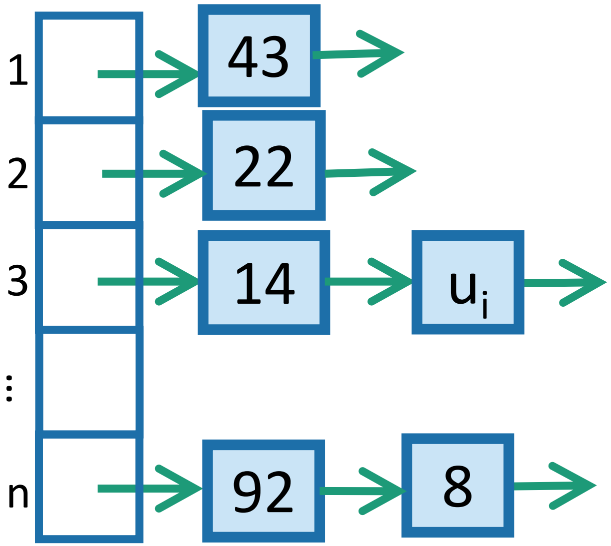
So we want **not that**.



More precisely

- We want:
  - For all u_i that the bad guy chose
  - E[number of items in u_i 's bucket] ≤ 2.
- If that were the case,
  - For each operation involving u_i
  - E[time of operation] = O(1)

We could replace "2" here with any constant; it would still be good.



So, in expectation, it would takes O(1) time per INSERT/DELETE/SEARCH operation.

So we want:

- For all i=1, …, n,

  E[number of items in u_i ‘s bucket] $\leq$ 2.

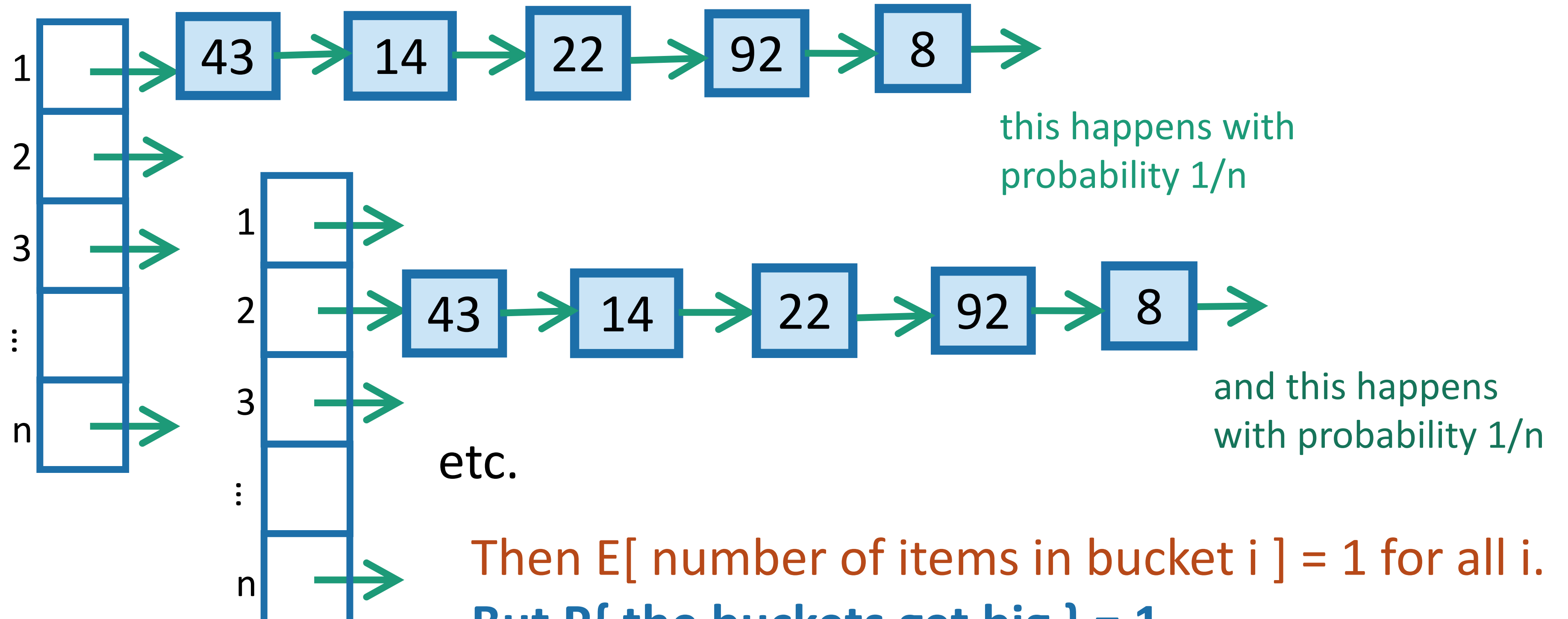
Aside: why not:

This slide skipped in class

- For all i=1,…,n:

E[number of items in bucket i] ≤ ___?

Suppose that:



Then E[number of items in bucket i] = 1 for all i.

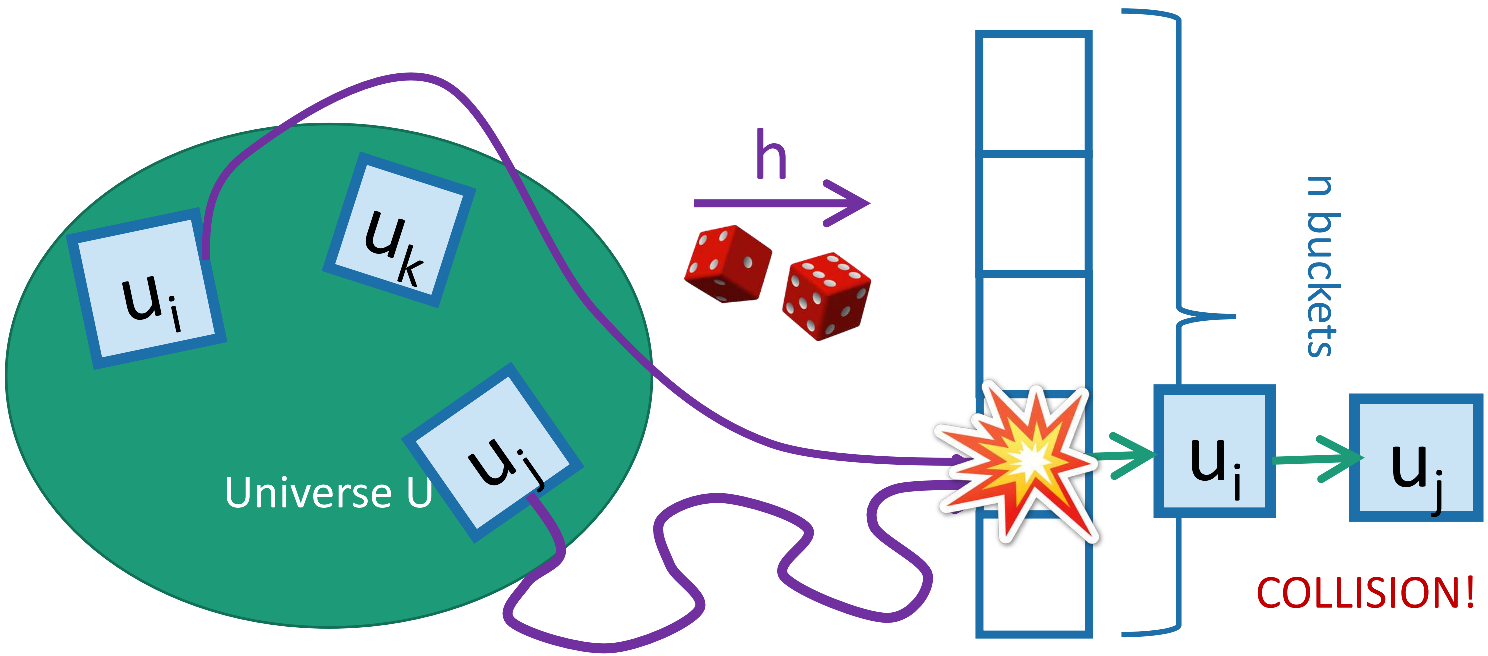
But P{ the buckets get big } = 1.

h is uniformly random

Expected number of items in u_i's bucket?

- $E[\ ] = \sum_{j=1}^{n} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} 1/n$ you will verify this on HW
- $\quad = 1 + \frac{n-1}{n} \leq 2.$

That's what we wanted.



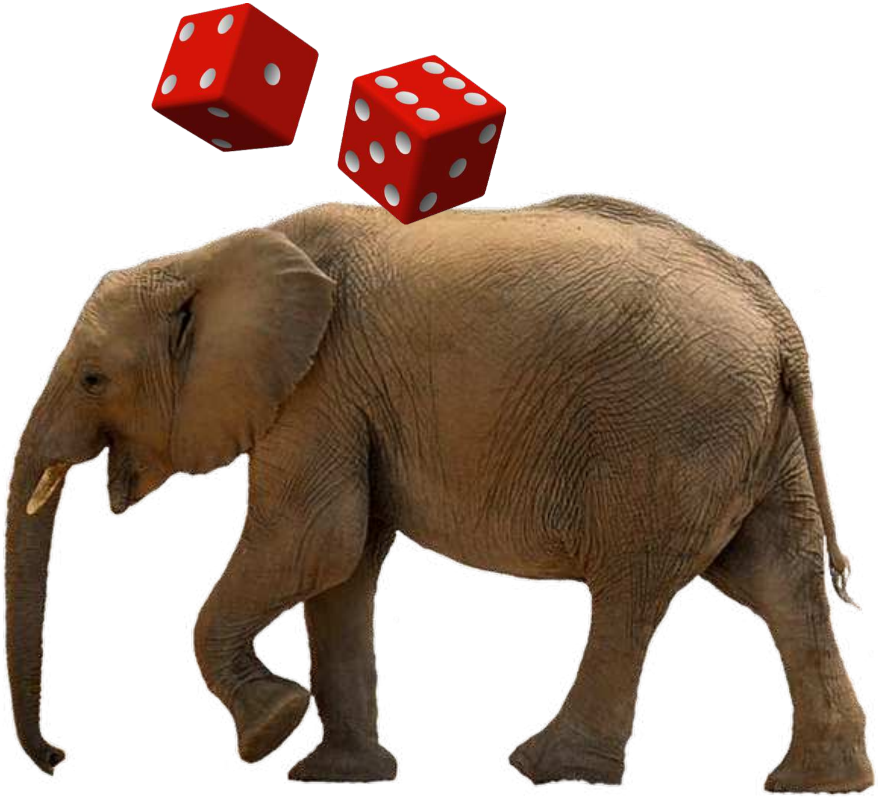
That's great!

- For all i=1, …, n,
  - E[number of items in u_i 's bucket] $\leq$ 2
- This implies (as we saw before):
  - For any sequence of INSERT/DELETE/SEARCH operations on any n elements of U, the expected runtime (over the random choice of h) is ***O(1) per operation.***

So, the solution is:

pick a uniformly random hash function.

The elephant in the room



The elephant in the room

"Pick a uniformly random hash function"

How do we do that?

Let’s implement this!

- IPython Notebook for Lecture 8

Let's *NOT* implement this!

Issues:

- Suppose **U = { all of the possible hashtags }**
- If we completely choose the random function up front, we have to iterate through all of U.
  - **128^{140} possible ASCII strings of length 140.**
  - (More than the number of particles in the universe)
- And even ignoring the time considerations
  - We have to store h(x) for every x.

Another thought…

- Just remember h on the relevant values

13 22 43 92 7

h(7) = 8

h(43) = 2

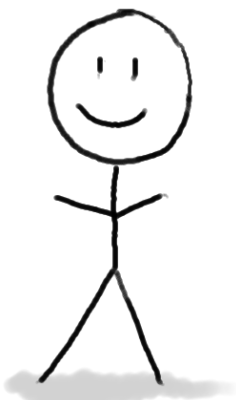
h(92) = 3

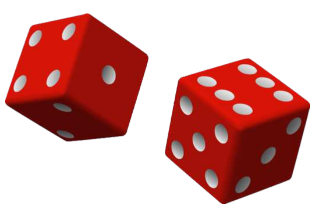
h(22) = 3

h(13) = 6

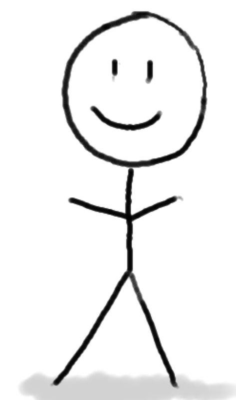
h(13) = 6

h(22) = 3





We need some way of storing keys and values with O(1) INSERT/DELETE/SEARCH…



Algorithm now

Algorithm later

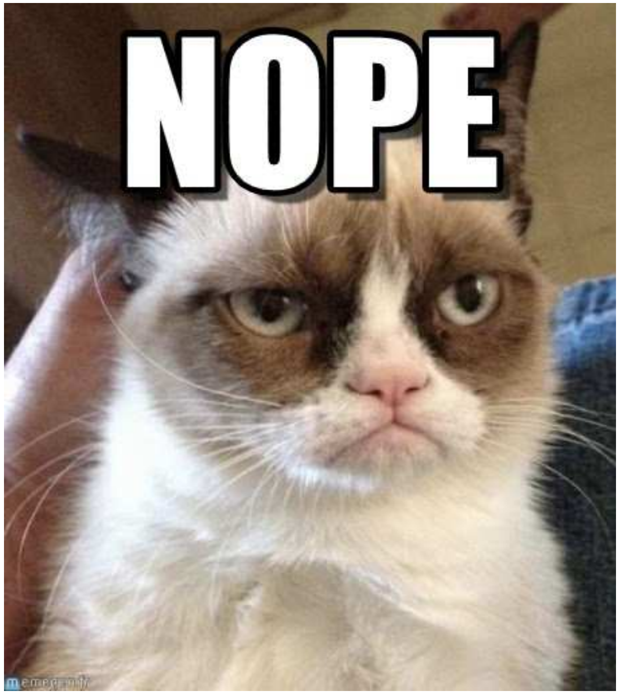
How much space does it take to store h?

- For each element x of U:
  - store h(x)
  - (which is a random number in {1,…,n}).

- Storing a number in {1,..,n} takes log(n) bits.
- So storing M of them takes Mlog(n) bits.
- In contrast, direct addressing would require M bits.

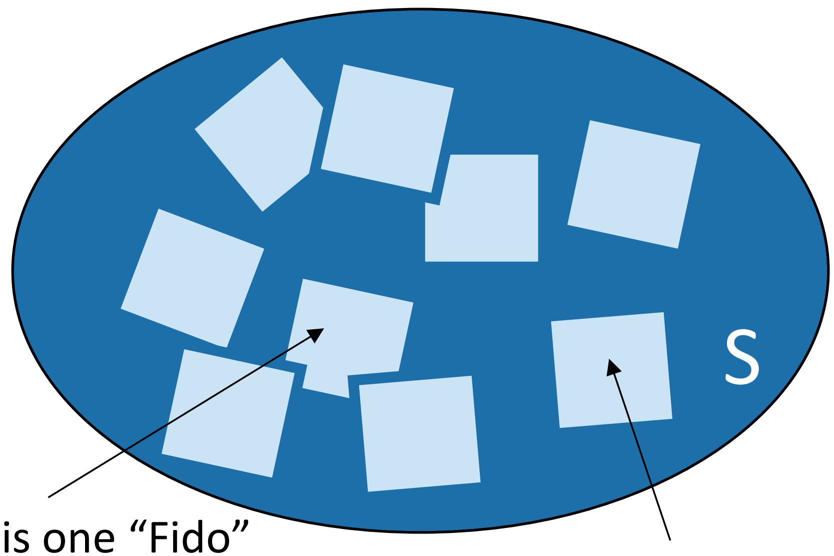
Hang on now

- Sure, **that** way of storing the function h won't work.
- But maybe there's another way?



Aside: description length

- Say I have a set S with s things in it.
- I get to write down the elements of S however I like.
  - (in binary)
- How many bits do I need?



I’ll call this one “Fido”

Or, 01101011

This one is named “Hercules”

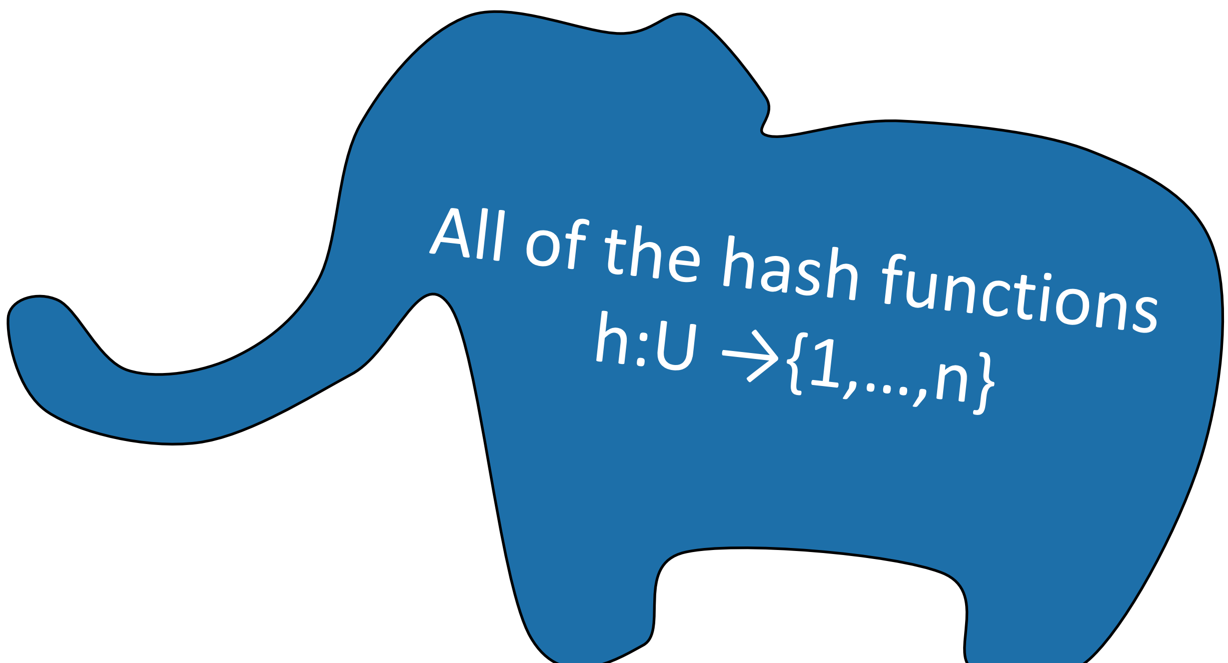
Or, 101

On board: the answer is log(s)

Space needed to store a random fn h?

- Say that this elephant-shaped blob represents the set of all hash functions.
- It has size n^M. (Really big!)

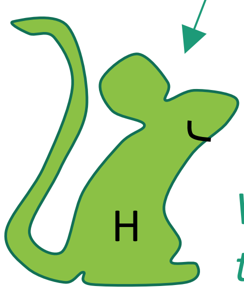
- To write down a random hash function, we need **$\log(n^M) = M\log(n)$** bits. ☹



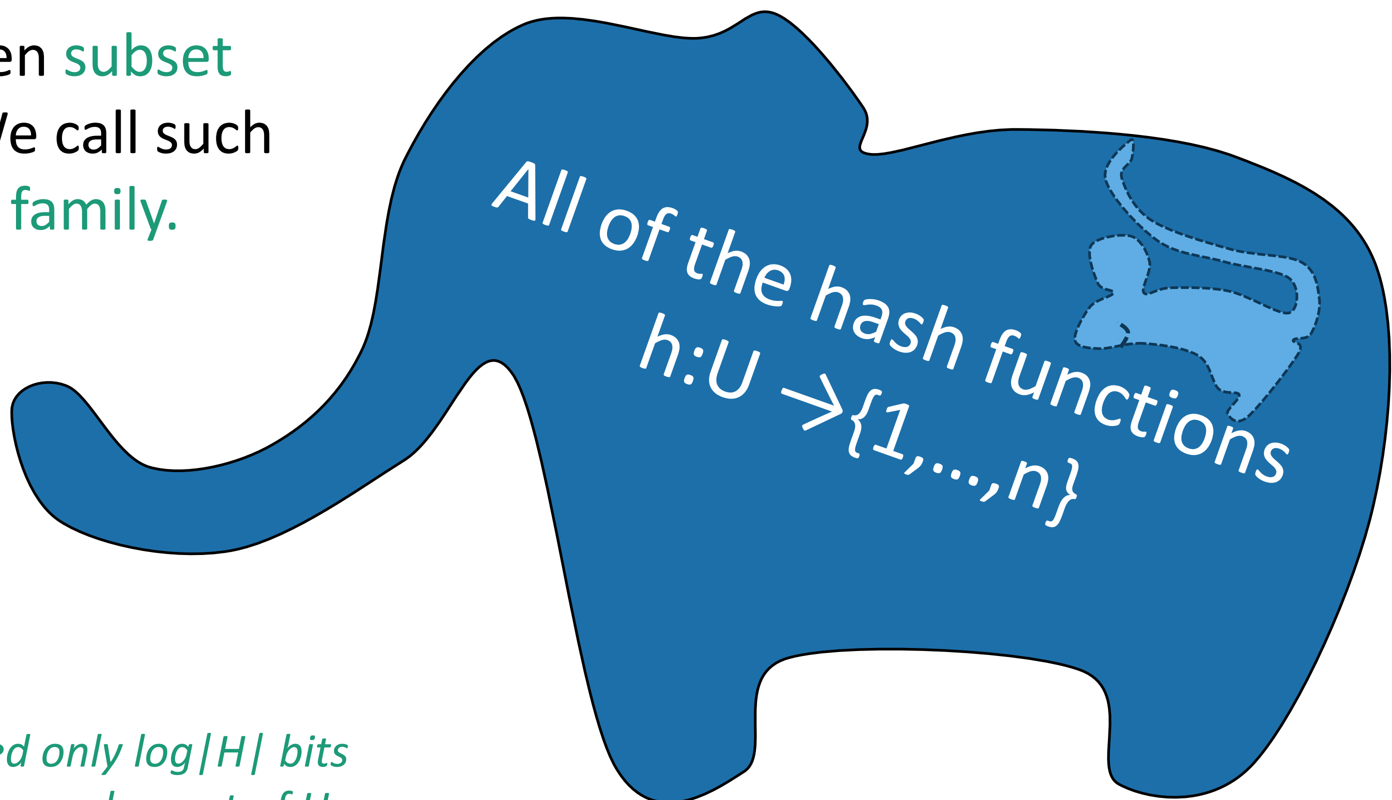
Solution

- Pick from a smaller set of functions.

A cleverly chosen subset of functions. We call such a subset a hash family.



We need only log|H| bits to store an element of H.



Outline

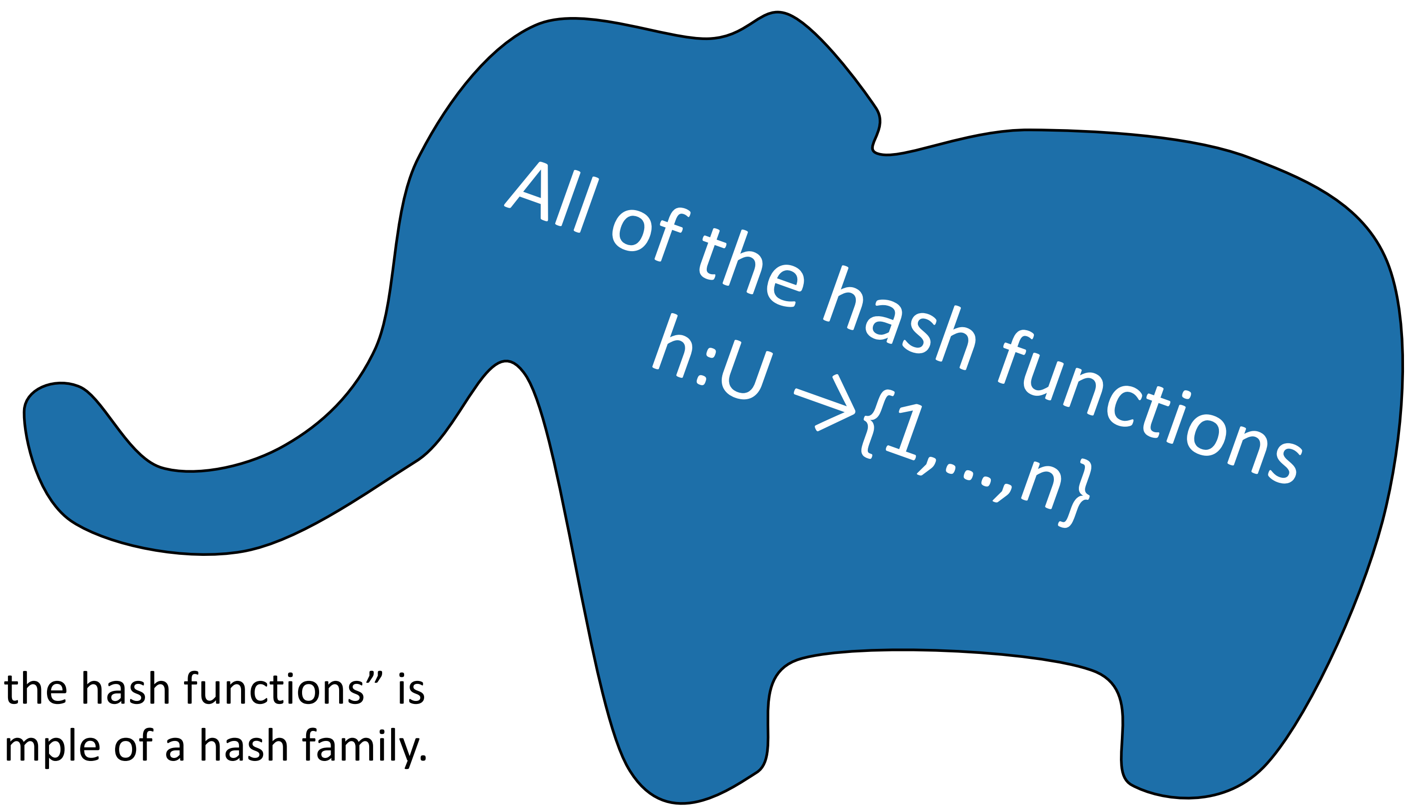
- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.

- **Hash families** are the magic behind hash tables.

- **Universal hash families** are even more magic.

Hash families

- A hash family is a collection of hash functions.

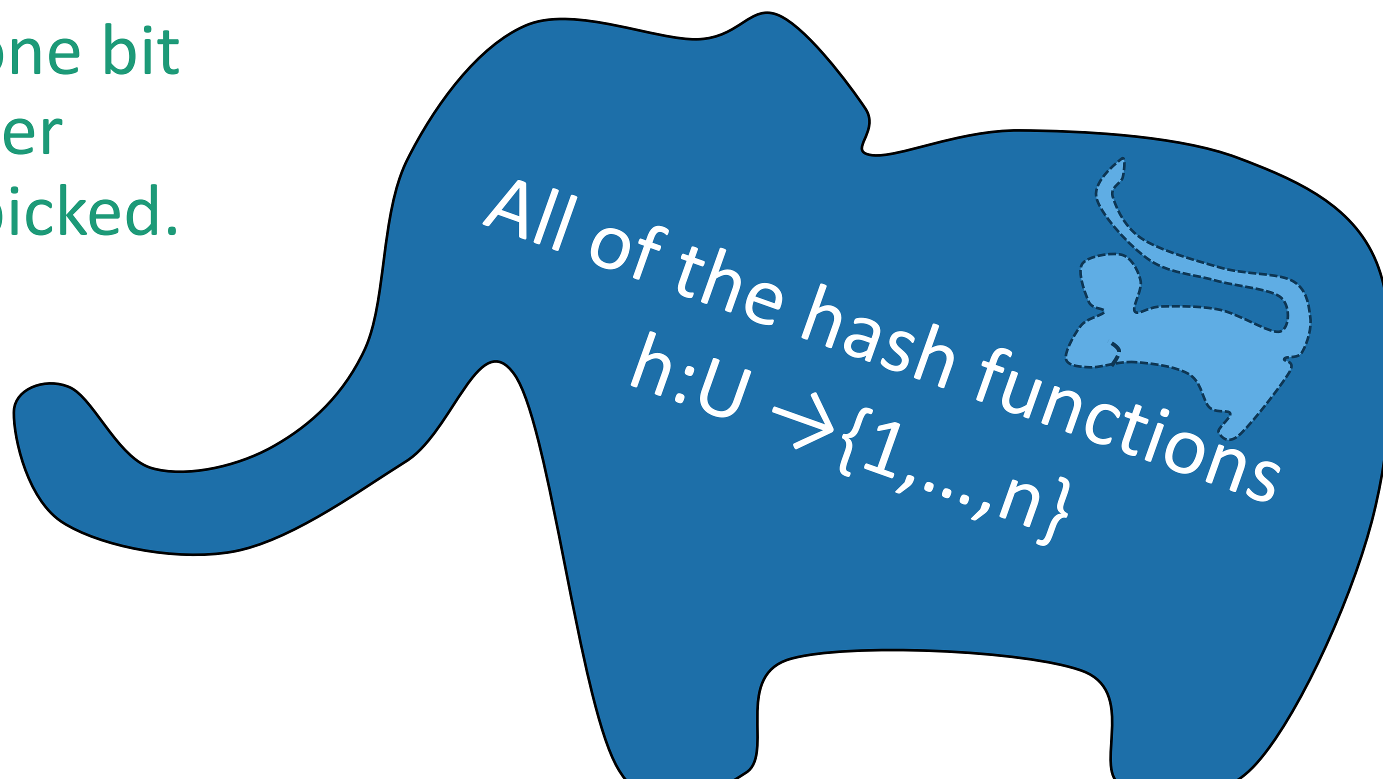


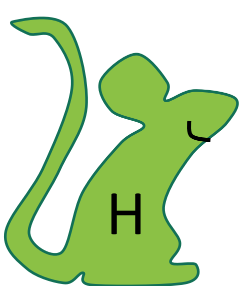
"All of the hash functions" is
an example of a hash family.

Example:
a smaller hash family

This is still a terrible idea!
Don't use this example!
For pedagogical purposes only!

- H = { function which returns the least sig. digit,
  function which returns the most sig. digit }
- Pick h in H at random.
- Store just one bit to remember which we picked.

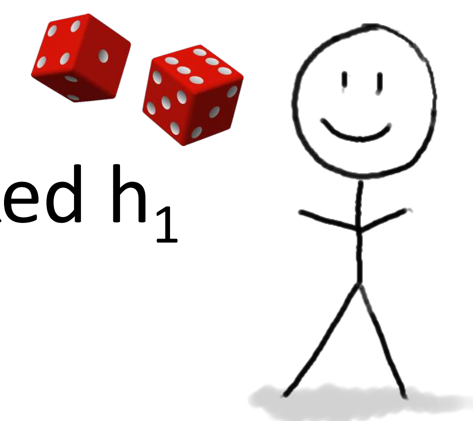




The game

h_0 = Most_significant_digit
h_1 = Least_significant_digit
$H = \{h_0, h_1\}$

2. You, the algorithm, chooses a **random** hash function $h: U \rightarrow \{0, \dots, 9\}$. Choose it randomly from H.

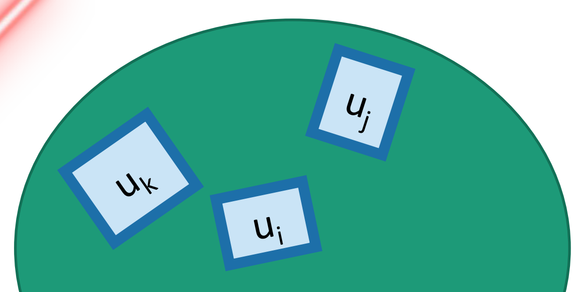


I picked h_1

1. An adversary (who knows H) chooses any n items $u_1, u_2, \dots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

19	22	42	92	0

INSERT 19, INSERT 22, INSERT 42, INSERT 92, INSERT 0, SEARCH 42, DELETE 92, SEARCH 0, INSERT 92



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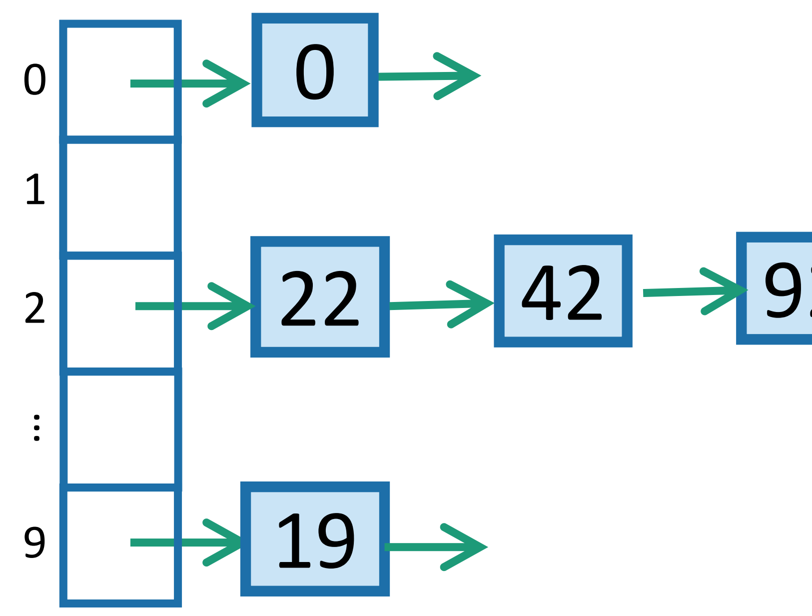

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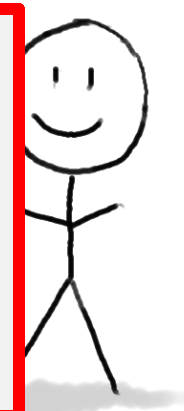
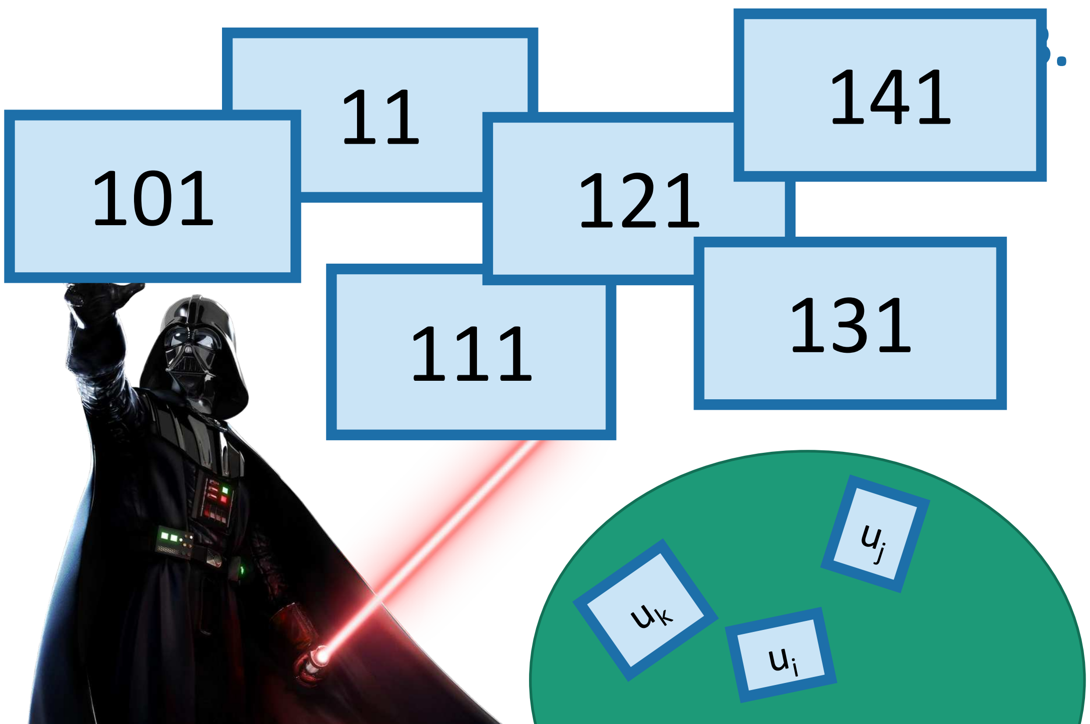
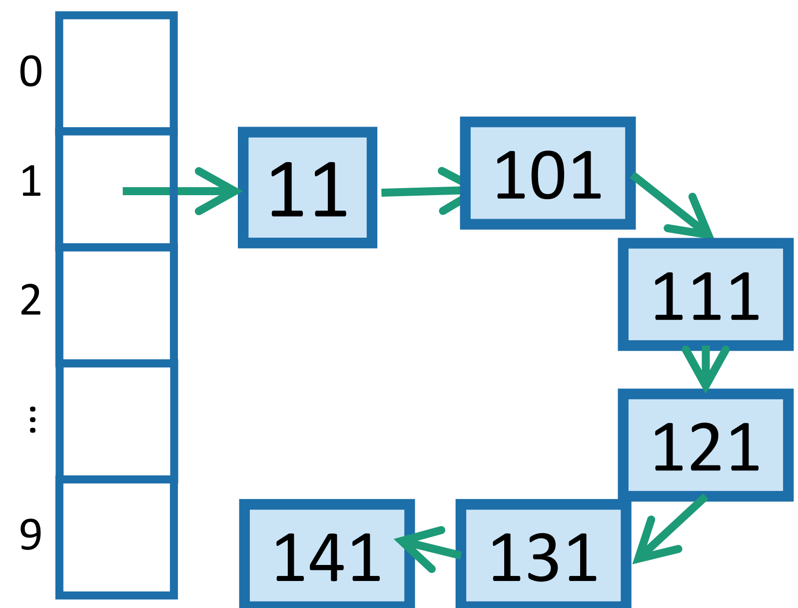
1. An adversary (who knows H) chooses any n items $u_1, u_2, \dots, u_n \in U$, and any sequence of INSERT/DELETE/SEARCH operations on those items.

This adversary could have been more adversarial!





3. **HASH IT OUT** #hashpuns

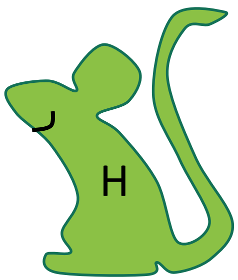


Outline

- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.
- **Hash families** are the magic behind hash tables.
- **Universal hash families** are even more magic.

How to pick the hash family?

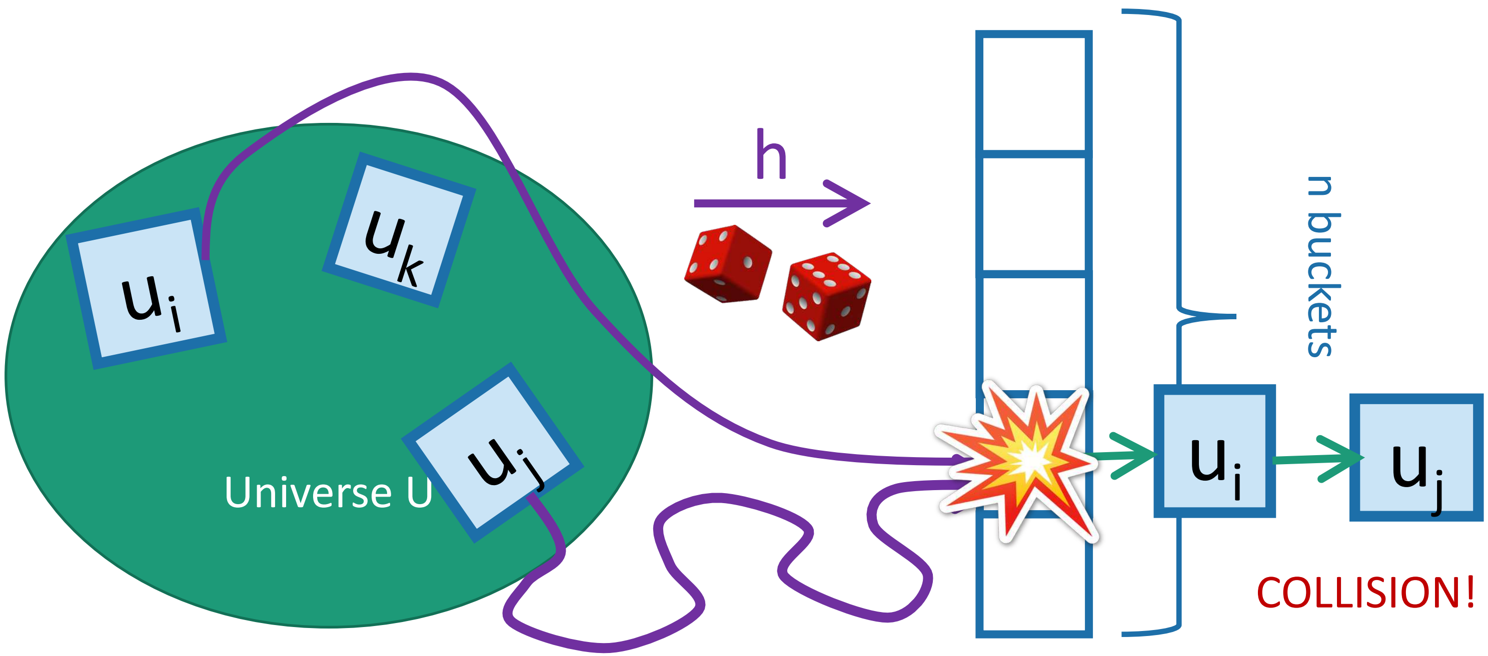
- Definitely not like in that example.
- Let’s go back to that computation from earlier….



Expected number of items in u_i's bucket?

- $E[\ ] = \sum_{j=1}^{n} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} P\{\, h(u_i) = h(u_j) \}$
- $\quad = 1 + \sum_{j \neq i} 1/n$ you will verify this on HW
- $\quad = 1 + \frac{n-1}{n} \leq 2.$

So the number of items in u_i's bucket is O(1).



How to pick the hash family?

- Let's go back to that computation from earlier….
- $E[\text{ number of things in bucket } h(u_i)\ ]$
- $= \sum_{j=1}^{n} P\{\ h(u_i) = h(u_j)\}$
- $= 1 + \sum_{j \neq i} P\{\ h(u_i) = h(u_j)\}$
- $\leq 1 + \sum_{j \neq i} 1/n$
- $= 1 + \frac{n-1}{n} \leq 2.$
- All we needed was that this $\leq$ 1/n.

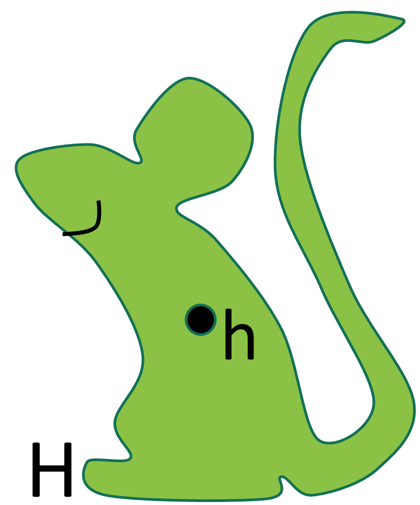
Strategy

- Pick a small hash family H, so that when I choose h randomly from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j) \} \leq \frac{1}{n}$$

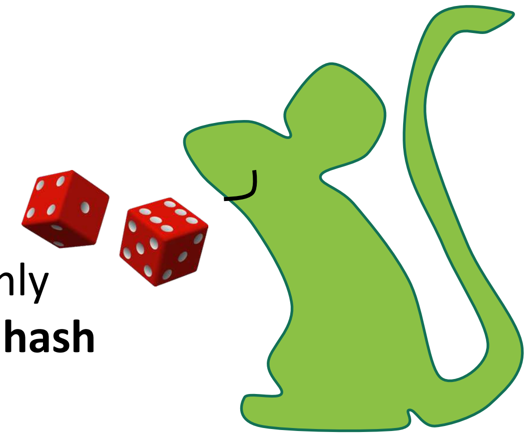
In English: fix any two elements of U. The probability that they collide under a random h in H is small.

- A hash family H that satisfies this is called a **universal hash family.**
- Then we still get O(1)-sized buckets in expectation.
- But now the space we need is log(|H|) bits.
    - Hopefully pretty small!

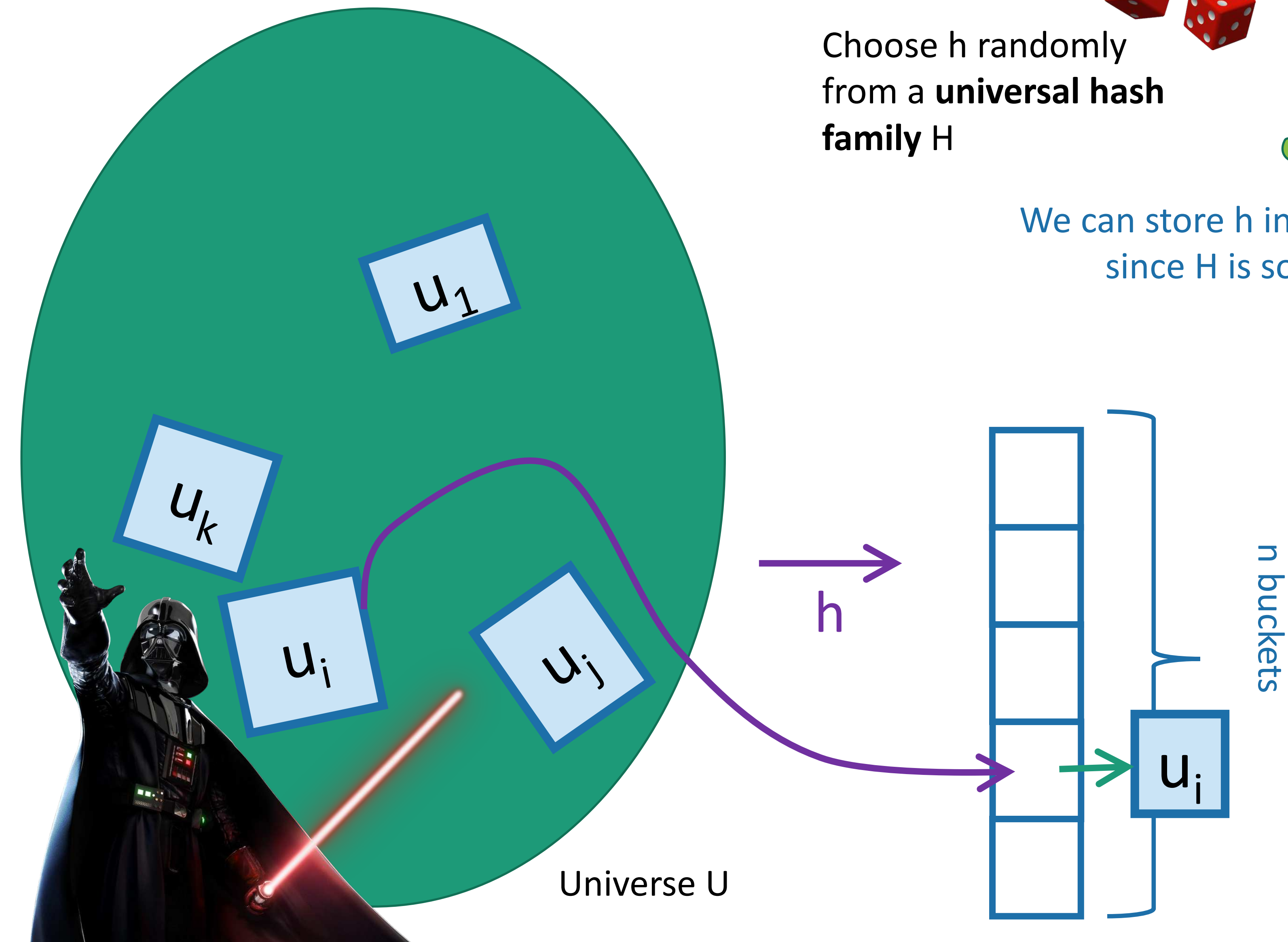


So the whole scheme will be

Choose h randomly from a **universal hash family** H



We can store h in small space since H is so small.



Probably these buckets will be pretty balanced.

Universal hash family

Let's stare at this definition

- H is a ***universal hash family*** if:
  - When h is chosen uniformly at random from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$

$$P_{h\in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

You actually saw this in your pre-lecture exercise!
Toads = hash fns
Ice cream = items
"Like" and "Dislike" = buckets

Check our understanding…

Slide (probably) ***skipped in class***

- H is a ***universal hash family*** if:
  - When h is chosen uniformly at random from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j) \} \leq \frac{1}{n}$$

- H is **[something else]** if:
  - When h is chosen uniformly at random from H,

$$\text{for all } u \in U, \text{for all } x \in \{0, \ldots, n-1\},$$
$$P_{h \in H}\{\, h(u_i) = x \} \leq \frac{1}{n}$$

Are these different?

Pre-lecture exercise

Slide skipped in class

Statement 1: P[random toad likes vanilla] = ½, P[random toad likes chocolate] = ½

P["vanilla" lands in the bucket "like"] = ½

Statement 2: P[random toad feels the same about chocolate and vanilla] = ½

P [vanilla and chocolate land in the same bucket] = ½

Universe = { vanilla, chocolate }

Buckets = { like, dislike }

Toads = different possible ways of distributing items

Pre-lecture exercise

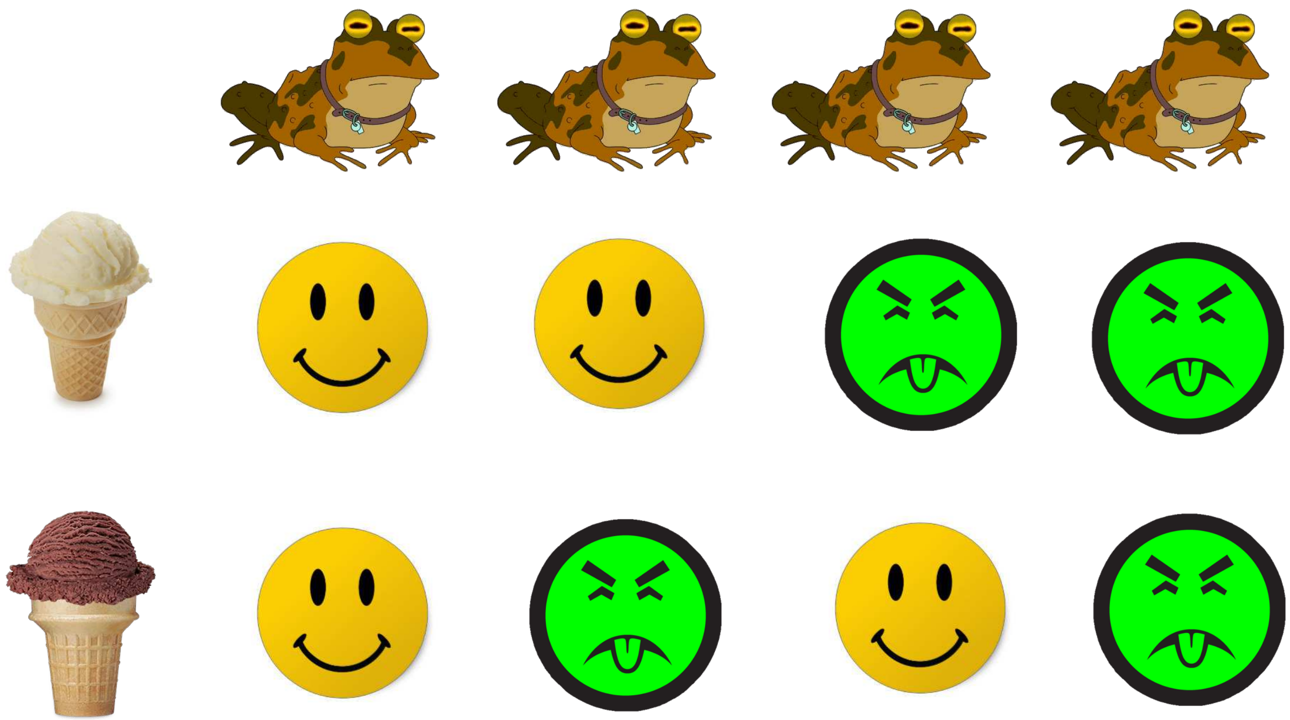
Slide skipped in class

Statement 1: P[random toad likes vanilla] = ½, P[random toad likes chocolate] = ½

P[“vanilla” lands in the bucket “like”] = ½

Statement 2: P[random toad feels the same about chocolate and vanilla] = ½

P [vanilla and chocolate land in the same bucket] = ½



Universe = { vanilla, chocolate }

Buckets = { like, dislike }

Toads = different possible ways of distributing items

Seem like they might be the same…?

Pre-lecture exercise

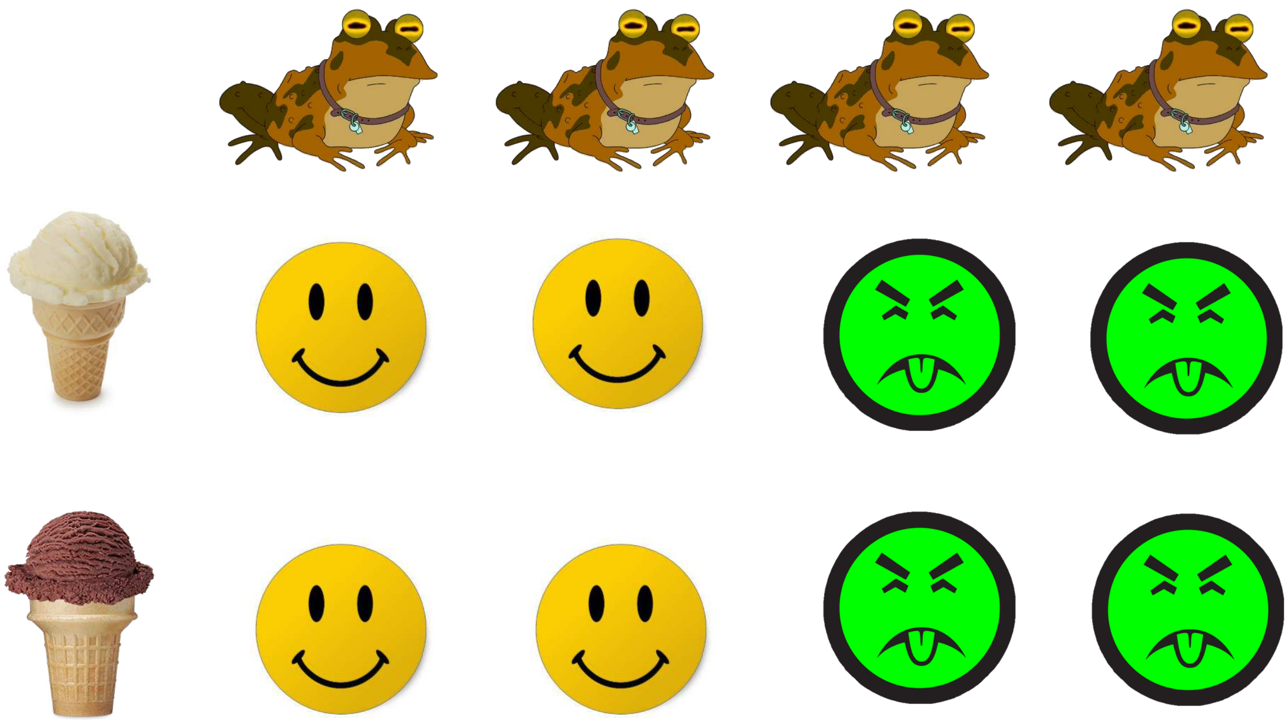
Slide skipped in class

Statement 1: P[random toad likes vanilla] = ½, P[random toad likes chocolate] = ½

P["vanilla" lands in the bucket "like"] = ½

Statement 2: P[random toad feels the same about chocolate and vanilla] = ½

P [vanilla and chocolate land in the same bucket] = ½



Universe = { vanilla, chocolate }

Buckets = { like, dislike }

Toads = different possible ways of distributing items

But no! 1 is true but 2 is not.

Slide skipped in class

Check our understanding...

- H is a universal hash family if:
  - When h is chosen uniformly at random from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

- H is **[something else]** if:
  - When h is chosen uniformly at random from H,

$$\text{for all } u \in U, \text{for all } x \in \{0, \ldots, n-1\},$$
$$P_{h \in H}\{\, h(u_i) = x\} \leq \frac{1}{n}$$

These are different!

- Pick a small hash family H, so that when I choose h randomly from H,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

Example

- Uniformly random hash function h
  - [We just saw this]
  - [Of course, this one has other downsides…]

- Pick a small hash family H, so that when I choose h randomly from H,

for all $u_i, u_j \in U$ with $u_i \neq u_j$,

$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

Non-example

- h_0 = Most_significant_digit
- h_1 = Least_significant_digit
- H = $\{h_0, h_1\}$
  - [discussion on board]

A small universal hash family??

- Here's one:
  - Pick a prime $p \geq M$.
  - Define

$$f_{a,b}(x) = ax + b \quad mod\ p$$

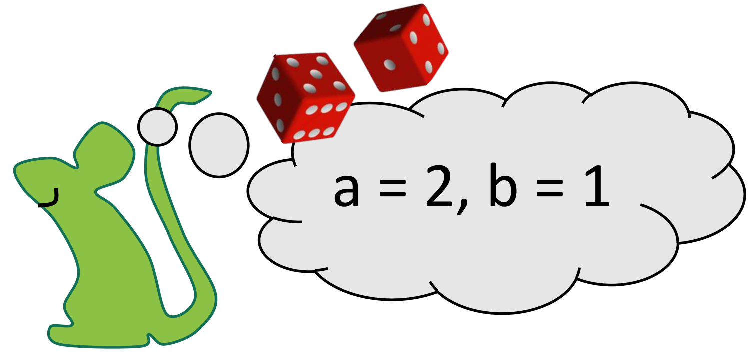
$$h_{a,b}(x) = f_{a,b}(x) \quad mod\ n$$

  - Claim:

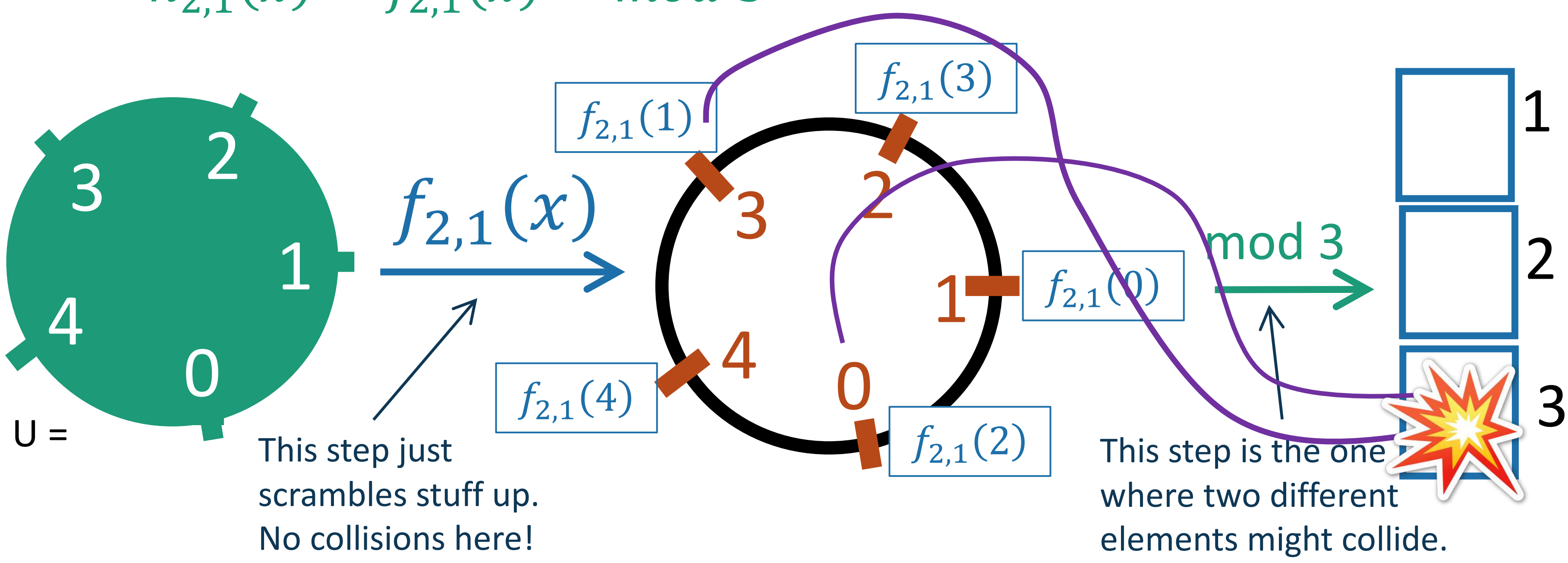
$$H = \{\ h_{a,b}(x)\ :\ a \in \{1, \dots, p-1\}, b \in \{0, \dots, p-1\}\ \}$$

is a universal hash family.

Say what?

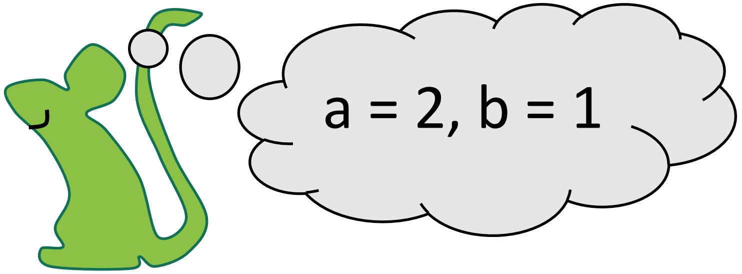


- Example: M = p = 5, n = 3
- To draw h from H:
  - Pick a random a in {1,…,4}, b in {0,…,4}
- As per the definition:
  - $f_{2,1}(x) = 2x + 1 \quad mod\ 5$
  - $h_{2,1}(x) = f_{2,1}(x) \quad mod\ 3$



Ignoring why this is a good idea

- Can we store h with small space?



- Just need to store two numbers:
  - a is in {1,…,p-1}
  - b is in {0,…,p-1}
  - So about 2log(p) bits
  - By our choice of p, that's O(log(M)) bits.

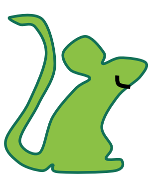
Compare: direct addressing was M bits!
Twitter example: log(M) = 140 log(128) = **980** vs M = **128^{140}**

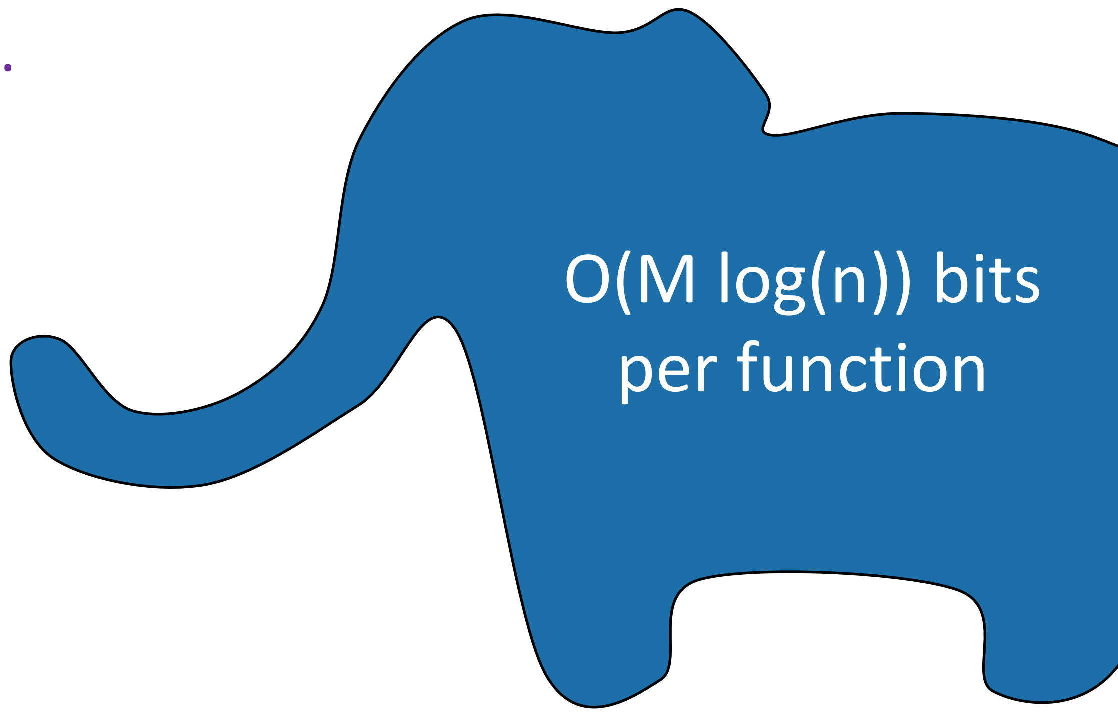
Another way to see this

using only the size of H

- We have p-1 choices for a, and p choices for b.
- So $|H| = p(p-1) = O(M^2)$
- Space needed to store an element h:
  - $\log(M^2) = O(\log(M))$.

O(log(M)) bits per function





Why does this work?

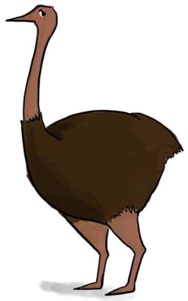
- This is actually a little complicated.
  - There are some hidden slides here about why.
  - Also see the lecture notes.

- The thing we have to show is that the collision probability is not very large.
- **Intuitively**, this is because:
  - for any (fixed, not random) pair $x \neq y$ in {0,….,p-1},
  - If a and b are random,
  - ax + b and ay + b are independent random variables. (why?)

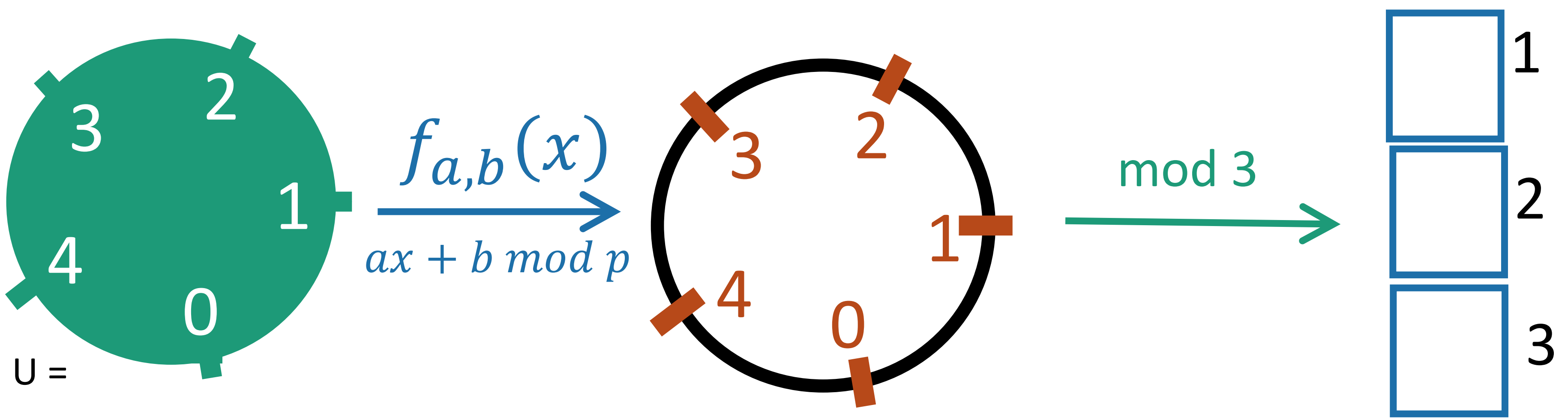
This slide skipped in class – here for reference!

Why does this work?

Convince yourself that it will be the same for any pair!



- Want to show:
  - for all $u_i, u_j \in U$ with $u_i \neq u_j$, $P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$
- aka, the probability of any two elements colliding is small.
- Let's just fix two elements and see an example.
  - Let's consider $u_i, = 0,\ \ u_j = 1.$



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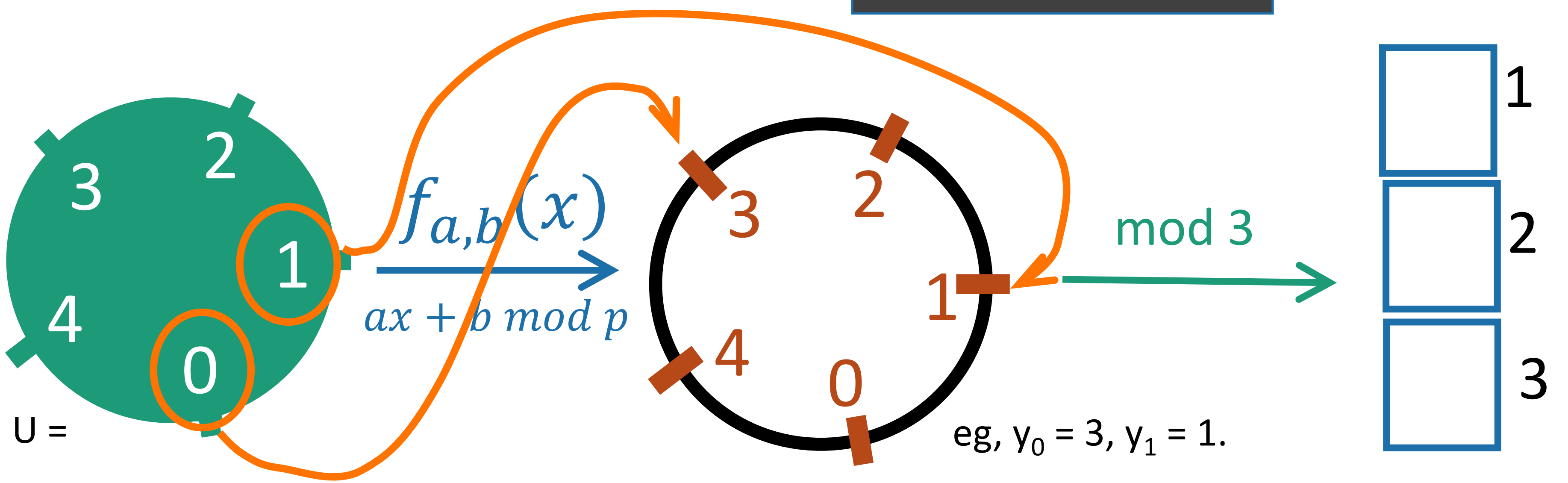
The probability that 0 and 1 collide is small

- Want to show:
  - $P_{h \in H}\{\, h(0) = h(1)\} \leq \frac{1}{n}$
- For any $y_0 \neq y_1 \in \{0,1,2,3,4\}$, how many a,b are there so that $f_{a,b}(0) = y_0$ and $f_{a,b}(1) = y_1$?
- Claim: it's exactly one.
  - Proof: solve the system of eqs. $\begin{array}{l} a \cdot 0 + b = y_0 \;\; mod\ p \\ a \cdot 1 + b = y_1 \;\; mod\ p \end{array}$ for a and b.



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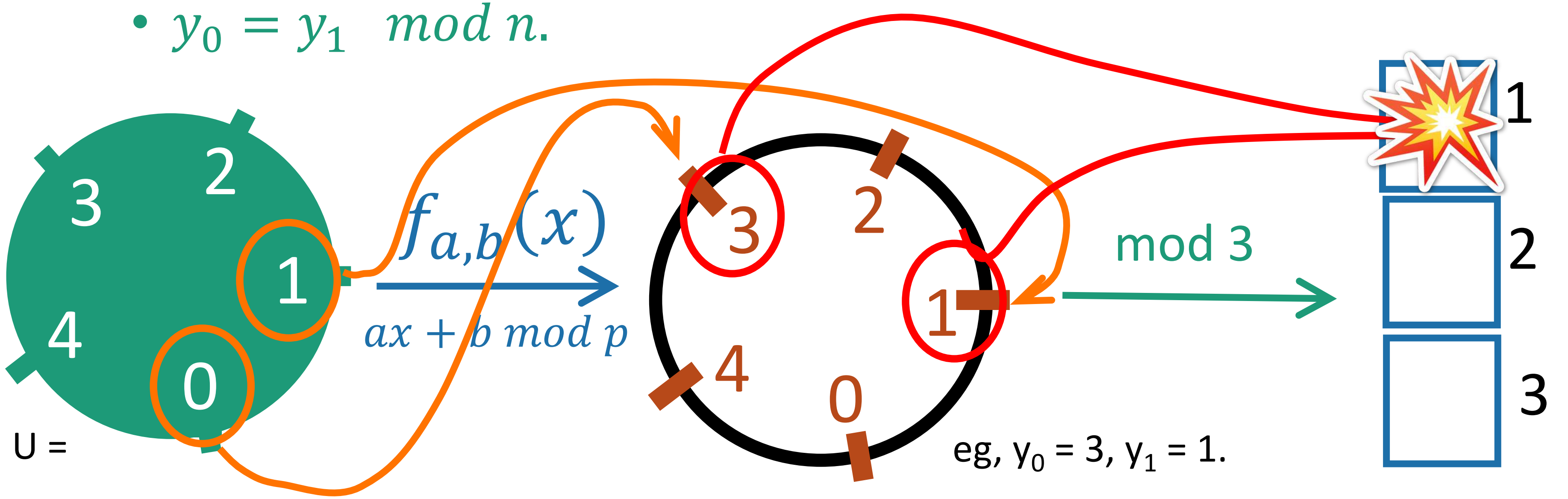
The probability that 0 and 1 collide is small

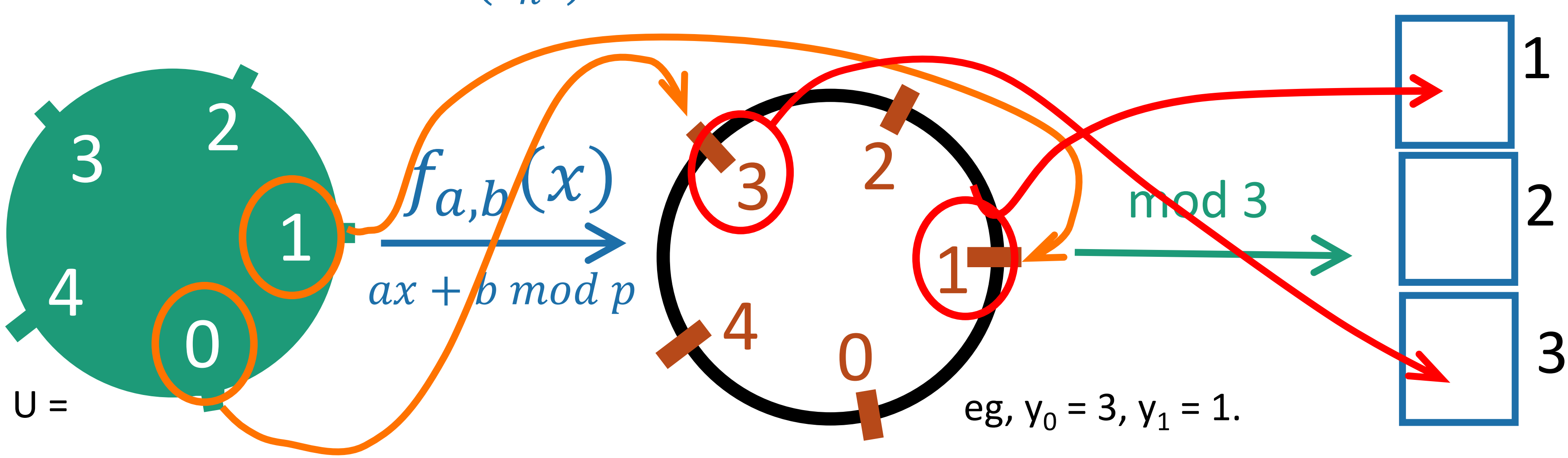
- Want to show:
  - $P_{h \in H}\{\, h(0) = h(1)\} \leq \frac{1}{n}$
- For any $y_0 \neq y_1 \in \{0,1,2,3,4\}$, **exactly one pair** a,b have $f_{a,b}(0) = y_0$ and $f_{a,b}(1) = y_1$.
- If 0 and 1 collide it's b/c there's some $y_0 \neq y_1$ so that:
  - $f_{a,b}(0) = y_0$ and $f_{a,b}(1) = y_1$.
  - $y_0 = y_1 \quad mod\ n$.



This slide skipped in class – here for reference!

The probability that 0 and 1 collide is small

- Want to show:
  - $P_{h\in H}\{\ h(0) = h(1)\} \leq \frac{1}{n}$
- The number of a,b so that 0,1 collide under $h_{a,b}$ is at most the number of $y_0 \neq y_1$ so that $y_0 = y_1 \quad mod\ n$.
- How many is that?
  - We have p choices for y_0, then at most 1/n of the remaining p-1 are valid choices for y_1…
  - So at most $p \cdot \left(\frac{p-1}{n}\right)$.



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The probability that 0 and 1 collide is small

- Want to show:
  - $P_{h \in H}\{\, h(0) = h(1)\} \leq \frac{1}{n}$
- The # of (a,b) so that 0,1 collide under $h_{a,b}$ is $\leq p \cdot \left(\frac{p-1}{n}\right)$.
- The probability (over a,b) that 0,1 collide under $h_{a,b}$ is:
- $P_{h \in H}\{\, h(0) = h(1)\} \;\leq \dfrac{p \cdot \left(\frac{p-1}{n}\right)}{|H|}$
- $= \dfrac{p \cdot \left(\frac{p-1}{n}\right)}{p(p-1)}$
- $= \dfrac{1}{n}$.

This slide skipped in class – here for reference!

The same argument goes for any pair

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h\in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

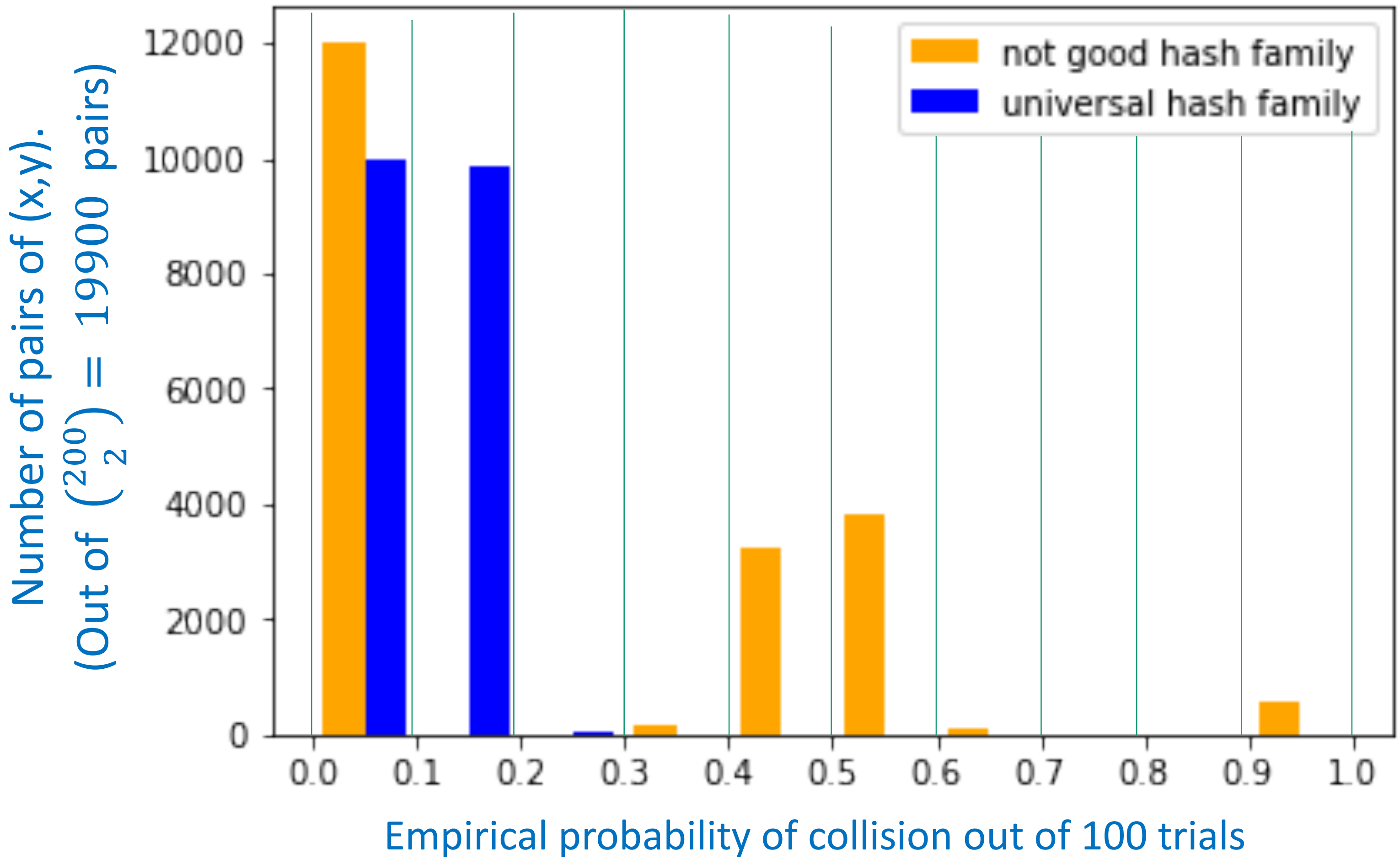
That’s the definition of a universal hash family.

So this family H indeed does the trick.

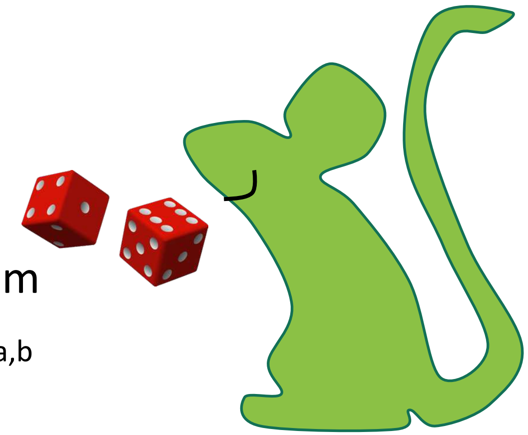
But let's check that it **does** work

- Back to IPython Notebook for Lecture 8…

M=200, n=10

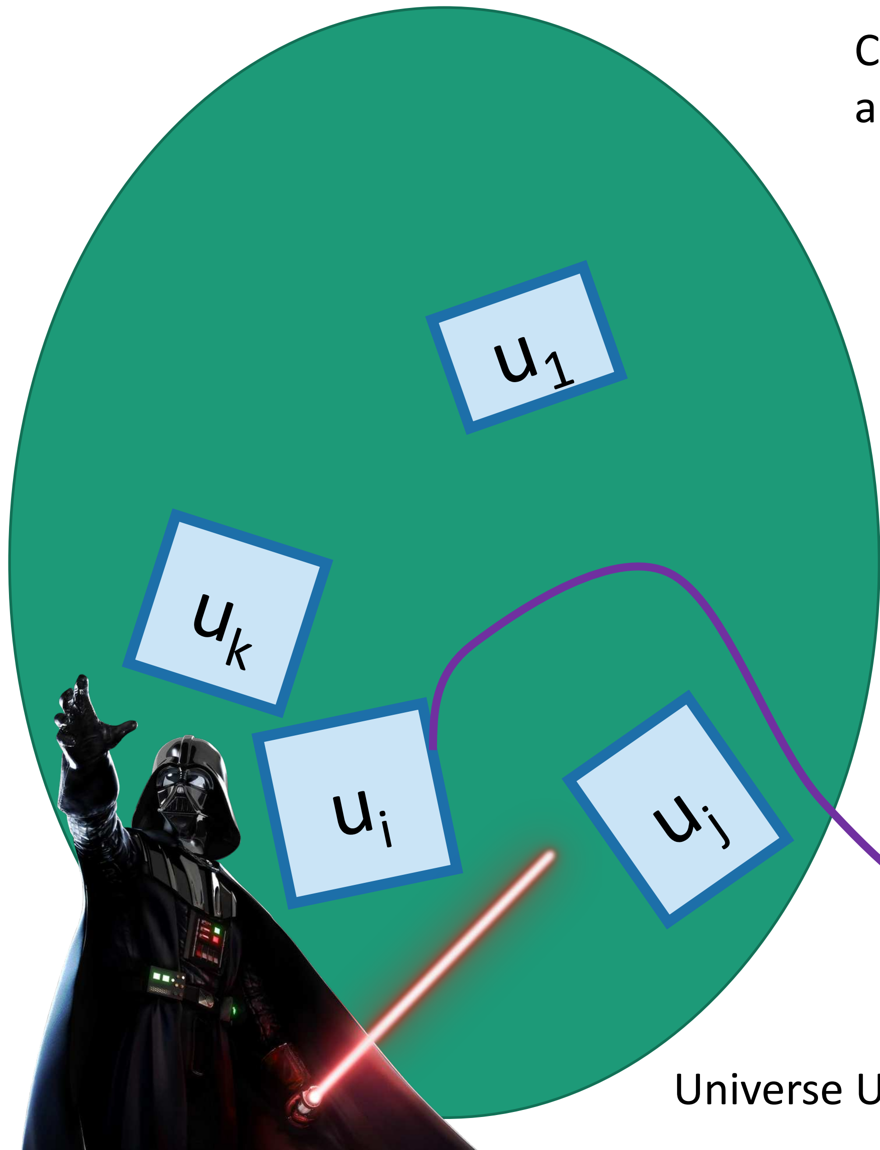


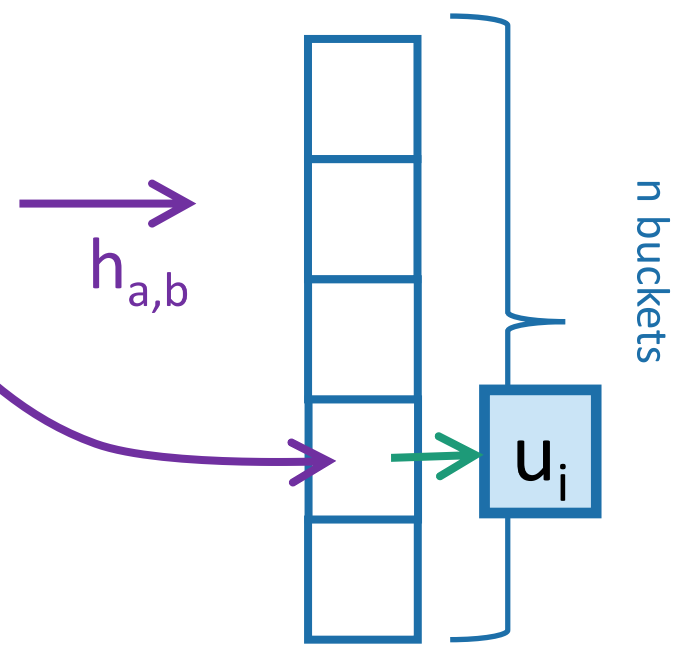
So the whole scheme will be



Choose a and b at random and form the function $h_{a,b}$

We can store h in space O(log(M)) since we just need to store a and b.





Probably these buckets will be pretty balanced.

Outline

- **Hash tables** are another sort of data structure that allows fast INSERT/DELETE/SEARCH.
  - like self-balancing binary trees
  - The difference is we can get better performance in expectation by using randomness.

- **Hash families** are the magic behind hash tables.

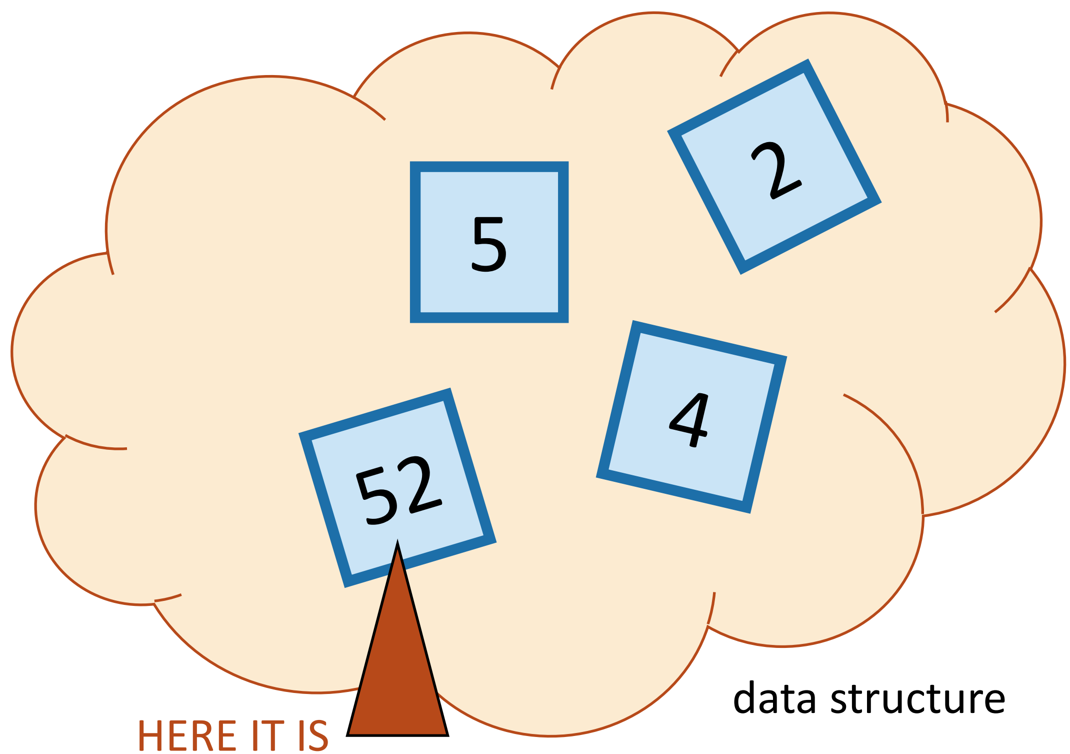
- **Universal hash families** are even more magic.

Recap

Want O(1)
INSERT/DELETE/SEARCH

- We are interesting in putting nodes with keys into a data structure that supports fast INSERT/DELETE/SEARCH.

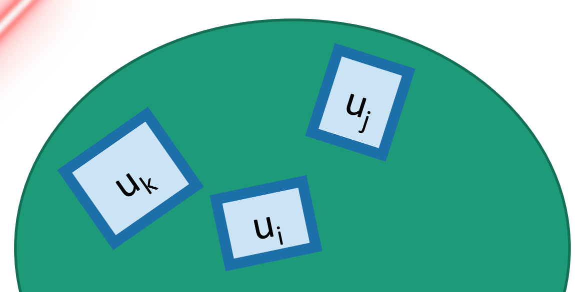
- INSERT 5
- DELETE 4
- SEARCH 52



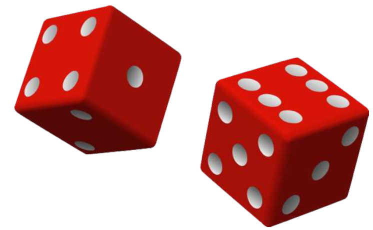
We studied this game

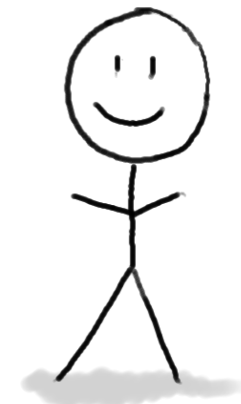
1. An adversary chooses any n items $u_1, u_2, \dots, u_n \in U$, and any sequence of L INSERT/DELETE/SEARCH operations on those items.

INSERT 13, INSERT 22, INSERT 43, INSERT 92, INSERT 7, SEARCH 43, DELETE 92, SEARCH 7, INSERT 92

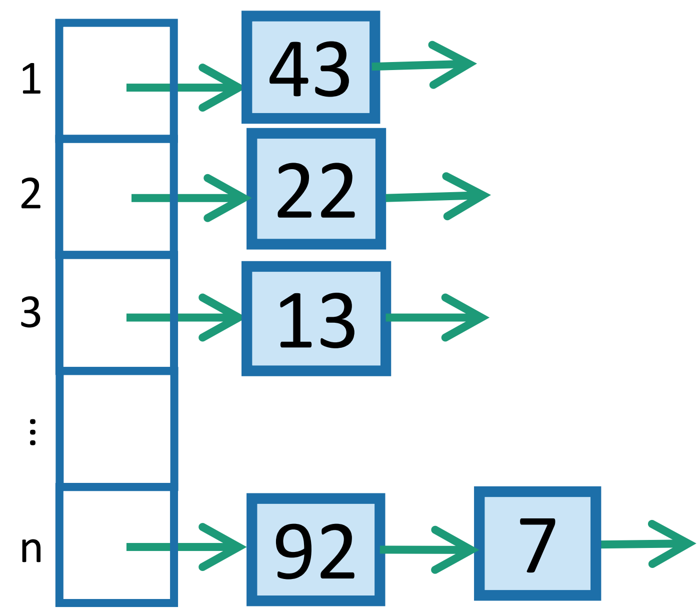


2. You, the algorithm, chooses a **random** hash function $h: U \to \{1, \dots, n\}$.





3. **HASH IT OUT**



Uniformly random h was good

aka, collision probability is small

- If we choose h uniformly at random,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$
$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

- That was enough to ensure that, in expectation, a bucket isn't too full.

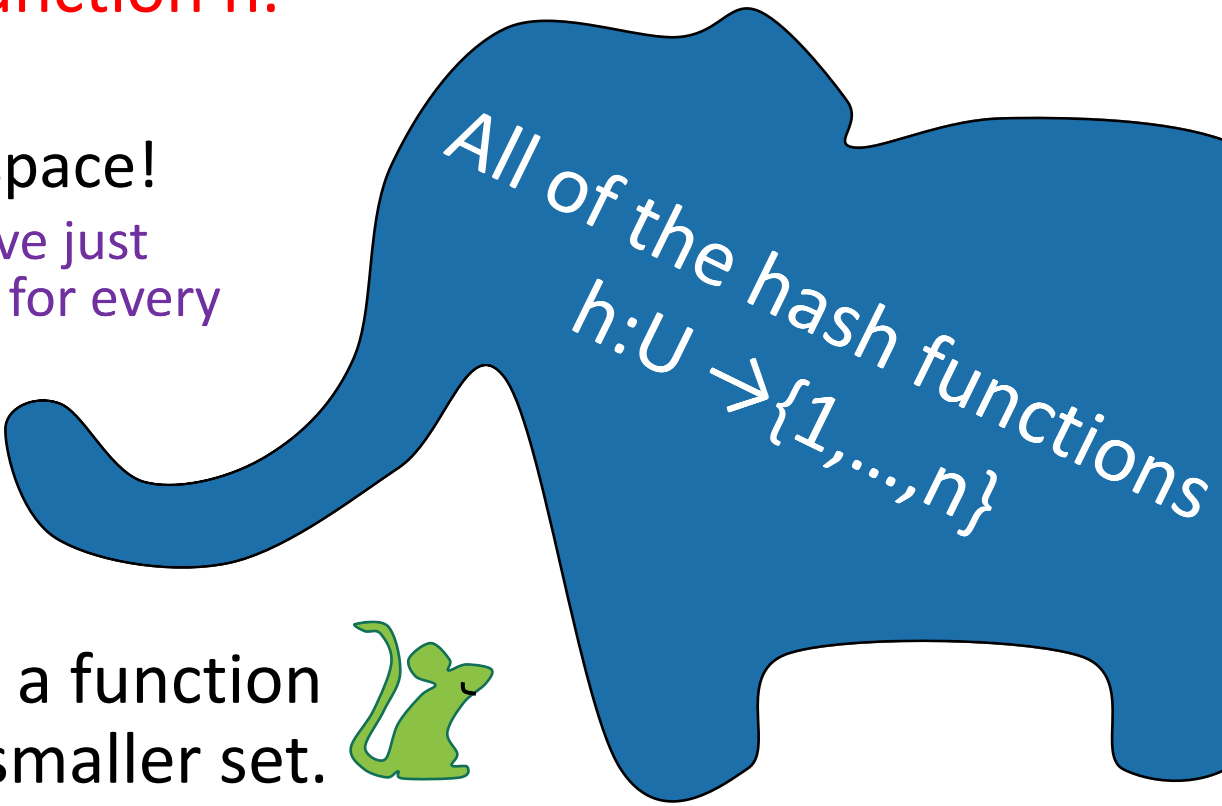
A bit more formally:

For any sequence of INSERT/DELETE/SEARCH operations on any n elements of U, the expected runtime (over the random choice of h) is O(1) per operation.

Uniformly random h was bad

- If we actually want to implement this, we have to store the hash function h.

- That takes a lot of space!
  - We may as well have just initialized a bucket for every single item in U.



- Instead, we chose a function randomly from a smaller set.

We needed a smaller set
that still has this property

- If we choose h uniformly at random,

$$\text{for all } u_i, u_j \in U \quad \text{with } u_i \neq u_j,$$

$$P_{h \in H}\{\, h(u_i) = h(u_j)\} \leq \frac{1}{n}$$

This was all we needed to make sure that the buckets were balanced in expectation!

- We call any set with that property a

universal hash family.

- We gave an example of a really small one ☺

Conclusion:

- We can build a hash table that supports INSERT/DELETE/SEARCH in O(1) expected time,
  - if we know that only n items are every going to show up, where n is waaaayyyyyy less than the size M of the universe.

- The space to implement this hash table is

  O(n log(M)) bits.

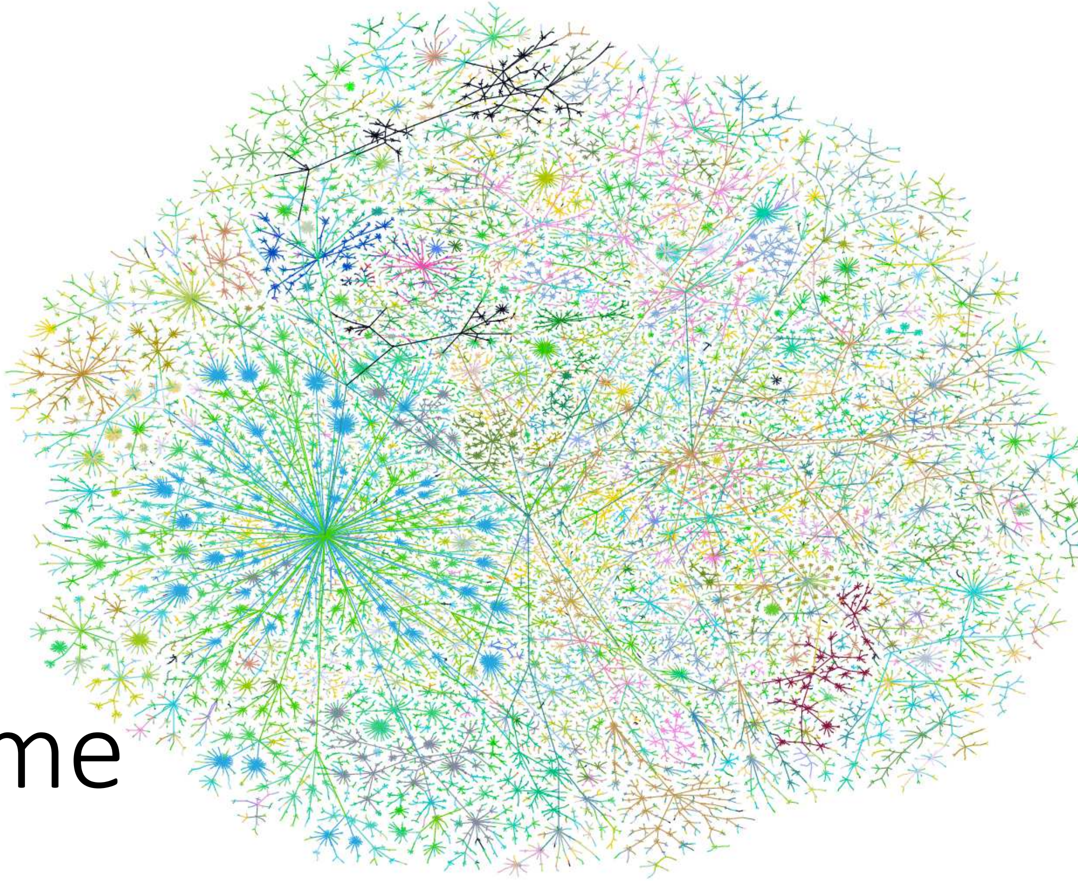
  - O(n) buckets
  - O(n) items with log(M) bits per item
  - O(log(M)) to store the hash fn.
- M is waaayyyyyy bigger than n, but log(M) probably isn't.

That's it for data structures (for now)

Now we can use these going forward!

Next Time

- Graph algorithms!



Before Next Time

- Pre-lecture exercise for Lecture 9
  - Intro to graphs