Binomial Heaps
Outline for this Week

- **Binomial Heaps (Today)**
  - A simple, flexible, and versatile priority queue.

- **Lazy Binomial Heaps (Today)**
  - A powerful building block for designing advanced data structures.

- **Fibonacci Heaps (Wednesday)**
  - A heavyweight and theoretically excellent priority queue.
Review: Priority Queues
Priority Queues

- A **priority queue** is a data structure that stores a set of elements annotated with *keys* and allows efficient extraction of the element with the least key.

- More concretely, supports these operations:
  - `pq.enqueue(v, k)`, which enqueues element `v` with key `k`;
  - `pq.find-min()`, which returns the element with the least key; and
  - `pq.extract-min()`, which removes and returns the element with the least key,
Priority queues are frequently implemented as binary heaps.

enqueue and extract-min run in time $O(\log n)$; find-min runs in time $O(1)$.

We're not going to cover binary heaps this quarter; I assume you've seen them before.
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Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - **meld**($pq_1, pq_2$): Destroy $pq_1$ and $pq_2$ and combine their elements into a single priority queue.
  - $pq$.decrease-key($v, k'$): Given a pointer to element $v$ already in the queue, lower its key to have new value $k'$.
  - $pq$.add-to-all($\Delta k$): Add $\Delta k$ to the keys of each element in the priority queue (typically used with *meld*).

  In lecture, we'll cover binomial heaps to efficiently support *meld* and Fibonacci heaps to efficiently support *meld* and *decrease-key*.

- After the TAs ensure that it's not too hard to do so, you'll design a priority queue supporting efficient *meld* and *add-to-all* on the problem set.
Meldable Priority Queues

- A priority queue supporting the *meld* operation is called a **meldable priority queue**.
- \( \text{meld}(pq_1, pq_2) \) destructively modifies \( pq_1 \) and \( pq_2 \) and produces a new priority queue containing all elements of \( pq_1 \) and \( pq_2 \).
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A priority queue supporting the *meld* operation is called a **meldable priority queue**.

*meld*(pq₁, pq₂) destructively modifies pq₁ and pq₂ and produces a new priority queue containing all elements of pq₁ and pq₂.
Efficiently Meldable Queues

- Standard binary heaps do not efficiently support *meld*.
- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.
Binomial Heaps

- The **binomial heap** is an efficient priority queue data structure that supports efficient melding.

- We'll study binomial heaps for several reasons:
  - Implementation and intuition is totally different than binary heaps.
  - Used as a building block in other data structures (Fibonacci heaps, soft heaps, etc.)
  - Has a beautiful intuition; similar ideas can be used to produce other data structures.
The Intuition: **Binary Arithmetic**
Adding Binary Numbers

• Given the binary representations of two numbers $n$ and $m$, we can add those numbers in time $\Theta(\max\{\log m, \log n\})$.

\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 1 & 0 \\
+ & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]
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\hline & & & & \text{1} \\
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```
  1 0 1 1 1 1 0
+ 1 1 1 1 1 1
  ___________  
  0 1 1 1 1 1 1
```
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```
1 1
+ 1 1 1
---
1 0 1
```
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```
+ 1 1
 1 0 1 1 1 0
  1 1 1 1 1 1
  ------------
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```
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```
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  1 0 1 1 0
  1 1 1 1 1
  ------------
  0 1 0 1 1
```
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```
+ 1 1 1
 1 0 1 1 0 0
 1 1 1 1 1 1
 0 1 0 1 1
```
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```
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 1 0 1 1 1 0
+ 1 1 1 1 1 1
 0 0 1 0 1 1
```
Adding Binary Numbers

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```
  1  1  1  1  1
 1  0  1  1  0
  +  1  1  1  1  1
  ____
  1  0  0  1  0  0  1
```
A Different Intuition

• Represent $n$ and $m$ as a collection of “packets” whose sizes are powers of two.

• Adding together $n$ and $m$ can then be thought of as combining the packets together, eliminating duplicates.
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```
\[\begin{array}{c}
16 \\
+ \\
8 \\
\hline
1
\end{array}\]
```
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Why This Works

• In order for this arithmetic procedure to work efficiently, the packets must obey the following properties:
  • The packets must be stored in ascending/descending order of size.
  • The packets must be stored such that there are no two packets of the same size.
  • Two packets of the same size must be efficiently “fusable” into a single packet.
Building a Priority Queue

- **Idea:** Adapt this approach to build a priority queue.
- Store elements in the priority queue in "packets" whose sizes are powers of two.
- Store packets in ascending size order.
- We'll choose a representation of a packet so that two packets of the same size can easily be fused together.
Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.
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As long as the packets provide $O(1)$ access to the minimum, we can execute $\text{find-min}$ in time $O(\log n)$. 
Building a Priority Queue

• What properties must our packets have?
  • Sizes must be powers of two.
  • Can efficiently fuse packets of the same size.
  • Can efficiently find the minimum element of each packet.
Inserting into the Queue

• If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.

• **Idea:** Meld together the queue and a new queue with a single packet.
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Time required: $O(\log n)$ fuses.
Deleting the Minimum

• Our analogy with arithmetic breaks down when we try to remove the minimum element.

• After losing an element, the packet will not necessarily hold a number of elements that is a power of two.
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- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.
Fracturing Packets

• If we have a packet with $2^k$ elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.

• **Fun fact**: $2^k - 1 = 1 + 2 + 4 + \ldots + 2^{k-1}$.

• **Idea**: “Fracture” the packet into $k - 1$ smaller packets, then add them back in.
Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.
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![Diagram of fracturing packets]
Fracturing Packets

- We can \textit{extract-min} by fracturing the packet containing the minimum and adding the fragments back in.

- Runtime is $O(\log n)$ fuses in \textit{meld}, plus fragment cost.
Building a Priority Queue

• What properties must our packets have?
  • Size must be a power of two.
  • Can efficiently fuse packets of the same size.
  • Can efficiently find the minimum element of each packet.
  • Can efficiently “fracture” a packet of $2^k$ nodes into packets of 1, 2, 4, 8, ..., $2^{k-1}$ nodes.

• What representation of packets will give us these properties?
Binomial Trees

- A **binomial tree of order** $k$ is a type of tree recursively defined as follows:

  A binomial tree of order $k$ is a single node whose children are binomial trees of order 0, 1, 2, ..., $k - 1$.

- Here are the first few binomial trees:
Binomial Trees

- **Theorem:** A binomial tree of order $k$ has exactly $2^k$ nodes.

- **Proof:** Induction on $k$. Assuming that binomial trees of orders 0, 1, 2, ..., $k - 1$ have $2^0$, $2^1$, $2^2$, ..., $2^{k-1}$ nodes, then the number of nodes in an order-$k$ binomial tree is

$$2^0 + 2^1 + ... + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k$$

So the claim holds for $k$ as well. ■
Binomial Trees

- A **heap-ordered binomial tree** is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.

- We will use heap-ordered binomial trees to implement our “packets.”
Binomial Trees

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Make the binomial tree with the larger root the first child of the tree with the smaller root.
Binomial Trees

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Make the binomial tree with the larger root the first child of the tree with the smaller root.
What properties must our packets have?

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Binomial Trees

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Binomial Trees

● What properties must our packets have?
  ● Size must be a power of two. ✔
  ● Can efficiently fuse packets of the same size. ✔
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  ● Can efficiently “fracture” a packet of $2^k$ nodes into packets of 1, 2, 4, 8, ..., $2^{k-1}$ nodes. ✔
The Binomial Heap

- A **binomial heap** is a collection of heap-ordered binomial trees stored in ascending order of size.

- Operations defined as follows:
  - **meld** \((pq_1, pq_2)\): Use addition to combine all the trees.
    - Fuses \(O(\log n)\) trees. Total time: \(O(\log n)\).
  - \(pq\).enqueue\((v, k)\): Meld \(pq\) and a singleton heap of \((v, k)\).
    - Total time: \(O(\log n)\).
  - \(pq\).find-min\(): Find the minimum of all tree roots.
    - Total time: \(O(\log n)\).
  - \(pq\).extract-min\(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: \(O(\log n)\).
Time-Out for Announcements!
Office Hours Update

- Keith's office hours are now moved to Gates 178 going forward – looks like we didn't actually have Hewlett 201 after lecture. 😊

- Thursday office hours changed from 7:30PM – 9:30PM, location TBA.

- As always, feel free to email us with questions!
Problem Set Two Graded

- Problem Set Two has been graded; will be returned at end of lecture.
- Rough solution sketches available up front!
Problem Set Three Clarification

- Many of you have questions about Q2 on Problem Set Three.
- For parts (iii) and (iv), assume the following:
  - The basic data structure can be constructed in worst-case time $O(n)$.
  - The cost of a cut is worst-case $O(\min\{|T_1|, |T_2|\})$.
- You don't need to justify these facts. We're mostly interested in seeing your amortized analyses.
Your Questions
“What's a popular data structure in place of map for military purposes, where guaranteed time of operations are required?”

**Red/black trees** are the gold standard here – they've got excellent worst-case performance and support fast insertions and deletions.

Hash tables have *expected* $O(1)$ operations, but that requires good hash functions. Search “HashDoS” for an attack on many programming languages' implementations of hash tables.
"How do you determine out of how many fewer points a problem set will be worth for people working alone vs. in pairs? Are you happy with how the optional pairs system has worked thus far?"

For PS1, about 25% the class worked in pairs. For PS2, about 50% of the class worked in pairs.

I'm hoping to encourage people to work in pairs without punishing people who choose not to. I'm still tuning the buffer amount.
"Can you write a CS-themed musical for us?"
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I'm thinking *Les Miserables* could be adapted for CS.

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"Can you write a CS-themed musical for us?"

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Some sample songs:

“Server in the Cloud”
"Can you write a CS-themed musical for us?"

I'm thinking *Les Miserables* could be adapted for CS. Some sample songs:

“Server in the Cloud”
“Red and Black”
"Can you write a CS-themed musical for us?"

I'm thinking *Les Miserables* could be adapted for CS. Some sample songs:

“Server in the Cloud”

“Red and Black”

“Do you Hear the Balanced Tree?”
Back to CS166!
Analyzing Insertions

- Each *enqueue* into a binomial heap takes time $O(\log n)$, since we have to meld the new node into the rest of the trees.

- However, it turns out that the amortized cost of an insertion is lower in the case where we do a series of $n$ insertions.
Adding One

• Suppose we want to execute $n++$ on the binary representation of $n$.

• Do the following:
  • Find the longest span of 1's at the right side of $n$.
  • Flip those 1's to 0's.
  • Set the preceding bit to 1.

```
1 0 1 1 1 0
```
Adding One

• Suppose we want to execute \(n++\) on the binary representation of \(n\).

• Do the following:
  • Find the longest span of 1's at the right side of \(n\).
  • Flip those 1's to 0's.
  • Set the preceding bit to 1.

1 0 1 1 1
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- Do the following:
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- Runtime: $\Theta(b)$, where $b$ is the number of bits flipped.
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.

- **Idea:** Use as a potential function the number of 1's in the number.

\[ \Phi = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \]
An Amortized Analysis

- **Claim**: Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea**: Use as a potential function the number of 1's in the number.

$\Phi = 1$

0 0 0 0 0 0 1
An Amortized Analysis

- **Claim**: Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea**: Use as a potential function the number of $1$'s in the number.

\[
\Phi = 1
\]

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

- Actual cost: 1
- $\Delta \Phi$: +1
- Amortized cost: 2
An Amortized Analysis

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$\Phi = 1$
An Amortized Analysis

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- **Idea:** Use as a potential function the number of 1's in the number.

\[ \Phi = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \]
An Amortized Analysis

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$\Phi = 1 \quad 0 \quad 0 \quad 0 \quad \boxed{1} \quad 0$
An Amortized Analysis

• **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.

• **Idea:** Use as a potential function the number of 1's in the number.

\[ \Phi = 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \]
An Amortized Analysis

- **Claim**: Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea**: Use as a potential function the number of 1's in the number.

\[
egin{array}{cccccccc}
\Phi &=& 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\text{Actual cost: 2} & \quad \Delta \Phi: 0 & \quad \text{Amortized cost: 2}
\end{array}
\]
An Amortized Analysis

- **Claim**: Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea**: Use as a potential function the number of 1's in the number.

\[ \Phi = 2 \]
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea:** Use as a potential function the number of 1's in the number.

<table>
<thead>
<tr>
<th>Φ</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Φ</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Actual cost: 1
ΔΦ: 1
Amortized cost: 2
An Amortized Analysis

• **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.

• **Idea:** Use as a potential function the number of 1's in the number.

\[ \Phi = 2 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \]
An Amortized Analysis

- **Claim**: Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea**: Use as a potential function the number of 1's in the number.

$\Phi = 1$

0 0 0 0 1 0
An Amortized Analysis

- **Claim:** Starting at zero, the amortized cost of adding one to the total is $O(1)$.
- **Idea:** Use as a potential function the number of 1's in the number.

$\Phi = 1$ 0 0 0 0 1 0
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\[ \Phi = 0 \]
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$$\Phi = 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0$$
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<table>
<thead>
<tr>
<th>Φ = 1</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual cost: 3</td>
<td>ΔΦ: -1</td>
<td>Amortized cost: 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Properties of Binomial Heaps

• Starting with an empty binomial heap, the amortized cost of each insertion into the heap is $O(1)$, assuming there are no deletions.

• **Rationale:** Binomial heap operations are isomorphic to integer arithmetic.

• Since the amortized cost of incrementing a binary counter starting at zero is $O(1)$, the amortized cost of enqueuing into an initially empty binomial heap is $O(1)$. 
Binomial vs Binary Heaps

- Interesting comparison:
  - The cost of inserting $n$ elements into a binary heap, one after the other, is $\Theta(n \log n)$ in the worst-case.
  - If $n$ is known in advance, a binary heap can be constructed out of $n$ elements in time $\Theta(n)$.
  - The cost of inserting $n$ elements into a binomial heap, one after the other, is $\Theta(n)$, even if $n$ is not known in advance!
A Catch

• This amortized time bound does not hold if \textit{enqueue} and \textit{extract-min} are intermixed.

• \textbf{Intuition:} Can force expensive insertions to happen repeatedly.
A Catch

- This amortized time bound does not hold if *enqueue* and *extract-min* are intermixed.

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- This amortized time bound does not hold if `enqueue` and `extract-min` are intermixed.
- **Intuition:** Can force expensive insertions to happen repeatedly.
**Question:** Can we make insertions amortized $O(1)$, regardless of whether we do deletions?
Where's the Cost?

- Why does *enqueue* take time $O(\log n)$?
- **Answer**: May have to combine together $O(\log n)$ different binomial trees together into a single tree.

- **New Question**: What happens if we don't combine trees together?
- That is, what if we just add a new singleton tree to the list?
Lazy Melding

- More generally, consider the following lazy melding approach:

To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$. 

```
3 7 5
6 4 8
8
```
Lazy Melding

- More generally, consider the following lazy melding approach:

  To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$. 

```
    3   7   5   1
  /   /   /   /
6  4  8  /   /
 /   /   /
8
```
Lazy Melding

• More generally, consider the following lazy melding approach:

To meld together two binomial heaps, just combine the two sets of trees together.

• If we assume the trees are stored in doubly-linked lists, this can be done in time $O(1)$. 
The Catch: Part One

- When we use eager melding, the number of trees is $O(\log n)$.
- Therefore, $\textit{find-min}$ runs in time $O(\log n)$.
- **Problem:** $\textit{find-min}$ no longer runs in time $O(\log n)$ because there can be $\Theta(n)$ trees.
A Solution

- Have the binomial heap store a pointer to the minimum element.
- Can be updated in time $O(1)$ after doing a meld by comparing the minima of the two heaps.
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- Can be updated in time $O(1)$ after doing a meld by comparing the minima of the two heaps.
The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.

- **Rationale:** Need to update the pointer to the minimum.
The Catch: Part Two

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The Catch: Part Two

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- **Rationale**: Need to update the pointer to the minimum.
Resolving the Issue

- **Idea:** When doing an *extract-min*, coalesce all of the trees so that there's at most one tree of each order.

- **Intuitively:**
  - The number of trees in a heap grows slowly (only during an insert or meld).
  - The number of trees in a heap drops rapidly after coalescing (down to $O(\log n)$).
  - Can backcharge the work done during an *extract-min* to *enqueue* or *meld*. 
Coalescing Trees

- Our eager melding algorithm assumes that
  - there is either zero or one tree of each order, and that
  - the trees are stored in ascending order.
- **Challenge:** When coalescing trees in this case, neither of these properties necessarily hold.
Wonky Arithmetic

• Let's turn back to arithmetic to get an intuition for how to solve this problem.
Wonky Arithmetic

• Let's turn back to arithmetic to get an intuition for how to solve this problem.

Sum: 19
Bits Needed: 5
Wonky Arithmetic

- Let's turn back to arithmetic to get an intuition for how to solve this problem.
Wonky Arithmetic

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Wonky Arithmetic

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**Wonky Arithmetic**

- Let's turn back to arithmetic to get an intuition for how to solve this problem.

```
8 4 2
2 1 1 1
```
Wonky Arithmetic

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Wonky Arithmetic

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Wonky Arithmetic

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![Diagram with numbers 16, 1, 1, 1]
Wonky Arithmetic

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Wonky Arithmetic

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Let's turn back to arithmetic to get an intuition for how to solve this problem.
Wonky Arithmetic

• Compute the number of bits necessary to hold the sum.
  • Only $O(\log n)$ bits are needed.
• Create an array of that size, initially empty.
• For each packet:
  • If there is no packet of that size, place the packet in the array at that spot.
  • If there is a packet of that size:
    - Fuse the two packets together.
    - Recursively add the new packet back into the array.
Now With Trees!

- Compute the number of *trees* necessary to hold the *nodes*.
  - Only $O(\log n)$ *trees* are needed.
- Create an array of that size, initially empty.
- For each *tree*:
  - If there is no *tree* of that size, place the *tree* in the array at that spot.
  - If there is a *tree* of that size:
    - Fuse the two *trees* together.
    - Recursively add the new *tree* back into the array.
Coalescing Trees
Coalescing Trees

Total number of nodes: 15
(Can compute in time $\Theta(T)$, where $T$ is the number of trees, if each tree is tagged with its order)

Bits needed: 4
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Coalescing Trees
Analyzing Coalesce

- Suppose there are $T$ trees.
- We spend $\Theta(T)$ work iterating across the main list of trees twice:
  - Pass one: Count up number of nodes (if each tree stores its order, this takes time $\Theta(T)$).
  - Pass two: Place each node into the array.
- Each merge takes time $O(1)$.
- The number of merges is $O(T)$.
- Total work done: $\Theta(T)$.
- In the worst case, this is $O(n)$. 
The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - *enqueue*: $O(1)$
  - *meld*: $O(1)$
  - *find-min*: $O(1)$
  - *extract-min*: $O(n)$.

- These are worst-case time bounds. What about an amortized time bounds?
An Observation

- The expensive step here is \textit{extract-min}, which runs in time proportional to the number of trees.
- Each tree can be traced back to one of three sources:
  - An \textit{enqueue}.
  - A \textit{meld} with another heap.
  - A tree exposed by an \textit{extract-min}.
- Let's use an amortized analysis to shift the blame for the \textit{extract-min} performance to other operations.
The Potential Method

• We will use the potential method in this analysis.

• When analyzing insertions with eager merges, we set $\Phi(D)$ to be the number of trees in $D$.

• Let's see what happens if we use this $\Phi$ here.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.
Analyzing an Insertion

- To enqueue a key, we add a new binomial tree to the forest and possibly update the min pointer.

Actual time: $O(1)$. \(\Delta \Phi: +1\)

Amortized time: $O(1)$.
Analyzing an Insertion

- To **enqueue** a key, we add a new binomial tree to the forest and possibly update the \textit{min} pointer.

![Diagram](image_url)
Analyzing an Insertion

- To *enqueue* a key, we add a new binomial tree to the forest and possibly update the *min* pointer.

Actual time: $O(1)$. $\Delta \Phi: +1$

Amortized time: $\mathcal{O}(1)$. 

![Diagram](attachment:image.png)
Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$. 

![Diagram of melded binomial heaps]

```
min
   ▼
   3
   ▼
   6
   ▼
   8

min
   ▼
   1
   ▼
   2
   ▼
   9
   ▼
   3
   ▼
   4
```

[Diagram showing meld operation]
Analyzing a Meld

• Suppose that we *meld* two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
Suppose that we **meld** two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
Analyzing a Meld

- Suppose that we **meld** two lazy binomial heaps $B_1$ and $B_2$. Actual cost: $O(1)$.
- Let $\Phi_{B_1}$ and $\Phi_{B_2}$ be the initial potentials of $B_1$ and $B_2$.
- The new heap $B$ has potential $\Phi_{B_1} + \Phi_{B_2}$ and $B_1$ and $B_2$ have potential 0.
- $\Delta\Phi$ is zero.
- Amortized cost: $O(1)$. 

\[ \text{min} \]

\[
\begin{array}{cccc}
3 & 7 & 5 & 1 \\
6 & 4 & 8 & 2 \\
8 & & & 9 \\
\end{array}
\]
Analyzing a Find-Min

- Each \textit{find-min} does $O(1)$ work and does not add or remove trees.
- Amortized cost: $O(1)$. 

\begin{center}
\begin{tikzpicture}
\node[draw,fill=red!30] (root) {\textit{min}};
\node[draw,fill=red!30] (value) {3} child {node[draw,fill=red!30] (left) {6} child {node[draw,fill=red!30] (left_left) {8}} child {node[draw,fill=red!30] (left_right) {4}}},
\node[draw,fill=red!30] (value_right) {7} child {node[draw,fill=red!30] (right) {5} child {node[draw,fill=red!30] (right_right) {8}} child {node[draw,fill=red!30] (right_left) {1}}},
\node[draw,fill=red!30] (value_right_right) {2} child {node[draw,fill=red!30] (right_right_right) {9}} child {node[draw,fill=red!30] (right_left_right) {3}} child {node[draw,fill=red!30] (right_right_left) {4}};
\end{tikzpicture}
\end{center}
Analyzing Extract-Min

- Initially, we expose the children of the minimum element. This takes time $O(\log n)$.
- Suppose that at this point there are $T$ trees. As we saw earlier, the runtime for the coalesce is $\Theta(T)$.
- When we're done merging, there will be $O(\log n)$ trees remaining, so $\Delta \Phi = -T + O(\log n)$.
- Amortized cost is
  
  $O(\log n) + \Theta(T) + O(1) \cdot (-T + O(\log n))$

  $= O(\log n) + \Theta(T) - O(1) \cdot T + O(1) \cdot O(\log n)$

  $= O(\log n)$. 
The Overall Analysis

- The *amortized* costs of the operations on a lazy binomial heap are as follows:
  - **enqueue**: $O(1)$
  - **meld**: $O(1)$
  - **find-min**: $O(1)$
  - **extract-min**: $O(\log n)$

- Any series of $e$ **enqueue**es mixed with $d$ **extract-min**s will take time $O(e + d \log e)$. 
Why This Matters

- Lazy binomial heaps are a powerful building block used in many other data structures.
- We'll see one of them, the *Fibonacci heap*, when we come back on Wednesday.
- Assuming the TAs think it's reasonable, you'll see another (supporting *add-to-all*) on the problem set.
Next Time

- **The Need for decrease-key**
  - A powerful and versatile operation on priority queues.

- **Fibonacci Heaps**
  - A variation on lazy binomial heaps with efficient decrease-key.

- **Implementing Fibonacci Heaps**
  - ... is harder than it looks!