

# Fibonacci Heaps

*You can submit  
Problem Set 3 in  
the box up front.*

# Outline for Today

- **Review from Last Time**
  - Quick refresher on binomial heaps and lazy binomial heaps.
- **The Need for *decrease-key***
  - An important operation in many graph algorithms.
- **Fibonacci Heaps**
  - A data structure efficiently supporting *decrease-key*.
- **Representational Issues**
  - Some of the challenges in Fibonacci heaps.

A Personal Note: **This is Exciting!**

# Review: (Lazy) Binomial Heaps

# Building a Priority Queue

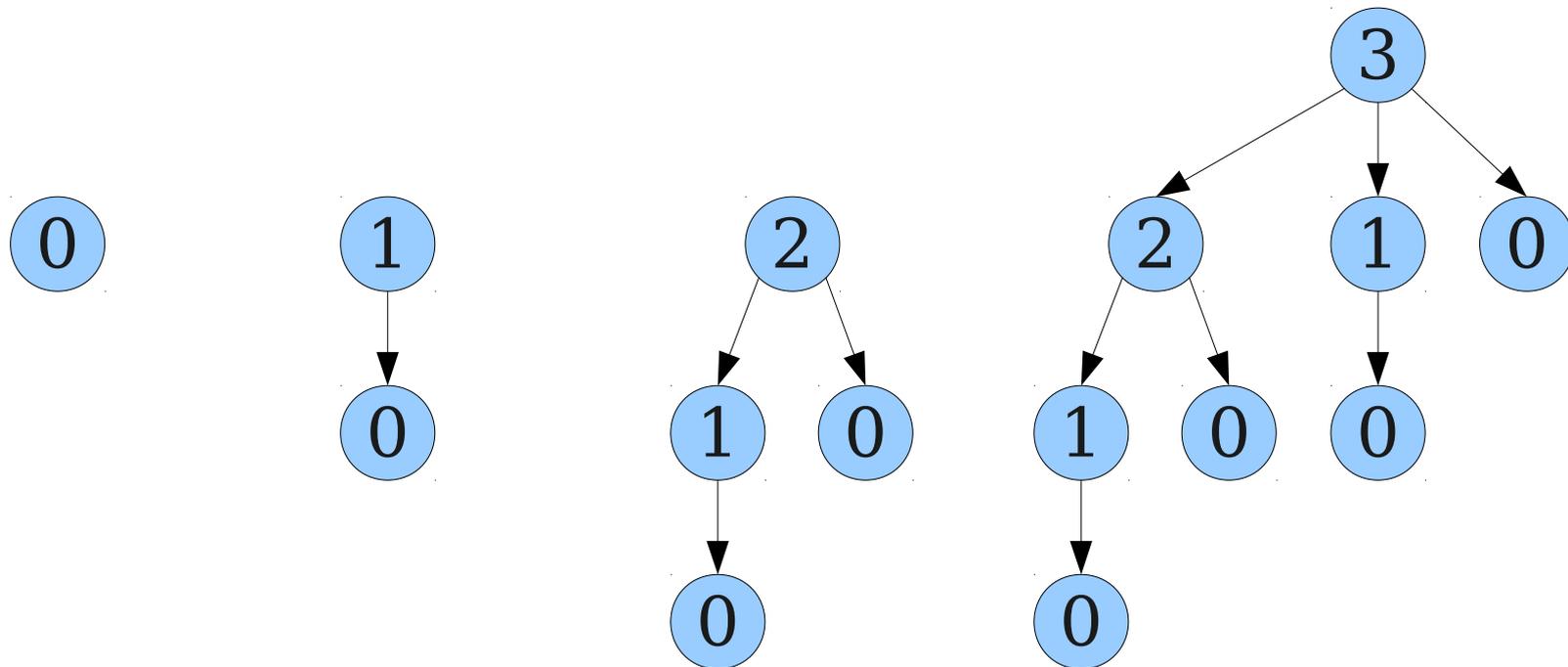
- Group nodes into “packets” with the following properties:
  - Size must be a power of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently “fracture” a packet of  $2^k$  nodes into packets of 1, 2, 4, 8, ...,  $2^{k-1}$  nodes.

# Binomial Trees

- A **binomial tree of order  $k$**  is a type of tree recursively defined as follows:

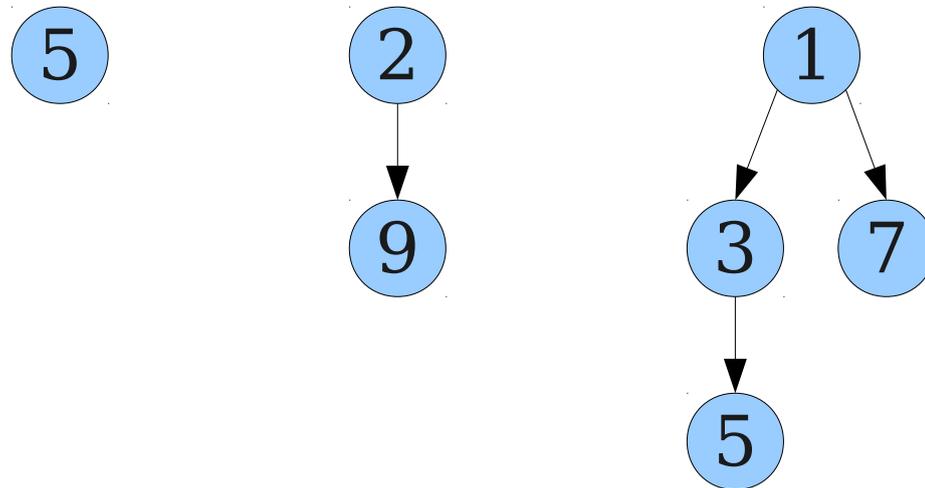
A binomial tree of order  $k$  is a single node whose children are binomial trees of order  $0, 1, 2, \dots, k - 1$ .

- Here are the first few binomial trees:



# Binomial Trees

- A **heap-ordered binomial tree** is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.
- We will use heap-ordered binomial trees to implement our “packets.”



# The Binomial Heap

- A **binomial heap** is a collection of heap-ordered binomial trees stored in ascending order of size.
- Operations defined as follows:
  - ***meld***( $pq_1, pq_2$ ): Use addition to combine all the trees.
    - Fuses  $O(\log n)$  trees. Total time:  $O(\log n)$ .
  - ***pq.enqueue***( $v, k$ ): Meld  $pq$  and a singleton heap of  $(v, k)$ .
    - Total time:  $O(\log n)$ .
  - ***pq.find-min***(): Find the minimum of all tree roots.
    - Total time:  $O(\log n)$ .
  - ***pq.extract-min***(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time:  $O(\log n)$ .

# Lazy Binomial Heaps

- A **lazy binomial heap** is a variation on a standard binomial heap in which *melds* are done lazily by concatenating tree lists together.
- Tree roots are stored in a doubly-linked list.
- An extra pointer is required that points to the minimum element.
- *extract-min* eagerly coalesces binomial trees together and runs in amortized time  $O(\log n)$ .

# The Overall Analysis

- Set  $\Phi(D)$  to be the number of trees in  $D$ .
- The *amortized* costs of the operations on a lazy binomial heap are as follows:
  - ***enqueue***:  $O(1)$
  - ***meld***:  $O(1)$
  - ***find-min***:  $O(1)$
  - ***extract-min***:  $O(\log n)$
- Details are in the previous lecture.
- Let's quickly review ***extract-min***'s analysis.

# Analyzing Extract-Min

- Initially, we expose the children of the minimum element. This takes time  $O(\log n)$ .
- Suppose that at this point there are  $T$  trees. The runtime for the coalesce is  $\Theta(T)$ .
- When we're done merging, there will be  $O(\log n)$  trees remaining, so  $\Delta\Phi = -T + O(\log n)$ .
- Amortized cost is

$$\begin{aligned} & O(\log n) + \Theta(T) + O(1) \cdot (-T + O(\log n)) \\ &= O(\log n) + \Theta(T) - O(1) \cdot T + O(1) \cdot O(\log n) \\ &= \mathbf{O(\log n)}. \end{aligned}$$

# A Detail in the Analysis

- The amortized cost of an *extract-min* is

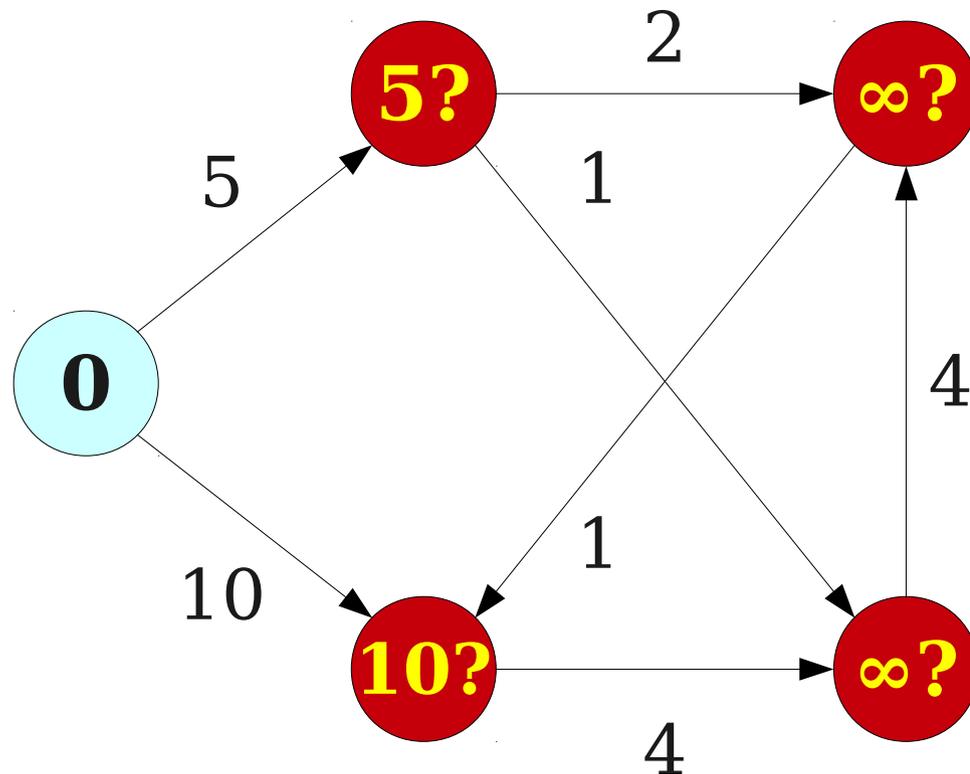
$$O(\log n) + \Theta(T) + O(1) \cdot (-T + O(\log n))$$

- Where do these  $O(\log n)$  terms come from?
  - First  $O(\log n)$ : Removing the minimum element might expose  $O(\log n)$  children, since the maximum order of a tree is  $O(\log n)$ .
  - Second  $O(\log n)$ : Maximum number of trees after a coalesce is  $O(\log n)$ .
- **A different intuition:** Let  $M(n)$  be the maximum possible order of a tree in a lazy binomial heap.
- Amortized runtime is  $O(M(n))$ .

The Need for *decrease-key*

# Review: Dijkstra's Algorithm

- Dijkstra's algorithm solves the single-source shortest paths (SSSP) problem in graphs with nonnegative edge weights.

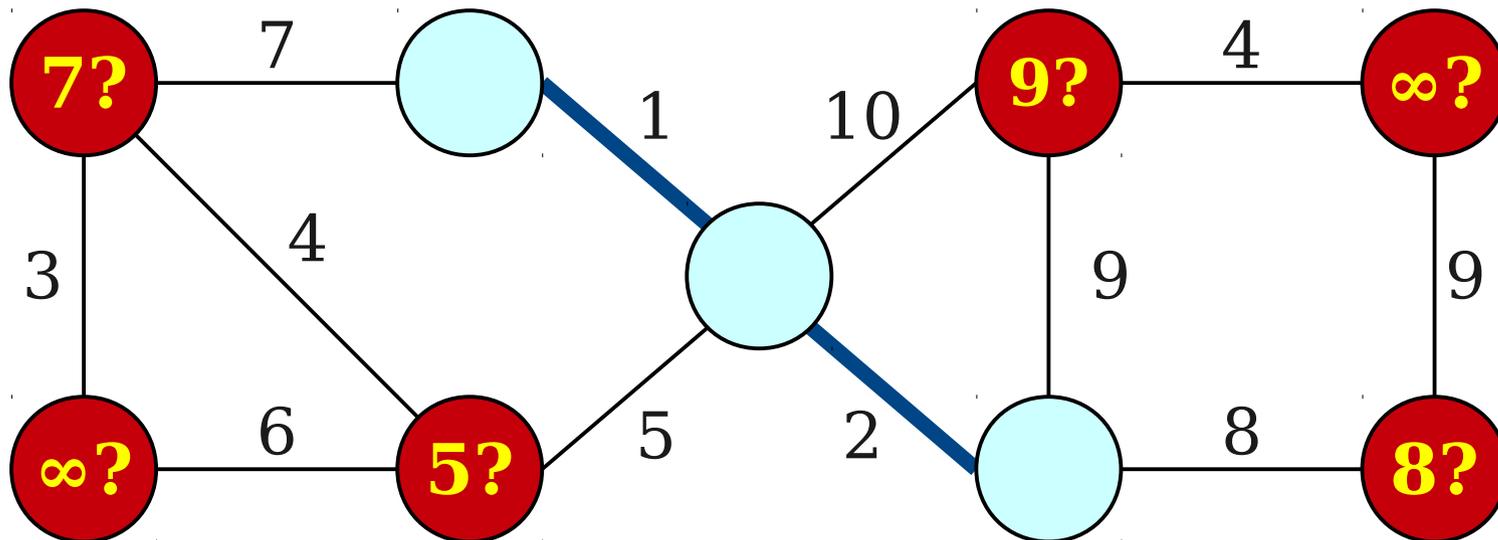


# Dijkstra and Priority Queues

- At each step of Dijkstra's algorithm, we need to do the following:
  - Find the node at  $v$  minimum distance from  $s$ .
  - Update the candidate distances of all the nodes connected to  $v$ . (Distances only decrease in this step.)
- This first step sounds like an ***extract-min*** on a priority queue.
- How would we implement the second step?

# Review: Prim's Algorithm

- Prim's algorithm solves the minimum spanning tree (MST) problem in undirected graphs.



# Prim and Priority Queues

- At each step of Prim's algorithm, we need to do the following:
  - Find the node  $v$  outside of the spanning tree with the lowest-cost connection to the tree.
  - Update the candidate distances from  $v$  to nodes outside the set  $S$ .
- This first step sounds like an ***extract-min*** on a priority queue.
- How would we implement the second step?

# The *decrease-key* Operation

- Some priority queues support the operation  $pq.\mathbf{decrease-key}(v, k)$ , which works as follows:  
*Given a pointer to an element  $v$  in  $pq$ , lower its key (priority) to  $k$ . It is assumed that  $k$  is less than the current priority of  $v$ .*
- This operation is crucial in efficient implementations of Dijkstra's algorithm and Prim's MST algorithm.

# Dijkstra and *decrease-key*

- Dijkstra's algorithm can be implemented with a priority queue using
  - $O(n)$  total *enqueues*,
  - $O(n)$  total *extract-mins*, and
  - $O(m)$  total *decrease-keys*.
- Dijkstra's algorithm runtime is

$$O(n T_{\text{enq}} + n T_{\text{ext}} + m T_{\text{dec}})$$

# Prim and *decrease-key*

- Prim's algorithm can be implemented with a priority queue using
  - $O(n)$  total *enqueues*,
  - $O(n)$  total *extract-mins*, and
  - $O(m)$  total *decrease-keys*.
- Prim's algorithm runtime is

$$O(n T_{\text{enq}} + n T_{\text{ext}} + m T_{\text{dec}})$$

# Standard Approaches

- In a binary heap, *enqueue*, *extract-min*, and *decrease-key* can be made to work in time  $O(\log n)$  time each.
- Cost of Dijkstra's / Prim's algorithm:  
$$O(n T_{\text{enq}} + n T_{\text{ext}} + m T_{\text{dec}})$$
$$= O(n \log n + n \log n + m \log n)$$
$$= \mathbf{O(m \log n)}$$

# Standard Approaches

- In a binomial heap,  $n$  **enqueues** takes time  $O(n)$ , each **extract-min** takes time  $O(\log n)$ , and each **decrease-key** takes time  $O(\log n)$ .
- Cost of Dijkstra's / Prim's algorithm:  
$$O(n T_{\text{enq}} + n T_{\text{ext}} + m T_{\text{dec}})$$
$$= O(n + n \log n + m \log n)$$
$$= \mathbf{O(m \log n)}$$

# Where We're Going

- The **Fibonacci heap** has these runtimes:
  - *enqueue*:  $O(1)$
  - *meld*:  $O(1)$
  - *find-min*:  $O(1)$
  - *extract-min*:  $O(\log n)$ , amortized.
  - *decrease-key*:  $O(1)$ , amortized.

- Cost of Prim's or Dijkstra's algorithm:

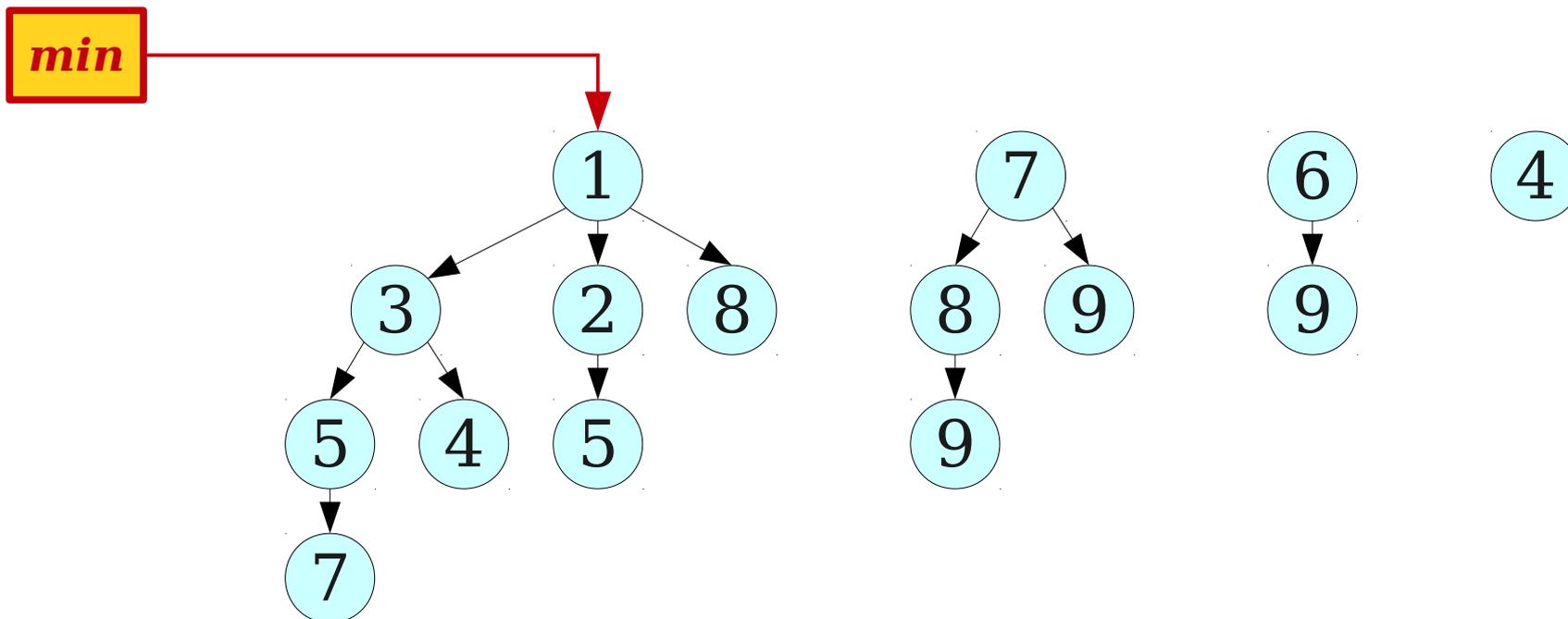
$$\begin{aligned} & O(n T_{\text{enq}} + n T_{\text{ext}} + m T_{\text{dec}}) \\ &= O(n + n \log n + m) \\ &= \mathbf{O(m + n \log n)} \end{aligned}$$

- This is theoretically optimal for a comparison-based priority queue in Dijkstra's or Prim's algorithms.

The Challenge of *decrease-key*

# A Simple Implementation

- It is possible to implement *decrease-key* in time  $O(\log n)$  using lazy binomial heaps.
- **Idea:** “Bubble” the element up toward the root of the binomial tree containing it and (potentially) update the *min* pointer.



# The Challenge

- **Goal:** Implement *decrease-key* in amortized time  $O(1)$ .
- Why is this hard?
  - Lowering a node's priority might break the heap property.
  - Correcting the imbalance  $O(\log n)$  layers deep in a tree might take time  $O(\log n)$ .
- We will need to change our approach.

**Time-Out for Announcements!**

# Problem Set Four

- Problem Set Four goes out today; it's due next Wednesday at the start of class.
- Play around with binomial heaps, lazy binomial heaps, Fibonacci heaps, and amortized analysis!
- We recommend working in pairs on this one; stick around after class if you're looking for a partner!

# Problem Set Three Feedback

- Thanks for the feedback on PS3!
- Forgot to give feedback? Please try to do so as soon as possible.
- I do take feedback seriously and I've been reading the responses already.

# Final OH Times/Places

- Keith's Monday OH will be from 3:30PM – 5:30PM in Gates 178.
- TA OH are Thursday from 7:30PM – 9:30PM in **160-318**.
- And, of course, you can always email us with questions!



# HOW WILL YOU RESPOND? COURAGEOUS CONVERSATIONS with Glenn Singleton

Stanford GSE Alum, award-winning author,  
former Ivy League Admissions Director,  
and President and Founder of Pacific Educational Group

CEMEX

Thursday, April 24th  
7:00 - 9:00 pm



Black Community Services Center  
Program in Feminist, Gender and Sexuality Studies

MEChA de Stanford  
FLIP

Clayman Institute for Gender Research  
Inter-Sorority Council

SPREES  
The Stanford Democrats

SAAAC

Program in Urban Studies  
EAST House

SAIO  
AASA

Metadiscussion about race. This is particularly relevant in STEM fields.

Thursday, 7PM - 9PM in CEMEX.

Your Questions

“What program(s) do you use to make the drawings on your lecture slides? It looks like it takes forever.”

I'm just using the default LibreOffice that comes with Linux. I keep thinking I should switch tools, but every animation and drawing is different!

“Can you post an 'as-is' copy of your slides before lecture starts? It's nice to be able to go back and review a slide after you've moved on from it.”

Sure! I keep meaning to do this, but sometimes when I'm heading over I forget to update the website.

“Can you implement an efficient stack using only two (or more) queues?”

You can implement a stack with two queues, but to the best of my knowledge there's no “efficient” (i.e. amortized  $O(1)$ ) way to do this. Let me know if I'm mistaken!

“Until what age I need to procrastinate my real life and study instead?”

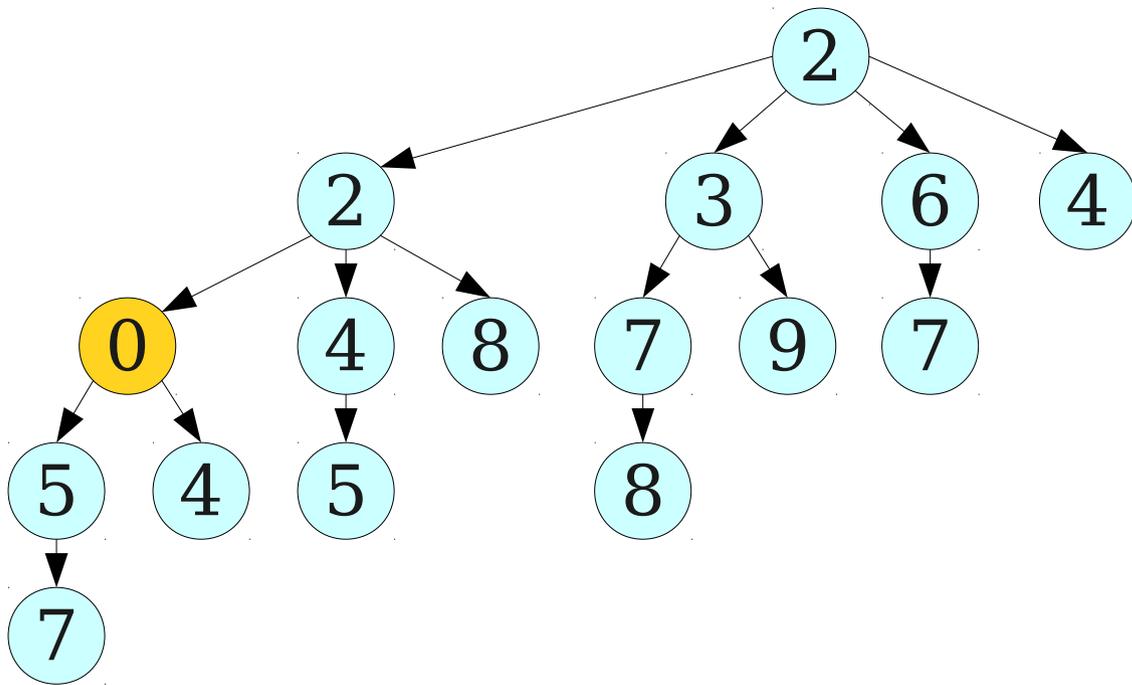
You don't need to feel this way! If you do, come talk to me and I can offer some Life Advice™.

Back to CS166!

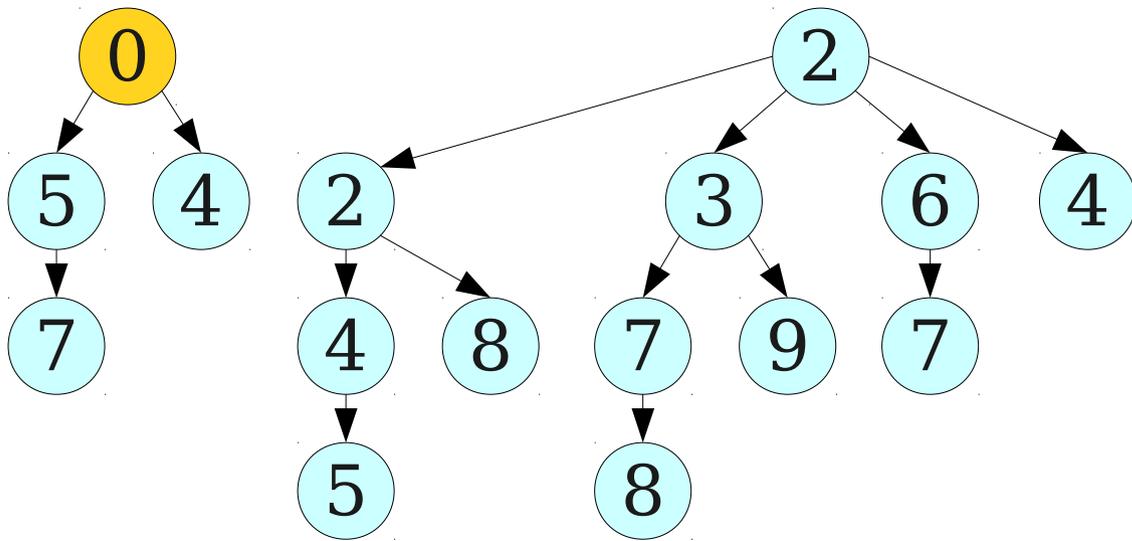
# The Challenge

- **Goal:** Implement *decrease-key* in amortized time  $O(1)$ .
- Why is this hard?
  - Lowering a node's priority might break the heap property.
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- We will need to change our approach.

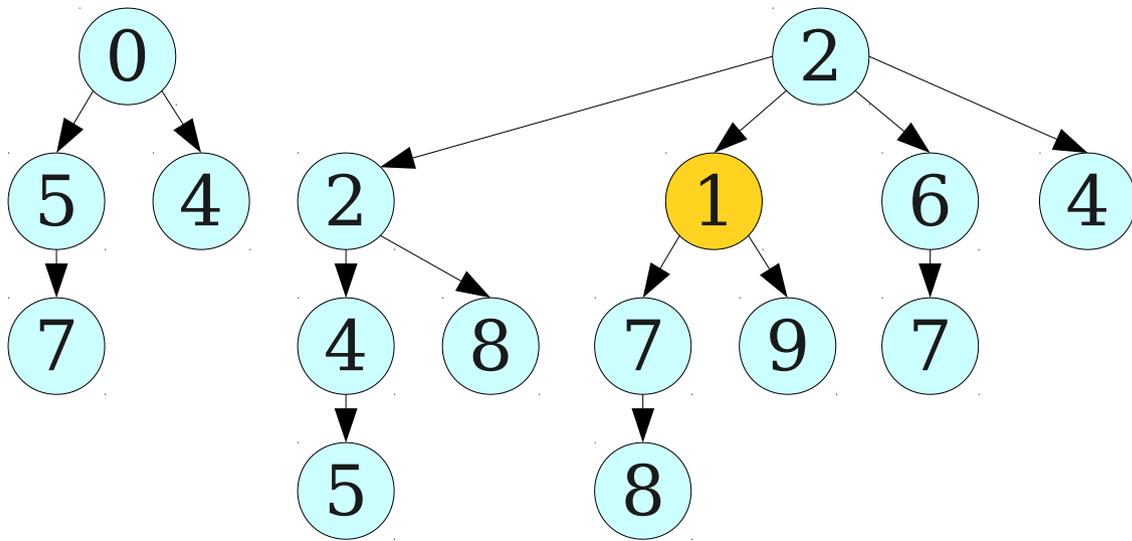
# A Crazy Idea



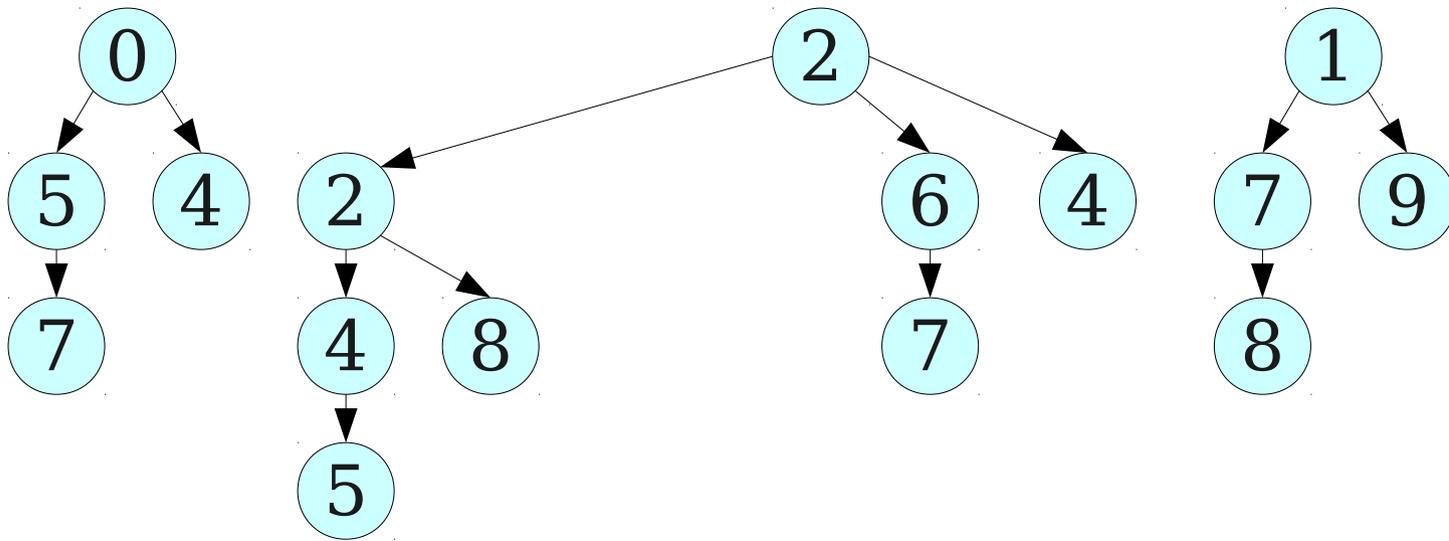
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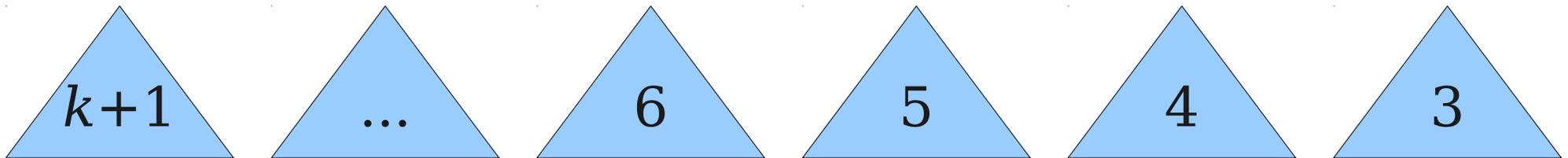
# A Crazy Idea

- To implement *decrease-key* efficiently:
  - Lower the key of the specified node.
  - If its key is greater than or equal to its parent's key, we're done.
  - Otherwise, cut that node from its parent and hoist it up to the root list, optionally updating the min pointer.
- Time required:  $O(1)$ .
  - This requires some changes to the tree representation; more details later.

# Tree Sizes and Orders

- **Recall:** A binomial tree of order  $k$  has  $2^k$  nodes and the root has  $k$  children.
- Going forward, we'll say that the **order** of a node is the number of children it has.
- Concern: If trees can be cut, a tree of order  $k$  might have many fewer than  $2^k$  nodes.

# The Problem



Number of nodes:  $\Theta(k^2)$

Number of trees:  $\Theta(n^{1/2})$

# The Problem

- **Recall:** The amortized cost of an *extract-min* is  $O(M(n))$ , where  $M(n)$  is the maximum order of a tree in the heap.
- With true binomial trees, this is  $O(\log n)$ .
- With our “damaged” binomial trees, this can be  $\Theta(n^{1/2})$ .
- We've lost our runtime bounds!

# The Problem

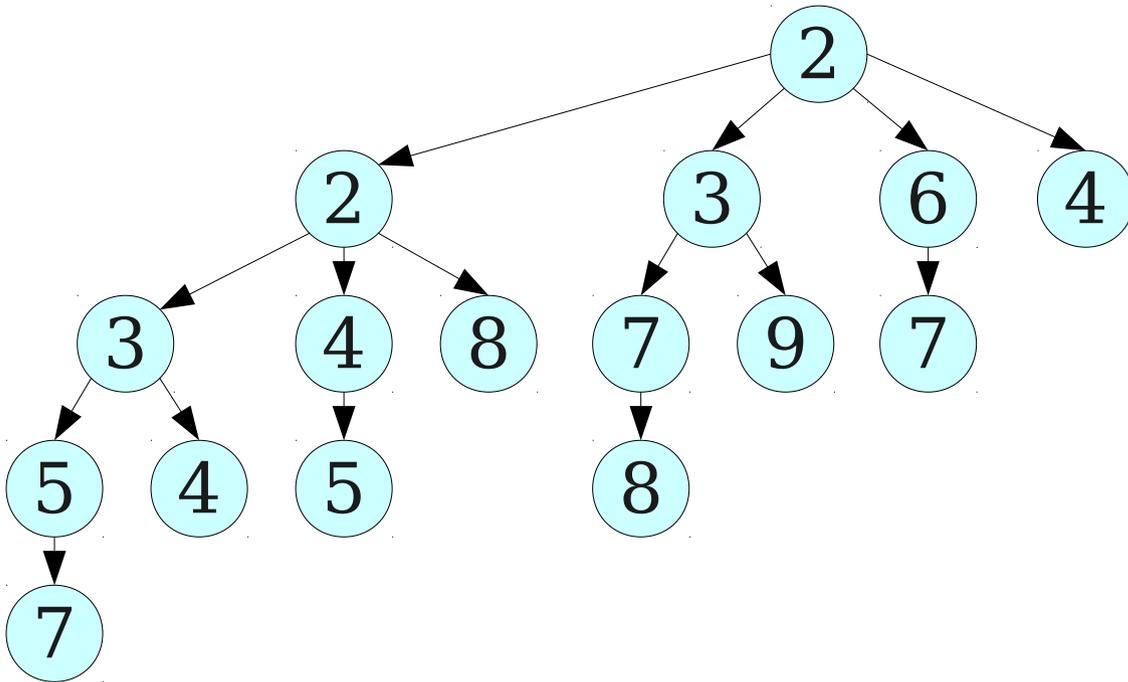
- This problem arises because we have lost one of the guarantees of binomial trees:  
    A binomial tree of order  $k$  has  $2^k$  nodes.
- When we cut low-hanging trees, the root node won't learn that these trees are missing.
- However, communicating this information up from the leaves to the root might take time  $O(\log n)$ !

# The Tradeoff

- If we don't impose any structural constraints on our trees, then trees of large order may have too few nodes.
  - Leads to  $M(n)$  getting too high, wrecking our runtime bounds for ***extract-min***.
- If we impose too many structural constraints on our trees, then we have to spend too much time fixing up trees.
  - Leads to ***decrease-key*** taking too long.

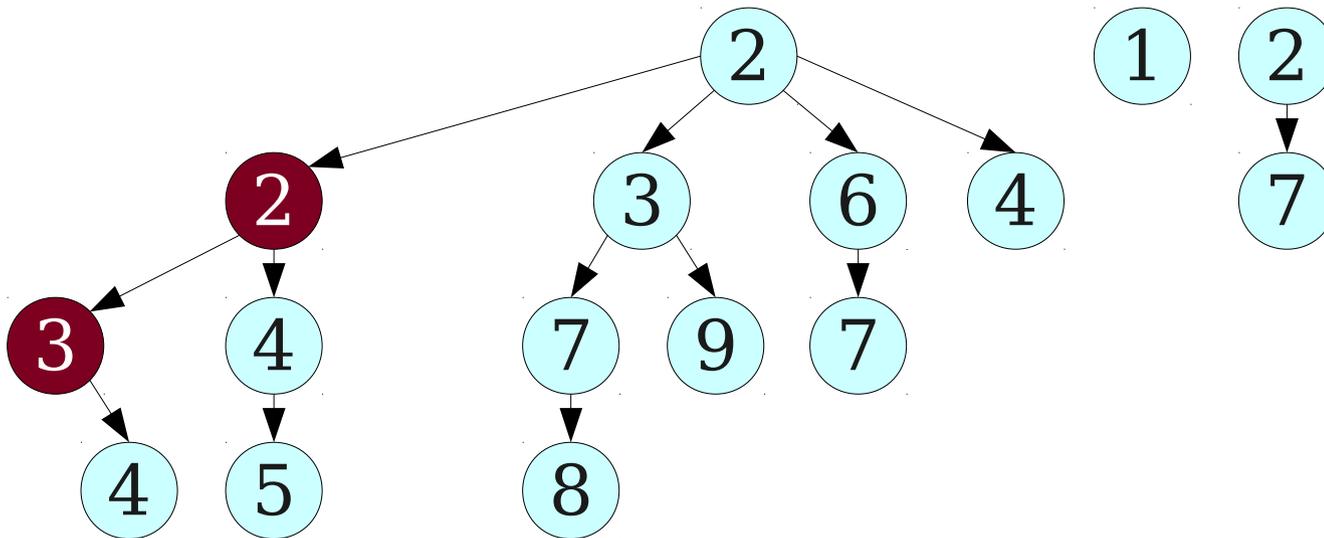
# The Compromise

- Every non-root node is allowed to lose at most one child.
- If a non-root node loses two children, we cut it from its parent. (This might trigger more cuts.)
- We will *mark* nodes in the heap that have lost children to keep track of this fact.



# The Compromise

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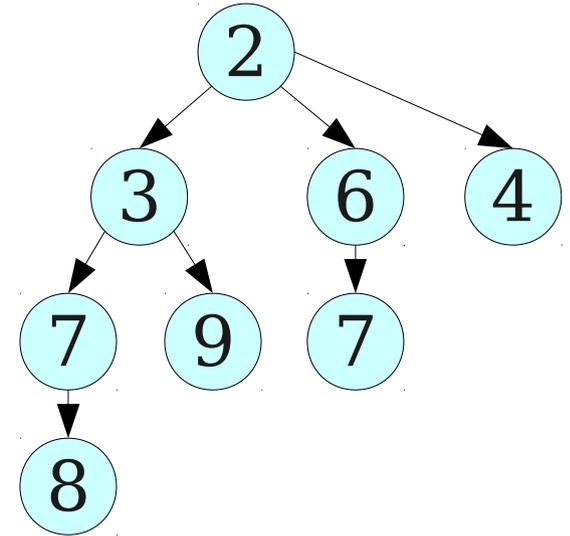


# The Compromise

- To cut node  $v$  from its parent  $p$ :
  - Unmark  $v$ .
  - Cut  $v$  from  $p$ .
  - If  $p$  is not already marked and is not the root of a tree, mark it.
  - If  $p$  was already marked, recursively cut  $p$  from its parent.

# The Compromise

- If we do a few **decrease-keys**, then the tree won't lose “too many” nodes.
- If we do many **decrease-keys**, the information propagates to the root.



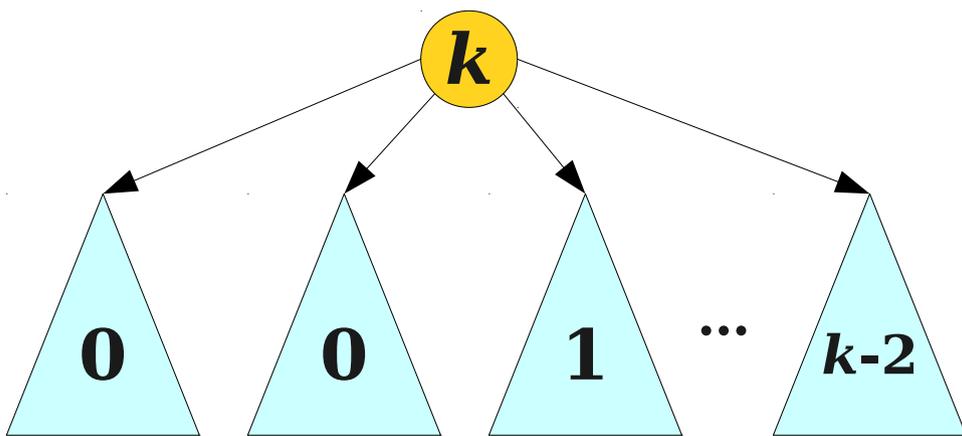
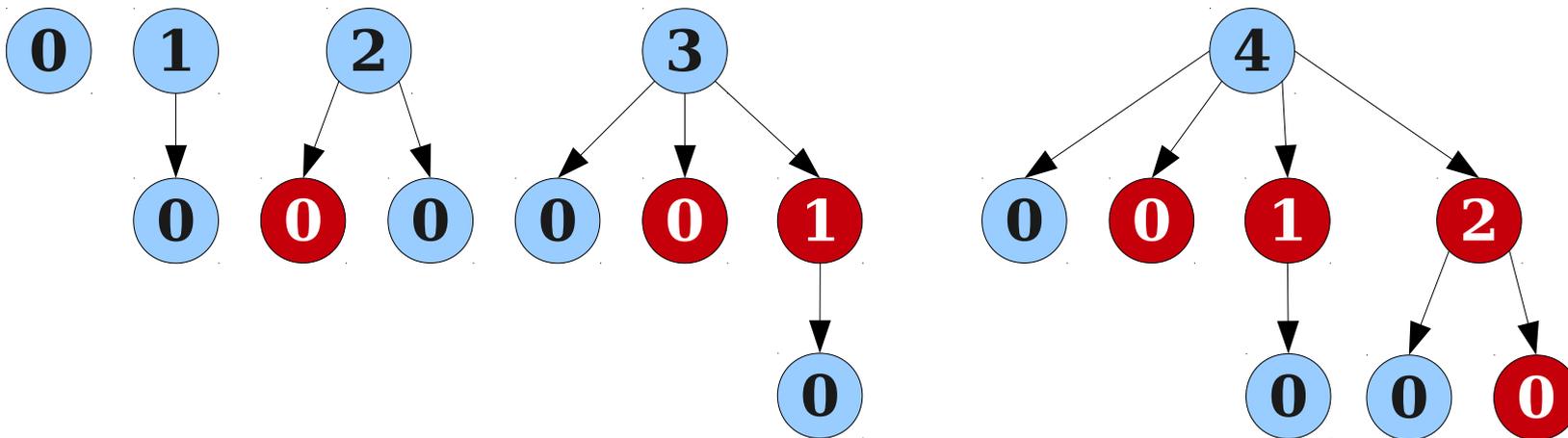
# Assessing the Impact

- The amortized cost of an *extract-min* is  $O(M(n))$ , where  $M(n)$  is the maximum possible order of a tree.
- This used to be  $O(\log n)$  because our trees had exponentially many nodes in them.
- What is it now?

# Two Extremes

- If we never do any *decrease-keys*, then the trees in our data structure are all binomial trees.
- Each tree of order  $k$  has  $2^k$  nodes in it, the maximum possible order is  $O(\log n)$ .
- On the other hand, suppose that all trees in the binomial heap have lost the maximum possible number of nodes.
- In that case, how many nodes will each tree have?

# Maximally-Damaged Trees



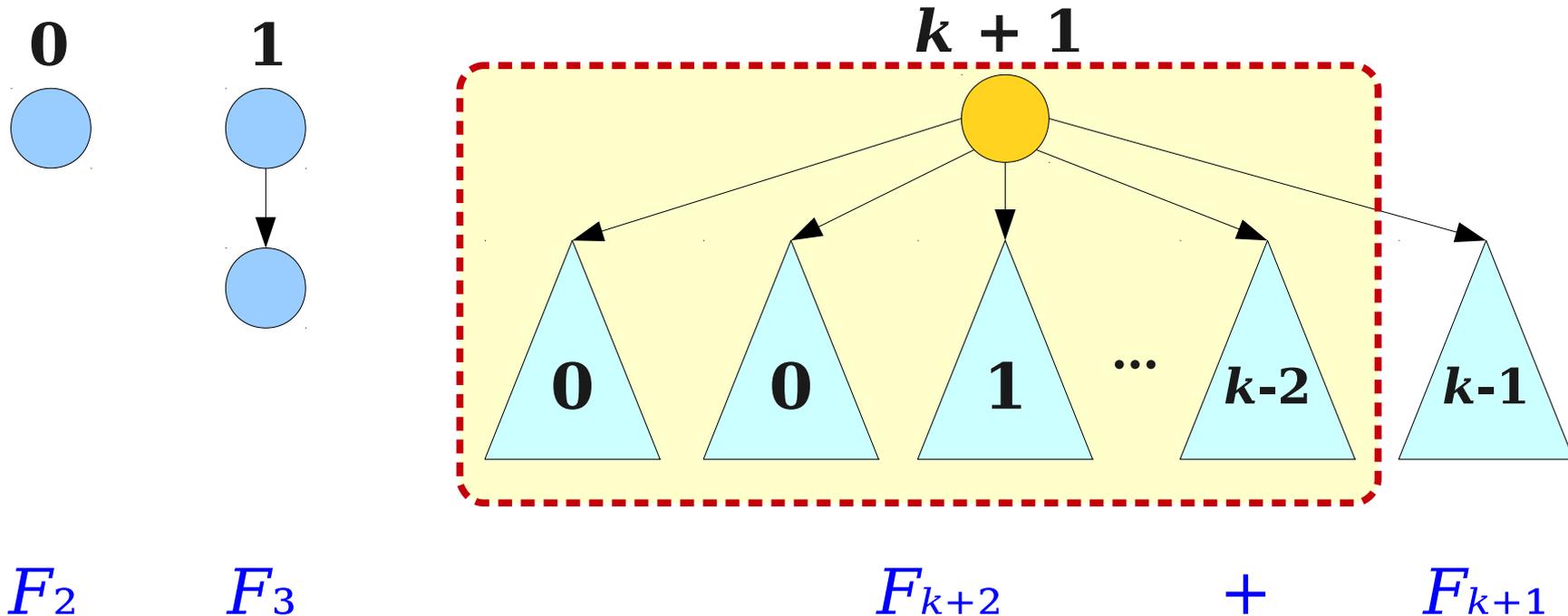
A *maximally-damaged tree of order  $k$*  is a node whose children are maximally-damaged trees of orders

$0, 0, 1, 2, 3, \dots, k - 2.$



# Maximally-Damaged Trees

- **Theorem:** The number of nodes in a maximally-damaged tree of order  $k$  is  $F_{k+2}$ .
- **Proof:** Induction.



# $\varphi$ -bonacci Numbers

- **Fact:** For  $n \geq 2$ , we have  $F_n \geq \varphi^{n-2}$ , where  $\varphi$  is the golden ratio:

$$\varphi \approx 1.61803398875\dots$$

- **Claim:** In our modified data structure, we have  $M(n) = O(\log n)$ .
- **Proof:** In a tree of order  $k$ , there are at least  $F_{k+2} \geq \varphi^k$  nodes. Therefore, the maximum order of a tree in our data structure is  $\log_{\varphi} n = O(\log n)$ . ■

# Fibonacci Heaps

- A **Fibonacci heap** is a lazy binomial heap where *decrease-key* is implemented using the earlier cutting-and-marking scheme.
- Operation runtimes:
  - *enqueue*:  $O(1)$
  - *meld*:  $O(1)$
  - *find-min*:  $O(1)$
  - *extract-min*:  $O(\log n)$  amortized
  - *decrease-key*: Up next!

# Analyzing *decrease-key*

- In the best case, *decrease-key* takes time  $O(1)$  when no cuts are made.
- In the worst case, *decrease-key* takes time  $O(C)$ , where  $C$  is the number of cuts made.
- What is the *amortized* cost of a *decrease-key*?

# Refresher: Our Choice of $\Phi$

- In our amortized analysis of lazy binomial heaps, we set  $\Phi$  to be the number of trees in the heap.
- With this choice of  $\Phi$ , we obtained these amortized time bounds:
  - ***enqueue***:  $O(1)$
  - ***meld***:  $O(1)$
  - ***find-min***:  $O(1)$
  - ***extract-min***:  $O(\log n)$

# Rethinking our Potential

- Intuitively, a cascading cut only occurs if we have a long chain of marked nodes.
- Those nodes were only marked because of previous *decrease-key* operations.
- **Idea:** Backcharge the work required to do the cascading cut to each preceding *decrease-key* that contributed to it.
- Specifically, change  $\Phi$  as follows:  
$$\Phi = \#trees + \#marked$$
- **Note:** Since only *decrease-key* interacts with marked nodes, our amortized analysis of all previous operations is still the same.

# The (New) Amortized Cost

- Using our new  $\Phi$ , a *decrease-key* makes  $C$  cuts, it
  - Marks one new node (+1),
  - Unmarks  $C$  nodes ( $-C$ ), and
  - Adds  $C$  tree to the root list ( $+C$ ).

- Amortized cost is

$$\begin{aligned} & \Theta(C) + O(1) \cdot \Delta\Phi \\ &= \Theta(C) + O(1) \cdot (1 - C + C) \\ &= \Theta(C) + O(1) \cdot 1 \\ &= \Theta(C) + O(1) \\ &= \Theta(C) \end{aligned}$$

- Hmm... that didn't work.

# The Trick

- Each ***decrease-key*** makes extra work for *two* future operations:
  - Future ***extract-mins*** that now have more trees to coalesce, and
  - Future ***decrease-keys*** that might have to do cascading cuts.
- We can make this explicit in our potential function:

$$\Phi = \#trees + 2 \cdot \#marked$$

# The (Final) Amortized Cost

- Using our new  $\Phi$ , a *decrease-key* makes  $C$  cuts, it
  - Marks one new node (+2),
  - Unmarks  $C$  nodes ( $-2C$ ), and
  - Adds  $C$  tree to the root list (+ $C$ ).

- Amortized cost is

$$\begin{aligned} & \Theta(C) + O(1) \cdot \Delta\Phi \\ &= \Theta(C) + O(1) \cdot (2 - 2C + C) \\ &= \Theta(C) + O(1) \cdot (2 - C) \\ &= \Theta(C) - O(C) \\ &= \Theta(1) \end{aligned}$$

- We now have amortized  $O(1)$  *decrease-key*!

# The Story So Far

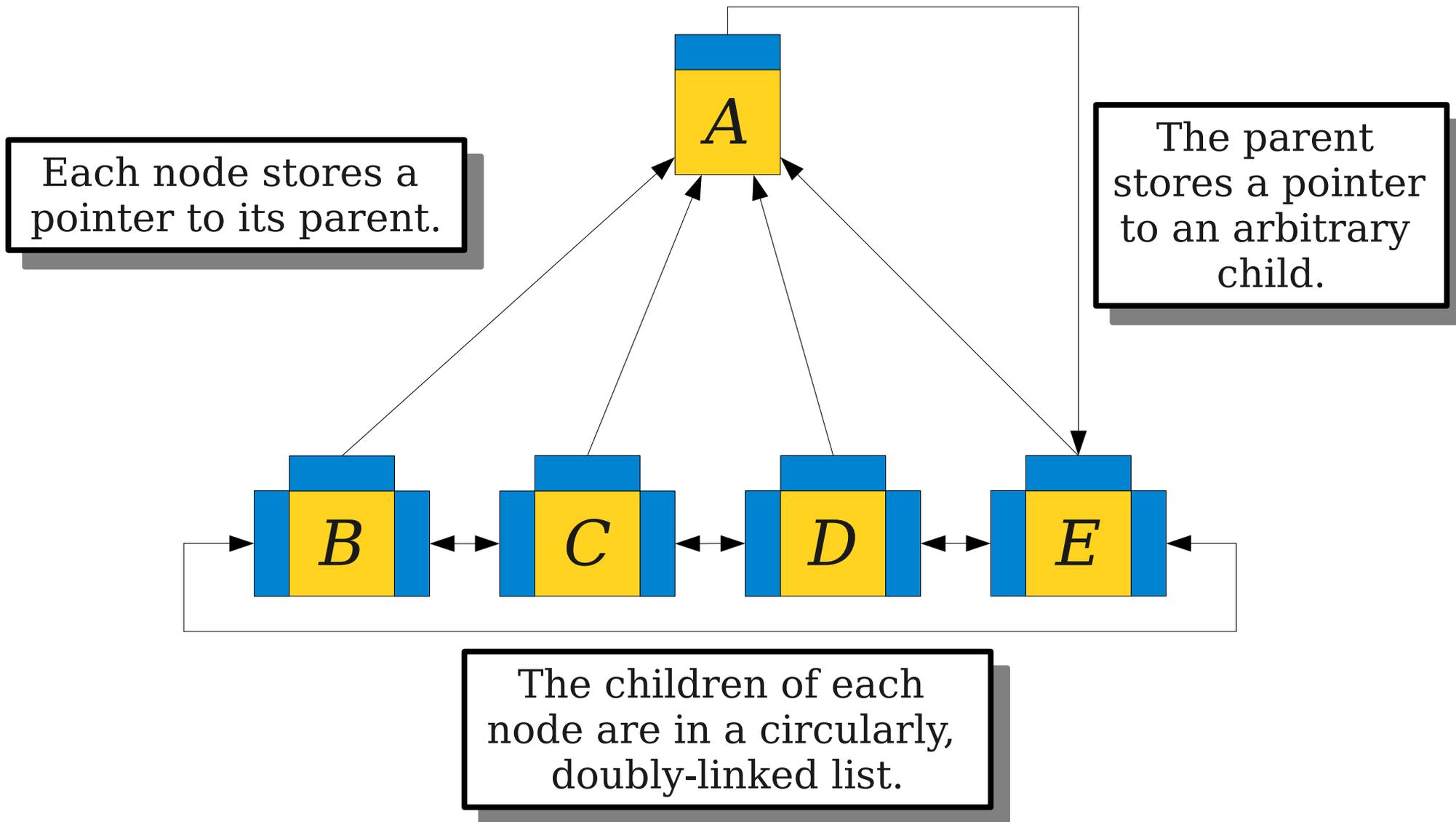
- The Fibonacci heap has the following amortized time bounds:
  - *enqueue*:  $O(1)$
  - *find-min*:  $O(1)$
  - *meld*:  $O(1)$
  - *decrease-key*:  $O(1)$  amortized
  - *extract-min*:  $O(\log n)$  amortized
- This is about as good as it gets!

# The Catch: Representation Issues

# Representing Trees

- The trees in a Fibonacci heap must be able to do the following:
  - During a merge: Add one tree as a child of the root of another tree.
  - During a cut: Cut a node from its parent in time  $O(1)$ .
- **Claim:** This is trickier than it looks.

# The Solution



# Awful Linked Lists

- Trees are stored as follows:
  - Each node stores a pointer to *some* child.
  - Each node stores a pointer to its parent.
  - Each node is in a circularly-linked list of its siblings.
- Awful, but the following possible are now possible in time  $O(1)$ :
  - Cut a node from its parent.
  - Add another child node to a node.
- This is the main reason Fibonacci heaps are so complex.

# Fibonacci Heap Nodes

- Each node in a Fibonacci heap stores
  - A pointer to its parent.
  - A pointer to the next sibling.
  - A pointer to the previous sibling.
  - A pointer to an arbitrary child.
  - A bit for whether it's marked.
  - Its order.
  - Its key.
  - Its element.

# In Practice

- In practice, Fibonacci heaps are *slower* than other heaps.
- Why?
  - Huge memory requirements per node.
  - High constant factors on all operations.

# In Theory

- That said, Fibonacci heaps are worth knowing about for several reasons:
  - Clever use of a two-tiered potential function shows up in lots of data structures.
  - Implementation of *decrease-key* forms the basis for many other advanced priority queues.
  - Gives the theoretically optimal comparison-based implementation of Prim's and Dijkstra's algorithms.

# Summary

- ***decrease-key*** is a useful operation in many graph algorithms.
- Implement ***decrease-key*** by cutting a node from its parent and hoisting it up to the root list.
- To make sure trees of high order have lots of nodes, add a marking scheme and cut nodes that lose two or more children.
- Represent the data structure using Awful Linked Lists.
- Can prove that the number of trees is  $O(\log n)$  by most maximally damaged trees in the heap.

# Next Time

- **Splay Trees**
  - Amortized-efficient balanced trees.
- **Static Optimality**
  - Is there a single best BST for a set of data?
- **Dynamic Optimality**
  - Is there a single best BST for a set of data if that BST can change over time?